Boolean Coverings of Quantum Observable Structure: A Setting for an Abstract Differential Geometric Mechanism

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Abstract

We develop the idea of employing localization systems of Boolean coverings, associated with measurement situations, in order to comprehend structures of Quantum Observables. In this manner, Boolean domain observables constitute structure sheaves of coordinatization coefficients in the attempt to probe the Quantum world. Interpretational aspects of the proposed scheme are discussed with respect to a functorial formulation of information exchange, as well as, quantum logical considerations. Finally, the sheaf theoretical construction suggests an opearationally intuitive method to develop differential geometric concepts in the quantum regime.

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1 Introduction

The main guiding idea in our investigation is based on the employment of objects belonging to the Boolean species of observable structure, as covers, for the understanding of the objects belonging to the quantum species of observable structure. The language of Category theory [1, 2] proves to be suitable for the implementation of this idea in a universal way. The conceptual essence of this scheme is the development of a sheaf theoretical perspective [3, 4] on Quantum observable structures.

The physical interpretation of the categorical framework makes use of the analogy with geometric manifold theory. Namely, it is associated with the development of a Boolean manifold picture, that takes place through the identification of Boolean charts in systems of localization for quantum event algebras with reference frames, relative to which the results of measurements can be coordinatized. In this sense, any Boolean chart in a localization system covering a quantum algebra of events, corresponds to a set of Boolean events which become realizable in the experimental context of a measurement situation. This identification amounts to the introduction of a relativity principle in Quantum theory, suggesting a contextual interpretation of its descriptive apparatus.

In quantum logical approaches the notion of event, associated with the measurement of an observable, is taken to be equivalent to a proposition describing the behavior of a physical system. This formulation of Quantum theory is based on the identification of propositions with projection operators on a complex Hilbert space. In this sense, the Hilbert space formalism of Quantum theory associates events with closed subspaces of a separable, complex Hilbert space corresponding to a quantum system. Then, the quantum event algebra is identified with the lattice of closed subspaces of the Hilbert space, ordered by inclusion and carrying an orthocomplementation operation which is given by the orthogonal complements of the closed subspaces [5-6]. Equivalently it is isomorphic to the partial Boolean algebra of closed subspaces of the Hilbert space of the system, or alternatively the partial Boolean algebra of projection operators of the system [7].

We argue that the set theoretical axiomatizations of quantum observable structures hides the intrinsic significance of Boolean localizing systems in the formation of these structures. Moreover, the operational procedures followed in quantum measurement are based explicitly in the employment of appropriate Boolean environments. The construction of these contexts of observation are related with certain abstractions and can be metaphorically considered as pattern recognition arrangements. In the categorical language we adopt, we can explicitly associate them with appropriate Boolean coverings of the structure of quantum events. In this way, the real significance of a quantum structure proves to be, not at the level of events, but at the level of gluing together observational contexts. The main thesis of this paper is that the objectification of a quantum observable structure takes place through Boolean reference frames that can be pasted together using category theoretical means. Contextual topos theoretical approaches to quantum structures have been considered, from a different viewpoint in [8,9], and discussed in [10-12].

In Section 2 we define event and observable structures in a category theoretical language. In Section 3 we introduce the functorial concepts of Boolean coordinatizations and Boolean observable presheaves, and also, develop the idea of fibrations over Boolean observables. In Section 4 we prove the existence of an adjunction between the topos of presheaves of Boolean observables and the category of Quantum observables. In Section 5 we define systems of localization for measurement of observables over a quantum event algebra. In Section 6 we talk about isomorphic representations of quantum algebras in terms of Boolean localization systems using the adjunction established. In Section 7 we examine the consequences of the scheme related to the interpretation of the logic of quantum propositions. In Section 8 we discuss the implications of covering systems in relation to the possibility of development of a differential geometric machinery suitable for the quantum regime. Finally, we summarize the conclusions in Section 9.

2 Event and Observable Structures as Categories

A **Quantum event structure** is a category, denoted by $\mathcal{L}$, which is called the category of Quantum event algebras.

Its objects, denoted by L, are Quantum algebras of events, that is orthomodular σ-orthoposets. More concretely, each object L in $\mathcal{L}$, is considered as a partially ordered set of Quantum events, endowed with a maximal element 1, and with an operation of orthocomplementation $[-]^* : L \longrightarrow L$, which satisfy, for all $l \in L$, the following conditions: [a] $l \leq 1$, [b] $l^{**} = l$, [c] $l \vee l^* = 1$, [d] $l \leq \acute{l} \Rightarrow \acute{l}^* \leq l^*$, [e] $l \bot \acute{l} \Rightarrow l \vee \acute{l} \in L$, [f] for $l, \acute{l} \in L, l \leq \acute{l}$ implies that l and $\acute{l}$ are compatible, where $0 := 1^*$, $l \bot \acute{l} := l \leq \acute{l}^*$, and the operations of meet $\wedge$ and join $\vee$ are defined as usually. We also recall that $l, \acute{l} \in L$ are compatible if the sublattice generated by $\{l, l^*, \acute{l}, \acute{l}^*\}$ is a Boolean algebra, namely if it is a Boolean sublattice. The σ-completeness condition, namely that the join of countable families of pairwise orthogonal events must exist, is also required in order to have a well defined theory of observables over L.

Its arrows are Quantum algebraic homomorphisms, that is maps $K \xrightarrow{H} L$, which satisfy, for all $k \in K$, the following conditions: [a] $H(1) = 1$, [b] $H(k^*) = [H(k)]^*$, [c] $k \leq \acute{k} \Rightarrow H(k) \leq H(\acute{k})$, [d] $k \perp \acute{k} \Rightarrow H(k \vee \acute{k}) \leq H(k) \vee H(\acute{k})$, [e] $H(\bigvee_n k_n) = \bigvee_n H(k_n)$, where $k_1, k_2, \ldots$ countable family of mutually orthogonal events.

A **Classical event structure** is a category, denoted by $\mathcal{B}$, which is called the category of Boolean event algebras. Its objects are σ-Boolean algebras of events and its arrows are the corresponding Boolean algebraic homomorphisms.

The notion of observable corresponds to a physical quantity that can be measured in the context of an experimental arrangement. In any measurement situation the propositions that can be made concerning a physical quantity are of the following type: the value of the physical quantity lies in some Borel set of the real numbers. A proposition of this form corresponds to an event as it is apprehended by an observer using his measuring instrument. An observable Ξ is defined to be an algebraic homomorphism from the Borel algebra of the real line $Bor(\mathbf{R})$ to the quantum event algebra L.

$$\Xi : Bor(\mathbf{R}) \to L$$

such that: [i] $\Xi(\emptyset) = 0, \Xi(\mathbf{R}) = 1$, [ii] $E \bigcap F = \emptyset \Rightarrow \Xi(E) \perp \Xi(F)$, for $E, F \in Bor(\mathbf{R})$, [iii] $\Xi(\bigcup_n E_n) = \bigvee_n \Xi(E_n)$, where $E_1, E_2, \ldots$ sequence of mutually disjoint Borel sets of the real line.

If L is isomorphic with the orthocomplemented lattice of orthogonal projections on a Hilbert space, then it follows from von Neumann's spectral theorem that the observables are in 1-1 correspondence with the hypermaximal Hermitian operators on the Hilbert space.

A **Quantum observable structure** is a category, denoted by $\mathcal{O}_Q$, which is called the category of Quantum observables. Its objects are the quantum observables $\Xi : Bor(\mathbf{R}) \to L$ and its arrows $\Xi \longrightarrow \Theta$ are the commutative triangles [Diagram 1], or equivalently the quantum algebraic homomorphisms $L \xrightarrow{H} K$ in $\mathcal{L}$, preserving by definition the join of countable families of pairwise orthogonal events, such that $\Theta = H \circ \Xi$ in [Diagram 1] is again a quantum observable.

Correspondingly, a **Boolean observable structure** is a category, denoted by $\mathcal{O}_B$, which is called the category of Boolean observables. Its objects are the Boolean observables $\xi : Bor(\mathbf{R}) \to B$ and its arrows are the Boolean algebraic homomorphisms $B \xrightarrow{h} C$ in $\mathcal{B}$, such that $\theta = h \circ \xi$ in [Diagram 2]

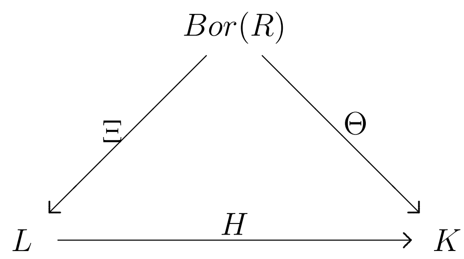

Diagram 1

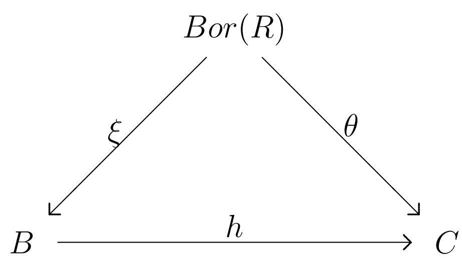

Diagram 2

is again a Boolean observable.

3 Functorial Formulation of Observables

3.1 Presheaves of Boolean Observables

If ${\mathcal{O}_B}^{op}$ is the opposite category of $\mathcal{O}_B$, then $\mathbf{Sets}^{{\mathcal{O}_B}^{op}}$ denotes the functor category of presheaves on Boolean observables. Its objects are all functors $\mathbf{X} : {\mathcal{O}_B}^{op} \longrightarrow \mathbf{Sets}$ and its morphisms are all natural transformations between such functors. Each object $\mathbf{X}$ in this category is a contravariant set-valued functor on $\mathcal{O}_B$, called a presheaf on $\mathcal{O}_B$.

For each Boolean observable ξ of $\mathcal{O}_B$, $\mathbf{X}(\xi)$ is a set, and for each arrow

$f:\theta \longrightarrow \xi$, $\mathbf{X}(f):\mathbf{X}(\xi)\longrightarrow \mathbf{X}(\theta)$ is a set function. If $\mathbf{X}$ is a presheaf on $\mathcal{O}_B$ and $x \in \mathbf{X}(\theta)$, the value $\mathbf{X}(f)(x)$ for an arrow $f:\theta\longrightarrow\xi$ in $\mathcal{O}_B$ is called the restriction of x along f and is denoted by $\mathbf{X}(f)(x) = x \circ f$.

Each object ξ of $\mathcal{O}_B$ gives rise to a contravariant Hom-functor $\mathbf{y}[\xi] := Hom_{\mathcal{O}_B}(-,\xi)$. This functor defines a presheaf on $\mathcal{O}_B$. Its action on an object θ of $\mathcal{O}_B$ is given by

$$\mathbf{y}[\xi](\theta) := Hom_{\mathcal{O}_B}(\theta,\xi)$$

whereas its action on a morphism $\eta \xrightarrow{w} \theta$, for $v:\theta\longrightarrow\xi$ is given by

$$\mathbf{y}[\xi](w) : Hom_{\mathcal{O}_B}(\theta,\xi)\longrightarrow Hom_{\mathcal{O}_B}(\eta,\xi)$$

$$\mathbf{y}[\xi](w)(v) = v \circ w$$

Furthermore $\mathbf{y}$ can be made into a functor from $\mathcal{O}_B$ to the contravariant functors on $\mathcal{O}_B$

$$\mathbf{y}:\mathcal{O}_B\longrightarrow \mathbf{Sets}^{{\mathcal{O}_B}^{op}}$$

such that $\xi \mapsto Hom_{\mathcal{O}_B}(-,\xi)$. This is an embedding and it is a full and faithful functor.

The functor category of presheaves on Boolean observables $\mathbf{Sets}^{{\mathcal{O}_B}^{op}}$ provides an instantiation of a structure known as topos. A topos exemplifies a well defined notion of variable set. It can be conceived as a local mathematical framework corresponding to a generalized model of set theory or as a generalized space. Moreover it provides a natural example of a many-valued truth structure, which remarkably is not ad hoc, but reflects genuine constraints of the surrounding universe.

3.2 The Grothendieck Fibration Technique

Since $\mathcal{O}_B$ is a small category, there is a set consisting of all the elements of all the sets $\mathbf{X}(\xi)$, and similarly there is a set consisting of all the functions $\mathbf{X}(f)$. This observation regarding $\mathbf{X} : {\mathcal{O}_B}^{op} \longrightarrow \mathbf{Sets}$ permits us to take the disjoint union of all the sets of the form $\mathbf{X}(\xi)$ for all objects ξ of $\mathcal{O}_B$. The elements of this disjoint union can be represented as pairs (ξ, x) for all objects ξ of $\mathcal{O}_B$ and elements $x \in \mathbf{X}(\xi)$. Thus the disjoint union of sets is made by labelling the elements. Now we can construct a category whose set of objects is the disjoint union just mentioned. This structure is called the category of elements of the presheaf $\mathbf{X}$, denoted by $\mathbf{G}(\mathbf{X}, \mathcal{O}_\mathbf{B})$. Its objects are all pairs (ξ, x), and its morphisms $(\acute{\xi}, \acute{x}) \longrightarrow (\xi, x)$ are those morphisms $u : \acute{\xi} \longrightarrow \xi$ of $\mathcal{O}_B$ for which $xu = \acute{x}$. Projection on the second coordinate of $\mathbf{G}(\mathbf{X}, \mathcal{O}_\mathbf{B})$ defines a functor $\mathbf{G}_\mathbf{X} : \mathbf{G}(\mathbf{X}, \mathcal{O}_\mathbf{B}) \longrightarrow \mathcal{O}_\mathbf{B}$. $\mathbf{G}(\mathbf{X}, \mathcal{O}_\mathbf{B})$ together with the projection functor $\mathbf{G}_\mathbf{X}$ is called the split discrete fibration induced by $\mathbf{X}$, and $\mathcal{O}_B$ is the base category of the fibration. We note that the fibration is discrete because the fibers are categories in which the only arrows are identity

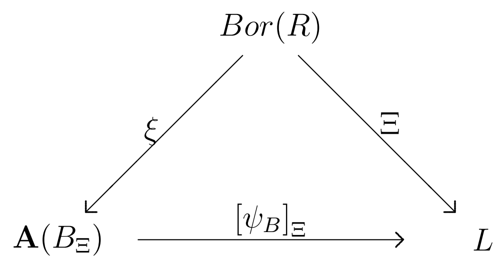

Diagram 3

arrows. If ξ is a Boolean observable object of $\mathcal{O}_B$, the inverse image under $\mathbf{G_X}$ of ξ is simply the set $\mathbf{X}(\xi)$, although its elements are written as pairs so as to form a disjoint union. The instantiation of the fibration induced by $\mathbf{X}$, is an application of the general Grothendieck construction [13].

3.3 Boolean Modelling Functor

We define a modelling or coordinatization functor $\mathbf{A} : \mathcal{O}_B \longrightarrow \mathcal{O}_Q$ which assigns to Boolean observables in $\mathcal{O}_B$ (that plays the role of the model category) the underlying Quantum observables from $\mathcal{O}_Q$, and to Boolean homomorphisms the underlying quantum algebraic homomorphisms. Hence $\mathbf{A}$ acts as a forgetful functor, forgetting the extra Boolean structure of $\mathcal{O}_B$.

Equivalently, the coordinatization functor can be characterized as, $\mathbf{A} : \mathcal{B} \longrightarrow \mathcal{L}$ which assigns to Boolean event algebras in $\mathcal{B}$ the underlying quantum event algebras from $\mathcal{L}$ and to Boolean homomorphisms the underlying quantum algebraic homomorphisms, such that [Diagram 3] commutes.

3.4 Functorial Relation of Event with Observable Algebras

The categories of Event algebras and Observables are related functorialy as follows: Under the action of a modelling functor, $Bor(\mathbf{R})$ may be considered as an object of $\mathcal{L}$. Hence, it is possible to construct the covariant representable functor $\mathbf{F} : \mathcal{L} \longrightarrow \mathbf{Sets}$, defined by $\mathbf{F} = Hom_{\mathcal{L}}(Bor(\mathbf{R}), -)$. The application of the fibration technique on the functor $\mathbf{F}$ provides the category of elements of this functor, which is the category of all arrows in $\mathcal{L}$ from the object $Bor(\mathbf{R})$, characterized equivalently as the comma category $[Bor(\mathbf{R})/\mathcal{L}]$. We conclude that the category of Quantum observables $\mathcal{O}_Q$ is actually the comma category $[Bor(\mathbf{R})/\mathcal{L}]$ or, equivalently, the category of elements of the functor $\mathbf{F} = Hom_{\mathcal{L}}(Bor(\mathbf{R}), -)$. Analogous comments hold for the category of Boolean observables.

4 Adjointness between Presheaves of Boolean Observables and Quantum Observables

We consider the category of quantum observables $\mathcal{O}_Q$ and the modelling functor $\mathbf{A}$, and we define the functor $\mathbf{R}$ from $\mathcal{O}_Q$ to the topos of presheaves

$$
\begin{array}{ccc}
\mathbf{X}(\xi) & \xrightarrow{\tau_\xi} & Hom_{\mathcal{O}_Q}(\mathbf{A}(\xi), \Xi) \\
\mathbf{X}(u) \Big\downarrow & & \Big\downarrow {}^{*}\mathbf{A}(u) \\
\mathbf{X}(\acute{\xi}) & \xrightarrow{\tau_\xi} & Hom_{\mathcal{O}_Q}(\mathbf{A}(\acute{\xi}), \Xi)
\end{array}
$$

Diagram 4

given by

$$\mathbf{R}(\Xi) : \xi \mapsto Hom_{\mathcal{O}_Q}(\mathbf{A}(\xi), \Xi)$$

A natural transformation τ between the topos of presheaves on the category of Boolean observables $\mathbf{X}$ and $\mathbf{R}(\Xi)$, $\tau : \mathbf{X} \longrightarrow \mathbf{R}(\Xi)$ is a family τ_ξ indexed by Boolean observables ξ of $\mathcal{O}_B$ for which each τ_ξ is a map

$$\tau_\xi : \mathbf{X}(\xi) \rightarrow Hom_{\mathcal{O}_Q}(\mathbf{A}(\xi), \Xi)$$

of sets, such that the diagram of sets [Diagram 4] commutes for each Boolean homomorphism $u : \acute{\xi} \rightarrow \xi$ of $\mathcal{O}_B$.

If we make use of the category of elements of the Boolean observables-variable set $\mathbf{X}$, being an object in the topos of presheaves, then the map τ_ξ, defined above, can be characterized as:

$$\tau_\xi : (\xi, p) \rightarrow Hom_{\mathcal{O}_Q}(\mathbf{A} \circ G_{\mathbf{X}}(\xi, p), \Xi)$$

Equivalently, such a τ can be seen as a family of arrows of $\mathcal{O}_Q$ which is being indexed by objects (ξ, p) of the category of elements of the presheaf of

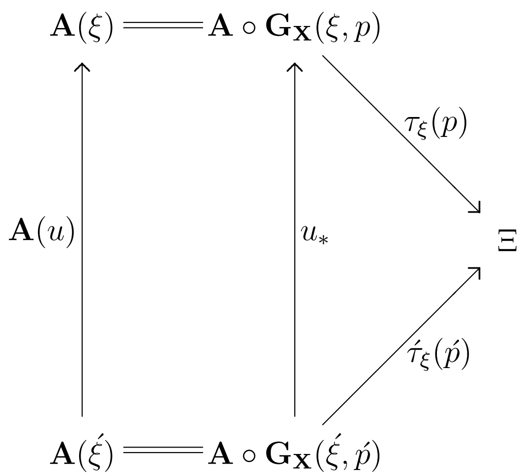

Diagram 5

Boolean observables $\mathbf{X}$, namely

$$\{\tau_\xi(p) : \mathbf{A}(\xi) \to \Xi\}_{(\xi,p)}$$

From the perspective of the category of elements of $\mathbf{X}$, the condition of the commutativity of [Diagram 4] is equivalent with the condition that for each Boolean homomorphism $u : \acute{\xi} \to \xi$ of $\mathcal{O}_B$, [Diagram 5] commutes.

From [Diagram 5] we can see that the arrows $\tau_\xi(p)$ form a cocone from the functor $\mathbf{A} \circ G_{\mathbf{X}}$ to the quantum observable algebra object Ξ. Making use of the definition of the colimit, we conclude that each such cocone emerges by the composition of the colimiting cocone with a unique arrow from the colimit $\mathbf{LX}$ to the quantum observable object Ξ. In other words, there is a bijection which is natural in $\mathbf{X}$ and Ξ

$$Nat(\mathbf{X}, \mathbf{R}(\Xi)) \cong Hom_{\mathcal{O}_Q}(\mathbf{LX}, \Xi)$$

From the above bijection we are driven to the conclusion that the functor $\mathbf{R}$ from $\mathcal{O}_Q$ to the topos of presheaves given by

$$\mathbf{R}(\Xi) : \xi \mapsto Hom_{\mathcal{O}_Q}(\mathbf{A}(\xi), \Xi)$$

has a left adjoint $\mathbf{L} : \mathbf{Sets}^{{\mathcal{O}_B}^{op}} \to \mathcal{O}_Q$, which is defined for each presheaf of Boolean observables $\mathbf{X}$ in $\mathbf{Sets}^{{\mathcal{O}_B}^{op}}$ as the colimit

$$\mathbf{L}(\mathbf{X}) = Colim\{\mathbf{G}(\mathbf{X}, \mathcal{O}_\mathbf{B}) \xrightarrow{\mathbf{G_X}} \mathcal{O}_\mathbf{B} \xrightarrow{\mathbf{A}} \mathcal{O}_\mathbf{Q}\}$$

Consequently there is a **pair of adjoint functors** $\mathbf{L} \dashv \mathbf{R}$ as follows:

$$\mathbf{L} : \mathbf{Sets}^{[[Bor(\mathbf{R})/\mathcal{B}]]^{op}} \rightleftarrows [Bor(\mathbf{R})/\mathcal{L}] : \mathbf{R}$$

The adjunction, which will be the main interpretational tool in the proposed scheme, consists of the functors $\mathbf{L}$ and $\mathbf{R}$, called left and right adjoints with respect to each other respectively, as well as the natural bijection

$$Nat(\mathbf{X}, \mathbf{R}(\Xi)) \cong Hom_{[Bor(\mathbf{R})/\mathcal{L}]}(\mathbf{LX}, \Xi)$$

As an application we may use as $\mathbf{X}$ the representable presheaf of the topos of Boolean observables $\mathbf{y}[\xi]$. Then, the bijection defining the adjunction takes the form:

$$Nat(\mathbf{y}[\xi], \mathbf{R}(\Xi)) \cong Hom_{\mathcal{O}_Q}(\mathbf{Ly}[\xi], \Xi)$$

Because the functor $\mathbf{X} = \mathbf{y}[\xi]$ is representable, the corresponding category of elements $\mathbf{G}(\mathbf{y}[\xi], \mathcal{O}_\mathbf{B})$ has a terminal object, that is, the element $1 : \xi \longrightarrow \xi$

$$\coprod_{v:\acute{\xi}\to\xi}\mathbf{A}(\acute{\xi}) \quad \overset{\zeta}{\underset{\eta}{\rightrightarrows}} \quad \coprod_{(\xi,p)}\mathbf{A}(\xi) \xrightarrow{\chi} \mathbf{X}\otimes_{\mathcal{O}_B}\mathbf{A}$$

Diagram 6

of $\mathbf{y}[\xi](\xi)$. Therefore, the colimit of the composite $\mathbf{A} \circ \mathbf{G}_{\mathbf{y}[\xi]}$ is going to be just the value of $\mathbf{A} \circ \mathbf{G}_{\mathbf{y}[\xi]}$ on the terminal object. Thus, we have

$$\mathbf{Ly}[\xi](\xi) \cong \mathbf{A} \circ \mathbf{G}_{\mathbf{y}[\xi]}(\xi, 1_\xi) = \mathbf{A}(\xi)$$

In this way we provide a characterization of $\mathbf{A}(\xi)$ as the colimit of the representable presheaf on the category of Boolean observables.

Furthermore, the categorical syntax provides a representation of a colimit as a coequalizer of a coproduct. This representation shows that the left adjoint functor of the adjunction is like the tensor product $-\otimes_{[Bor\mathbf{R}/\mathcal{B}]}\mathbf{A}$ [14]. More specifically, the coequalizer representation of the colimit $\mathbf{LX}$ [Diagram 6] shows that the elements of $\mathbf{X}\otimes_{\mathcal{O}_B}\mathbf{A}$, considered as a set endowed with a quantum algebraic structure, are all of the form $\chi(p, q)$, or in a suggestive notation,

$$\chi(p, q) = p \otimes q, \quad p \in \mathbf{X}(\xi), q \in \mathbf{A}(\xi)$$

satisfying the coequalizer condition $pv \otimes \acute{q} = p \otimes v\acute{q}$.

5 System Of Measurement Localizations For Quantum Observables

The notion of a system of localizations for a quantum observable, which will be defined subsequently, is based on the categorical idea that the quantum object Ξ in $\mathcal{O}_Q$ is possible to be comprehended by means of appropriate covering maps $\xi \longrightarrow \Xi$ having as their domains locally defined Boolean observables ξ in $\mathcal{O}_B$. It is obvious that any single map from any modelling Boolean observable to a quantum observable is not sufficient to determine it entirely and hence, it is a priori destined to contain only a limited amount of information about it. This problem may be tackled only if we employ many structure preserving maps from the modelling Boolean observables to a quantum observable simultaneously to cover it completely.

A **system of prelocalizations** for quantum observable Ξ in $\mathcal{O}_Q$ is a subfunctor of the Hom-functor $\mathbf{R}(\Xi)$ of the form $\mathbf{S} : {\mathcal{O}_B}^{op} \to \mathbf{Sets}$, namely for all ξ in $\mathcal{O}_B$ it satisfies $\mathbf{S}(\xi) \subseteq [\mathbf{R}(\Xi)](\xi)$. Hence a system of prelocalizations for quantum observable Ξ in $\mathcal{O}_Q$ is a set $\mathbf{S}(\xi)$ of quantum algebraic homomorphisms of the form

$$\psi_\xi : \mathbf{A}(\xi) \longrightarrow \Xi, \qquad \xi \in \mathcal{O}_B$$

such that $\langle \psi_\xi : \mathbf{A}(\xi) \longrightarrow \Xi$ in $\mathbf{S}(\xi)$, and $\mathbf{A}(v) : \mathbf{A}(\acute{\xi}) \to \mathbf{A}(\xi)$ in $\mathcal{O}_Q$ for

$v : \acute{\xi} \rightarrow \xi$ in $\mathcal{O}_B$, implies $\psi_\xi \circ \mathbf{A}(v) : \mathbf{A}(\acute{\xi}) \longrightarrow \mathcal{O}_Q$ in $\mathbf{S}(\xi)\rangle$.

According to the above definition, the functional role of the Hom-functor $\mathbf{R}(\Xi)$ is equivalent to depicting a set of algebraic homomorphisms, in order to provide local coverings of a quantum observable by coordinatizing Boolean objects. We may characterize the maps $\psi_\xi : \mathbf{A}(\xi) \longrightarrow \Xi, \quad \xi \in \mathcal{O}_B$ in a system of prelocalizations for quantum observable Ξ as Boolean domain covers. Their domains B_Ξ provide Boolean coefficients associated with measurement situations. The introduction of the notion of a system of prelocalizations is forced on the basis of operational physical arguments . According to Kochen-Specker theorem it is not possible to understand completely a quantum mechanical system with the use of a single system of Boolean devices. On the other side, in every concrete experimental context, the set of events that have been actualized in this context forms a Boolean algebra. Consequently, any Boolean domain object $(B_\Xi, [\psi_B]_\Xi : \mathbf{A}(B_\Xi) \longrightarrow L)$ in a system of prelocalizations for quantum event algebra, making [Diagram 7] commutative, corresponds to a set of Boolean events that become actualized in the experimental context of B. These Boolean objects play the role of localizing devices in a quantum event structure, that are induced by measurement situations. The above observation is equivalent to the statement that a measurement-induced Boolean algebra serves as a reference frame, in a topos-theoretical environment, relative to which a measurement result is being coordinatized.

$$
\begin{array}{ccc}
Bor(\mathbf{R}) & \xrightarrow{\acute{\xi}} & \mathbf{A}(\acute{B}_{\Xi}) \\
\downarrow \xi & \searrow \Xi & \downarrow [\psi_{\acute{B}}]_{\Xi} \\
\mathbf{A}(B_{\Xi}) & \xrightarrow{[\psi_B]_{\Xi}} & L
\end{array}
$$

Diagram 7

A family of Boolean observable covers $\psi_{\xi} : \mathbf{A}(\xi) \longrightarrow \Xi, \quad \xi \in \mathcal{O}_B$ is the generator of the system of prelocalization $\mathbf{S}$ if this system is the smallest among all that contain that family. It is evident that a quantum observable, and correspondingly the quantum event algebra over which it is defined, can have many systems of measurement prelocalizations, that, remarkably, form an ordered structure. More specifically, systems of prelocalization constitute a partially ordered set under inclusion. We note that the minimal system is the empty one, namely $\mathbf{S}(\xi) = \emptyset$ for all $\xi \in \mathcal{O}_B$, whereas the maximal system is the Hom-functor $\mathbf{R}(\Xi)$ itself, or equivalently, all quantum algebraic homomorphisms $\psi_{\xi} : \mathbf{A}(\xi) \longrightarrow \Xi$.

The transition from a system of prelocalizations to a system of localizations for a quantum observable, can be realized if certain compatibility conditions are satisfied on the overlap of the modelling Boolean domain covers. In order to accomplish this it is necessary to introduce the categorical concept of pullback in $\mathcal{O}_Q$ [Diagram 8].

The pullback of the Boolean domain covers $\psi_{\xi} : \mathbf{A}(\xi) \longrightarrow \Xi, \xi \in \mathcal{O}_B$ and

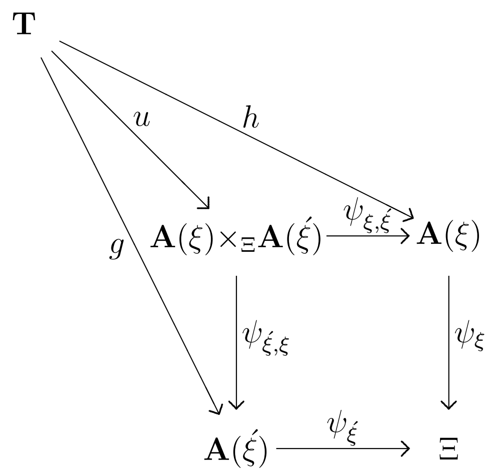

Diagram 8

$\psi_{\acute{\xi}} : \mathbf{A}(\acute{\xi}) \longrightarrow \Xi, \acute{\xi} \in \mathcal{O}_B$ with common codomain the quantum observable Ξ, consists of the object $\mathbf{A}(\xi) \times_{\Xi} \mathbf{A}(\acute{\xi})$ and two arrows $\psi_{\xi\acute{\xi}}$ and $\psi_{\acute{\xi}\xi}$, called projections, as shown in [Diagram 8]. The square commutes and for any object T and arrows h and g that make the outer square commute, there is a unique $u : T \longrightarrow \mathbf{A}(\xi) \times_{\Xi} \mathbf{A}(\acute{\xi})$ that makes the whole diagram commutative. Hence we obtain the condition: $\psi_{\acute{\xi}} \circ g = \psi_{\xi} \circ h$.

We emphasize that if ψ_{ξ} and $\psi_{\acute{\xi}}$ are injective maps, then their pullback is isomorphic with the intersection $\mathbf{A}(\xi) \cap \mathbf{A}(\acute{\xi})$. Then we can define the pasting map, which is an isomorphism, as follows:

$$\Omega_{\xi,\acute{\xi}} : \psi_{\acute{\xi}\xi}(\mathbf{A}(\xi) \times_{\Xi} \mathbf{A}(\acute{\xi})) \longrightarrow \psi_{\xi\acute{\xi}}(\mathbf{A}(\xi) \times_{\Xi} \mathbf{A}(\acute{\xi}))$$

by putting

$$\Omega_{\xi,\acute{\xi}} = \psi_{\xi\acute{\xi}} \circ {\psi_{\acute{\xi}\xi}}^{-1}$$

The following conditions hold: [i] $\Omega_{\xi,\xi} = 1_\xi, 1_\xi := id_\xi$, [ii] $\Omega_{\xi,\acute{\xi}} \circ \Omega_{\acute{\xi},\acute{\acute{\xi}}} = \Omega_{\xi,\acute{\acute{\xi}}}$ if $\mathbf{A}(\xi) \cap \mathbf{A}(\acute{\xi}) \cap \mathbf{A}(\acute{\acute{\xi}}) \neq 0$, and [iii] $\Omega_{\xi,\acute{\xi}} = \Omega_{\acute{\xi},\xi}$ if $\mathbf{A}(\xi) \cap \mathbf{A}(\acute{\xi}) \neq 0$.

The pasting map assures that $\psi_{\acute{\xi}\xi}(\mathbf{A}(\xi) \times_\Xi \mathbf{A}(\acute{\xi}))$ and $\psi_{\xi\acute{\xi}}(\mathbf{A}(\xi) \times_\Xi \mathbf{A}(\acute{\xi}))$ are going to cover the same part of the quantum observable in a compatible way.

Given a system of measurement prelocalizations for quantum observable $\Xi \in \mathcal{O}_Q$, and correspondingly for the Quantum event algebra over which it is defined, we call it a **system of localizations** if the above conditions are satisfied, and moreover, the quantum algebraic structure is preserved.

We assert that the above compatibility conditions provide the necessary relations for understanding a system of measurement localizations for a quantum observable as a structure sheaf or sheaf of Boolean coefficients consisting of local Boolean observables. This is connected to the fact that systems of measurement localizations are actually subfunctors of the representable Hom-functor $\mathbf{R}(\Xi)$ of the form $\mathbf{S} : {\mathcal{O}_B}^{op} \rightarrow \mathbf{Sets}$, namely for all ξ in $\mathcal{O}_B$ satisfy $\mathbf{S}(\xi) \subseteq [\mathbf{R}(\Xi)](\xi)$. In this sense the pullback compatibility conditions express gluing relations on overlaps of Boolean domain covers and convert a presheaf subfunctor of the Hom-functor into a sheaf. The concept of sheaf expresses exactly the pasting conditions that local modelling objects have to satisfy, namely, the way by which local data, providing Boolean coefficients obtained in measurement situations, can be collated.

The comprehension of a measurement localization system as a sheaf of Boolean coefficients permits the conception of a Quantum observable (or of its associated quantum event algebra) as a generalized manifold, obtained by pasting the $\psi_{\acute{\xi}\xi}(\mathbf{A}(\xi)\times_{\Xi}\mathbf{A}(\acute{\xi}))$ and $\psi_{\xi\acute{\xi}}(\mathbf{A}(\xi)\times_{\Xi}\mathbf{A}(\acute{\xi}))$ covers together by the transition functions $\Omega_{\xi,\acute{\xi}}$. In this perspective the generalized manifold, which represents categorically a quantum observable object, is understood as a colimit in the category of elements of a sheaf of Boolean coefficients, that contains compatible families of modelling Boolean observables.

6 Isomorphic Representations of Quantum Observables by Boolean Localization Systems

The ideas developed in the previous section may be used to provide the basis for the representation of Quantum observables and their associated Quantum event algebras in terms of Boolean covering systems, if we pay attention to the counit of the established adjunction, denoted by the vertical map in [Diagram 9].

The diagram suggests that the representation of a quantum observable Ξ in $\mathcal{O}_Q$ and, subsequently, of a quantum event algebra L in $\mathcal{L}$, in terms of a coordinatization system of measurement localizations, consisting of Boolean

$$\coprod_{v:\acute{\xi}\to\xi}\mathbf{A}(\acute{\xi}) \underset{\eta}{\overset{\zeta}{\rightrightarrows}} \coprod_{(\xi,p)}\mathbf{A}(\xi)\to[\mathbf{R}(\Xi)](-)\otimes_{\mathcal{O}_B}\mathbf{A} \overset{\epsilon_{\Xi}}{\longrightarrow} \Xi$$

Diagram 9

coefficients, is full and faithful, if and only if the counit of the established adjunction, restricted to that system, is an isomorphism, that is, structure-preserving, 1-1 and onto [14]. It is easy to see that the counit of the adjunction, restricted to a system of measurement localizations is a quantum algebraic isomorphism, iff the right adjoint functor is full and faithful, or equivalently, iff the cocone from the functor $\mathbf{A} \circ G_{\mathbf{R}(\Xi)}$ to the quantum observable Ξ is universal for each object Ξ in $\mathcal{O}_Q$ [2, 3]. In the latter case we characterize the coordinatization functor $\mathbf{A} : \mathcal{O}_B \longrightarrow \mathcal{O}_Q$ or, equivalently, the functor $\mathbf{A} : \mathcal{B} \longrightarrow \mathcal{L}$ such that [Diagram 3] commutes, a proper modelling functor. As a consequence if we consider as $\mathcal{B}$ the category of Boolean subalgebras of a quantum event algebra L of ordinary Quantum Mechanics, that is an orthomodular σ-orthoposet of orthogonal projections of a Hilbert space, together with a proper modelling inclusion functor $\mathbf{A} : \mathcal{B} \longrightarrow \mathcal{L}$, the counit of the established adjunction restricted to a system of measurement localizations is an isomorphism.

The physical significance of this representation lies on the fact that the whole information content in a Quantum event algebra is preserved by every covering Boolean system, qualified as a system of measurement localizations. The preservation property is established by the counit isomorphism. It is remarkable that the categorical notion of adjunction provides the appropriate formal tool for the formulation of invariant properties, giving rise to preservation principles of a physical character.

If we return to the intended representation, we realize that the surjective property of the counit guarantees that the Boolean domain covers, being themselves objects in the category of elements $\mathbf{G}(\mathbf{R}(L), B)$, cover entirely the quantum event algebra L, whereas its injective property guarantees that any two covers are compatible in a system of measurement localizations. Moreover, since the counit is also a homomorphism, it preserves the algebraic structure.

In the physical state of affairs, each cover corresponds to a set of Boolean events actualized locally in a measurement situation. The equivalence classes of Boolean domain covers represent quantum events in L through compatible coordinatizations by Boolean coefficients. Consequently, the structure of a quantum event algebra is being generated by the information that its structure preserving maps, encoded as Boolean covers in measurement localization systems, carry as well as their compatibility relations.

7 Implications for Quantum Logic

The covering process leads naturally to a contextual description of quantum events (or quantum propositions) with respect to Boolean reference frames of measurement and finally to a representation of them as equivalence classes of unsharp Boolean events. The latter term is justified by the fact that, in case, L signifies a truth-value structure, each cover can be interpreted as an unsharp Boolean algebra of events corresponding to measurement of observable Ξ. More concretely, since covers are maps $[\psi_B]_\Xi : \mathbf{A}(B_\Xi) \longrightarrow L$, each Boolean event realized in the domain B_Ξ, besides its true or false truth value assignment in a measurement context related to the outcome of an experiment that has taken place, is also assigned a truth value representing its relational information content for the comprehension of the coherence of the whole quantum structure, measured by the degrees in the poset L or, equivalently, by the degrees assigned to its poset structure of localization systems.

Between these two levels of truth value assignment there exists an intermediate level, revealed by the instantiation of the Boolean power construction in the context of the Grothendieck fibration technique. This intermediate level refers to a truth value assignment to propositions describing the possible behavior of a quantum system in a specified Boolean context of observation without having passed yet an experimental test.

We may remind that the fibration induced by a presheaf of Boolean algebras $\mathbf{P}$ provides the category of elements of $\mathbf{P}$, denoted by $\mathbf{G}(\mathbf{P}, \mathcal{B})$. Its objects are all pairs (B, p), and its morphisms $(\acute{B}, \acute{p}) \longrightarrow (B, p)$ are those morphisms $u : \acute{B} \longrightarrow B$ of $\mathcal{B}$ for which $pu = \acute{p}$. Projection on the second coordinate of $\mathbf{G}(\mathbf{P}, \mathcal{B})$ defines a functor $\mathbf{G_P} : \mathbf{G}(\mathbf{P}, \mathcal{B}) \longrightarrow \mathcal{B}$. If B is an object of $\mathcal{B}$, the inverse image under $\mathbf{G_P}$ of B is simply the set $\mathbf{P}(\mathrm{B})$. As we have explained, the objects of the category of elements $\mathbf{G}(\mathbf{R}(L), B)$ constitute Boolean domain covers for measurement and have been identified as Boolean reference frames on a quantum observable structure.

We notice that the set of objects of $\mathbf{G}(\mathbf{R}(L), \mathcal{B})$ consists of all the elements of all the sets $\mathbf{R}(L)(B)$ and, more concretely, has been constructed from the disjoint union of all the sets of the above form, by labeling the elements. The elements of this disjoint union are represented as pairs $(B, \psi_B : \mathbf{A}(B) \longrightarrow L)$ for all objects B of $\mathcal{B}$ and elements $\psi_B \in \mathbf{R}(L)(B)$.

Taking into account the projection functor, defined above, this set is actually a fibered structure. Each fiber is a set defined over a Boolean algebra relative to which a measurement result is being coordinatized. If we denote by (ψ_B, q) the elements of each fiber, with $\psi_B \in \mathbf{R}(L)(B)$ and $q \in \mathbf{A}(B)$, then the set of maps

$$(\psi_B, q) \longrightarrow q$$

can be interpreted as the Boolean power of the set

$$\Upsilon_B = \{(\psi_B, q), \psi_B \in \mathbf{R}(L)(B), q \in \mathbf{A}(B)\}$$

with respect to the underlying Boolean algebra B [15].

The Boolean power construction forces an interpretation of the Boolean algebra relative to which a measurement result is being coordinatized, as a domain of local truth values with respect to a measurement that has not taken place yet. Moreover the set of local measurement covers defined over B is considered as a Boolean-valued set. In this sense, the local coordinates corresponding to a Boolean domain of measurement may be considered as Boolean truth values.

We further observe that the set of objects of $\mathbf{G}(\mathbf{R}(L), B)$ consists of the disjoint union of all the fibers Υ_B, denoted by $\Upsilon = \coprod_B \Upsilon_B$. This set can also acquire a Boolean power interpretation as follows:

We define a binary relation on the set Υ according to:

$(\psi_{\acute{B}}, \acute{q}) \bigotimes (\psi_B, q)$ iff $\exists\ \eta : \psi_{\acute{B}} \longrightarrow \psi_B : \eta(\acute{q}) = q, \psi_{\acute{B}} = \psi_B \circ \eta$.

It is evident that for any $\eta : \acute{B} \longrightarrow B$ we obtain: $(\psi_B \circ \eta, \acute{q}) \bigotimes (\psi_B, \eta(\acute{q}))$. Furthermore, we require the satisfaction of the compatibility relations that are valid in a system of localizations. Then it is possible to define the Boolean power of the set Υ with respect to the maximal Boolean algebra belonging to such a compatible system of localizations. We may say that the Boolean coordinates, interpreted as local Boolean truth values via the Boolean power construction, reflect a relation of indistinguishability due to overlapping of the corresponding covers.

The viewpoint of Boolean valued sets has far reaching consequences regarding the interpretation of quantum logic and will be discussed in detail in a future work from the perspective of Lawvere's topoi [16]. At the present stage, we may say that the logical interpretation of the Boolean fibration method, seems to substantiate Takeuti's and Davis's approach to the foundations of quantum logic [17, 18], according to whom, quantization of a proposition of classical physics is equivalent to interpreting it in a Boolean extension of a set theoretical universe, where B is a complete Boolean algebra of projection operators on a Hilbert space. In the perspective of the present analysis, we may argue that the fibration technique in the presheaf of Boolean algebras $\mathbf{G}(\mathbf{R}(L), B)$ provides the basis for a natural interpretation of the logic of quantum propositions, referring to the possible behavior of a quantum system in a concrete localization context with respect to an experimental test that has not been actualized yet, in terms of a truth value assignment, assuming existence in the corresponding Boolean context of a covering system, and realized in terms of local valuations on the Boolean coordinates of the specified cover.

8 Differential Geometry in the Quantum Regime

The application of Stone representation theorem for Boolean algebras permits the replacement of Boolean algebras by fields of subsets of a measurement space, providing in this manner a natural operationalization of the meaning of Boolean covers. Thus, if we replace each Boolean algebra B in $\mathcal{B}$ by its set-theoretical representation $[\mathbf{\Sigma}, B_\Sigma]$, consisting of a local measurement space $\mathbf{\Sigma}$ and its local field of subsets B_Σ, it is possible to define local measurement space charts $(B_\Sigma, \psi_{B_\Sigma} : \mathbf{A}(B_\Sigma) \longrightarrow L)$ and corresponding space localization systems for quantum observable Ξ over quantum event algebra L in $\mathcal{L}$. Topologically, each local space is considered as a compact Hausdorff space, the compact open subsets of which are the maximal filters or the prime ideals of the underlying Boolean algebra.

From local measurement space charts $(B_\Sigma, \psi_{B_\Sigma} : \mathbf{A}(B_\Sigma) \longrightarrow L)$ we may form their equivalence classes which, modulo the conditions for compatibility on overlaps, will represent a single quantum event in L. Under these circumstances, we may interpret the equivalence classes of local space charts $\psi_{B_\Sigma} \otimes a$, $a \in \mathbf{A}(B_\Sigma)$ as the experimental actualizations of the quantum events in L, corresponding to measurement of observables Ξ. In the operational framework two local space representations of a quantum observable satisfy the compatibility condition on overlapping regions, iff their associated measurements are equivalent to measurements sharing the same experimental arrangement.

We also observe that the inverse of a local space representation of a quantum observable plays the role of a random variable on this local space Σ. Consequently, every quantum observable may be considered locally, as a measurable function defined over the local measurement space Σ. Phrased differently, random variables defined over local spaces provide Boolean coordinatizations for a quantum observable and moreover satisfy compatibility conditions on the overlaps of their local domains of definition. Subsequently, if we consider the collection of measurable functions defined over the category of local spaces we obtain a sheaf of Boolean coefficients for the measurement of a quantum observable, such that the latter is represented by a colimit construction in the category of elements of this sheaf. Addition and multiplication over $\mathbf{R}$ induce the structure of a sheaf of $\mathbf{R}$-algebras (or a sheaf of rings). A natural question that arises in this setting is if it could be possible to consider the above sheaf of $\mathbf{R}$-algebras as the structure algebra sheaf of a generalized space. From a physical point of view, this move would reflect the appropriate generalization of the arithmetics, or sheaves of coefficients, that have to be used in the transition from the classical to the quantum regime. The appropriate framework to accommodate structure sheaves of the above form is Abstract Differential Geometry (ADG), developed by Mallios in [19, 20]. ADG is an extension of classical Differential Geometry according to which, instead of smooth functions, one starts with a general sheaf of algebras. The important thing is that these sheaves of algebras, which in our perspective correspond to quantum observables, can be interrelated with appropriate differentials, interpreted as Leibniz sheaf morphisms. This interpretation is suited to the development of Differential Geometry in the Quantum regime and will be carried out at a later stage.

9 Conclusions

The conceptual root of the proposed relativistic perspective on quantum structure, established by systems of Boolean measurement localization systems, is located on the physical meaning of the adjunction between presheaves of Boolean observables and quantum observables.

Let us consider that $\mathbf{Sets}^{\mathcal{B}^{op}}$ is the universe of Boolean observable event structures modelled in **Sets**, or else the world of Boolean windows, and $\mathcal{L}$ that of Quantum event structures. In the proposed interpretation the functor $\mathbf{L} : \mathbf{Sets}^{\mathcal{B}^{op}} \longrightarrow \mathcal{L}$ can be comprehended as a translational code from Boolean windows to the Quantum species of event structure, whereas the functor $\mathbf{R} : \mathcal{L} \longrightarrow \mathbf{Sets}^{\mathcal{B}^{op}}$ as a translational code in the inverse direction. In general, the content of the information is not possible to remain completely invariant translating from one language to another and back. However, there remain two ways for a Boolean-event algebra variable set $\mathbf{P}$ to communicate a message to a quantum event algebra L. Either the information is given in Quantum terms with $\mathbf{P}$ translating, which can be represented as the quantum homomorphism $\mathbf{LP} \longrightarrow L$ or the information is given in Boolean terms with L translating, that in turn, can be represented as the natural transformation $\mathbf{P} \longrightarrow \mathbf{R}(L)$. In the first case, from the perspective of L information is being received in quantum terms, while in the second, from the perspective of $\mathbf{P}$ information is being sent in Boolean terms. The natural bijection then corresponds to the assertion that these two distinct ways of communicating are equivalent. Thus, the physical meaning of the adjoint situation signifies a two-way dependency of the involved languages in communication with respect to the variation of the information collected in localization contexts of measurement. More remarkably, the representation of a quantum observable as a categorical colimit, resulting from the same adjunctive relation, reveals an entity that can admit a multitude of instantiations, specified mathematically by different coordinatizing Boolean coefficients in Boolean localization systems.

The underlying invariance property specified by the adjunction is associated with the informational content of all these phenomenically different instantiations in distinct measurement contexts, and can be formulated as follows: the informational content of a quantum observable structure remains invariant with respect to Boolean domain coordinatizations if and only if the counit of the adjunction, restricted to covering systems, qualified as Boolean localization systems, is an isomorphism. Thus, the counit isomorphism provides a categorical equivalence, signifying an invariance in the translational code of communication between Boolean windows, acting as localization devices for measurement, and quantum systems.

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References

1 F. W. Lawvere and S. H. Schanuel, *Conceptual Mathematics*, Cambridge University Press, Cambridge, 1997.

2 S. MacLane, *Categories for the Working Mathematician*, Springer-Verlag, New York, 1971.

3 S. MacLane and I. Moerdijk, *Sheaves in Geometry and Logic*, Springer-Verlag, New York, 1992.

4 J. L. Bell, *Toposes and Local Set Theories*, Oxford University Press, Oxford, 1988.

5 V. S. Varadarajan, *Geometry of Quantum Mechanics*, Vol. 1, Van Nostrand, Princeton, New Jersey, 1968.

6 G. Birkhoff and J. von Neumann, The logic of Quantum Mechanics, *Annals of Mathematics* **37**, 823 (1936).

7 S. Kochen and E. Specker, The problem of hidden variables in Quantum Mechanics, *J. Math. and Mech.* **17**, 59 (1967).

8 J. Butterfield and C. J. Isham, A Topos Perspective on the Kochen-Specker Theorem: I. Quantum States as Generalized Valuations, *International Journal of Theoretical Physics* **37**, 2669 (1998).

9 J. Butterfield and C. J. Isham, A Topos Perspective on the Kochen-Specker Theorem: II. Conceptual Aspects and Classical Analogues, *International Journal of Theoretical Physics* **38**, 827 (1999).

10 J. P. Rawling and S. A. Selesnick, Orthologic and Quantum Logic. Models and Computational Elements, *Journal of the Association for Computing Machinery* **47**, 721 (2000).

11 I. Raptis, Presheaves, Sheaves, and their Topoi in Quantum Gravity and Quantum Logic, gr-qc/0110064.

12 J. Butterfield and C. J. Isham, Some Possible Roles for Topos Theory in Quantum Theory and Quantum Gravity, *Foundations of Physics* **30**, 1707 (2000).

13 M. Artin, A. Grothendieck, and J. L. Verdier, *Theorie de topos et cohomologie etale des schemas*, (Springer LNM 269 and 270, Springer-Verlag, Berlin, 1972).

14 E. Zafiris, Topos Theoretical Reference Frames on the Category of Quantum Observables, quant-ph/0202057.

15 J. L. Bell, *Boolean-Valued Models and Independence Proofs in Set Theory* , Oxford University Press, Oxford, 1985.

16 F. W. Lawvere, Continuously Variable Sets: Algebraic Geometry=Geometric Logic, *Proceedings of the Logic Colloquium in Bristol*, North-Holland, Amsterdam (1975).

17 G. Takeuti, Two Applications of Logic to Mathematics, *Mathematical Society of Japan* **13**, Kano Memorial Lectures 3 (1978).

18 M. Davis, A Relativity Principle in Quantum Mechanics, *International Journal of Theoretical Physics* **16**, 867 (1977).

19 A. Mallios, *Geometry of Vector Sheaves: An axiomatic Approach to Differential Geometry* , vols 1-2, Kluwer Academic Publishers, Dordrecht (1998).

20 A. Mallios, Remarks on Singularities, gr-qc/0202028.