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Spectral Stability under L^p-Perturbation of the Second-Order Coefficients*

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INTRODUCTION

Let H be a second order elliptic operator acting on a domain $\Omega\subset\mathbf{R}^N$. There has been a lot of work on the stability of the spectrum and the resolvent of H under various sorts of perturbations. Perturbations of the 0th-order term are well studied. See for example [7]. There are also several results on perturbations of the higher-order terms. Classical perturbation theory can be applied in the case of uniform or asymptotic perturbations (see [5]). P. Deift [4] has obtained results for measurable perturbations in the context of scattering theory. More recent results ([3]) deal with boundary perturbations. In this paper we study L^p-perturbations of the second-order terms.

We work on a bounded Euclidean domain $\Omega\subset\mathbf{R}^N$ which we assume initially to have a C^1 boundary. The operators involved are uniformly elliptic with real measurable coefficients and satisfy Dirichlet boundary conditions. We first prove eigenvalue stability, but our main aim is the stability of the resolvent in trace classes.

If one is only interested in the fact of convergence rather than controlling the rate, then a simple approach based upon the monotone or dominated convergence theorem exists. However, quantitative control by such methods is not possible.

We emphasize the fact that we deal with operators with measurable coefficients. If one restricts attention to the smooth coefficient case, then standard methods of perturbation theory involve assuming uniform bounds on the first derivatives of the coefficients. Such bounds are not needed in our approach and are not always available in applications: apart from being more general, the measurable coefficients hypothesis is necessary for the study of the heat transport in a body with randomly distributed impurities. In particular we study the asymptotic form of the heat diffusion

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in a uniform medium containing a large number of impurities each of small volume; see Proposition 16.

We make use of a formula of Deift [4] that was used in the late seventies in scattering theory. Our main result is Theorem 9, where we establish Lipschitz continuity of the resolvent in trace ideals as the coefficients vary in L^p spaces. It turns out that the proof depends heavily on L^p bounds on the gradients of the eigenfunctions of the operators involved. Such bounds are available from [6] and that is where boundary regularity is needed. We show that the C^1 condition on the boundary can be weakened to a Lipschitz condition. In the last part of the paper we apply our main results to examine how the heat transport in a body is affected by small impurities. Finally, we identify the limit operators that describe the heat diffusion when the conductivity of the impurities becomes infinite or zero and we study the latter.

In a paper that will appear soon, we shall generalize these results in three directions: we shall be working on Riemannian manifolds, with weighted Laplace-Beltrami operators and Neumann and mixed boundary conditions.

The Technical Setting

Let $\Omega \subset \mathbf{R}^N$ be bounded with a C^1 boundary. For a positive definite matrix $a = \{a_{ij}(x)\}$ depending measurably upon $x \in \Omega$ we denote by H_a the self-adjoint operator on $L^2(\Omega)$ given formally by

$$H_a = -\sum_i \frac{\partial}{\partial x_i} \left\{ a_{ij}(x) \frac{\partial}{\partial x_j} \right\}$$

subject to Dirichlet boundary conditions.

We assume from now on that H_a is uniformly elliptic, so that defining

$$d\colon W_0^{1,2}(\Omega) \to \bigoplus_{r=1}^{N} L^2(\Omega), \qquad df = \nabla f,$$

we have

$$H_a = d^* a d. \tag{1}$$

Equivalently, $f \in \mathrm{Dom}(H_a)$ if and only if $f \in W_0^{1,2}(\Omega)$ and there exists $h \in L^2(\Omega)$ such that

$$\int_\Omega a \nabla f \cdot \nabla \phi \, dx = \int_\Omega h \phi \, dx \qquad \text{all} \quad \phi \in W_0^{1,2}(\Omega) \; (\text{or } C_c^\infty(\Omega))$$

in which case we define $H_a f = h$.

It is well known that such an operator has a discrete spectrum $0<\lambda_{1,a}<\lambda_{2,a}\leqslant\lambda_{3,a}\leqslant\cdots$ and that there exists a complete orthonormal set of eigenfunctions $\{\phi_{n,a}\}$, $H_a\phi_{n,a}=\lambda_{n,a}\phi_{n,a}$.

We also set $R_a=(H_a+1)^{-1}$. Finally, for any symmetric matrix w defined on Ω and bounded away from $-\infty$ and $+\infty$ we denote by Q_w the quadratic form

$$Q_w(f,g)=\int_\Omega w\nabla f\cdot\nabla g\ dx$$

with $\mathrm{Dom}(Q_w)=W_0^{1,2}(\Omega)$ so that, in particular, Q_a is the quadratic form associated to H_a.

We quote from [6] the following regularity result, which will turn out to be crucial for our results. See also [8, p 90] for a proof of the fact that the C^1 condition is sufficient.

THEOREM 1. *Let $\Omega\subset\mathbf{R}^N$ be bounded with a C^1 boundary and let H_a be uniformly elliptic on $L^2(\Omega)$. There exists a Q, $2<Q<\infty$, depending only on the ellipticity constants of H_a such that for any $2\leqslant q<Q$ the equation*

$$H_a u=f$$

with $f\in L^{Nq/(N+q)}$ has a unique solution $u\in W_0^{1,q}$ and

$$\|u\|_{W^{1,q}}\leqslant c_q\|f\|_{Nq/(N+q)}. \tag{2}$$

In fact, Q depends only on the ratio of the ellipticity constants of H_a. When the ratio is very large Q is close to 2, while for a ratio very close to one Q is close to $+\infty$.

EIGENVALUE STABILITY

In this section we show how a simple application of the min-max principle yields stability estimates for the eigenvalues of a uniformly elliptic operator when the coefficients vary in L^p spaces. We shall need the following

LEMMA 2. *Let H be a positive self-adjoint operator with compact resolvent and let Q be the corresponding quadratic form. If $\lambda_1\leqslant\lambda_2\leqslant\lambda_3\leqslant\cdots$ are the eigenvalues of H, then for any $k\in\mathbf{N}$*

$$\sum_{n=1}^{k}\lambda_n=\inf\sum_{n=1}^{k}Q(f_n) \tag{3}$$

where the infimum is taken over all orthonormal sets $\{f_1, ..., f_k\} \subset$ $\mathrm{Dom}(\mathrm{H}^{1/2})$.

Proof. It is clear that the left-hand side in (3) is larger than the right-hand side. Let $\{\phi_n\}_{n=1}^{\infty}$ be a complete orthonormal system of eigenfunctions of H with $H\phi_n = \lambda_n \phi_n$. Let $\{f_n\}_{n=1}^{k}$ be an orthonormal subset of $\mathrm{Dom}(H^{1/2})$ and write

$$f_n = \sum_{m=1}^{\infty} a_{nm}\phi_m, \qquad n = 1, ..., k.$$

Then

$$\sum_{n=1}^{k} Q(f_n) = \sum_{n=1}^{k} \sum_{m=1}^{x} |a_{nm}|^2 \lambda_m$$

and the result follows since the orthonormality of $\{f_n\}$ implies

$$\sum_{n=1}^{k} |a_{nm}|^2 \leqslant 1. \quad \blacksquare$$

Let H_a, H_b be uniformly elliptic. By c_a we shall denote any constant that depends only on the ellipticity constants of H_a. More generally, if w is any scalar- or matrix-valued function on Ω, a subindex w shall indicate dependence only on $\|w\|_\infty$ and $\|w^{-1}\|_\infty$.

We have the following

THEOREM 3. *There exists a number* $P = P_{a,b}$, $2 < P < +\infty$, *such that for all* $p > P$ *and all* $k \in \mathbf{N}$ *we have*

$$\left|\sum_{n=1}^{k} \lambda_{n,b} - \lambda_{n,a}\right| \leqslant c_{p,a,b} \|b-a\|_p \, k^{1+2/N+1/P}. \tag{4}$$

Proof. From the lemma we have

$$\sum_{n=1}^{k} Q_a(\phi_{n,a}) = \sum_{n=1}^{k} \lambda_{n,a} \leqslant \sum_{n=1}^{k} Q_a(\phi_{n,b}),$$

$$\sum_{n=1}^{k} Q_b(\phi_{n,b}) = \sum_{n=1}^{k} \lambda_{n,b} \leqslant \sum_{n=1}^{k} Q_b(\phi_{n,a}).$$

Subtracting we get

$$\sum_{n=1}^{k} Q_{b-a}(\phi_{n,b}) \leqslant \sum_{n=1}^{k} \lambda_{n,b} - \lambda_{n,a} \leqslant \sum_{n=1}^{k} Q_{b-a}(\phi_{n,a}) \tag{5}$$

and hence

$$\left|\sum_{n=1}^{k} \lambda_{n,b}-\lambda_{n,a}\right| \leqslant \max\left\{\left|\sum_{n=1}^{k} Q_{b-a}(\phi_{n,a})\right| \cdot \left|\sum_{n=1}^{k} Q_{b-a}(\phi_{n,b})\right|\right\}$$

$$\leqslant \sum_{n=1}^{k} \{|Q_{b-a}(\phi_{n,a})|+|Q_{b-a}(\phi_{n,b})|\}. \tag{6}$$

Denoting by ϕ_n either $\phi_{n,a}$ or $\phi_{n,b}$ we have for any $1<p<\infty$

$$|Q_{b-a}(\phi_n)| = \left|\int_\Omega (b-a)\,\nabla\phi_n\cdot\nabla\phi_n\;dx\right|$$

$$\leqslant \|b-a\|_p\,\|\nabla\phi_n\|_{2p'}^2 \tag{7}$$

where $p'=p/(p-1)$.

If the constants Q_a, Q_b are as in Theorem 1 and we set $Q_0=\min\{Q_a, Q_b\}$, then (2) implies

$$\|d\phi_n\|_q \leqslant c_q \lambda_n \|\phi_n\|_{Nq/(N+q)}, \qquad \text{all} \quad q<Q_0. \tag{8}$$

Now, it is a standard ultracontractivity result (see [2, p. 63]) that for any $2\leqslant s\leqslant\infty$ the semigroup $e^{-H_a t}$ maps L^2 into L^s and in fact

$$\|e^{-H_a t} f\|_s \leqslant c t^{-N(s-2)/4s}, \qquad \text{all} \quad f\in L^2. \tag{9}$$

In particular

$$\|\phi_n\|_s \leqslant c e^{\lambda_n t} t^{-N(s-2)/4s}$$

so that optimising over $t>0$ we conclude that

$$\|\phi_n\|_s \leqslant c\lambda_n^{N(s-2)/4s}, \qquad 2\leqslant s\leqslant\infty.$$

It follows from (8) that

$$\|d\phi_n\|_q \leqslant c_q \lambda_n^{(N+2-2N/q)/4}, \qquad \text{all} \quad q<Q_0 \tag{10}$$

and so

$$\|d\phi_n\|_{2p'} \leqslant c_p \lambda_n^{(1/2)+(N/4P)}, \qquad \text{all} \quad p>P \tag{11}$$

where P is such that $2P'=Q_0$.

Since $\lambda_n \leqslant c_{a,b} n^{2/N}$, (11) implies

$$\|d\phi_n\|_{2p'} \leqslant c_{p,a,b} n^{(1/N)+(1/2P)}, \qquad \text{all} \quad p>P.$$

It follows that

$$\left|\sum_{n=1}^{k} \lambda_{n,b} - \lambda_{n,a}\right| \leqslant c_{p,a,b} \|b-a\|_p \sum_{n=1}^{k} n^{(2/N)+(1/p)}, \qquad \text{all} \quad n \in \mathbf{N}$$

which proves the theorem since

$$\sum_{n=1}^{k} n^{(2/N)+(1/p)} \sim k^{1+(2/N)+(1/p)} \qquad \text{as} \quad k \to \infty. \quad \blacksquare$$

Here and below, the symbol $\sim$ indicates that either the ratio of two quantities converges to one, or that it is bounded away from zero and infinity. The meaning intended will be clear from the context.

Standard perturbation theory arguments can also be used to estimate the differences $\lambda_{n,b} - \lambda_{n,a}$, and one can prove that there exists $\hat{P} = \hat{P}_{a,b}$ such that

$$|\lambda_{n,b} - \lambda_{n,a}| \leqslant c_{p,a,b} \|b-a\|_p \, n^{(2/N)+(1/P)}, \qquad \text{all} \quad p > P.$$

This, of course, implies (4) with P replaced by $\hat{P}$.

This theorem says a lot about the stability of eigenvalues, but there are further questions that one can pose. In our main theorem we establish stability of the resolvents in trace classes. Such a result not only implies eigenvalue stability, it also yields stability of the spectral projections and, hence, of eigenspaces and eigenfunctions.

Preliminary Results

In order to compare the resolvents of two operators we shall need the following well known result, a proof of which can be found in [4].

Proposition 4. *Let* $T\colon \mathscr{H}_1 \to \mathscr{H}_2$ *be closed and densely defined. Let* $H = T^*T$, $F = TT^*$. *Then* $Sp(H) \cup \{0\} = Sp(F) \cup \{0\}$ *and if* $\mu \notin Sp(H)$, $\mu \neq 0$, *then*

$$\text{(i)} \qquad \mu(H+\mu)^{-1} + T^*(F+\mu)^{-1}T = 1$$

$$\text{(ii)} \qquad (F+\mu)^{-1}T = T(H+\mu)^{-1}$$

and dually

$$\text{(i)}' \qquad \mu(F+\mu)^{-1} + T(H+\mu)^{-1}T^* = 1$$

$$\text{(ii)}' \qquad (H+\mu)^{-1}T^* = T^*(F+\mu)^{-1}.$$

Moreover, $\mu \neq 0$ is an eigenvalue of H if and only if it is an eigenvalue of F and if so the two multiplicities coincide.

Note that not only we may have $0 \in \mathrm{Sp}(F)$ while $0 \notin \mathrm{Sp}(H)$, it may also be the case that $\mathrm{Ker}(F)$ is infinite dimensional. This will turn out to be an important feature of this problem.

Equation (12) below is an analogue of the resolvent formula

$$(-\Delta + V_1)^{-1} - (-\Delta + V_2)^{-1} = -(-\Delta + V_1)^{-1} (V_1 - V_2)(-\Delta + V_2)^{-1}$$

which is useful for the study of 0th-order perturbations. We set

$$r_a = a^{1/2} d \colon L^2(\Omega) \to \bigoplus L^2(\Omega),$$

so that $H_a = r_a^* r_a$ by (1), and define

$$F_a = r_a r_a^* \colon \bigoplus L^2(\Omega) \to \bigoplus L^2(\Omega).$$

Lemma 5. *There exist partial isometries U_a, U_b: $L^2(\Omega) \to \bigoplus L^2(\Omega)$ such that*

$$R_b - R_a = U_b^* G(F_b)\, b^{1/2}(b^{-1} - a^{-1})\, a^{1/2} G(F_a)\, U_a \tag{12}$$

where $G(t) = t^{1/2}/(t+1)$.

Proof. From Proposition 4 and we have

$$\begin{aligned} R_b - R_a &= -r_b^*(F_b + 1)^{-1}\, r_b + r_a^*(F_a + 1)^{-1}\, r_a \\ &= -d^*[(dd^* + b^{-1})^{-1} - (dd^* + a^{-1})^{-1}]\, d \\ &= d^*(dd^* + b^{-1})^{-1}\, (b^{-1} - a^{-1})(dd^* + a^{-1})^{-1}\, d \\ &= r_b^*(F_b + 1)^{-1}\, b^{1/2}(b^{-1} - a^{-1})\, a^{1/2}(F_a + 1)^{-1}\, r_a. \end{aligned}$$

Using polar decomposition we can write $r_a = |r_a^*|\, U_a$ and observe that $|r_a^*| = F_a^{\ 1/2}$. ∎

As we shall see later, the fact that $G(0) = 0$ is crucial, in view of the fact that the kernels of F_a and F_b are infinite dimensional.

For $1 \leqslant r < \infty$ let $\mathscr{C}^r$ denote the trace ideal

$$\mathscr{C}^r = \{A \in \mathscr{B}(L^2(\Omega)) \mid \operatorname{tr} |A|^r < \infty\}$$

normed by

$$\|A\|_{\mathscr{C}^r} = (\operatorname{tr} |A|^r)^{1/r}$$

or, equivalently,

$$\|A\|_{\mathscr{C}^r} = \left\{\sum_n \mu_n(A)^r\right\}^{1/r} \tag{13}$$

where $\{\mu_n(A)\}$ are the singular values of A.

Lemma 5 yields at once the following

COROLLARY 6. *For any $r > N/2$ there exists a constant $c_{r,a,b}$ such that*

$$\|R_b - R_a\|_{\mathscr{C}^r} \leqslant c_{r,a,b} \|b-a\|_\infty. \tag{14}$$

Proof. Using Holder's inequality for trace ideals, we have for any $1 \leqslant r \leqslant \infty$,

$$\|R_b - R_a\|_r \leqslant \|G(F_b)\|_{2r} \, \|b^{1/2}(b^{-1} - a^{-1})a^{1/2}\|_\infty \, \|G(F_a)\|_{2r}$$

Since $G(0) = 0$, Proposition 4 and (13) imply that

$$\|G(F_a)\|_{2r} = \left(\sum_n G(\lambda_{n,a})^{2r}\right)^{1/2r}$$

where $\{\lambda_{n,a}\}$ are the eigenvalues of H_a. Since $\lambda_{n,a} \sim n^{2/N}$ as $n \to \infty$, we conclude that $\|G(F_a)\|_{2r} < \infty$ if and only if $r > N$, and the same holds for $\|G(F_b)\|_{2r}$. (14) follows if we note that

$$\|b^{1/2}(b^{-1} - a^{-1})a^{1}/2\|_\infty < c_{a,b}\|b-a\|_\infty. \quad \blacksquare$$

What we are interested in is to obtain estimates of this type but with the L^∞ norm being replaced by some other L^p norm, with p being as small as possible. See Proposition 16 for a physical interpretation of this requirement.

A NEGATIVE RESULT

Before continuing, let us see why a certain approach which seems, probably, more natural and efficient does not actually work in the problem we are interested in.

A theorem of Birman and Solomjak [1] asserts that the eigenvalues $\{\lambda_n\}$ of H_a satisfy

$$\lambda_n \sim c_N \left\{\int_\Omega (\det a^{-1})^{1/2}\right\}^{-2/N} n^{2/N} \qquad \text{as} \quad n \to \infty \tag{15}$$

so that, in particular,

$$\lambda_n^{-1} \leqslant c_N \|a^{-1}\|_{N/2}\, n^{2/N}.$$

Hence, for any $q > N/2$,

$$\begin{aligned}\|H_a^{-1}\|_q &= \Big(\sum_n \lambda_n^{-q}\Big)^{1/q} \\ &\leqslant c_N \|a^{-1}\|_{N/2} \Big(\sum_n n^{-2q/N}\Big)^{1/q} \\ &= c_{N,\,q} \|a^{-1}\|_{N/2}.\end{aligned} \tag{16}$$

If the map $a^{-1} \mapsto H_a^{-1} = (d^*ad)^{-1}$ were linear, which seems as though it could be in view of what the formula

$$(H_b + \mu)^{-1} - (H_a + \mu)^{-1} = d^*(dd^* + \mu b^{-1})^{-1} (b^{-1} - a^{-1})(dd^* + \mu a^{-1})^{-1} d$$

looks like as $\mu \to 0$, then we would conclude that

$$\|H_b^{-1} - H_a^{-1}\|_q \leqslant c_q \|b^{-1} - a^{-1}\|_{N/2}, \qquad \text{all} \quad q > N/2,$$

a much better result than Theorem 9 in that the range of the parameters is better and no regularity of the boundary is needed. However, we have the following

Lemma 7. *The map $a^{-1} \mapsto H_a^{-1}$ is not linear.*

Proof. For $\mu > 0$ we have

$$(H_b + \mu)^{-1} - (H_a + \mu)^{-1} = U_b^* G_\mu(F_b)\, b^{1/2} (b^{-1} - a^{-1})\, a^{1/2} G_\mu(F_a) U_a$$

where $G_\mu(t) = t^{1/2}/(t + \mu)$.

Defining $P_a \colon \bigoplus L^2 \to \bigoplus L^2$ by

$$P_a f = \begin{cases} 0 & \text{if} \quad f \in \operatorname{Ker} F_a \\ \lambda_n^{-1/2} f & \text{if} \quad f \in \operatorname{Ker}(F_a - \lambda_n) \end{cases}$$

we can easily check that

$$\|G_\mu(F_a) - P_a\| \to 0 \qquad \text{as} \quad \mu \to 0$$

and conclude that

$$H_b^{-1} - H_a^{-1} = T_b^* (b^{-1} - a^{-1})\, T_a \tag{17}$$

where $T_a = a^{1/2} P_a U_a$. Replacing by kb, $k \in \mathbf{N}$, and observing that $T_b = T_{kb}$ we have

$$\begin{aligned} H_{kb}^{-1} - H_a^{-1} &= T_{kb}^*(k^{-1}b^{-1} - a^{-1})\,T_a \\ &= k^{-1}T_b^* b^{-1} T_a - T_b^* a^{-1} T_a \end{aligned}$$

and letting $k \to \infty$

$$H_a^{-1} = T_b^* a^{-1} T_a = T a^{-1} T_a$$

where $T =: T_l$. Now, suppose that T_a does not depend on a. Then

$$\begin{array}{ll} T_a T_a^* & \text{is independent of } a \\ \Rightarrow a^{1/2} P_a^2 a^{1/2} & \text{is independent of } a \\ \Rightarrow P_a^2 = a^{-1/2}\, Q a^{-1/2}, & \text{some } Q \\ \Rightarrow a^{-1/2}\, Q a^{-1/2} f = 0, & \text{all } f \in \operatorname{Ker} F_a \\ \Rightarrow Qg = 0, & \text{all } g \text{ such that } d^*ag = 0 \end{array}$$

which is a contradiction. ∎

Resolvent Stability

To prove our main result, Theorem 9, we need some trace estimates for operators of the form $\tilde{V}\tilde{G}(F_a)$ acting on $\bigoplus L^2(\Omega)$.

Let $2 < p < \infty$ be a parameter. We think of $\operatorname{Sp}(F_a)$ as a measure space with each eigenvalue $\lambda_{n,a}$ carrying a weight $\lambda_{n,a}^{N/p} \times m(\lambda_{n,a})$, where m stands for the multiplicity of the eigenvalue, while to 0 we asign weight $+\infty$ reflecting the fact that $\operatorname{Ker}(F_a)$ is infinite dimensional. Associated to this discrete measure space are the corresponding l^q spaces, $1 \leqslant q \leqslant \infty$, defined by

$$l^q = \left\{ \tilde{G}\colon \operatorname{Sp}(F_a) \to \mathbf{R} \mid \tilde{G}(0) = 0, \sum_n |\tilde{G}(\lambda_{n,a})|^q \,\lambda_{n,a}^{N/p} < \infty \right\}, \qquad 1 \leqslant q < \infty$$

and

$$l^\infty = \{ \tilde{G}\colon \operatorname{Sp}(F_a) \to \mathbf{R} \mid \sup_n |\tilde{G}(\lambda_{n,a})| < \infty \}.$$

We shall denote the corresponding norm simply by $\|\cdot\|_q$ although it also depends on a and p.

LEMMA 8. *Let* $\tilde{V}$ *be a measurable matrix-valued map and* $\tilde{G}\colon \mathrm{Sp}(F_a) \to \mathbf{R}$. *There exists* $P_a < \infty$ *such that for* $p > P_a$ *and* $1 \leqslant r \leqslant \infty$, $\tilde{V} \in L^{pr}(\Omega)$ *and* $\tilde{G} \in l^{2r}(Sp(F_a), \lambda_n^{N/p})$ *imply* $\tilde{V}\tilde{G}(F_a) \in \mathscr{C}^{2r}(L^2(\Omega))$ *and*

$$\| \tilde{V}\tilde{G}(F_a)\|_{2r} \leqslant c_{p,a} \| \tilde{V}\|_{pr} \|\tilde{G}\|_{2r}.$$

Proof. It is enough to prove the result for $r = 1$, ∞ since we can then use interpolation. The case $r = \infty$ is trivial.

Let $\{\phi_{n,a}\}$ be a complete orthonormal system of eigenfunctions of H_a, say $H_a\phi_{n,a} = \lambda_{n,a}\phi_{n,a}$. Then

$$\psi_{n,a} =: \lambda_{n,a}^{-1/2} r_a \phi_n, \qquad n \in \mathbf{N}$$

is an orthonormal system in $\oplus L^2(\Omega)$ that satisfies $F_a\psi_{n,a} = \lambda_{n,a}\psi_{n,a}$ and spans $(\mathrm{Ker}F_a)^{\perp}$. Since $\tilde{G}(0) = 0$, we have

$$\begin{aligned}
\| \tilde{V}\tilde{G}(F_a)\|_2^2 &= \sum_n \| \tilde{V}\tilde{G}(F_a)\,\psi_{n,a}\|_2^2 \\
&= \sum_n |\tilde{G}(\lambda_{n,a})|^2 \| \tilde{V}\psi_{n,a}\|_2^2 \\
&\leqslant \| \tilde{V}\|_p^2 \sum_n |\tilde{G}(\lambda_{n,a})|^2 \, \|\psi_{n,a}\|_{2p/(p-2)}^2 \\
&\leqslant c_a \| \tilde{V}\|_p^2 \sum_n |\tilde{G}(\lambda_{n,a})|^2 \, \lambda_{n,a}^{-1} \| d\phi_{n,a}\|_{2p/(p-2)}^2.
\end{aligned}$$

Setting $P_a = 2Q_a/(Q_a - 2)$ it follows from (10) that

$$\|d\phi_{n,a}\|_{2P/(p-2)} \leqslant c_{p,a}\lambda_{n,a}^{(p+N)/2p}, \qquad \text{all} \quad p > P_a$$

and we conclude that

$$\| \tilde{V}\tilde{G}(F_a)\|_2^2 \leqslant c_{p,a} \| \tilde{V}\|_p^2 \sum_n |\tilde{G}(\lambda_{n,a})|^2 \, \lambda_{n,a}^{N/p}$$

as required. ∎

Now we can prove the following

THEOREM 9. *There exists* P_0 *satisfying* $2 < P_0 < \infty$ *and depending only on* Ω *and the ellipticity constants of* H_a *and* H_b *such that if*

$$\text{(i)} \qquad p > P_0 \tag{18}$$

and

$$\text{(ii)} \qquad r > \frac{N}{2} + \frac{N}{p}, \tag{19}$$

then

$$\|R_b - R_a\|_r \leqslant c_{p,r,a,b} \|b-a\|_{pr/2}. \tag{20}$$

Proof. We may assume that $W =: b-a \geqslant 0$ and hence write $W = V^2$, since otherwise we can write $W = W_+ - W_-$ and use the fact that $\|W_\pm\|_q \leqslant \|W\|_q$.

For any $1 \leqslant r \leqslant \infty$ we then have from Lemma 5

$$\|R_b - R_a\|_r \leqslant \|G(F_b)\, b^{-1/2} V\|_{2r} \|Va^{-1/2} G(F_a)\|_{2r}$$

so that if $p > P_0 =: \max\{P_a, P_b\}$, then

$$\|R_b - R_a\|_r \leqslant c_{p,a,b} \|V\|_{pr}^2 \;\|G(F_b)\|_{2r} \;\|G(F_a)\|_{2r}$$

by Lemma 8. We have $G(0) = 0$ and, in fact,

$$\begin{aligned}
\|G\|_{2r} < \infty &\Leftrightarrow \sum_n |G(\lambda_{n,a})|^{2r} \lambda_n^{N/p} < \infty \\
&\Leftrightarrow \sum_n \lambda_{n,a}^{-r + (N/p)} < \infty \\
&\Leftrightarrow \sum_n n^{2/N(-r+(N/p))} < \infty \\
&\Leftrightarrow \frac{1}{2} + \frac{1}{p} < \frac{r}{N}
\end{aligned}$$

which proves the theorem. ∎

Remark. If the boundary $\partial\Omega$ and the matrices a and b are sufficiently smooth, then the constant Q in Theorem 1 can be taken to be equal to $+\infty$. This implies that the index P_0 in the Theorem can be taken to be equal to 2 in that case.

Remark. If we do not make any regularity assumptions on $\partial\Omega$, then a variation of Theorem 1 exists (Theorem 2 of [6]) that involves local rather than global Sobolev estimates. Hence the conclusion of Lemma 8, and hence of Theorem 9, is still valid under the additional assumption that the difference $b-a$ has compact support in Ω.

Our theorem establishes the Lipschitz continuity of the map

$$L^{pr/2} \ni a \mapsto R_a \in \mathscr{C}^r$$

when p and r satisfy (i) and (ii) and the L^∞ norms of the matrices a and a^{-1} are bounded away from infinity. It is possible however to improve the range of both parameters p and r at the cost of replacing Lipschitz continuity by Hölder continuity.

COROLLARY 10. *If p, r, s and γ satisfy*

$$p > P_0, \qquad r > \frac{N}{2} + \frac{N}{p}, \qquad \frac{N}{2} < s < r, \qquad \gamma < \frac{r(2s-N)}{s(2r-N)} \tag{21}$$

then there exists a constant $c_{p,r,s,\gamma,a,b} < \infty$ such that

$$\|R_b - R_a\|_s \leqslant c_{p,r,s,\gamma,a,b} \, \|b-a\|_t^{2t\gamma/pr} \tag{22}$$

for all $1 \leqslant t \leqslant pr/2$.

Proof. From (17) we have

$$\|R_b - R_a\|_q \leqslant c_{q,a,b}, \qquad \text{all} \quad q > N/2. \tag{23}$$

Interpolation between (20) and (23) yields

$$\|R_b - R_a\|_s \leqslant c_{p,r,s,\gamma,a,\,b} \, \|b-a\|_{pr/2}^{\gamma} \tag{24}$$

and the result follows if we apply the formula

$$\|u\|_p \leqslant \|u\|_\infty^{(p-t)/p} \, \|u\|_t^{t/p}$$

which is valid for all $1 \leqslant t \leqslant p$. ∎

From the above estimate we can immediately obtain stability estimates for the eigenvalues: let $\{\lambda_{n,a}\}$, $\{\lambda_{n,b}\}$ be the eigenvalues of H_a and H_b respectively.

COROLLARY 11. *If p, r, s, γ and t are as in Corollary 10 then*

$$\left(\sum_n |\lambda_{n,b} - \lambda_{n,a}|^s \, n^{-4s/N} \right)^{1/s} \leqslant c_{p,r,s,a,b} \, \|b-a\|_t^{2t\gamma/pr}. \tag{25}$$

Proof. This follows from the fact that if A, B are any two compact operators acting on a Hilbert space $\mathscr{H}$, then

$$\left(\sum_n |\mu_n(A) - \mu_n(B)|^s \right)^{1/s} \leqslant \|A-B\|_{\mathscr{C}^s(\mathscr{H})}$$

where $\{\mu_n(A)\}$, $\{\mu_n(B)\}$ are the singular values of the operators. See [9, p. 20] for a proof. ∎

A comparison of Theorem 3 with Theorem 9 is natural. First, we note that the constant P of Theorem 3 is equal to $P_0/2$, where P_0 is as in Theorem 9.

Hence, while (4) says that

$$\left|\sum_{n=1}^{k} \lambda_{n,b} - \lambda_{n,a}\right| \leqslant c\, \|b-a\|_p\, k^{1+(1/N)+(1/P)}, \qquad \text{all} \quad p > P_0/2. \tag{26}$$

from (25) we have

$$\sum_{n=1}^{k} |\lambda_{n,b} - \lambda_{n,a}| \leqslant c\, \|b-a\|_{pr/2}\, k^{1+(1/N)-(1/r)} \tag{27}$$

whenever $p > P_0$ and $r > N/2 + N/p$. Choosing any $p > P_0$ and

$$r = \frac{N}{2} + \frac{N}{p} + \varepsilon$$

(27) gives

$$\sum_{n=1}^{k} |\lambda_{n,b} - \lambda_{n,a}| \leqslant c\, \|b-a\|_{(pN/4)+(N/2)+\varepsilon}\, k^{1+(1/N)-(2p/N(2+p))+\varepsilon}$$

while (26) for $q = pN/4 + N/2$ gives

$$\left|\sum_{n=1}^{k} \lambda_{n,b} - \lambda_{n,a}\right| \leqslant c\, \|b-a\|_{(pN/4)+(N/2)}\, k^{1+(2/N)+(4/pN+2N)}.$$

Interestingly enough, one observes that

$$1 + \frac{4}{N} - \frac{2p}{N(2+p)} = 1 + \frac{2}{N} + \frac{4}{pN+2N},$$

so that, in some sense, the two methods yield equally good results. However, while (26) has the disadvantage that it involves first summing and then taking absolute values, it is better in that q can be smaller than $pr/2$ in (27) and in that it does not produce the factor $k\epsilon$ that (27) gives.

Lipschitz Domains

One of the assumptions of Theorem 9 was that the boundary $\partial\Omega$ is C^1. At the cost of a larger P_0 it is possible to replace that hypothesis by a weaker one.

Definition. Let $D \subset \mathbf{R}^N$ be bounded. We say that Ω has the *global Lipschitz property* if there exists $\pi: D \to \mathbf{R}^N$ bi-Lipschitz such that $\pi(D) =: \tilde{D}$ has a C^1 boundary.

Suppose that Ω is globally Lipschitz. There are two ways to proceed. One is to establish $W^{1,p}$ bounds on the eigenfunctions of the operators using Lemma 12 below and then proceed as before to prove Theorem 14. The second, which we follow, is to make use of Theorem 9. In both cases the proof is based on the fact that uniform ellipticity is preserved under globally Lipschitz transformations.

Let M_1, M_2 be the two Lipschitz constants of Ω:

$$\frac{1}{M_2}|x-y| \leqslant |\pi(x)-\pi(y)| \leqslant M_1|x-y|, \qquad \text{all} \quad x, y \in \Omega.$$

For $x \in \Omega$ set $\tilde{x} = \pi(x)$ and for a function (or a matrix) $f \in L^1_{loc}(\Omega)$ define $\tilde{f}$ on Ω by

$$\tilde{f}(\tilde{x}) = f(x), \qquad \text{all} \quad x \in \Omega.$$

Let $J\pi$, $J\pi^{-1}$ be the Jacobian matrices of π and π^{-1} respectively, so that

$$\|J\pi\|_\infty \leqslant cM_1, \qquad \|J\pi^{-1}\|_\infty \leqslant cM_2 \tag{28}$$

and

$$\|\det(J\pi)\|_\infty \leqslant cM_1^N, \qquad \|\det(J\pi^{-1})\|_\infty \leqslant cM_2^N. \tag{29}$$

Using (28), (29) and the chain rule one can easily check that for $1 \leqslant p \leqslant \infty$

$$f \in W^{1,p}(\Omega) \Leftrightarrow \tilde{f} \in W^{1,p}(\Omega)$$

and

$$\frac{1}{cM_1^{N/p}M_2}\|f\|_{1,p} \leqslant \|\tilde{f}\|_{1,p} \leqslant cM_1M_2^{N/p}\|f\|_{1,p}. \tag{30}$$

The following lemma describes the operator on $L^2(\Omega)$ induced by H_a. Set $\delta_\pi = |\det(J\pi)|^{-1}$.

Lemma 12. *Define the matrix* a_1 *on* Ω *by*

$$a_1 = \delta_\pi (J\pi)^* a(J\pi).$$

and set $\hat{a} = \tilde{a}_1$. *Then*

$$f \in \mathrm{Dom}(H_a) \Leftrightarrow \tilde{f} \in \mathrm{Dom}(H_{\hat{a}})$$

and if $f \in \mathrm{Dom}(H_a)$ *then*

$$H_{\hat{a}} \tilde{f} = (\delta_\pi H_a f)^{\sim}. \tag{31}$$

Proof. First we note that $H_{\hat{a}}$ is uniformly elliptic and in fact

$$\|\hat{a}\|_\infty \leqslant \|\delta_\pi\|_\infty \, \|J_\pi\|_\infty^2 \, \|a\|_\infty, \tag{32}$$

$$\|\hat{a}^{-1}\|_\infty \leqslant \|\delta_\pi^{-1}\|_\infty \, \|J_\pi^{-1}\|_\infty^2 \, \|a^{-1}\|_\infty. \tag{33}$$

It is enough to prove the one implication since the statement is symmetric. So, let $f \in \mathrm{Dom}(H_a)$, $H_a f = h$. Then

$$\int_\Omega a \nabla f \cdot \nabla \phi \, dx = \int_\Omega h\phi \, dx, \qquad \text{all} \quad \phi \in C_c^\infty(\Omega)$$

$$\Leftrightarrow \int_\Omega a_1 (J\pi)^{-1} \nabla f \cdot (J\pi)^{-1} \nabla \phi \, |\det(J\pi)| \, dx = \int_\Omega h\phi \, dx, \qquad \text{all} \quad \phi \in C_c^\infty(\Omega)$$

$$\Leftrightarrow \int_{\tilde{\Omega}} \tilde{a}_1 \nabla \tilde{f} \cdot \nabla \bar{\phi} \, dx = \int_{\tilde{\Omega}} \delta_\pi \tilde{h} \bar{\phi} \, d\tilde{x}, \qquad \text{all} \quad \phi \in C_c^\infty(\Omega)$$

and the result follows since $\phi \in C_c^\infty(\Omega)$ implies $\tilde{\phi} \in W_c^{1,\infty}(\Omega) \subset W_0^{1,2}(\tilde{\Omega})$. ∎

If for $g \in L^2(\Omega)$ we define $\hat{g} = \tilde{\delta}_\pi \tilde{g} \in L^2(\tilde{\Omega})$, it follows from Lemma 12 that

$$(R_a g)^{\sim} = R_{\hat{a}} \hat{g}, \qquad \text{all} \quad g \in L^2(\Omega). \tag{34}$$

LEMMA 13. *For any* $1 \leqslant r \leqslant \infty$ *we have*

$$\|R_b - R_a\|_r \leqslant c M_1^{N/2} M_2^{3N/2} \, \|R_{\hat{b}} - R_{\hat{a}}\|_r.$$

Proof. For any $g \in L^2(\Omega)$ we have

$$\begin{aligned} \|(R_b - R_a)\, g\| &\leqslant c M_2^{N/2} \|\{(R_b - R_a)\, g\}^{\sim}\| \\ &= c M_2^{N/2} \|(R_{\hat{b}} - R_{\hat{a}})\, \hat{g}\| \end{aligned}$$

and so

$$\frac{\|(R_b - R_a)\, g\|}{\|g\|} \leqslant c M_1^{N/2} M_2^{3N/2} \frac{\|(R_{\hat{b}} - R_{\hat{a}})\, \hat{g}\|}{\|\hat{g}\|}. \tag{35}$$

Since $g \mapsto \hat{g}$ is invertible, for any subspace L of $L^2(\Omega)$ we have

$$\dim L = \dim \{\hat{g} \mid g \in L\}$$

and so the lemma follows from (35) using the min-max theorem. ∎

THEOREM 14. *There exists* $P_0<\infty$, $P=P_{a,b,J\pi}$, *such that if*

$$\text{(i)} \qquad p>P_0$$

and

$$\text{(ii)} \qquad r>\frac{N}{2}+\frac{N}{p}.$$

then

$$\|R_b-R_a\|_r \leqslant c_{r,p,a,b,J\pi}\,\|b-a\|_{pr/2}.$$

Proof. This follows directly from Lemmas 12 and 13 and Theorem 9. ▮

COROLLARY 15. *Corollaries* 10 *and* 11 *are also valid if* Ω *is globally Lipschitz.*

APPLICATIONS AND A LIMIT CASE

A typical application of the above results examines how the heat diffusion on a body is affected by the existence of impurities of different conductivity. Using Theorem 14 we establish spectral stability as the impurities become small in volume.

Let $\Omega\subset\mathbf{R}^N$ be bounded with a globally Lipschitz boundary. Let H denote the Dirichlet Laplacian on Ω so that if Ω is homogeneous of conductivity one, then the Dirichlet heat diffusion is discribed by the equation $\partial u/\partial t=-Hu$.

Now suppose instead that there are disjoint connected sets $S_k\subset\Omega$, $k=1,2,\ldots,r$, of conductivity $\alpha_k\neq 1$. Then the heat equation becomes

$$\frac{\partial u}{\partial t}=-H_a u \tag{36}$$

where $a(x)$ is the scalar matrix given by

$$a(x)=\begin{cases}1 & \text{if } \; x\in\Omega\backslash\bigcup S_k\\ \alpha_k & \text{if } \; x\in S_k.\end{cases}$$

Set $S=\bigcup S_k$ and let $|S|$ denote the Lebesgue measure of S. From Theorem 14 we immediately deduce the following

PROPOSITION 16. *For* P_0, p *and* r *as in Theorem* 10, *we have*

$$\|R-R_a\|_r \leqslant c(\max_k|1-\alpha_k|)\,|S|^{2/pr}. \tag{37}$$

Note that the fact that p in (16) can be strictly smaller than $+\infty$ is crucial in establishing stability of the resolvent as S shrinks to a set of measure zero.

For the sake of simplicity we assume from now on that $\alpha_1 = \alpha_2 = \cdots = \alpha_r =: \alpha > 1$.

EXAMPLE 1. Suppose that there are M disjoint balls $B(u_k, \rho)$, $k = 1, ..., M$, with centre u_k and the same small radius ρ. The following expresses the intuition that the effect of many small randomly distributed spherical impurities depends upon balancing the number M against the radius ρ: if P_0 is as above and we fix $r > N/2$, then

$$\|R - R_a\|_r \leqslant c\,|1 - \alpha|\; o(1) \qquad \text{as} \quad \rho \to 0$$

provided $M = o(\rho^{-N})$.

EXAMPLE 2. More detailed information can be obtained if we are only interested in eigenvalue stability. If $\{\lambda_n\}$, $\{\lambda_{n,a}\}$ are the corresponding eigenvalues, equation (5) and a simple argument yield

$$0 \leqslant \lambda_{n,a} - \lambda_n \leqslant 2(\alpha - 1) \int_S (|\nabla\phi_1|^2 + \cdots + |\nabla\phi_n|^2)\, dx \tag{38}$$

where $\{\phi_n\}$ are the eigenfunctions of H.

This estimate can be improved if additional information is available. If Ω is regular enough so that $\|\nabla\phi_k\|_\infty < \infty$, $1 \leqslant k \leqslant n$, (38) gives

$$0 \leqslant \lambda_{n,a} - \lambda_n \leqslant 2(\alpha - 1)(\|\nabla\phi_1\|_2^\infty + \cdots + \|\nabla\phi_n\|_\infty{}^2)|S|. \tag{39}$$

If we assume that S shrinks uniformly in the sense that it is contained in the ball of fixed centre x_0 and radius ϵ, then taking the Taylor expansion of ϕ_n around x_0 yields the asymptotic inequality

$$0 \leqslant \lambda_{n,a} - \lambda_n \leqslant 2(\alpha - 1)\,\omega_N(|\nabla\phi_1(x_0)|^2 + \cdots + |\nabla\phi_n(x_0)|^2)\varepsilon^N + O(\varepsilon^{N+2}) \tag{40}$$

as $\varepsilon \to 0$, where ω_N denotes the volume of the unit ball.

EXAMPLE 3. Let Ω be the unit ball in $\mathbf{R}^N$ and let

$$S = \{x \in \Omega \mid |x| < \varepsilon\}.$$

Let H_ε denote the corresponding operators so that $H_0 = -\Delta$. The bottom eigenvalue λ_ε of H_ε as well as the ground state can be calculated explicitly using separation of variables and one can see that

$$\lambda_\varepsilon = \lambda_0 + k_N(1 - \alpha^{-1})\,\varepsilon^{N+2} + O(\varepsilon^{N+4}), \qquad \text{as} \quad \varepsilon \to 0. \tag{41}$$

The constant k_N can be explicitly computed: if

$$J_\nu(y) = y^\nu \sum_{i=0}^{\infty} c_{\nu,i}\, y^{2i}, \qquad \nu \in \mathbf{R}$$

denotes the Bessel functions of the first kind of order ν, then for N odd we get

$$k_N = \frac{\gamma_{(N-2)/2}}{g'_N(\lambda_0)}$$

where

$$\gamma_\nu = \frac{c_{\nu,1}^2 - 2c_{\nu,0}c_{\nu,2}}{\nu c_{\nu,0}c_{-\nu,0}}$$

and

$$g_N(y) = \frac{J_{(N-2)/2}(y^{1/2})}{J_{-(N-2)/2}(y^{1/2})}\, y^{-(N+2)/2}.$$

λ_0 being the first eigenvalue of H_0 or, equivalently, the least positive solution of $J_{(N-2)/2}(y^{1/2}) = 0$. The result is also valid for N even with the only difference that the functions $J_{-\nu}$ must be replaced by Weber's Bessel functions of the second kind N_ν. The reason why we have an ε^{N+2} in (41) and not an ε^N as (40) suggests is that $\nabla\phi_1(0) = 0$ in this case.

A similar formula can be obtained for higher-order eigenvalues in the same way. The calculations as well as the results will involve additional parameters related to the angular momentum of the corresponding eigenfunctions.

Degenerate Operators. There are further questions that one can pose concerning the operators H_a above. It is natural to ask what happens when the constant α converges either to zero or to infinity.

We assume from now on that S is an open subset of Ω with locally Lipschitz boundary and that $\bar{S} \subset \Omega$. We also set $U = \Omega \setminus \bar{S}$.

For $0 < \alpha < \infty$, let Q_a be the quadratic form on $L^2(\Omega)$ associated with the operator H_a, so that

$$\mathrm{Dom}(Q_a) = W_0^{1,2}(\Omega).$$

$$Q_a(f) = \int_\Omega a(x)\, |\nabla f|^2\, dx, \qquad \text{all} \quad x \in W_0^{1,2}.$$

Let Q_∞ be the limit of the Q_a's as $\alpha \to \infty$. Clearly

$$\mathrm{Dom}(Q_\infty) = \{f \in W_0^{1,2}(\Omega) \mid f \text{ is constant on each } S_k\}$$

and

$$Q_\infty(f) = \int_U |\nabla f|^2 \, dx, \qquad \text{all} \quad f \in \mathrm{Dom}(Q_\infty).$$

This form is closed but not densely defined. It is however densely defined if considered as acting on the Hilbert space $L^2(A, \mu) \subset L^2(\Omega)$ where $A =: \{\sigma_1\} \cup \cdots \cup \{\sigma_r\} \cup U$ is the measure space defined by

$$\begin{cases} \mu(\sigma_k) = |S_k|, & k = 1, ..., r \\ \mu(E) = |E|, & \text{all} \quad E \subset U \end{cases}$$

and one easily checks that the Lipschitz condition implies that the space

$$\mathscr{D} =: \mathrm{Dom}(Q_\infty) \cap C_c^\infty(\Omega)$$

is a core of Q_∞.

The study of the operator H_∞ on $L^2(A)$ associated to the form Q_∞ and its dependence on the S_k's and their relative position is a very interesting problem which is under investigation.

The case $\alpha \to 0$ is much simpler and in fact the Lipschitz condition on S can be weakened to the condition $|\partial S| = 0$. Let $Q'_n = \lim_{\alpha \to 0} Q_a$ so that

$$\mathrm{Dom}(Q'_n) = W_0^{1,2}(\Omega)$$

and

$$Q'_n(f) = \int_U |\nabla f|^2 \, dx, \qquad \text{all} \quad f \in \mathrm{Dom}(Q'_n).$$

Let V be the closure in $W^{1,2}(U)$ of the set

$$V_0 = \{f|_U \mid f \in C_c^\infty(\Omega)\}$$

and define the quadratic form Q_n on $L^2(\Omega)$ by

$$\mathrm{Dom}(Q_n) = \{f \in L^2(\Omega) \mid f|_U \in V\}$$

and

$$Q_n(f) = \int_U |\nabla f|^2 \, dx, \qquad \text{all} \quad f \in \mathrm{Dom}(Q_n).$$

The proof of the following proposition is simple and is omitted:

PROPOSITION 17. (i) *Q_n is densely defined and closed and* (ii) *$C_c^\infty(\Omega)$ is a core of Q_n and* (*hence*) *Q_n is the form closure of Q'_n.*

Let H_n be the self-adjoint operator on $L^2(\Omega)$ associated with the form Q_n. Formally, one can write

$$H_n = -\sum_i \frac{\partial}{\partial x_i} \left\{ a_0(x) \frac{\partial}{\partial x_i} \right\}$$

where

$$a_0(x) = \begin{cases} 1 & \text{if} \quad x \in U \\ 0 & \text{if} \quad x \in S. \end{cases}$$

Let H_V be the self-adjoint operator on $L^2(U)$ associated with the quadratic form Q_V, where

$$\mathrm{Dom}(Q_V) = V$$

and

$$Q_V(f) = \int_U |\nabla f|^2 \, dx, \qquad \text{all} \quad f \in V.$$

H_V is the Laplacian on U satisfying Dirichlet boundary conditions on $\partial\Omega$ and Neumann boundary conditions on ∂S.

In accordance with our intuition we have the following

THEOREM 18. *The spaces $L^2(S)$ and $L^2(U)$ are invariant under the action of H_n and writing*

$$L^2(\Omega) = L^2(S) \oplus L^2(U) \tag{42}$$

we have

(i) $$\mathrm{Dom}(H_n) = L^2(S) \oplus \mathrm{Dom}(H_V) \tag{43}$$

(ii) $$H_n = 0 \oplus H_V. \tag{44}$$

Proof. First, we note that whether or not a function in $L^2(\Omega)$ lies in $\mathrm{Dom}(Q_n)$ or $\mathrm{Dom}(H_n)$ depends only on its restriction on U. Now, let $f \in \mathrm{Dom}(H_n)$, $H_n f = g$. Then $f \in \mathrm{Dom}(Q_n)$ and

$$\int_U \nabla f \cdot \nabla \phi \, dx = \int_\Omega g \phi \, dx, \qquad \text{all} \quad \phi \in C_c^\infty(\Omega).$$

It follows that $g|_S = 0$ and

$$\int_U \nabla f \cdot \nabla \phi \, dx = \int_U g\phi \, dx, \qquad \text{all} \quad \phi \in V_0. \tag{45}$$

Since $f \in \mathrm{Dom}(Q_n)$ implies $f|_U \in V$, it follows that $f|_U \in \mathrm{Dom}(H_V)$ and

$$H_V(f|_U) = g|_U. \tag{46}$$

Conversely, let $f \in L^2(\Omega)$ be such that $f|_U =: f_1 \in \mathrm{Dom}(H_V)$, $H_V f_1 =: g$, say. Then $f_1 \in V$ and

$$\int_U \nabla f_1 \cdot \nabla \phi \, dx = \int_U g\phi \, dx, \qquad \text{all} \quad \phi \in V$$

which clearly implies that $f \in \mathrm{Dom}(H_n)$ and $H_n f = 0 \oplus g$. ∎

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