SHARP HEAT KERNEL BOUNDS AND FINSLER-TYPE METRICS

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1. Introduction

THE aim of this paper is to prove sharp heat kernel bounds for a class of higher order uniformly elliptic homogeneous operators acting on a domain $\Omega \subset \mathbf{R}^N$ and satisfying Dirichlet boundary conditions.

Heat kernels estimates for higher order elliptic operators acting on Riemannian manifolds have existed for some time now under various assumptions on the highest order coefficients such as C_b^∞ or C^m. See **[12, 13, 14, 16]** and references therein. In the case of Euclidean domains those estimates take the form

$$|K(t,x,y)| \leqslant c_1 t^{-N/2m} \exp\left\{-c_2 \frac{|x-y|^{2m/(2m-1)}}{t^{1/(2m-1)}} + c_3 t\right\} \tag{1}$$

for some positive constants c_i and all $t > 0$ and $x, y \in \Omega$. This was generalized to operators with measurable coefficients by Davies **[7]**, who then used the estimate to show that e^{-Ht} defines a strongly continuous semigroup on $L^p(\Omega)$, $1 \leqslant p < \infty$, and that the corresponding generator H_p has spectrum that is p-independent.

The heat kernel estimates obtained in **[7]** depended heavily upon the assumption $2m > N$, the condition under which the Sobolev embedding $W_0^{m,2}(\Omega) \subset C_0(\Omega)$ is valid. In the case $2m = N$ estimate (1) remains true, as can be shown by following the method of **[3]**. This is also proved in **[9]**, where the analysis is carried out in a more general Lie group context. For the case $2m < N$ there are examples that show that Gaussian-type estimates are not always valid **[2, 8]**. However the off-diagonal decay of the semigroup can be expressed in a suitable operator sense **[7]**. The assumption $2m > N$ is made throughout the present paper.

The question of estimates on the constant c_2 was first addressed in **[4]**. There it is proved that if H is homogeneous with measurable coefficients and $(-\Delta)^m \leqslant H \leqslant (1+\mu)(-\Delta)^m$, then one can take

$$c_2 = \sigma_m - O(\mu) \quad (\text{small } \mu) \tag{2}$$

where

$$\sigma_m = (2m-1)(2m)^{-2m/(2m-1)} \sin\left(\frac{\pi}{4m-2}\right) \tag{3}$$

is a constant that cannot be improved.

This is sharp as long as one is only interested in bounds involving the Euclidean metric in the Gaussian term; naturally, as in the second order case, the Euclidean distance is not the one that best describes the off-diagonal decay of the heat kernel. This is seen in results on asymptotic estimates of the heat kernel. Working with constant coefficient operators on $\mathbf{R}^N$ that satisfy the so-called *strong convexity* condition (see definition below), Evgrafov and Postnikov [**10**] gave explicit short time asymptotics for the heat kernel in terms of a certain Finsler metric induced canonically by the operator. This was later generalized by Tintarev [**17**] to operators with smooth coefficients but under the additional assumption that the spatial variables x and y are close enough. [In that paper however Tintarev wrongly quoted a result from [**10**] and thus did not make the required strong convexity assumption.] The conjecture was put forward in [**10**] that convexity (which is implied by strong convexity [**10**]) is sufficient for the short time asymptotics proved there; but a later example by the same authors [**11**] strongly suggests that the conjecture is actually false.

The metric that appears in the Gaussian-type term in these papers is not Riemannian, in general at least. It is an example of a class of metrics known as Finsler metrics, that appear naturally in different problems of mathematical physics. We refer to [**5, 1**] for more details.

The scope of this paper is to obtain heat kernel bounds that involve both the constant σ_m and the Finsler metric $d(x, y)$. As a corollary of our main theorem we prove the estimate

$$|K(t, x, y)| \leqslant c_{\varepsilon,M} t^{-N/2m} \exp\left\{-(\sigma_m - \varepsilon)\frac{d_M(x, y)^{2m/(2m-1)}}{t^{1/(2m-1)}} + c_3 t\right\} \quad (4)$$

for all $\varepsilon, M > 0$ and all x, y in Ω and $t > 0$ (Corollary 9). Here $d_M(x, y)$ is an appropriate Finsler-type metric on Ω that converges to the actual metric $d(x, y)$ as $M \to \infty$. This bound, unlike that in [**4**], is good for any operator, irrespectively of the ellipticity constant $1 + \mu$. This however is achieved at the cost of a local regularity assumption on the coefficients which are now assumed to lie in $W^{m,\infty}(\Omega)$; we also assume that the coefficients $a_{\alpha\beta}$ of H are real, as opposed to complex in [**4**]; the precise assumptions are made at the beginning of the next section. We emphasize nevertheless that bounds of this type did not exist even for operators with smooth coefficients. Moreover, in our case, unlike [**17**], the spatial variables x and y do not have to be close.

The fact that (4) is valid for any domain Ω (rather than $\mathbf{R}^N$ only) gives an additional advantage to our bounds as opposed to those that involve the Euclidean metric. There are domains, such as horse-shoe-type domains, for which the Finsler-type metric is much larger than the Euclidean one and therefore provides better bounds. In other words, the Finsler-type metric 'feels' better not only the operator H but also the geometry of Ω. Owen [**15**] has performed a very detailed analysis of the metric $d_M(x, y)$ for the operator Δ^2 acting on a C^2 domain in$\mathbf{R}^2$, and he proved that it converges to the geodesic distance on Ω uniformly in x, y. See Remark 3 at the end of this paper.

2. Preliminaries

Given a multi-index $\alpha = (\alpha_1, ..., \alpha_N)$ and a vector $u = (u_1, ..., u_N)$, we use the standard notation D^α and u^α for the differential operator $(\partial/\partial_1)^{\alpha_1} \dots (\partial/\partial_N)^{\alpha_N}$ and the number $u_1^{\alpha_1} \dots u_N^{\alpha_N}$ correspondingly. For $k \geqslant 0$ we shall denote by $\nabla^k f$ the vector $(D^\alpha f)_{|\alpha|=k}$. Given two multi-indices α and γ with $\gamma \leqslant \alpha$, we also set $c_\gamma^\alpha = \alpha!/\gamma!(\alpha-\gamma)!$ and $c_\alpha^{|\alpha|} = |\alpha|!/\alpha!$. Moreover, given r, $0 \leqslant r \leqslant m$, we shall denote by $\zeta_{r,m}$ the seminorms

$$\zeta_{r,m}(g) = \sup_{r \leqslant k \leqslant m} \|\nabla^k g\|_\infty$$

whenever the right-hand side is finite.

Let $A(x) = \{a_{\alpha\beta}(x)\}_{|\alpha|,|\beta|=m}$ be a bounded real symmetric matrix defined on Ω, whose entries lie in $W^{m,\infty}(\Omega) := \{g \in L^\infty(\Omega) \mid \nabla^k g \in L^\infty(\Omega),\ 1 \leqslant k \leqslant m\}$. We assume that there exists a constant $c > 0$ such that

$$c^{-1}\|(-\Delta)^{m/2} f\|_2^2 \leqslant \int_\Omega \sum_{\substack{|\alpha|=m \\ |\beta|=m}} a_{\alpha\beta}(x) D^\alpha f(x) D^\beta \bar{f}(x) dx \leqslant c\|(-\Delta)^{m/2} f\|_2^2$$

for all $f \in C_c^\infty(\Omega)$. We also assume that the matrix $\{a_{\alpha\beta}(x)\}$ is positive semi-definite for all $x \in \Omega$,

$$\sum a_{\alpha\beta}(x) p_\alpha \bar{p}_\beta \geqslant 0, \quad \text{all } (p_\alpha)_{|\alpha|=m}. \tag{5}$$

Under these assumptions we define the operator H, given formally by

$$Hf(x) = (-1)^m \sum_{\substack{|\alpha|=m \\ |\beta|=m}} D^\alpha \{a_{\alpha\beta}(x) D^\beta f(x)\} \tag{6}$$

and satisfying Dirichlet boundary conditions, to be the self-adjoint operator associated to the closed form Q with domain $W_0^{m,2}(\Omega)$ given by

$$Q(f) = \int_\Omega \sum_{\substack{|\alpha|=m \\ |\beta|=m}} a_{\alpha\beta}(x) D^\alpha f(x) D^\beta \bar{f}(x) dx.$$

Following [7] we shall call such operators *homogeneous superelliptic* operators.

Given a superelliptic operator H we are going to associate it to a family of Finsler-type metrics $d_M(x, y)$ where M is a large parameter. Let

$$a(x,\xi) = \sum_{\substack{|\alpha|=m \\ |\beta|=m}} a_{\alpha\beta}(x) \xi^{\alpha+\beta}, \quad x \in \Omega, \ \ \xi \in \mathbf{R}^N,$$

be the principal symbol of the operator, which we assume satisfies

$$c^{-1}|\xi|^{2m} \leqslant a(x,\xi) \leqslant c|\xi|^{2m}, \tag{7}$$

for some constant $c \geqslant 1$ and all $x \in \Omega$, $\xi \in \mathbf{R}^N$. We define the class

$$\mathcal{E}_m(\Omega) = \{\phi \in C^\infty(\Omega) \mid \|D^\alpha \phi\|_\infty < +\infty, \ \ 0 \leqslant |\alpha| \leqslant m\}$$

and we will be particularly interested in the subclasses

$$\mathcal{E}_{a,M} = \{\phi \in \mathcal{E}_m(\Omega) \mid a(x, \nabla\phi(x)) \leqslant 1, \quad \|D^\gamma \phi\|_\infty \leqslant M, \quad 2 \leqslant |\gamma| \leqslant m\}$$

and

$$\mathcal{E}_a = \bigcup_{M>0} \mathcal{E}_{a,M}.$$

The distance $d_M(x, y)$ is then defined to be

$$d_M(x, y) = \sup\{\phi(x) - \phi(y) \mid \phi \in \mathcal{E}_{a,M}\}. \tag{8}$$

Our main heat kernel estimate, Theorem 8, involves Gaussian-type estimates in terms of the distance $d_M(x, y)$ for arbitrarily large M. We are not going to investigate the convergence of $d_M(x, y)$ to the actual Finsler metric

$$d(x, y) := \sup\{\phi(x) - \phi(y) \mid \phi \in C^\infty(\Omega), \quad a(z, \nabla\phi(z)) \leqslant 1, \quad z \in \Omega\}.$$

This is a geometrical problem whose nature is entirely different to that of this paper. See also Remark 3 at the end of the paper.

Our proof makes use of the twisting technique first introduced in [**6**]. Given $\phi \in \mathcal{E}_m(\Omega)$ the (multiplication) operator e^ϕ leaves $W_0^{m,2}(\Omega)$ invariant so that one can define the non-symmetric form Q_ϕ by

$$Q_\phi(f, g) = Q(e^\phi f, e^{-\phi} g)$$

for all $f, g \in \mathrm{Dom}(Q_\phi) := W_0^{m,2}(\Omega)$. Then Q_ϕ and Q have the same highest order terms and standard interpolation shows that for any $\varepsilon > 0$ there exists a constant c_ε such that

$$|Q_\phi(f) - Q(f)| < \varepsilon Q(f) + c_\varepsilon\{\zeta_{1,m}(\phi) + \zeta_{1,m}(\phi)^{2m}\}\|f\|_2^2 \tag{9}$$

for all $f \in C_c^\infty(\Omega)$. See Lemma 2 of [**7**] for a detailed proof. We denote by H_ϕ the (non-symmetric) operator associated with Q_ϕ, so that

$$H_\phi f = e^{-\phi} H e^\phi f \tag{10}$$

for all $f \in \mathrm{Dom}(H_\phi) = \{f \in L^2 | e^\phi f \in \mathrm{Dom}(H)\}$.

The following proposition was proved in [**4**] for operators that are not necessarily homogeneous and whose highest order coefficients need only be measurable.

PROPOSITION 1. *Assume $2m > N$. Let $\phi \in \mathcal{E}_m(\Omega)$ and let the constant $k > 0$ be such that*

$$\mathrm{Re}\langle H_\phi f, f\rangle \geqslant -k\|f\|_2^2 \tag{11}$$

for all $f \in C_c^\infty(\Omega)$. Then for any $\delta > 0$ there exists a constant c_δ such that

$$|K(t, x, y)| \leqslant c_\delta t^{-N/2m} \exp\{\phi(x) - \phi(y) + (1 + \delta)kt + \delta t\} \tag{12}$$

for all $x, y \in \Omega$ and all $t > 0$.

Proof. This is Proposition 2.5 of [**4**].

3. The main estimates

Let

$$a(x,\xi)=\sum_{\substack{|\alpha|=m\\|\beta|=m}} a_{\alpha\beta}(x)\xi^{\alpha+\beta}$$

be the principal symbol of the operator H. We shall also write this in a slightly different way, namely we define the functions a_γ, $|\gamma|=2m$, by

$$a(x,\xi)=\sum_{|\gamma|=2m} c_\gamma^{2m} a_\gamma(x)\xi^\gamma$$

[so $a_{\alpha\beta}\neq a_{\alpha+\beta}$ in general]. For each $x\in\Omega$ we then define a quadratic form $\Gamma(x,\cdot)$ on the space $\mathbf{C}^\nu$, $\nu=\nu(N,m)$ being the number of multi-indices of length m, by

$$\Gamma(x,p)=\sum_{\substack{|\alpha|=m\\|\beta|=m}} a_{\alpha+\beta}(x)p_\alpha\bar{p}_\beta,\quad \text{all } p=(p_\alpha)\in\mathbf{C}^\nu. \tag{13}$$

DEFINITION. *The symbol $a(x,\xi)$ is strongly convex if the form Γ is non-negative for all $x\in\Omega$.*

The notion of strong convexity was first introduced in **[10]**. It is proved there that a strongly convex symbol is necessarily convex; the converse however is not always true, although it is in the second order case.

Let

$$k_m=\left[\sin\left(\frac{\pi}{4m-2}\right)\right]^{-2m+1}$$

and

$$S(x,\xi)=\operatorname{Re} a(x,\xi-i\nabla\phi(x))+k_m a(x,\nabla\phi(x)). \tag{14}$$

The following lemma was proved in **[10]**.

LEMMA 2. *There exist positive real numbers $w_0, w_1, \dots, w_{m-2}$ such that for all $x\in\Omega$ and $\xi\in\mathbf{R}^N$ we have*

$$S(x,\xi)=\sum_{s=0}^{m-2} w_s\Gamma(x,p_{x,\xi}^{(s)}) \tag{15}$$

where $p_{x,\xi}^{(s)}$ is the vector in $\mathbf{R}^\nu$ defined for all $x\in\Omega$, $\xi\in\mathbf{R}^N$ by requiring that

$$\sum_{|\alpha|=m} c_\alpha^m p_{x,\xi,\alpha}^{(s)}u^\alpha=(\xi\cdot u)^{m-s-2}(\nabla\phi(x)\cdot u)^s\{(\xi\cdot u)^2-[\cos(\pi/(4m-2))]^2(\nabla\phi\cdot u)^2\}$$

for all $u\in\mathbf{R}^N$. [So $p_{x,\xi}^{(x)}$ depends on x only via $\nabla\phi(x)$.]

Proof. See [**10**].

We shall also need the following lemma.

LEMMA 3. *Let P be a sesquilinear form on $C_c^\infty(\mathbf{R}^N)$ given by*

$$P(u,v)=\int\int p(\xi,\eta)u(\xi)\overline{v(\eta)}d\xi d\eta$$

for all $u, v \in C_c^\infty(\mathbf{R}^N)$. If the integral kernel $p(\xi,\eta)$ satisfies

$$p(\xi,\eta)=p(\eta,\xi), \quad \textit{all } \ \xi,\eta\in\mathbf{R}^N,$$

then

$$\mathrm{Re}P(u,u)=\int\int \mathrm{Re}[p(\xi,\eta)]u(\xi)\overline{u(\eta)}d\xi d\eta, \quad u\in C_c^\infty(\mathbf{R}^N).$$

Proof. For $u, v \in C_c^\infty(\mathbf{R}^N)$ we have

$$\begin{aligned}\overline{P(u,v)}&=\int\int \overline{p(\xi,\eta)u(\xi)}v(\eta)d\xi d\eta\\&=\int\int \overline{p(\eta,\xi)u(\xi)}v(\eta)d\xi d\eta\\&=\int\int \overline{p(\xi,\eta)}v(\xi)\overline{u(\eta)}d\xi d\eta\end{aligned}$$

and therefore

$$\begin{aligned}\mathrm{Re}P(u,u)&=\frac{1}{2}[P(u,u)+\overline{P(u,u)}]\\&=\int\int\frac{1}{2}[p(\xi,\eta)+\overline{p(\xi,\eta)}]u(\xi)\overline{u(\eta)}d\xi d\eta\\&=\int\int \mathrm{Re}[p(\xi,\eta)]u(\xi)\overline{u(\eta)}d\xi d\eta\end{aligned}$$

as required.

We now define a new symbol $a(x,\xi,\eta)$ of two complex variables by

$$a(x,\xi,\eta)=\sum_{\substack{|\alpha|=m\\|\beta|=m}} a_{\alpha\beta}(x)\xi^\alpha\bar{\eta}^\beta, \quad x\in\Omega, \ \ \xi,\eta\in\mathbf{C}^N.$$

Following (14) and (13) we also set

$$S(x,\xi,\eta)=\mathrm{Re}[a(x,\xi-i\nabla\phi(x),\eta+i\nabla\phi(x))]+k_m a(x,\nabla\phi(x)) \tag{16}$$

and

$$\Gamma(x,p,q)=\sum_{\substack{|\alpha|=m\\|\beta|=m}} a_{\alpha+\beta}(x)p_\alpha\bar{q}_\beta,\qquad p,q\in\mathbf{C}^\nu. \tag{17}$$

We shall denote by $\hat{f}$ the Fourier transform of a function $f\in C_c^\infty(\Omega)$,

$$\hat{f}(\xi)=(2\pi)^{-N/2}\int_\Omega e^{-i\xi\cdot x}f(x)dx,$$

and for a positive real number s we use the notation

$$\|\nabla^s f\|_2^2=\int_{\mathbf{R}^N}|\xi|^{2s}|\hat{f}(\xi)|^2 d\xi.$$

From it follows immediately the Sobolev interpolation inequality

$$\|\nabla^s f\|_2\leqslant c(\|\nabla^r f\|_2+\|\nabla^t f\|_2), \tag{18}$$

which is valid for all $r\leqslant s\leqslant t$ and all $f\in C_c^\infty(\Omega)$.

Given $\phi\in\mathcal{E}_m(\Omega)$ and a multi-index γ we define the function

$$P_{\gamma,\phi}(x)=e^{-\phi(x)}[D^\gamma e^{\phi(x)}].$$

This is a polynomial in $D^\kappa\phi$, $\kappa\leqslant\gamma$, which we write as

$$P_{\gamma,\phi}=:(\nabla\phi)^\gamma+E_{\gamma,\phi}.$$

We point out that (i) $E_{0,\phi}=0$ and (ii) $E_{\gamma,\phi}$ is a linear combination of products of derivatives of ϕ and that each term involves at least one derivative of order $\geqslant 2$. We also define the functions

$$E_{\gamma,\delta,\phi}=(\nabla\phi)^\gamma E_{\delta,-\phi}+(-\nabla\phi)^\delta E_{\gamma,\phi}+E_{\gamma,\phi}E_{\delta,-\phi}.$$

Finally, we define the quadratic form $Q_{1,\phi}$ on $W_0^{m,2}(\Omega)$ by

$$Q_{1,\phi}(f)=\int_\Omega\sum_{\substack{|\alpha|=m\\|\beta|=m}}\sum_{\substack{\gamma\leqslant\alpha\\ \delta\leqslant\beta}} a_{\alpha\beta}c_\gamma^\alpha c_\delta^\beta(\nabla\phi)^\gamma(-\nabla\phi)^\delta D^{\alpha-\gamma}f D^{\beta-\delta}\bar{f}dx.$$

LEMMA 4. *For any $\phi\in\mathcal{E}_m$ there exists a quadratic form $R_{1,\phi}(f)$ of order less than $2m$ such that*

$$Q_\phi(f)=Q_{1,\phi}(f)+R_{1,\phi}(f),\quad \textit{all}\ f\in C_c^\infty(\Omega).$$

The form $R_{1,\phi}(f)$ satisfies

$$|R_{1,\phi}(f)|\leqslant cW(A,\phi)(\|f\|_2^2+\|\nabla^{m-(1/2)}f\|_2^2),$$

where

$$W(A,\phi)=\sum_{\substack{|\gamma|\leqslant m\\|\delta|\leqslant m}}\|E_{\gamma,\delta,\phi}\|_\infty+\sup_{\substack{\|\alpha|=m\\|\beta|=m}}\sup_{k\leqslant 2m}\zeta_{1,m}(a_{\alpha\beta}|\nabla\phi|^k).$$

Proof. We have

$$\begin{aligned} Q_\phi(f) &= \int_\Omega \sum_{\substack{|\alpha|=m\\|\beta|=m}} a_{\alpha\beta} D^\alpha(e^\phi f) D^\beta(e^{-\phi}\bar{f})dx \\ &= \int_\Omega \sum_{\substack{|\alpha|=m\\|\beta|=m}} a_{\alpha\beta} \sum_{\substack{\gamma\leqslant\alpha\\ \delta\leqslant\beta}} c_\gamma^\alpha c_\delta^\beta P_{\gamma,\phi} P_{\delta,-\phi} D^{\alpha-\gamma} f D^{\beta-\delta}\bar{f}dx \\ &= Q_{1,\phi}(f) + R_{1,\phi}(f), \end{aligned}$$

where

$$R_{1,\phi}(f) = \int_\Omega \sum_{\substack{|\alpha|=m\\|\beta|=m}} \sum_{\substack{\gamma\leqslant\alpha\\ \delta\leqslant\beta}} c_\gamma^\alpha c_\delta^\beta a_{\alpha\beta} E_{\gamma,\delta,\phi} D^{\alpha-\gamma} f D^{\beta-\delta}\bar{f}dx.$$

Recalling that $E_{\gamma,\delta,\phi} = 0$ when $\gamma = \delta = 0$ and using (18) we obtain the estimate

$$|R_{1,\phi}(f)| < c\{\sup_{\substack{|\gamma|\leqslant m\\|\delta|\leqslant m}} \|E_{\gamma,\delta,\phi}\|_\infty\}(\|f\|_2^2 + \|\nabla^{m-(1/2)} f\|_2^2)$$

for all $f \in C_c^\infty(\Omega)$.

LEMMA 5. *We have*

$$\begin{aligned} &\mathrm{Re}Q_{1,\phi}(f) + k_m \int_\Omega a(x, \nabla\phi(x))|f(x)|^2 dx = \\ &= (2\pi)^{-N} \int_\Omega \int_{\mathbf{R}^N} \int_{\mathbf{R}^N} S(x,\xi,\eta) e^{i(\xi-\eta)-x} \hat{f}(\xi)\overline{\hat{f}(\eta)} d\xi d\eta dx \end{aligned}$$

for all $f \in C_c^\infty(\Omega)$.

Proof. Using elementary properties of the Fourier transform $f \mapsto \hat{f}$ we have

$$\begin{aligned} &Q_{1,\phi}(f) \\ &= (2\pi)^{-N} \int_\Omega \int_{\mathbf{R}^N} \int_{\mathbf{R}^N} \sum_{\substack{|\alpha|=m\\|\beta|=m}} a_{\alpha\beta} \sum_{\substack{\gamma\leqslant\alpha\\ \delta\leqslant\beta}} c_\gamma^\alpha c_\delta^\beta (-i\nabla\phi)^\gamma (-i\nabla\phi)^\delta \times \\ &\qquad \times \xi^{\alpha-\gamma}\eta^{\beta-\delta} e^{i(\xi-\eta)\cdot x} \hat{f}(\xi)\overline{\hat{f}(\eta)} d\xi d\eta dx \\ &= (2\pi)^{-N} \int_\Omega \int_{\mathbf{R}^N} \int_{\mathbf{R}^N} \sum_{\substack{|\alpha|=m\\|\beta|=m}} a_{\alpha\beta} (\xi - i\nabla\phi)^\alpha (\eta - i\nabla\phi)^\beta e^{i(\xi-\eta)\cdot x} \hat{f}(\xi)\overline{\hat{f}(\eta)} d\xi d\eta dx \\ &= (2\pi)^{-N} \int_\Omega \int_{\mathbf{R}^N} \int_{\mathbf{R}^N} a(x, \xi - i\nabla\phi(x), \eta + i\nabla\phi(x)) e^{i(\xi-\eta)\cdot x} \hat{f}(\xi)\overline{\hat{f}(\eta)} d\xi d\eta dx. \end{aligned}$$

Now let $x \in \Omega$ be fixed and let

$$p(\xi,\eta) = a(x, \xi - i\nabla\phi(x), \eta + i\nabla\phi(x)).$$

The fact that the coefficients $\{a_{\alpha\beta}\}$ are real implies that $p(\xi, \eta) = p(\eta, \xi)$. Applying Lemma 3 to the function $u(\xi) = e^{i\xi\cdot x}\hat{f}(\xi)$ we obtain

$$\mathrm{Re}\int_{\mathbf{R}^N}\int_{\mathbf{R}^N} a(x, \xi - i\nabla\phi(x), \eta + i\nabla\phi(x))e^{i(\xi-\eta)\cdot x}\hat{f}(\xi)\overline{\hat{f}(\eta)}d\xi d\eta =$$
$$\int_{\mathbf{R}^N}\int_{\mathbf{R}^N} \mathrm{Re}[a(x, \xi - i\nabla\phi(x), \eta + i\nabla\phi(x)]e^{i(\xi-\eta\cdot x}\hat{f}(\xi)\overline{\hat{f}(\eta)}d\xi d\eta.$$

Integrating over x and using definition (16) we conclude that

$$\mathrm{Re}Q_{1,\phi}(f) + k_m\int a(x, \nabla\phi(x))|f(x)|^2dx =$$
$$(2\pi)^{-N}\int_\Omega\int_{\mathbf{R}^N}\int_{\mathbf{R}^N} S(x, \xi, \eta)e^{i(\xi-\eta)\cdot x}\hat{f}(\xi)\overline{\hat{f}(\eta)}d\xi d\eta dx,$$

as required.

LEMMA 6. *Given $\phi \in \mathcal{E}_m$ there exists a quadratic form R_ϕ of order smaller than $2m$ such that*

$$\mathrm{Re}Q_\phi(f) + k_m\int_\Omega a(x, \nabla\phi(x))|f(x)|^2dx =$$
$$(2\pi)^{-N}\int_\Omega\int_{\mathbf{R}^N}\int_{\mathbf{R}^N}\sum_{s=0}^{m-2} w_s\Gamma(x, p^{(s)}_{x,\xi}, p^{(s)}_{x,\eta})e^{i(\xi-\eta)\cdot x}\hat{f}(\xi)\overline{\hat{f}(\eta)}d\xi d\eta dx +$$
$$R_\phi(f). \tag{19}$$

The form $R_\phi(f)$ satisfies

$$|R_\phi(f)| < cW(A, \phi)(\|f\|_2^2 + \|\nabla^{m-(1/2)}f\|_2^2). \tag{20}$$

Proof. From the last two lemmas we have

$$\mathrm{Re}Q_\phi(f) + k_m\int_\Omega a(x, \nabla\phi(x))|f(x)|^2dx$$
$$= \mathrm{Re}Q_{1,\phi}(f) + k_m\int_\Omega a(x, \nabla\phi(x))|f(x)|^2dx + \mathrm{Re}R_{1,\phi}(f)$$
$$= (2\pi)^{-N}\int_\Omega\int_{\mathbf{R}^N}\int_{\mathbf{R}^N} S(x, \xi, \eta)e^{i(\xi-\eta)\cdot x}\hat{f}(\xi)\overline{\hat{f}(\eta)}d\xi d\eta dx + \mathrm{Re}R_{1,\phi}(f).$$

Now, although (15) tells us that

$$S(x, \xi, \xi) = \sum_{s=0}^{m-2} w_s\Gamma(x, p^{(s)}_{x,\xi}, p^{(s)}_{x,\xi}), \quad x \in \Omega,\ \xi \in \mathbf{R}^N,$$

simple examples show that

$$S(x,\xi,\eta) \neq \sum_{s=0}^{m-2} w_s \Gamma(x, p_{x,\xi}^{(s)}, p_{x,\eta}^{(s)})$$

in general when $\xi \neq \eta$. We shall see however that the triple integrals of these two functions (after being multiplied by $e^{i(\xi-\eta)\cdot x}\hat{f}(\xi)\overline{\hat{f}(\eta)}$) have a difference which is small in the sense appropriate for the lemma. For this we shall need the following

CLAIM. For any three multi-indices γ, δ and κ and for any smooth function $b(x)$ that satisfies $\|D^{\kappa_1} b\|_\infty < \infty$, $\kappa_1 \leqslant \kappa$, we have

$$(2\pi)^{-N}\int_\Omega\int_{\mathbf{R}^N}\int_{\mathbf{R}^N} b(x)\xi^\gamma \eta^{\delta+\kappa} e^{i(\xi-\eta)\cdot x}\hat{f}(\xi)\overline{\hat{f}(\eta)}d\xi d\eta dx =$$
$$(2\pi)^{-N}\int_\Omega\int_{\mathbf{R}^N}\int_{\mathbf{R}^N} b(x)\xi^{\gamma+\kappa} \eta^{\delta} e^{i(\xi-\eta)\cdot x}\hat{f}(\xi)\overline{\hat{f}(\eta)}d\xi d\eta dx + S(f)$$

for all $f \in C_c^\infty$, where $S(f)$ is a form of order $|\gamma+\delta+\kappa|-1$ satisfying

$$|S(f)| \leqslant c\{\sup_{0<\kappa_1\leqslant\kappa} \|D^{\kappa_1} b\|_\infty\}(\|\nabla^{(|\gamma+\delta|)/2} f\|_2^2 + \|\nabla^{(|\gamma+\delta+\kappa|-1)/2} f\|_2^2). \tag{21}$$

Proof of claim. We have

$$\begin{aligned}\text{LHS} &= (-i)^{|\gamma|} i^{|\delta+\kappa|}\int_\Omega b(x) D^\gamma f D^{\delta+\kappa}\bar{f}dx \\ &= (-i)^{|\gamma+\kappa|} i^{|\delta|}\int_\Omega D^\kappa\{b(x)D^\gamma f\} D^\delta \bar{f}dx \\ &= (-i)^{|\gamma+\kappa|} i^{|\delta|}\int_\Omega b(x) D^{\gamma+\kappa} f D^\delta \bar{f}dx + S(f) \\ &= \text{RHS} + S(f)\end{aligned}$$

where

$$S(f) = (-i)^{|\gamma+\kappa|} i^{|\delta|}\int_\Omega \sum_{0<\kappa_1\leqslant\kappa} c_{\kappa_1}^\kappa D^{\kappa_1} b D^{\gamma+\kappa-\kappa_1} f D^\delta \bar{f} dx$$

is a lower order term that satisfies (21) because of (18). Hence the claim is proved. We now return to the form $Q_{1,\phi}(f)$ which we write as

$$Q_{1,\phi}(f) = \sum_{\substack{|\alpha|=m\\|\beta|=m}}\sum_{\substack{\gamma\leqslant\alpha\\\delta\leqslant\beta}} Q_{1,\phi,\alpha\beta\gamma\delta}(f),$$

where

$$Q_{1,\phi,\alpha\beta\gamma\delta}(f) = (2\pi)^{-N}\int_\Omega\int_{\mathbf{R}^N}\int_{\mathbf{R}^N} b_{\alpha\beta\gamma\delta}(x)\xi^{\alpha-\gamma}\eta^{\beta-\delta} e^{i(\xi-\eta)\cdot x}\hat{f}(\xi)\overline{\hat{f}(\eta)}d\xi d\eta dx$$

with

$$b_{\alpha\beta\gamma\delta}(x) = c_\gamma^\alpha c_\delta^\beta a_{\alpha\beta}(x)(-i\nabla\phi(x))^{\gamma+\delta}.$$

From the claim we have

$$\begin{aligned}
&Q_{1,\phi}(f) \\
&= (2\pi)^{-N} \int_\Omega \int_{\mathbf{R}^N} \int_{\mathbf{R}^N} \sum_{\substack{|\alpha|=m \\ |\beta|=m}} \sum_{\substack{\gamma\leqslant\alpha \\ \delta\leqslant\beta}} b_{\alpha\beta\gamma\delta}\xi^{\alpha-\gamma}\eta^{\beta-\delta} e^{i(\xi-\eta)\cdot x} \hat{f}(\xi)\overline{\hat{f}(\eta)} d\xi d\eta dx \\
&= (2\pi)^{-N} \int_\Omega \int_{\mathbf{R}^N} \int_{\mathbf{R}^N} \sum_{\substack{|\alpha|=m \\ |\beta|=m}} \sum_{\substack{\gamma\leqslant\alpha \\ \delta\leqslant\beta}} b_{\alpha\beta\gamma\delta}\xi^{\alpha+\beta-\gamma-\delta} e^{i(\xi-\eta)\cdot x} \hat{f}(\xi)\overline{\hat{f}(\eta)} d\xi d\eta dx \\
&+ \sum_{\substack{|\alpha|=m \\ |\beta|=m}} \sum_{\substack{\gamma\leqslant\alpha \\ \delta\leqslant\beta}} R_{2,\phi,\alpha\beta\gamma\delta},
\end{aligned}$$

where

$$\begin{aligned}
|R_{2,\phi,\alpha\beta\gamma\delta}| &\leqslant \sup_{0<\kappa_1\leqslant\beta-\delta} \|D^{\kappa_1} b_{\alpha\beta\gamma\delta}\|_\infty \{\|\nabla^{(|\alpha-\gamma|)/2} f\|_2^2 + \|\nabla^{(2m-|\gamma+\delta|-1)/2} f\|_2^2\} \\
&\leqslant c \sup_{\substack{|\alpha|=m \\ |\beta|=m}} \sup_{|\kappa|\leqslant 2m} \zeta_{1,m}(a_{\alpha\beta}\nabla\phi^\kappa)(\|f\|_2^2 + \|\nabla^{m-(1/2)} f\|_2^2).
\end{aligned}$$

Therefore, using the fact that

$$\sum_{\substack{|\alpha|=m \\ |\beta|=m}} \sum_{\substack{\gamma\leqslant\alpha \\ \delta\leqslant\beta}} b_{\alpha\beta\gamma\delta}(x)\xi^{\alpha-\gamma}\eta^{\beta-\delta} = a(x, \xi - i\nabla\phi(x), \eta + i\nabla\phi(x))$$

as well as (19), we have

$$\begin{aligned}
&\mathrm{Re} Q_{1,\phi}(f) + k_m \textstyle\int_\Omega a(x, \nabla\phi(x))|f(x)|^2 dx \\
&= (2\pi)^{-N} \int_\Omega \int_{\mathbf{R}^N} \int_{\mathbf{R}^N} S(x,\xi,\xi) e^{i(\xi-\eta)\cdot x} \hat{f}(\xi)\overline{\hat{f}(\eta)} d\xi d\eta dx + R_{2,\phi}(f) \\
&= (2\pi)^{-N} \int_\Omega \int_{\mathbf{R}^N} \int_{\mathbf{R}^N} \sum_{s=0}^{m-2} w_s \Gamma(x, p_{x,\xi}^{(s)}, p_{x,\xi}^{(s)}) e^{i(\xi-\eta)\cdot x} \hat{f}(\xi)\overline{\hat{f}(\eta)} d\xi d\eta dx + R_{2,\phi}(f),
\end{aligned}$$

where

$$R_{2,\phi}(f) = \sum_{\substack{|\alpha|=m \\ |\beta|=m}} \sum_{\substack{\gamma\leqslant\alpha \\ \delta\leqslant\beta}} \mathrm{Re} R_{2,\phi,\alpha\beta\gamma\delta}(f),$$

satisfies

$$|R_{2,\phi}(f)| \leqslant c W(A,\phi)(\|f\|_2^2 + \|\nabla^{m-(1/2)} f\|_2^2).$$

By another application of the claim, we have

$$(2\pi)^{-N}\int_\Omega\int_{\mathbf{R}^N}\int_{\mathbf{R}^N}\sum_{s=0}^{m-2} w_s\Gamma(x, p^{(s)}_{x,\xi}, p^{(s)}_{x,\xi})e^{i(\xi-\eta)\cdot x}\hat{f}(\xi)\overline{\hat{f}(\eta)}d\xi d\eta dx$$
$$= (2\pi)^{-N}\int_\Omega\int_{\mathbf{R}^N}\int_{\mathbf{R}^N}\sum_{s=0}^{m-2} w_s\Gamma(x, p^{(s)}_{x,\xi}, p^{(s)}_{x,\eta})e^{i(\xi-\eta)\cdot x}\hat{f}(\xi)\overline{\hat{f}(\eta)}d\xi d\eta dx + R_3(f),$$

where

$$|R_3(f)| \leqslant c \sup_{\substack{|\alpha|=m\\|\beta|=m}} \sup_{|\kappa|\leqslant 2m} \zeta_{1,m}(a_{\alpha\beta}\nabla\phi^\kappa)(\|f\|_2^2 + \|\nabla^{(m-1)/2}f\|_2^2).$$

Combining the above results we conclude that (19) is valid for a form $R_\phi(f)$ that satisfies the required estimate (20).

From now on we consider functions in $\mathcal{E}_m(\Omega)$ that are of the form $\lambda\phi$, with $\lambda > 0$ and $\phi \in \mathcal{E}_a$.

PROPOSITION 7. *Let the symbol $a(x,\xi)$ be strongly convex. Then for $\phi \in \mathcal{E}_a$ and $\lambda > 0$ we have*

$$\mathrm{Re}Q_{\lambda\phi}(f) \geqslant -k_m\lambda^{2m}\|f\|_2^2 - R_{\lambda\phi}(f), \tag{22}$$

where

$$|R_{\lambda\phi}(f)| < cW(A,\lambda\phi)(\|f\|_2^2 + \|\nabla^{m-(1/2)}f\|_2^2). \tag{23}$$

Proof. Assume first that $\lambda = 1$. Let $x \in \Omega$ and $0 \leqslant s \leqslant m-2$ be fixed and let $f \in C_c^\infty(\Omega)$. We define the vector $q(x) \in \mathbf{C}^\nu$ by

$$q_\alpha(x) = \int_{\mathbf{R}^N} p^{(s)}_{x,\xi,\alpha}e^{i\xi\cdot x}\hat{f}(\xi)d\xi.$$

[Note that the integral converges absolutely since $f \in C_c^\infty(\Omega)$.] Assumption (5) and (17) then imply

$$0 \leqslant \sum_{\substack{|\alpha|=m\\|\beta|=m}} a_{\alpha\beta}(x)q_\alpha(x)\overline{q_\beta(x)}$$
$$= \sum_{\substack{|\alpha|=m\\|\beta|=m}} a_{\alpha\beta}(x)\left\{\int_{\mathbf{R}^N} p^{(s)}_{x,\xi,\alpha}e^{i\xi\cdot x}\hat{f}(\xi)d\xi\right\}\left\{\int_{\mathbf{R}^N} p^{(s)}_{x,\eta,\beta}e^{\eta\cdot x}\overline{\hat{f}(\eta)d\eta}\right\}$$
$$= \int_{\mathbf{R}^N}\int_{\mathbf{R}^N}\sum_{\substack{|\alpha|=m\\|\beta|=m}} a_{\alpha\beta}(x)p^{(s)}_{x,\xi,\alpha}p^{(s)}_{x,\eta,\beta}e^{i(\xi-\eta)\cdot x}\hat{f}(\xi)\overline{\hat{f}(\eta)}d\xi d\eta$$
$$= \int_{\mathbf{R}^N}\int_{\mathbf{R}^N}\Gamma(x, p^{(s)}_{x,\xi}, p^{(s)}_{x,\eta})e^{i(\xi-\eta)\cdot x}\hat{f}(\xi)\overline{\hat{f}(\eta)}d\xi d\eta.$$

Multiplying by w_s and summing over all $0 \leqslant s \leqslant m-2$ yields

$$\int_{\mathbf{R}^N}\int_{\mathbf{R}^N}\sum_{s=0}^{m-2} w_s\Gamma(x, p_{x,\xi}^{(s)}, p_{x,\eta}^{(s)})e^{i(\xi-\eta)\cdot x}\hat{f}(\xi)\overline{\hat{f}(\eta)}d\xi d\eta \geqslant 0$$

for all $x \in \Omega$. Integration with respect to x then shows that the triple integral in the right-hand side of (19) is non-negative, and therefore

$$\mathrm{Re}Q_\phi(f) \geqslant -k_m\int_\Omega a(x, \nabla\phi(x))|f(x)|^2dx + R_\phi(f).$$

Formula (22) for $\lambda = 1$ then follows by recalling that $\phi \in \mathcal{E}_a$ imples $a(x, \nabla\phi(x)) \leqslant 1$ for all $x \in \Omega$.

If $\lambda \neq 1$ then we have

$$\int_\Omega a(x, \lambda\nabla\phi(x))|f(x)|^2dx = \lambda^{2m}\int_\Omega a(x, \nabla\phi(x))|f(x)|^2dx \leqslant \lambda^{2m}\|f\|_2^2$$

and therefore k_m is replaced by $\lambda^{2m}k_m$ when $\phi \in \mathcal{E}_a$ is replaced by $\lambda\phi$. Otherwise the proof remains the same.

We can now state and prove our main theorem. The operator H acts on $L^2(\Omega)$, is subject to Dirichlet boundary conditions and satisfies all the assumptions stated at the beginning of Section 2. The constant σ_m is as in (3).

THEOREM 8. *Let $2m > N$ and assume that the principal symbol $a(x, \xi)$ of H is strongly convex. Let*

$$q_M = q_M(t, x, y) = d_M(x, y)^{2m/(2m-1)}t^{-1/(2m-1)}.$$

Then given $\varepsilon > 0$ there exists $c_\varepsilon < \infty$ such that

$$|K(t, x, y)| \leqslant c_\varepsilon t^{-N/2m}\exp\{-(\sigma_m - \varepsilon)q_M(1 + cM^{4m}q_M^{-1/2m}t^{1/2m})\} \tag{24}$$

for all large M and all $x, y \in \Omega$ and $t > 0$ such that $q_M(t, x, y) > t$ [or equivalently, such that $d_M(x, y) > t$.]

Proof. First we note that

$$\|f\|_2^2 + \|\nabla^{m-(1/2)}f\|_2^2 < c\{Q(f) + \|f\|_2^2\}, \quad f \in C_c^\infty. \tag{25}$$

Now let $\phi \in \mathcal{E}_a$ be given. It follows from (9), with $\varepsilon = 1/2$, that

$$Q(f) < 2\mathrm{Re}Q_{\lambda\phi}(f) + c\{\zeta_{1,m}(\lambda\phi) + \zeta_{1,m}(\lambda\phi)^{2m}\}\|f\|_2^2.$$

Hence we have from Proposition 7

$$\begin{aligned}\mathrm{Re}Q_{\lambda\phi}(f) &\geqslant -k_m\lambda^{2m}\|f\|_2^2 - R_{\lambda\phi}(f)\\ &\geqslant -k_m\lambda^{2m}\|f\|_2^2 - cW(A, \lambda\phi)\{\mathrm{Re}Q_{\lambda\phi}(f) + (1 + \zeta_{1,m}(\lambda\phi)^{2m})\|f\|_2^2\}.\end{aligned}$$

Distinguishing two cases according to whether Re $Q_{\lambda\phi}(f)$ is positive or negative, we conclude that

$$\mathrm{Re} Q_{\lambda\phi}(f) \geqslant -\{k_m\lambda^{2m} + cW(A,\lambda\phi)(1+\lambda^{2m}\zeta_{1,m}(\phi)^{2m})\}\|f\|_2^2.$$

We now apply Proposition 1. Setting $S(t,x,y) = \log\{t^{N/2m}|K(t,x,y)|\}$ we have

$$S(t,x,y) \leqslant c_\delta + \lambda\phi(x) - \lambda\phi(y) + \\ +(1+\delta)\{\lambda^{2m}k_m + cW(A,\lambda\phi)(1+\lambda^{2m}\zeta_{1,m}(\phi)^{2m})\}t + \delta t \quad (26)$$

for all $0 < \delta < 1$, $\phi \in \mathcal{E}_a$, $\lambda > 0$, $t > 0$ and $x, y \in \Omega$. We next scale our operator. We fix a parameter $s > 1$ and denote by U_s the unitary operator from $L^2(s\Omega)$ onto $L^2(\Omega)$ [where $s\Omega := \{sx|x \in \Omega\}$] given by

$$U_s f(x) = s^{N/2} f(sx), \quad x \in \Omega.$$

We then define $H' = s^{-2m}U_s^{-1}HU_s$ on $L^2(s\Omega)$ so that the coefficient matrix and heat kernel of H' are given by

$$A'(x) = A(s^{-1}x)$$

and

$$K'(t,x,y) = s^{-N}K(s^{-2m}t, s^{-1}x, s^{-1}y)$$

respectively. Morevoer the function

$$\phi'(x) := s\phi(s^{-1}x)$$

lies in $\mathcal{E}_{a'}$, where $a'(x,\xi)$ is the symbol of H'. We also note that for $s > 1$ we have

$$\zeta_{1,m}(a'_{\alpha\beta}(\nabla\phi')^\kappa) \leqslant cs^{-1}\zeta_{1,m}(a_{\alpha\beta}(\nabla\phi)^\kappa)$$

and

$$\sup_{\substack{|\gamma|\leqslant m \\ |\delta|\leqslant m}} \|E_{\gamma,\delta,\phi'}\|_\infty \leqslant cs^{-1} \sup_{\substack{|\gamma|\leqslant m \\ |\delta|\leqslant m}} \|E_{\gamma,\delta,\phi}\|_\infty$$

so that

$$W(A',\lambda\phi') \leqslant cs^{-1}W(A,\lambda\phi) \leqslant cs^{-1}(\lambda\zeta_{1,m}(\phi) + \lambda^{2m}\zeta_{2,m}(\phi)^{2m}).$$

We now apply (26) to $S'(t,x,y)$ (with ϕ replaced by ϕ'). For this we first note that the constants c_δ and c in (12) and (23), and therefore also in (26), are independent of the domain Ω. This is due to the fact that we work with Dirichlet boundary conditions and, therefore, constants that are good for $\Omega = \mathbf{R}^N$ are good for any

Ω. We therefore have

$$\begin{aligned}S(t,x,y) &= S'(s^{2m}t, sx, sy)\\ &\leqslant c_\delta + \lambda s[\phi(x)-\phi(y)] + (1+\delta)[\lambda^{2m}k_m +\\ &\quad cW(A',\lambda\phi')(1+\lambda^{2m}\zeta_{1,m}(\phi)^{2m}]s^{2m}t + \delta s^{2m}t\\ &\leqslant c_\delta + \lambda s[\phi(x)-\phi(y)] + (1+\delta)[\lambda^{2m}k_m +\\ &\quad cs^{-1}(\lambda\zeta_{1,m}(\phi) + \lambda^{4m}\zeta_{1,m}(\phi)^{4m}]s^{2m}t + \delta s^{2m}t\end{aligned}$$

for all $0<\delta<1$, $\lambda>0$, $\phi\in\mathcal{E}_a$, $s>1$ and $t>0$, $x,y\in\Omega$.

Taking the infimum of the right-hand side over all $\phi\in\mathcal{E}_{a,M}$, M large, yields

$$S(t,x,y)\leqslant c_\delta - \lambda s d_M(x,y) + (1+\delta)\lambda^{2m}s^{2m}k_m + cM^{4m}(\lambda+\lambda^{4m})s^{2m-1}t + \delta s^{2m}t.$$

[Note that because of the uniform ellipticity of H the condition $a(x,\nabla\phi(x))\leqslant 1$ automatically implies $|\nabla\phi(x)|\leqslant M$ if M is large enough.] We now choose

$$\lambda = s^{-1}\left(\frac{d_M(x,y)}{2mk_m t}\right)^{1/(2m-1)},$$

and observe that we then have

$$-\lambda s d_M(x,y) + \lambda^{2m}s^{2m}k_m t = -\sigma_m q_M(t,x,y).$$

Since we also have

$$\lambda s^{2m-1}t = cs^{2m-2}q_M^{1/2m}t^{(2m-1)/2m},\quad \lambda^{4m}s^{2m-1}t = cs^{-2m-1}q_M^2 t^{-1},$$

we have, after the substitution,

$$\begin{aligned}S(t,x,y)&\leqslant c_\delta - (\sigma_m - c\delta)q_M +\\ &+cM^{4m}s^{2m-2}q_M^{1/2m}t^{(2m-1)/2m} + cM^{4m}s^{-2m-1}q_M^2 t^{-1} + \delta s^{2m}t.\end{aligned}$$

Choosing $s = q_M^{1/2m}t^{-1/2m}$ (so $s>1$) we have

$$s^{2m-2}q_M^{1/2m}t^{(2m-1)/2m} = s^{-2m-1}q_M^2 t^{-1} = q_M^{(2m-1)/2m}t^{1/2m},\quad s^{2m}t = q_M,$$

and therefore the result follows upon putting $\delta = c_1\varepsilon$ for a suitable value of c_1.

COROLLARY 9. *Given $M>0$ and $\varepsilon>0$ there exists $k>0$ and $c_{\varepsilon,M}$ such that*

$$|K(t,x,y)|\leqslant c_{\varepsilon,M}t^{-N/2m}\exp\{-(\sigma_m-\varepsilon)d_M^{2m/(2m-1)}(x,y)t^{-1/(2m-1)} + kt\} \tag{27}$$

for all $x,y\in\Omega$ and $t>0$.

Proof. If $t<q_M(x,y)$ then (27) follows from Theorem 8; if $t>q_M(x,y)$ then the dominent term in the exponential in (1) is $c_3 t$ and (27) again follows.

Remarks. 1. This improves upon earlier results of this type. Moreover, the estimates in [**17**] are only valid for x and y sufficiently close and t near zero. The range of validity of our estimates is significantly wider in both Theorem 8 and Corollary 9.

2. As mentioned at the beginning of Section 2, Theorem 8 remains true if we consider operators acting not only on the whole of $\mathbf{R}^N$ but on any domain $\Omega \subset \mathbf{R}^N$. This is particularly useful when dealing with domains that are 'highly non-convex', such as spirals or domains that result from removing hypersurfaces from other domains. In such cases the Finsler-type distance can be arbitrarily larger than the Euclidean.

3. Owen [**15**] recently proved for the case of Δ^2 on a C^2 Euclidean domain Ω that

$$(1 - c_M)d(x, y) \leqslant d_M(x, y), \quad \text{all } x, y \in \Omega$$

where $c_M \to 0$ as $M \to \infty$ and $d(x, y)$ is the actual Finsler metric (Riemannian in this case). The same should be true for the general case of higher order operators with variable coefficients, so that the Finsler-type metric in Theorem 8 is replaced by the actual Finsler metric. This is currently being investigated.

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