Quantum Observables Algebras and Abstract Differential Geometry: The Topos-Theoretic Dynamics of Diagrams of Commutative Algebraic Localizations

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Abstract

We construct a sheaf-theoretic representation of quantum observables algebras over a base category equipped with a Grothendieck topology, consisting of epimorphic families of commutative observables algebras, playing the role of local arithmetics in measurement situations. This construction makes possible the adaptation of the methodology of Abstract Differential Geometry (ADG), *à la Mallios*, in a topos-theoretic environment, and hence, the extension of the "mechanism of differentials" in the quantum regime. The process of gluing information, within diagrams of commutative algebraic localizations, generates dynamics, involving the transition from the classical to the quantum regime, formulated cohomologically in terms of a functorial quantum connection, and subsequently, detected via the associated curvature of that connection.

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1 PROLOGUE

The working understanding of contemporary physical theories is grounded on the notion of observables. Observables are associated with physical quantities that, in principle, can be measured. In this sense physical systems are completely described by the collection of all observed data determined by adequate devices in appropriate measurement situations. The mathematical formalization of this procedure relies on the idea of expressing the observables by functions corresponding to measuring devices. Usually it is also stipulated that quantities admissible as measured results must be real numbers. It is a common belief that the resort to real numbers has the advantage of making our empirical access secure. Hence the underlying assumption on the basis of physical theories postulates that our form of observation is expressed by real number representability, and subsequently, observables are modelled by real-valued functions corresponding to measuring devices. At a further stage of development of this idea, two further assumptions are imposed on the structure of observables: the first of them specifies the algebraic nature of the set of all observables used for the description of a physical system, by assuming the structure of a commutative unital algebra $\mathcal{A}$ over the real numbers. The second assumption restricts the content of the set of real-valued functions corresponding to physical observables to those that admit a mathematical characterization as measurable, continuous or smooth. Thus, depending on the means of description of a physical system, observables are modelled by $\mathcal{R}$-algebras of measurable, continuous or smooth functions corresponding to suitably specifiable in each case measurement environments. Usually the smoothness assumption is postulated because it is desirable to consider derivatives of observables and effectively set-up a kinematical framework of description in terms of differential equations. Moreover, since we have initially assumed that real-number representability constitutes our form of observation in terms of the readings of measuring devices, the set of all $\mathcal{R}$-algebra homomorphisms $\mathcal{A} \to \mathcal{R}$, assigning to each observable in $\mathcal{A}$, the reading of a measuring device in $\mathcal{R}$, encapsulates all the information collected about a physical system in measurement situations in terms of algebras of real-valued observables. Mathematically, the set of all $\mathcal{R}$-algebra homomorphisms $\mathcal{A} \to \mathcal{R}$ is identified as the $\mathcal{R}$-spectrum of the unital commutative algebra of observables $\mathcal{A}$. The physical semantics of this connotation denotes the set that can be $\mathcal{R}$-observed by means of this algebra. It is well known that, in case $\mathcal{A}$ stands for a smooth algebra of real-valued observables, $\mathcal{R}$-algebra homomorphisms $\mathcal{A} \to \mathcal{R}$ can be legitimately identified with the points of a space that can be observed by means of $\mathcal{A}$, namely the points of a differential manifold that, in turn, denote the states of the observed physical system. From this perspective state spaces in general are derivative notions referring to sets of points $\mathcal{R}$-observed, through the lenses of corresponding algebras of observables.

An equally important notion referring to the conceptualization of physical observables is related with the issue of localization. Usually the operationalizations of measurement situations assume their existence locally and the underlying assumption is that the information gathered about observables in different measurement situations can be collated together by appropriate means. The notion of local requires the specification of a topology on an assumed underlying measurement space over which algebras of observables may be localized. The net effect of this localization procedure of algebras of observables together with the requirement of compatible information collation along localizations are formalized by the notion of sheaf. A sheaf of commutative unital $\mathcal{R}$-algebras of observables incorporates exactly the conditions for the transition from locally collected observable data to globally defined ones. In case of smooth observational procedures the notion of a sheaf of smooth $\mathcal{R}$-algebras of observables $\mathcal{A}$, means that locally $\mathcal{A}$ is like the $\mathcal{R}$-algebra $\mathcal{C}^\infty(\mathcal{R}^n)$ of infinitely differentiable functions on $\mathcal{R}^n$.

The interpretative power of this modeling scheme, based on sheaves of algebras of observables, has been recently vastly enhanced by the development of Abstract Differential Geometry (ADG) *à la Mallios* [1-8], which generalizes the differential geometric mechanism of smooth manifolds. Remarkably, it shows that most of the usual differential geometric constructions can be carried out by purely algebraic means without any use of any sort of C^∞-smoothness or any of the conventional calculus that goes with it. This conclusion is important because it permits the legitimate use of any appropriate $\mathcal{R}$-algebra sheaf of observables suited to a measurement situation, even Rosinger's singular algebra sheaf of generalized functions [7, 8], without loosing the differential mechanism, prior believed to be solely associated with smooth manifold state-spaces. Most significantly, ADG has made us realize that the differential geometric mechanism in its abstract algebraic sheaf theoretical formulation expressing from a physical viewpoint the kinematics and dynamics of information propagation through observables, is independent of the localization method employed for the extraction and subsequent coordinatization of its content. Thus, algebra sheaves of smooth real-valued functions together with their associated by measurement manifold $\mathcal{R}$-spectrums are by no means unique coordinatizations of the universal physical mechanism of qualitative information propagation through observables.

The major foundational difference between classical and quantum physical systems from the perspective of the modeling scheme by observables is a consequence of a single principle that can be termed principle of simultaneous observability. According to this, in the classical description of physical systems all their observables are theoretically compatible, or else, they can be simultaneously specified in a single local measurement context. On the other side, the quantum description of physical systems is based on the assertion of incompatibility of all theoretical observables in a single local measurement context, and as a consequence quantum-theoretically the simultaneous specification of all observables is not possible. The conceptual roots of the violation of the principle of simultaneous observability in the quantum regime is tied with Heisenberg's uncertainty principle and Bohr's principle of complementarity of physical descriptions [9-13]. A natural question that arises in this setting is whether one could express algebras of quantum observables in terms of structured families of local commutative algebras of classical observables capable of carrying all the information encoded in the former. Of course, the notion of local has to be carefully redesigned in this formulation, as it will become clear at a later stage. From a category-theoretic standpoint, the transition from a classical to a quantum description can be made simply equivalent to a transition from a category of commutative algebra sheaves of observables to a category of diagrams of commutative algebra sheaves of observables. The advantage of this formulation in comparison to global non-commutative axiomatizations of operator algebras of quantum observables is twofold: firstly, it makes transparent the construction of an algebra of quantum observables from the interconnection of locally defined commutative algebras of classically conceived observables, and secondly, it makes possible the extension of the differential geometric mechanism of ADG in the quantum regime, thus, in effect, of the classical one, as well.

According to the above line of reasoning, we are guided in expressing a globally non-commutative object, like an algebra of quantum observables, in terms of structured families of commutative algebras of observables, which have to satisfy certain compatibility relations, and also, a closure constraint. Hence, commutative algebras of real-valued observables are used locally, in an appropriate manner, accomplishing the task of providing partial congruent relations with globally non-commutative observable algebras, the internal structure of which, is being effectively expressed in terms of the interconnecting machinery binding the local objects together. This point of view stresses the contextual character of quantum theory and establishes a relation with commutative algebras associated with typical measurement situations. In order to proceed a suitable mathematical language has to be used. The criterion for choosing an appropriate language is rather emphasis in the form of the structures and the universality of the constructions involved. The ideal candidate for this purpose is provided by category theory [14-20]. Subsequently, we will see that sheaf theory [21-23], (yet, see also [1]), is the appropriate mathematical vehicle to carry out the program implied by the proposed methodology.

The concept of sheaf expresses essentially gluing conditions, namely, the way by which local data can be collated into global ones. It is the mathematical abstraction suited to formalizing the relations between covering systems and properties, and, furthermore, provides the means for studying the global consequences/information from locally defined properties. The notion of local is characterized “geometrically”, viz. by using a topology (in the general case a Grothendieck topology on a category), the axioms of which express closure conditions on the collection of covers. Essentially, a map which assigns a set to each object of a topology is called a sheaf if the map is locally defined, or else the value of the map on an object can be uniquely obtained from its values on any cover of that object. Categorically speaking, besides mapping each object to a set, a sheaf maps each arrow in the topology to a restriction function in the opposite direction. We stress the point that the transition from locally defined properties to global consequences/information happens via a compatible family of elements over a covering system of the global object. A covering system of the global object can be viewed as providing a decomposition of that object into local objects. The sheaf assigns a set to each element of that system, or else, to each local piece of the original object. A choice of elements from these sets, one for each piece, forms a compatible family if the choice respects the mappings by the restriction functions and if the elements chosen agree whenever two pieces of the covering system overlap. If such a locally compatible choice induces a unique choice for the object being covered, viz. a global choice, then the condition for being a sheaf is satisfied. We note that in general, there will be more locally defined or partial choices than globally defined ones, since not all partial choices need be extendible to global ones, but a compatible family of partial choices uniquely extends to a global one, or in other words, any presheaf uniquely defines a sheaf; thus, see e.g. A. Mallios [24], for an "information/choice"-theoretic formulation of the same.

In the following sections we shall see that a quantum observables algebra can be understood as a sheaf for a suitable Grothendieck topology on the category of commutative subalgebras of it. The idea is based on extension and elaboration of previous works of the author, communicated, both conceptually and technically, in the literature [25-29]. In all these papers, the focus has been shifted from point-set to topological localization models of quantum algebraic structures, that effectively, induce a transition in the semantics of observables from a set-theoretic to a sheaf-theoretic one. The primary physical motivation behind this strategy, has been generated by investigating the possibility of mathematically implementing localization processes referring to physical observation, concerning in particular quantum phenomena, that is not necessarily based on the existence of an underlying structure of points on the real line.

It is also instructive to mention that, contextual topos theoretical approaches to quantum structures have been also developed, from a different viewpoint in [30-34]. Moreover, the necessity of implementation of a sheaf-theoretic framework for overcoming the problems of singularities has been thoroughly discussed recently in [35]. Finally, the central thesis of [36], according to which, quantum physics at a fundamental level may itself be realized as a species of quantum computation, is strongly embraced by the author.

2 CATEGORIES OF OBSERVABLES

Category theory provides a general apparatus for dealing with mathematical structures and their mutual relations and transformations. The basic categorical principles that we adopt in the subsequent analysis are summarized as follows:

[i]. To each kind of mathematical structure, there corresponds a **category** whose objects have that structure, and whose morphisms preserve it.

[ii]. To any natural construction on structures of one kind, yielding structures of another kind, there corresponds a **functor** from the category of the first kind to the category of the second.

[iii]. To each translation between constructions of the above form there corresponds a **natural transformation**.

2.1 Classical and Quantum Observables Structures

A **Classical Observables structure** is a small category, denoted by $\mathcal{A}_C$, and called the *category of Classical Observables algebras*, or of *classical arithmetics*. Its objects are commutative unital $\mathcal{R}$-algebras of observables, and its arrows are unit preserving $\mathcal{R}$-algebras morphisms. Thus, $\mathcal{A}_C$ is a subcategory of that one of commutative unital $\mathcal{R}$-algebras and unit preserving $\mathcal{R}$-algebra morphisms.

Examples: [i] We consider the Boolean algebra of events B associated with the measurement of a physical system. In any experiment performed by an observer, the propositions that can be made concerning a physical quantity are of the type, which asserts that, the value of the physical quantity lies in some Borel set of the real numbers. The proposition that the value of a physical quantity lies in a Borel set of the real line corresponds to an event in the ordered event structure B, as it is apprehended by an observer. Thus we obtain a mapping $A_C : Bor(\mathbf{R}) \to B$ from the Borel sets of the real line to the event structure which captures precisely the notion of observable. Most importantly the above mapping is required to be a homomorphism. In this representation $Bor(\mathbf{R})$ stands for the algebra of events associated with a measurement device interacting with a physical system. The homomorphism assigns to every empirical event in $Bor(\mathbf{R})$ a proposition or event in B, that states, a measurement fact about the physical system interacting with the measuring apparatus. According to Stone's representation theorem for Boolean algebras, it is legitimate to replace Boolean algebras by fields of subsets of a measurement space. Hence we may replace the Boolean algebra B by its set-theoretical representation $[\mathbf{\Sigma}, B_\Sigma]$, consisting of a measurement space $\mathbf{\Sigma}$ and its field of subsets B_Σ. Then observables ξ are in injective correspondence with inverses of random variables $f : \mathbf{\Sigma} \to \mathcal{R}$. In this setting we may also identify a classical observables algebra with the $\mathcal{R}$-algebra of measurable functions defined on the measurement space $\mathbf{\Sigma}$.

[ii] We assume that the measurement space $[\mathbf{\Sigma}, B_\Sigma]$ above, is identified with the σ-algebra of Borel subsets of a topological space X. In this setting we could consider as a classical observables algebra the $\mathcal{R}$-algebra of continuous functions defined on X.

[iii] We assume that the topological space X above is paracompact and Hausdorff, and furthermore that can be endowed with the structure of a differential manifold. In this setting we could consider as a classical observables algebra the $\mathcal{R}$-algebra of smooth functions on X.

A **Quantum Observables structure** is a small category, denoted by $\mathcal{A}_Q$, and called the *category of Quantum Observables algebras*, or of *Quantum arithmetics*. Its objects are thus unital $\mathcal{R}$-algebras of observables, and its arrows are unit preserving $\mathcal{R}$-algebras morphisms. Hence, in other words, $\mathcal{A}_Q$ is a subcategory of the category of unital $\mathcal{R}$-algebras and unit preserving algebra morphisms.

Examples: [i] We consider the algebra of events L associated with the measurement of a quantum system. In this case L is not a Boolean algebra, but an orthomodular σ-orthoposet. A quantum observable Ξ is defined to be an algebra morphism from the Borel algebra of the real line $Bor(\mathbf{R})$ to the quantum event algebra L [13, 25-26, 28].

$$\Xi : Bor(\mathbf{R}) \to L$$

such that: [i] $\Xi(\emptyset) = 0, \Xi(\mathbf{R}) = 1$, [ii] $E \bigcap F = \emptyset \Rightarrow \Xi(E) \perp \Xi(F)$, for $E, F \in Bor(\mathbf{R})$, [iii] $\Xi(\bigcup_n E_n) = \bigvee_n \Xi(E_n)$, where $E_1, E_2, \ldots$ sequence of mutually disjoint Borel sets of the real line. Addition and multiplication on $\mathcal{R}$ induce on the set of quantum observables the structure of a partial commutative algebra over $\mathcal{R}$. In most of the cases the stronger assumption of a non-commutative algebra of quantum observables is adopted.

[ii] If L is isomorphic with the orthocomplemented lattice of orthogonal projections on a Hilbert space, then it follows from von Neumann's spectral theorem that the quantum observables are in 1-1 correspondence with the hypermaximal Hermitian operators on the Hilbert space.

[iii] An algebra of quantum observables can be made isomorphic to the partial algebra of Hermitian elements in a C^*-algebra.

The crucial observation that the development of this paper will be based on, has to do with the fact that a globally non-commutative or partial algebra of quantum observables determines an underlying diagram of commutative subalgebras. Then each commutative subalgebra can be locally identified, in a sense that will be made clear later, with an algebra of classical observables. Thus the information that is contained in an algebra of quantum observabes can be recovered by a gluing construction referring to its commutative subalgebras. This construction is also capable of extending the differential geometric mechanism to the regime of quantum systems via *ADG sheaf-theoretical methodology* and collating information appropriately.

2.2 The Notion of Differential Triad

A Differential Triad is a concept introduced by A. Mallios in an axiomatic approach to Differential Geometry [1]. The notion of Differential Triad replaces the assumptions on the local structure of a topological space X for its specification as a manifold, namely charts and atlases, with assumptions on the existence of a derivative ("flat connection") on an arbitrary sheaf of algebras on X, playing the equivalent role of the structure sheaf of germs of smooth functions on X. The major novelty of this notion relies on the fact that any sheaf of algebras may be regarded as the structural sheaf of a differential triad capable of providing a differential geometric mechanism, independent thus of any manifold concept, analogous, however, to the one supported by smooth manifolds.

The significance of the notion of Differential Triad for the purposes of the present work can be made clear in a procedure consisting of two levels:

The first level considers the localization of a commutative unital $\mathcal{R}$-algebra of observables over a topological measurement space X. The general methodology of localization, by means of an arbitrary topological commutative algebra has been discussed extensively in [24]. The localization procedure provides a sheaf of unital, commutative $\mathcal{R}$-algebras of observables over X. Having at our disposal this localized structure we may set up a differential triad associated with the sheaf of commutative algebras of observables A_C as follows: Let Ω be an A_C-module, that is, Ω stands for a sheaf of $\mathcal{R}$-vector spaces over X, such that $\Omega(U)$ is an $A_C(\mathrm{U})$ module, for every U in the topology τ_X of X. Besides, let $\vartheta := (\vartheta_U) : A_C \to \Omega$ be a sheaf morphism. Then the triplet $\Delta = (A_C, \vartheta, \Omega)$ constitutes a differential triad, if it satisfies the following conditions:

[i] ϑ is $\mathcal{R}$-linear, and

[ii] ϑ satisfies the Leibniz rule: for every pair (ξ_1, ξ_2) in $A_C \times_X A_C$ it holds that

$$\vartheta(\xi_1 \cdot \xi_2) = \xi_1 \cdot \vartheta(\xi_2) + \xi_2 \cdot \vartheta(\xi_1)$$

In this manner for every localized commutative unital $\mathcal{R}$-algebra sheaf of observables suited to a measurement situation we may associate a differential triad $\Delta = (A_C, \vartheta, \Omega)$ as above, that is capable of expressing according to ADG a generalized differential geometric mechanism referred to the propagation of information encoded in the sheaf $\mathbf{A}_C$, that in turn, instantiates a coordinatized arithmetic suited for the study of a physical system associated with a measurement environment. It is instructive to emphasize here that classically speaking, viz. for a classical physical system, all the observables are theoretically compatible, or simultaneously detectable, thus a single differential triad is enough for the complete determination of the former mechanism.

The second level of the proposed scheme has the purpose of extending the differential geometric mechanism to globally non-commutative algebras of quantum observables. We may remind that according to the principle of non-simultaneous observability in the quantum regime, as above, the observables of quantum systems are not compatible. Thus, a single differential triad associated with a commutative sector of an algebra of quantum observables is not possible to encode the totality of the information required for the set-up of a generalized differential geometric mechanism in the quantum case. What is needed is a procedure of gluing together differential triads attached to local commutative sectors. In this case the notion of local is distinguished from the classical case and is naturally provided by the definition of an appropriate Grothendieck topology over the opposite category of commutative subalgebras of a quantum algebra of observables. In this perspective an algebra of quantum observables, or quantum arithmetic, can be made isomorphic with a sheaf of locally commutative algebras of observables for this Grothendieck topology. Thus the differential geometric mechanism, following ADG, can be now applied locally in the quantum regime, as well, by referring to the aforementioned sheaf of locally commutative algebras of observables. In the sequel, it will become clear that the transition to the quantum regime involves considering diagrams of differential triads attached to commutative subalgebras of an algebra of quantum observables, together with a generalized conception of locality in the Grothendieck sense, that permits collation of local information in a sheaf-theoretic manner among these diagrams.

Since a quantum algebra of observables could be theoretically built up from diagrams of commutative algebras of observables, each one of them carrying a differential triad, it is necessary to specify their morphisms in a category-theoretic language. Let us consider that $\Delta_X = ({A^X}_C, \vartheta_X, \Omega_X)$, $\Delta_Y = ({A^Y}_C, \vartheta_Y, \Omega_Y)$ are differential triads associated with measurement situations that take place over the topological spaces X, Y respectively. A morphism from Δ_X to Δ_Y is a triplet (z, z_{A_C}, z_Ω) such that:

[i]: $z : X \rightarrow Y$ is a continuous map,

[ii]: $z_{A_C} : A^Y{}_C \rightarrow z_*(A^X{}_C)$ is a unit preserving morphism of classical sheaves of $\mathcal{R}$-algebras over Y, where $z_* : Sh_X \rightarrow Sh_Y$ denotes the push-out functor,

[iii]: $z_\Omega : \Omega_Y \rightarrow z_*(\Omega_X)$ is a morphism of sheaves of $\mathcal{R}$-vector spaces over Y, such that $z_\Omega(\xi\omega) = z_{A_C}(\xi) z_\Omega(\omega)$ $\forall(\xi, \omega)$ in $\mathbf{A}^Y{}_C \times_Y \Omega_Y$,

[iv]: the diagram below, denoting push-out operations of differential triads, commutes;

$$\begin{array}{ccc} A^Y{}_C & \xrightarrow{z_{A_C}} & z_*(A^X{}_C) \\ \downarrow{\scriptstyle \vartheta_Y} & & \downarrow{\scriptstyle z_*(\vartheta_X)} \\ \Omega_Y & \xrightarrow{z_\Omega} & z_*(\Omega_X) \end{array}$$

In the sequel, the category of differential triads associated with the subcategory of commutative algebras of a quantum algebra of observables will be used only when we discuss the extension of the differential geometric mechanism in the quantum regime. The description of localization of a quantum algebra of observables with respect to an appropriate Grothendieck topology on the opposite subcategory of its commutatives algebras will be based solely on the definitions provided in Section 2.1 for reasons of simplicity in the exposition of the method. Of course, it is obvious that a commutative subalgebra of a quantum observables algebra once localized itself over a measurement topological space X becomes a sheaf. Hence, as we shall see in detail in what follows, a differential triad can be appropriately associated with it.

3 FUNCTOR OF POINTS OF A QUANTUM OBSERVABLES ALGEBRA

The development of the ideas contained in the proposed scheme are based on the notion of the functor of points of a quantum observables algebra, so it is worthwhile to explicate its meaning in detail. The ideology behind this notion has its roots in the work of Grothendieck in algebraic geometry. If we consider the opposite of the category of algebras of quantum observables, that is, the category with the same objects but with arrows reversed ${\mathcal{A}_Q}^{op}$, each object in that context can be thought of as the locus of a quantum observables algebra, or else it carries the connotation of space. The crucial observation is that, any such space is determined, up to canonical isomorphism, if we know all morphisms into this locus from any other locus in the category. For instance, the set of morphisms from the one-point locus to A_Q in the categorial context of ${\mathcal{A}_Q}^{op}$ determines the set of points of the locus A_Q. The philosophy behind this approach amounts to considering any morphism in ${\mathcal{A}_Q}^{op}$ with target the locus A_Q as a generalized point of A_Q. For our purposes we consider the description of a locus A_Q in terms of all possible morphisms from all other objects of ${\mathcal{A}_Q}^{op}$ as redundant. For this reason we may restrict the generalized points of A_Q to all those morphisms in ${\mathcal{A}_Q}^{op}$ having as domains spaces corresponding to commutative subablgebras of a quantum observables algebra. Variation of generalized points over all domain-objects of the subcategory of ${\mathcal{A}_Q}^{op}$ consisting of commutative algebras of observables produces the functor of points of A_Q restricted to the subcategory of commutative objects, identified, in what follows, with ${\mathcal{A}_C}^{op}$. This functor of points of A_Q is made then an object in the category of presheaves $\mathbf{Sets}^{{\mathcal{A}_C}^{op}}$, representing a quantum observables algebra -(in the sequel for simplicity we talk of an algebra and its associated locus tautologically)- in the environment of the topos of presheaves over the category of its commutative subalgebras. This methodology will prove to be successful if it could be possible to establish an isomorphic representation of A_Q in terms of its generalized points $A_C \to A_Q$, considered as morphisms in the same category, collated together by sheaf-theoretical means.

From a physical point of view, the domains of generalized points of A_Q

specify precisely the kind of loci of variation that are used for individuation of observable events in the physical continuum in a quantum measurement situation, accomplishing an instantiation of Bohr's conception of a *phenomenon*, as referring exclusively to observations obtained under specific circumstances that constitute a physical descriptive context. Thus, the methodological underpinning of the introduction of generalized points $A_C \to A_Q$ adapt Bohr's concept of phenomenon as a referent of the assignment of an observable quantity to a system, in the context of a commutative domain considered appropriately as a local environment of measurement. In this sense generalized points play the equivalent role of *generalized reference frames*, such that reference to concrete events of the specified kind can be made possible only with respect to the former. In the trivial case the only locus is a point serving as a unique idealized measure of localization, and moreover, the only kind of reference frame is the one attached to a point. This kind of reference frames are used in classical physics, but prove to be insufficient for handling information related with quantum measurement situations due to the principle of non-simultaneous observability explicated previously. Hence, generalized points $A_C \to A_Q$ constitute reference frames only in a local sense by means of a *Grothendieck topology*, to be introduced at a latter stage, and information collected in different or overlapping commutative local domains A_C can be collated appropriately in the form of sheaf theoretical localization systems of A_Q. The net effect of this procedure, endowed in the above sense with a solid operational meaning, is the isomorphic representation of a quantum observable structure via a *Grothendieck topos*, understood as a *sheaf for a Grothendieck topology*. The notion of topos is essential and indispensable to the comprehension of the whole scheme, because it engulfs the crucial idea of a well-defined variable structure, admitting localizations over a multiplicity of generalized reference domains of coordinatizing coefficients, such that information about observable attributes collected in partially overlapping domains can be pasted together in a meaningful manner. Pictorially, the instantiation of such a topos theoretical scheme can be represented as a *fibered structure*, which we may call, as we shall see, a quantum observable structure, that fibers over a base category of varying reference loci, consisting of locally commutative coefficients, specified by operational means and standing for physical contexts of quantum measurement. We may formalize the ideas exposed above as follows:

We make the basic assumption that, there exists a *coordinatization functor*, $\mathbf{M} : \mathcal{A}_C \longrightarrow \mathcal{A}_Q$, which assigns to commutative observables algebras in $\mathcal{A}_C$, that instantiates a model category, *the underlying quantum algebras* from $\mathcal{A}_Q$, and to commutative algebras morphisms the underlying quantum algebraic morphisms.

If ${\mathcal{A}_C}^{op}$ is the opposite category of $\mathcal{A}_\mathcal{C}$, then $\mathbf{Sets}^{{\mathcal{A}_C}^{op}}$ denotes the *functor category of presheaves* of commutative observables algebras, with objects all functors $\mathbf{P} : {\mathcal{A}_C}^{op} \longrightarrow \mathbf{Sets}$, and morphisms all natural transformations between such functors. Each object $\mathbf{P}$ in this category is a contravariant set-valued functor on $\mathcal{A}_C$, called a *presheaf* on $\mathcal{A}_C$. The functor category of presheaves on commutative observables algebras $\mathbf{Sets}^{{\mathcal{A}_C}^{op}}$, exemplifies a well defined notion of a universe of *variable sets*, and is characterized as a *topos*. We recall that a topos is a category which has a terminal object, pullbacks, exponentials, and a subobject classifier, that is conceived as an object of generalized truth values. In this sense, a topos can be conceived as a local mathematical framework corresponding to a generalized model of set theory, or as a generalized space.

For each commutative algebra A_C of $\mathcal{A}_C$, $\mathbf{P}(A_C)$ is a set, and for each arrow $f : C_C \longrightarrow A_C$, $\mathbf{P}(f) : \mathbf{P}(A_C) \longrightarrow \mathbf{P}(C_C)$ is a set-function. If $\mathbf{P}$ is a presheaf on $\mathcal{A}_C$ and $p \in \mathbf{P}(A_C)$, the value $\mathbf{P}(f)(p)$ for an arrow $f : C_C \longrightarrow A_C$ in $\mathcal{A}_C$ is called the restriction of p along f and is denoted by $\mathbf{P}(f)(p) = p \cdot f$.

Each object A_C of $\mathcal{A}_C$ gives rise to a contravariant Hom-functor $\mathbf{y}[A_C] := Hom_{\mathcal{A}_C}(-, A_C)$. This functor defines a presheaf on $\mathcal{A}_C$. Its action on an object C_C of $\mathcal{A}_C$ is given by

$$\mathbf{y}[A_C](C_C) := Hom_{\mathcal{A}_\mathcal{C}}(C_C, A_C)$$

whereas its action on a morphism $D_C \xrightarrow{x} C_C$, for $v : C_C \longrightarrow A_C$ is given by

$$\mathbf{y}[A_C](x) : Hom_{\mathcal{A}_C}(C_C, A_C) \longrightarrow Hom_{\mathcal{A}_C}(D_C, A_C)$$

$$\mathbf{y}[A_C](x)(v) = v \circ x$$

Furthermore $\mathbf{y}$ can be made into a functor from $\mathcal{A}_C$ to the contravariant functors on $\mathcal{A}_C$

$$\mathbf{y} : \mathcal{A}_C \longrightarrow \mathbf{Sets}^{\mathcal{A}_C{}^{op}}$$

such that $A_C \mapsto Hom_{\mathcal{A}_C}(-, A_C)$. This is called the *Yoneda embedding* and it is a full and faithful functor.

Next we construct the *category of elements* of $\mathbf{P}$, denoted by $\mathbf{G}(\mathbf{P}, \mathcal{A}_\mathbf{C})$. Its objects are all pairs (A_C, p), and its morphisms $(\acute{A}_C, \acute{p}) \rightarrow (A_C, p)$ are those morphisms $u : \acute{A}_C \rightarrow A_C$ of $\mathcal{A}_C$ for which $pu = \acute{p}$. Projection on the second coordinate of $\mathbf{G}(\mathbf{P}, \mathcal{A}_\mathbf{C})$, defines a functor $\mathbf{G}(\mathbf{P}) : \mathbf{G}(\mathbf{P}, \mathcal{A}_\mathbf{C}) \rightarrow \mathcal{A}_\mathcal{C}$. $\mathbf{G}(\mathbf{P}, \mathcal{A}_\mathbf{C})$ together with the projection functor $\mathbf{G}(\mathbf{P})$ is called the *split discrete fibration* induced by $\mathbf{P}$, and $\mathcal{A}_C$ is the base category of the fibration. We note that the fibers are categories in which the only arrows are identity arrows. If A_C is an object of $\mathcal{A}_C$, the inverse image under $\mathbf{G}(\mathbf{P})$ of A_C is simply the set $\mathbf{P}(A_C)$, although its elements are written as pairs so as to form a disjoint union. The construction of the fibration induced by $\mathbf{P}$, is called the *Grothendieck construction* [13].

$$\begin{array}{ccc} \mathbf{G}(\mathbf{P}, \mathcal{A}_\mathcal{C}) & & \\ \Big\downarrow {\scriptstyle \mathbf{G}(\mathbf{P})} & & \\ \mathcal{A}_\mathcal{C} & \xrightarrow{\mathbf{P}} & \mathbf{Sets} \end{array}$$

Now, if we consider the category of quantum observables algebras $\mathcal{A}_Q$ and the coefficient functor $\mathbf{M}$, we can define the functor;

$$\mathbf{R} : \mathcal{A}_Q \rightarrow \mathbf{Sets}^{\mathcal{A}_C{}^{op}}$$

from $\mathcal{A}_Q$ to the category of presheaves of commutative observables algebras given by:

$$\mathbf{R}(A_Q) : A_C \mapsto \mathbf{R}(A_Q)(A_C) := Hom_{\mathcal{A}_Q}(\mathbf{M}(A_C), A_Q)$$

According to the philosophy of the *functor of points of a quantum observables algebra*, the objects of the category of elements $\mathbf{G}(\mathbf{R}(A_Q), \mathcal{A}_C)$

constitute *generalized points* of A_Q in the environment of presheaves of commutative observables algebras A_C.

We notice that the set of objects of $\mathbf{G}(\mathbf{R}(A_Q), \mathcal{A}_C)$, considered as a small category, consists of all the elements of all the sets $\mathbf{R}(A_Q)(A_C)$, and more concretely, has been constructed from the disjoint union of all the sets of the above form, by labeling the elements. The elements of this disjoint union are represented as pairs $(A_C, \psi_{A_C} | (\psi_{A_C} : \mathbf{M}(A_C) \longrightarrow A_Q))$ for all objects A_C of $\mathcal{A_C}$ and elements $\psi_{A_C} \in \mathbf{R}(A_Q)(A_C)$.

It is finally instructive to clarify that the functor of points of a quantum observables algebra, can be also legitimately made an object in the category of presheaves of modules $\mathbf{Mod}^{\mathcal{A_C}^{op}}$, under the requirement that its composition with the forgetful functor $\mathbf{F}r : \mathbf{Mod} \to \mathbf{Sets}$ is the presheaf of sets functor of points as determined above.

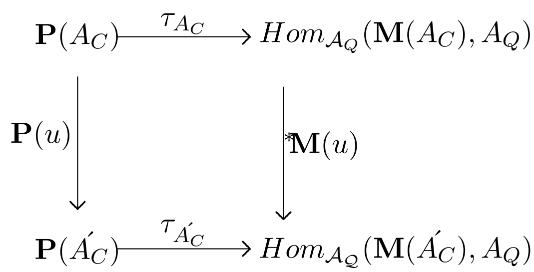
4 THE ADJOINT FUNCTORIAL CLASSICAL-QUANTUM RELATION

The existence of an adjunctive correspondence between the commutative and quantum observables algebras, which will be proved in what follows, provides the conceptual ground, concerning the *representation of quantum observables algebras in terms of sheaves of structured families of commutative observables algebras*; this is based on a categorical construction of colimits over categories of elements of presheaves of commutative algebras.

A natural transformation τ between the presheaves on the category of commutative algebras $\mathbf{P}$ and $\mathbf{R}(A_Q)$, $\tau : \mathbf{P} \longrightarrow \mathbf{R}(A_Q)$, is a family τ_{A_C} indexed by commutative algebras A_C of $\mathcal{A}_C$ for which each τ_{A_C} is a map of sets,

$$\tau_{A_C} : \mathbf{P}(A_C) \rightarrow Hom_{\mathcal{A}_Q}(\mathbf{M}(A_C), A_Q) \equiv \mathbf{R}(A_Q)(A_C)$$

such that the diagram of sets below commutes for each commutative algebras morphism $u : \acute{A_C} \to A_C$ of $\mathcal{A}_C$.



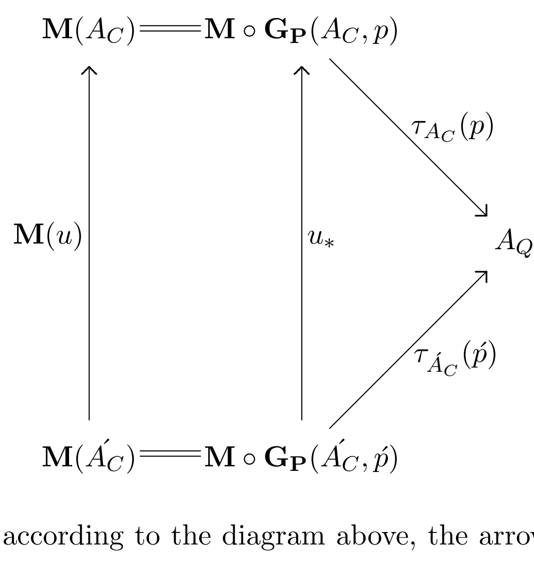
From the perspective of the category of elements of the commutative algebras-variable set P the map τ_{A_C}, defined above, is identical with the map:

$$\tau_{A_C} : (A_C, p) \rightarrow Hom_{\mathcal{A}_Q}(\mathbf{M} \circ G_{\mathbf{P}}(A_C, p), A_Q)$$

Subsequently such a τ may be represented as a family of arrows of $\mathcal{A}_Q$ which is being indexed by objects (A_C, p) of the category of elements of the presheaf of commutative algebras $\mathbf{P}$, namely

$$\{\tau_{A_C}(p) : \mathbf{M}(A_C) \rightarrow A_Q\}_{(A_C, p)}$$

Thus, according to the point of view provided by the category of elements of $\mathbf{P}$, the condition of the commutativity of the previous diagram, is equivalent to the condition that for each arrow u the following diagram commutes.



Consequently, according to the diagram above, the arrows $\tau_{A_C}(p)$ form a cocone from the functor $\mathbf{M} \circ G_{\mathbf{P}}$ to the quantum observables algebra A_Q. The categorical definition of colimit, points to the conclusion that each such cocone emerges by the composition of the colimiting cocone with a unique arrow from the colimit **LP** to the quantum observables algebra object A_Q. Equivalently, we conclude that there is a bijection, natural in **P** and A_Q, as follows:

$$Nat(\mathbf{P}, \mathbf{R}(A_Q)) \cong Hom_{\mathcal{A}_Q}(\mathbf{LP}, A_Q)$$

The established bijective correspondence, interpreted functorially, says that the functor of points **R** from $\mathcal{A}_Q$ to presheaves given by

$$\mathbf{R}(A_Q) : A_C \mapsto Hom_{\mathcal{A}_Q}(\mathbf{M}(A_C), A_Q)$$

has a left adjoint $\mathbf{L} : \mathbf{Sets}^{\mathcal{A}_C{}^{op}} \to \mathcal{A}_Q$, which is defined for each presheaf of commutative algebras **P** in $\mathbf{Sets}^{\mathcal{A}_C{}^{op}}$ as the colimit

$$\mathbf{L}(\mathbf{P}) = Colim\{\mathbf{G}(\mathbf{P}, \mathcal{A}_{\mathbf{C}}) \xrightarrow{\mathbf{G_P}} \mathcal{A}_{\mathbf{C}} \xrightarrow{\mathbf{M}} \mathcal{A}_{\mathbf{Q}}\}$$

Consequently there is a pair of adjoint functors $\mathbf{L} \dashv \mathbf{R}$ as follows:

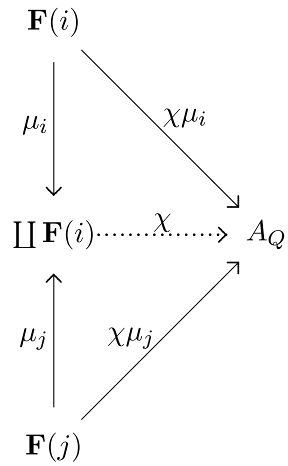
$$\mathbf{L} : \mathbf{Sets}^{\mathcal{A}_C{}^{op}} \underset{\longrightarrow}{\longleftarrow} \mathcal{A}_Q : \mathbf{R}$$

Thus we have constructed an adjunction which consists of the functors **L** and **R**, called left and right adjoints with respect to each other respectively, as well as, the natural bijection;

$$Nat(\mathbf{P}, \mathbf{R}(A_Q)) \cong Hom_{\mathcal{A}_Q}(\mathbf{LP}, A_Q)$$

The content of the *adjunction between the topos of presheaves of commutative observables algebras and the category of quantum observables algebras* can be further developed, if we make use of the categorical construction of the colimit defined above, as a coequalizer of a coproduct. We consider the colimit of any functor $\mathbf{F} : I \longrightarrow \mathcal{A}_Q$ from some index category **I** to $\mathcal{A}_Q$, called a diagram of a quantum observables algebra. Let $\mu_i : \mathbf{F}(i) \to \amalg_i \mathbf{F}(i)$, $i \in I$, be the injections into the coproduct. A morphism from this coproduct, $\chi : \amalg_i \mathbf{F}(i) \to A_Q$, is determined uniquely by the set of its components $\chi_i = \chi\mu_i$. These components χ_i are going to form a cocone over **F** to the quantum observable vertex A_Q only when, for all arrows $v : i \longrightarrow j$ of the index category I, the following conditions are satisfied;

$$(\chi\mu_j)\mathbf{F}(v) = \chi\mu_i$$

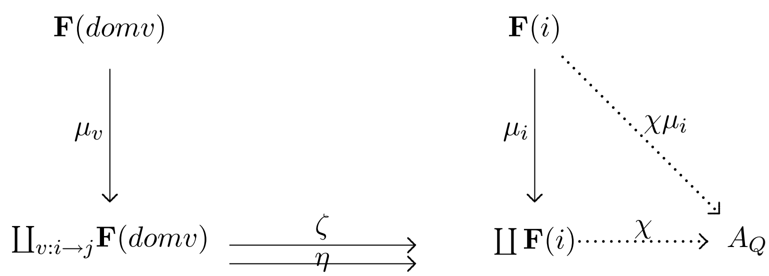


So we consider all $\mathbf{F}(domv)$ for all arrows v with its injections ν_v and obtain their coproduct $\amalg_{v:i\to j}\mathbf{F}(domv)$. Next we construct two arrows ζ and η, defined in terms of the injections ν_v and μ_i, for each $v : i \longrightarrow j$ by the conditions

$$\zeta\nu_v = \mu_i$$

$$\eta\nu_v = \mu_j\mathbf{F}(v)$$

as well as their coequalizer χ



The coequalizer condition $\chi\zeta = \chi\eta$ tells us that the arrows $\chi\mu_i$ form a cocone over $\mathbf{F}$ to the quantum observable vertex $\mathcal{A}_Q$. We further note that since χ is the coequalizer of the arrows ζ and η this cocone is the colimiting cocone for the functor $\mathbf{F} : I \to \mathcal{A}_Q$ from some index category I to $\mathcal{A}_Q$. Hence the colimit of the functor $\mathbf{F}$ can be constructed as a coequalizer of coproducts according to

$$\coprod_{v:i\to j}\mathbf{F}(domv) \underset{\eta}{\overset{\zeta}{\rightrightarrows}} \coprod \mathbf{F}(i) \xrightarrow{\chi} Colim\mathbf{F}$$

In the case considered the index category is the category of elements of the presheaf of commutative observables algebras $\mathbf{P}$ and the functor $\mathbf{M} \circ G_{\mathbf{P}}$ plays the role of the diagram of quantum observables algebras $\mathbf{F} : I \longrightarrow \mathcal{A}_Q$. In the diagram above the second coproduct is over all the objects (ξ, p) with $p \in \mathbf{P}(A_C)$ of the category of elements, while the first coproduct is over all the maps $v : (\acute{A_C}, \acute{p}) \longrightarrow (A_C, p)$ of that category, so that $v : \acute{A_C} \longrightarrow A_C$ and the condition $pv = \acute{p}$ is satisfied. We conclude that the colimit $\mathbf{L}_M(P)$ can be equivalently presented as the coequalizer:

$$\coprod_{v:\acute{A_C} \to A_C} \mathbf{M}(\acute{A_C}) \underset{\eta}{\overset{\zeta}{\rightrightarrows}} \coprod_{(A_C,p)} \mathbf{M}(A_C) \xrightarrow{\chi} \mathbf{P} \otimes_{\mathcal{A}_C} \mathbf{M}$$

The coequalizer presentation of the colimit shows that the Hom-functor $\mathbf{R}(A_Q)$ has a left adjoint which can be characterized categorically as the tensor product $- \otimes_{\mathcal{A}_C} \mathbf{M}$.

In order to clarify the above observation, we forget for the moment that the discussion concerns the category of quantum observables $\mathcal{A}_Q$, and we consider instead the category **Sets**. Then the coproduct $\amalg_p \mathbf{M}(A_C)$ is a coproduct of sets, which is equivalent to the product $\mathbf{P}(A_C) \times \mathbf{M}(A_C)$ for $A_C \in \mathcal{A}_C$. The coequalizer is thus the definition of the tensor product $\mathcal{P} \otimes \mathcal{A}$ of the set- valued functors:

$$\mathbf{P} : \mathcal{A}_C{}^{op} \longrightarrow \mathbf{Sets}, \qquad \mathbf{M} : \mathcal{A}_C \longrightarrow \mathbf{Sets}$$

$$\coprod_{A_C, \acute{A_C}} \mathbf{P}(A_C) \times Hom(\acute{A_C}, A_C) \times \mathbf{M}(\acute{A_C}) \underset{\eta}{\overset{\zeta}{\rightrightarrows}} \coprod_{A_C} \mathbf{P}(A_C) \times \mathbf{M}(A_C) \xrightarrow{\chi} \mathbf{P} \otimes_{\mathcal{A}_C} \mathbf{M}$$

According to the diagram above for elements $p \in \mathbf{P}(A_C)$, $v : \acute{A_C} \to A_C$ and $\acute{q} \in \mathbf{M}(\acute{A_C})$ the following equations hold:

$$\zeta(p, v, \acute{q}) = (pv, \acute{q}), \qquad \eta(p, v, \acute{q}) = (p, v\acute{q})$$

symmetric in $\mathbf{P}$ and $\mathbf{M}$. Hence, the elements of the set $\mathbf{P} \otimes_{\mathcal{A}_C} \mathbf{M}$ are all of the form $\chi(p, q)$. This element can be written as

$$\chi(p, q) = p \otimes q, \quad p \in \mathbf{P}(A_C), q \in \mathbf{M}(A_C)$$

Thus if we take into account the definitions of ζ and η above, we obtain

$$pv \otimes \acute{q} = p \otimes v\acute{q}$$

Furthermore, if we define the arrows

$$k_{A_C} : \mathbf{P} \otimes_{\mathcal{A}_C} \mathbf{M} \longrightarrow A_Q, \qquad l_{A_C} : \mathbf{P}(A_C) \longrightarrow Hom_{\mathcal{A}_Q}(\mathbf{M}(A_C), A_Q)$$

they are related under the fundamental adjunction by

$$k_{A_C}(p,q) = l_{A_C}(p)(q), \qquad A_C \in \mathcal{A}_C, p \in \mathbf{P}(A_C), q \in \mathbf{M}(A_C)$$

Here we consider k as a function on $\amalg_{A_C} \mathbf{P}(A_C) \times \mathbf{M}(A_C)$ with components $k_{A_C} : \mathbf{P}(A_C) \times \mathbf{M}(A_C) \longrightarrow A_Q$, satisfying the relation;

$$k_{\acute{A_C}}(pv, q) = k_{A_C}(p, vq)$$

in agreement with the equivalence relation defined above.

Now we replace the category **Sets** by the category of quantum observables $\mathcal{A}_Q$ under study. The element q in the set $\mathbf{M}(A_C)$ is replaced by a generalized element $q : \mathbf{M}(J_C) \to \mathbf{M}(A_C)$ from some modelling object $\mathbf{M}(J_C)$ of $\mathcal{A}_Q$. Then we consider k as a function $\amalg_{(A_C,p)} \mathbf{M}(A_C) \longrightarrow A_Q$ with components $k_{(A_C,p)} : \mathbf{M}(A_C) \to A_Q$ for each $p \in \mathbf{P}(A_C)$, that, for all arrows $v : \acute{A_C} \longrightarrow A_C$ satisfy;

$$k_{(\acute{A_C}, pv)} = k_{(A_C,p)} \circ \mathbf{M}(v)$$

Then the condition defining the bijection holding by virtue of the fundamental adjunction is given by

$$k_{(A_C,p)} \circ q = l_{A_C}(p) \circ q : \mathbf{M}(J_C) \to A_Q$$

This argument, being natural in the object $\mathbf{M}(J_C)$, is determined by setting $\mathbf{M}(J_C) = \mathbf{M}(A_C)$ with q being the identity map. Hence, the bijection takes the form $k_{(A_C,p)} = l_{A_C}(p)$, where $k : \amalg_{(A_C,p)} \mathbf{M}(A_C) \longrightarrow A_Q$, and $l_{A_C} : \mathbf{P}(A_C) \longrightarrow Hom_{\mathcal{A}_Q}(\mathbf{M}(A_C), A_Q)$.

The physical meaning of the adjunction between presheaves of commutative observables algebras and quantum observables algebras is made transparent if we consider that the adjointly related functors are associated with the process of encoding and decoding information relevant to the structural form of their domain and codomain categories. If we think of $\mathbf{Sets}^{\mathcal{A}_C{}^{op}}$ as the topos of variable commutative algebras modelled in **Sets**, and of $\mathcal{A}_Q$ as the universe of quantum observable structures, then the functor $\mathbf{L} : \mathbf{Sets}^{\mathcal{A}_C{}^{op}} \longrightarrow \mathcal{A}_Q$ signifies a translational code of information from the topos of commutative observables structures to the universe of globally non-commutative ones, whereas the functor $\mathbf{R} : \mathcal{A}_Q \longrightarrow \mathbf{Sets}^{\mathcal{A}_C{}^{op}}$ a translational code in the inverse direction. In general, the content of the information is not possible to remain completely invariant with respect to translating transformations from one universe to another and back. However, there remain two alternatives for a variable set over commutative observables algebras $\mathbf{P}$ to exchange information with a quantum observables algebra A_Q. Either the content of information is exchanged in non-commutative terms with $\mathbf{P}$ translating, represented as the quantum morphism $\mathbf{LP} \longrightarrow A_Q$, or the content of information is exchanged in commutative terms with A_Q translating, represented correspondingly as the natural transformation $\mathbf{P} \longrightarrow \mathbf{R}(A_Q)$. In the first case, from the perspective of A_Q information is being received in quantum terms, while in the second, from the perspective of $\mathbf{P}$ information is being sent in commutative algebras terms. The natural bijection then corresponds to the assertion that these two distinct ways of communicating are equivalent. Thus, the fact that these two functors are adjoint, expresses a relation of variation regulated by two poles, with respect to the meaning of the information related to observation. We claim that the totality of the content of information included in quantum observables structures remains *invariant under commutative algebras encodings*, corresponding to local commutative observables algebras, if and only if the adjunctive correspondence can be appropriately restricted to an equivalence of the functorially correlated categories. In the following sections we will show that this task can be accomplished by defining an appropriate Grothendieck topology on the category of commutative observables algebras, that, essentially permits the comprehension of *a quantum observables structure, as a sheaf of locally commutative ones over an appropriately specified covering system.* Subsequently, the categorical equivalence that will be established in the sequel, is going to be interpreted, as the denotator of an informational invariance property, referring to the translational code of communication between variable commutative observables algebras and globally non-commutative ones.

5 TOPOLOGIES ON CATEGORIES

5.1 Motivation

Our purpose is to show that the functor $\mathbf{R}$ from $\mathcal{A}_Q$ to presheaves given by

$$\mathbf{R}(A_Q) : A_C \mapsto Hom_{\mathcal{A}_Q}(\mathbf{M}(A_C), A_Q)$$

sends quantum observables algebras A_Q in $\mathcal{A}_Q$ not just into presheaves, but actually into sheaves for a suitable Grothendieck topology $\mathbf{J}$ on the category of commutative observables algebras $\mathcal{A}_C$, so that the fundamental adjunction will restrict to an equivalence of categories $\mathbf{Sh}(\mathcal{A}_C, \mathbf{J}) \cong \mathcal{A}_Q$. From a physical perspective the above can be understood as a topos theoretical formulation of Bohr's correspondence, or as a generalized "complementarity principle".

We note at this point that the usual notion of sheaf, in terms of coverings, restrictions, and collation, can be defined and used not just in the spatial sense, namely on the usual topological spaces, but in a generalized spatial sense, on more general topologies (Grothendieck topologies). In the usual definition of a sheaf on a topological space we use the open neighborhoods U of a point in a space X; such neighborhoods are actually monic topological maps $U \to X$. The neighborhoods U in topological spaces can be replaced by maps $V \to X$ not necessarily monic, and this can be done in any category with pullbacks. In effect, a covering by open sets can be replaced by a new notion of covering provided by a family of maps satisfying certain conditions.

For an object A_C of $\mathcal{A}_C$, we consider indexed families

$$\mathbf{S} = \{\psi_i : A_{Ci} \to A_C, i \in I\}$$

of maps to A_C (viz. maps with common codomain A_C), and we assume that, for each object A_C of $\mathcal{A}_C$, we have a set $\mathbf{\Lambda}(A_C)$ of certain such families satisfying conditions to be specified later. These families play the role of coverings of A_C, under those conditions. Based on such coverings, it is possible to construct the analogue of the topological definition of a sheaf, where as presheaves on $\mathcal{A}_C$ we consider the functors $\mathbf{P} : {\mathcal{A}_C}^{op} \to \mathbf{Sets}$. According to the topological definition of a sheaf on a space, we demand that for each open cover $\{U_i, i \in I\}$ of some U, every family of elements

$\{p_i \in \mathbf{P}(U_i)\}$ that satisfy the compatibility condition on the intersections $U_i \cap U_j, \forall i, j$, are pasted together, as a unique element $p \in \mathbf{P}(U)$. Imitating the above procedure for any covering $\mathbf{S}$ of an object A_C, and replacing the intersection $U_i \cap U_j$ by the pullback $A_{Ci} \times_{A_C} A_{Cj}$ in the general case, according to the diagram

$$\begin{array}{ccc} A_{Ci} \times_{A_C} A_{Cj} & \xrightarrow{g_{ij}} & A_{Cj} \\ {\scriptstyle h_{ij}} \downarrow & & \downarrow {\scriptstyle \psi_j} \\ A_{Ci} & \xrightarrow{\psi_i} & A_C \end{array}$$

we effectively obtain for a given presheaf $\mathbf{P} : \mathcal{A_C}^{op} \to \mathbf{Sets}$ a diagram of sets as follows

$$\begin{array}{ccc} \mathbf{P}(A_{Ci} \times_{A_C} A_{Cj}) & \xrightarrow{\mathbf{P}(g_{ij})} & \mathbf{P}(A_{Cj}) \\ {\scriptstyle \mathbf{P}(h_{ij})} \downarrow & & \downarrow {\scriptstyle \mathbf{P}(\psi_j)} \\ \mathbf{P}(A_{Ci}) & \xrightarrow{\mathbf{P}(\psi_i)} & \mathbf{P}(A_C) \end{array}$$

In this case the compatibility condition for a sheaf takes the form: if $\{p_i \in \mathbf{P}_i, i \in I\}$ is a family of compatible elements, namely satisfy $p_i h_{ij} = p_j g_{ij}, \forall i, j$, then a unique element $p \in \mathbf{P}(A_C)$ is being determined by the family such that $p \cdot \psi_i = p_i, \forall i \in I$, where the notational convention $p \cdot \psi_i = \mathbf{P}(\psi_i)(p)$ has been used . Equivalently, this condition can be expressed in the categorical terminology by the requirement that in the diagram

$$\prod_{i,j} \mathbf{P}(A_{Ci} \times_{A_C} A_{Cj}) \leftleftarrows \prod_i \mathbf{P}(A_{Ci}) \xleftarrow{\quad e \quad} \mathbf{P}(A_C)$$

the arrow e, where $e(p) = (p \cdot \psi_i, i \in I)$ is an equalizer of the maps $(p_i, i \in I) \to (p_i h_{ij}; i, j \in I \times I)$ and $(p_i, i \in I) \to (p_i g_{ij}; i, j \in I \times I)$, correspondingly.

The specific conditions that the elements of the set $\Lambda(A_C)$, or else the coverings of A_C, have to satisfy, naturally lead to the notion of a Grothendieck topology on the category $\mathcal{A}_C$.

5.2 Grothendieck topologies

We start our discussion by explicating the notion of a pretopology on the category of commutative observables algebras $\mathcal{A}_C$.

A *pretopology* on A_C is a function $\mathbf{\Lambda}$ where for each object A_C there is a set $\mathbf{\Lambda}(A_C)$. Each $\mathbf{\Lambda}(A_C)$ contains indexed families of $\mathcal{A}_C$-morphisms

$$\mathbf{S} = \{\psi_i : A_{Ci} \to A_C, i \in I\}$$

of maps to A_C such that the following conditions are satisfied:

(1) For each A_C in $\mathcal{A}_C$, $\{id_{A_C}\} \in \mathbf{\Lambda}(A_C)$;

(2) If $C_C \to A_C$ in $\mathcal{A}_C$ and $\mathbf{S} = \{\psi_i : A_{Ci} \to A_C, i \in I\} \in \mathbf{\Lambda}(A_C)$ then $\{\psi_1 : C_C \times_{A_C} A_{Ci} \to A_C, i \in I\} \in \mathbf{\Lambda}(C_C)$. Note that ψ_1 is the pullback in $\mathcal{A}_C$ of ψ_i along $C_C \to A_C$;

(3) If $\mathbf{S} = \{\psi_i : A_{Ci} \to A_C, i \in I\} \in \mathbf{\Lambda}(A_C)$, and for each $i \in I$, $\{{\psi_{ik}}^i : C_{Cik} \to A_{Ci}, k \in K_i\} \in \mathbf{\Lambda}(A_{Ci})$, then $\{{\psi_{ik}}^i \circ \psi_i : C_{Cik} \to A_{Ci} \to A_C, i \in I; k \in K_i\} \in \mathbf{\Lambda}(A_C)$. Note that C_{Cik} is an example of a double indexed object rather than the intersection of C_{Ci} and C_{Ck}.

The notion of a pretopology on the category of commutative algebras $\mathcal{A}_C$ is a categorical generalization of a system of set-theoretical covers on a topology $\mathbf{T}$, where a cover for $U \in \mathbf{T}$ is a set $\{U_i : U_i \in \mathbf{T}, \mathbf{i} \in \mathbf{I}\}$ such that $\cup_i U_i = U$. The generalization is achieved by noting that the topology ordered by inclusion is a poset category and that any cover corresponds to a collection of inclusion arrows $U_i \to U$. Given this fact, any family of arrows contained in $\mathbf{\Lambda}(A_C)$ of a pretopology is a cover, as well.

The passage from a pretopology to a categorical or Grothendieck topology on the category of commutative unital $\mathcal{R}$-algebras takes place through the introduction of appropriate covering devices, called *covering sieves.* For an object A_C in $\mathcal{A}_C$, an A_C-sieve is a family R of $\mathcal{A}_C$-morphisms with codomain A_C, such that if $C_C \to A_C$ belongs to R and $D_C \to C_C$ is any $\mathcal{A}_C$-morphism, then the composite $D_C \to C_C \to A_C$ belongs to R.

A *Grothendieck topology* on the category of commutative algebras $\mathcal{A}_C$, is a system J of sets, $J(A_C)$, for each A_C in $\mathcal{A}_C$, where each $J(A_C)$ consists of a set of A_C-sieves, (called the covering sieves), that satisfy the following conditions:

(1) For any A_C in $\mathcal{A}_C$ the maximal sieve $\{g : cod(g) = A_C\}$ belongs to $J(A_C)$ (maximality condition).

(2) If R belongs to $J(A_C)$ and $f : C_C \to A_C$ is an $\mathcal{A}_\mathcal{C}$-morphism, then $f^*(R) = \{h : C_C \to A_C, f \cdot h \in R\}$ belongs to $J(C_C)$ (stability condition).

(3) If R belongs to $J(A_C)$ and S is a sieve on C_C, where for each $f : C_C \to A_C$ belonging to R, we have $f^*(S)$ in $J(C_C)$, then S belongs to $J(A_C)$ (transitivity condition).

The small category $\mathcal{A}_C$ together with a Grothendieck topology $\mathbf{J}$, is called a *site*. A *sheaf on a site* $(\mathcal{A}_C, \mathbf{J})$ is defined to be any contravariant functor $\mathbf{P} : {\mathcal{A}_C}^{op} \to \mathbf{Sets}$, satisfying the equalizer condition expressed in terms of covering sieves S for A_C, as in the following diagram in $\mathbf{Sets}$:

$$\prod_{f \cdot g \in S} \mathbf{P}(domg) \quad \leftleftarrows \quad \prod_{f \in S} \mathbf{P}(domf) \quad \xleftarrow{\;e\;} \quad \mathbf{P}(A_C)$$

If the above diagram is an equalizer for a particular covering sieve S, we obtain that $\mathbf{P}$ satisfies the sheaf condition with respect to the covering sieve S.

A **Grothendieck topos** over the small category $\mathcal{A}_C$ is a category which is equivalent to the category of sheaves $\mathbf{Sh}(\mathcal{A}_C, \mathbf{J})$ on a site $(\mathcal{A}_\mathcal{C}, \mathbf{J})$. The site can be conceived as a system of generators and relations for the topos. We note that a category of sheaves $\mathbf{Sh}(\mathcal{A}_C, \mathbf{J})$ on a site $(\mathcal{A}_\mathcal{C}, \mathbf{J})$ is a full subcategory of the functor category of presheaves $\mathbf{Sets}^{{\mathcal{A}_C}^{op}}$.

The basic properties of a Grothendieck topos are the following:

(1). It admits finite projective limits; in particular, it has a terminal object, and it admits fibered products.

(2). If $(B_i)_{i \in I}$ is a family of objects of the topos, then the sum $\coprod_{i \in I} B_i$ exists and is disjoint.

(3). There exist quotients by equivalence relations and have the same good properties as in the category of sets.

6 GROTHENDIECK TOPOLOGY ON $\mathcal{A}_C$

6.1 $\mathcal{A}_C$ as a generating subcategory of $\mathcal{A}_Q$

We consider $\mathcal{A}_C$ as a *full subcategory* of $\mathcal{A}_Q$, whose set of objects $\{A_{Ci} : i \in I\}$, with I an index set, *generate* $\mathcal{A}_Q$; that is, for any diagram in $\mathcal{A}_Q$,

$$A_{Ci} \xrightarrow{\ w\ } A_Q \ \underset{u}{\overset{v}{\rightrightarrows}}\ \acute{A}_Q$$

the identity $v \circ w = u \circ w$, for every arrow $w : A_{Ci} \to A_Q$, and every A_{Ci}, implies that $v = u$. We notice that, for every pair of different parallel morphisms of A_Q, with common domain, there is a separating morphism of A_Q, with domain in $A_{Ci} \hookrightarrow A_Q$ and codomain the previous common domain. Equivalently, we can say that the set of all arrows $w : A_{Ci} \to A_Q$, constitute an epimorphic family. We may verify this claim, if we take into account the adjunction and observe that objects of $\mathcal{A}_Q$ are being constructed as *colimits* over categories of elements of presheaves over $\mathcal{A}_C$. Since objects of $\mathcal{A}_Q$ are constructed as colimits of this form, whenever two parallel arrows

$$A_Q \ \underset{u}{\overset{v}{\rightrightarrows}}\ \acute{A}_Q$$

are different, there is an arrow $l : A_{Ci} \to A_Q$ from some A_{Ci} in $\mathcal{A}_C$, such that $v \circ l \neq u \circ l$.

Since we assume that $\mathcal{A}_C$ is a full subcategory of $\mathcal{A}_Q$ we omit the explicit presence of the coordinatization functor $\mathbf{M}$ in the subsequent discussion.

The consideration that $\mathcal{A}_C$ is a generating subcategory of $\mathcal{A}_Q$ points exactly to the depiction of the appropriate Grothendieck topology on $\mathcal{A}_C$, that accomplishes our purpose of comprehending quantum observables algebras as sheaves on $\mathcal{A}_C$.

We assert that a sieve S on a commutative algebra A_C in $\mathcal{A}_{\mathcal{C}}$ is to be a covering sieve of A_C, when the arrows $s : C_C \to A_C$ belonging to the sieve S together form an epimorphic family in $\mathcal{A}_Q$. This requirement may be equivalently expressed in terms of a map

$$G_S : \coprod_{\{s: C_C \to A_C\} \in S} C_C \to A_C$$

being an epi in $\mathcal{A}_Q$.

6.2 The Grothendieck topology of Epimorphic Families

We will show in the sequel, that *covering sieves on commutative algebras in $\mathcal{A}_C$, being epimorphic families in $\mathcal{A}_Q$, indeed define, a Grothendieck topology on $\mathcal{A}_C$.*

First of all we notice that the maximal sieve on each commutative algebra A_C, includes the identity $A_C \to A_C$, thus it is a covering sieve. Next, the transitivity property of the depicted covering sieves is obvious. It remains to demonstrate that the covering sieves remain stable under pullback. For this purpose we consider the pullback of such a covering sieve S on A_C along any arrow $h : {A_C}' \to A_C$ in $\mathcal{A}_C$

$$\begin{array}{ccc} \coprod_{s \in S} C_C \times_{A_C} \acute{A}_C & \longrightarrow & \acute{A}_C \\ \Big\downarrow & & \Big\downarrow h \\ \coprod_{s \in S} C_C & \xrightarrow{\ G\ } & A_C \end{array}$$

The commutative algebras A_C in $\mathcal{A}_C$ generate the category of quantum observables algebras $\mathcal{A}_Q$, hence, there exists for each arrow $s : D_C \to A_C$ in S, an epimorphic family of arrows $\coprod [A_C]^s \to D_C \times_{A_C} \acute{A}_C$, or equivalently $\{{[A_C]^s}_j \to D_C \times_{A_C} \acute{A}_C\}_j$, with each domain $[A_C]^s$ a commutative algebra. Consequently the collection of all the composites:

$${[A_C]^s}_j \to D_C \times_{A_C} \acute{A}_C \to \acute{A}_C$$

for all $s : D_C \to A_C$ in S, and all indices j together form an epimorphic family in $\mathcal{A}_Q$, that is contained in the sieve $h^*(S)$, being the pullback of S along $h : A_C \to \acute{A}_C$. Therefore the sieve $h^*(S)$ is a covering sieve.

It is important to construct the representation of covering sieves within the category of commutative observables algebras $\mathcal{A}_C$. This is possible, if we first observe that for an object C_C of $\mathcal{A}_C$, and a covering sieve for the defined Grothendieck topology on $\mathcal{A}_C$, the map

$$G_S : \coprod_{(s:C_C \to A_C) \in S} C_C \to A_C$$

being an epi in $\mathcal{A}_Q$, can be equivalently presented as the coequalizer of its kernel pair, or else the pullback of G_S along itself

$$\begin{array}{ccc}
\coprod_{\acute{s}} \acute{D}_C \times_{C_C} \coprod_s D_C & \longrightarrow & \coprod_s D_C \\
\Big\downarrow & & \Big\downarrow G_S \\
\coprod_{\acute{s}} \acute{D}_C & \xrightarrow{G_S} & C_C
\end{array}$$

Furthermore, using the fact that pullbacks in $\mathcal{A}_Q$ preserve coproducts, the epimorhic family associated with a covering sieve on C_C, admits the following coequalizer presentation

$$\coprod_{\acute{s},s} \acute{D}_C \times_{C_C} D_C \quad \underset{q_2}{\overset{q_1}{\rightrightarrows}} \quad \coprod_s D_C \xrightarrow{G} C_C$$

Moreover, since $\mathcal{A}_C$ is a generating subcategory of $\mathcal{A}_Q$, for each pair of arrows $s : D_C \to C_C$ and $\acute{s} : \acute{D}_C \to C_C$ in the covering sieve S on the commutative algebra C_C, there exists an epimorphic family $\{A_C \to \acute{D}_C \times_{C_C} D_C\}$, such that each domain A_C is a commutative algebra object in $\mathcal{A}_C$.

Consequently, each element of the epimorphic family, associated with a covering sieve S on a commutative algebra C_C is represented by a commutative diagram in $\mathcal{A}_C$ of the following form:

$$\begin{array}{ccc}
A_C & \xrightarrow{l} & D_C \\
\Big\downarrow k & & \Big\downarrow s \\
\acute{D}_C & \xrightarrow{\acute{s}} & C_C
\end{array}$$

At a further step we may compose the representation of epimorphic families by commutative squares in $\mathcal{A}_C$, obtained previously, with the coequalizer presentation of the same epimorphic families. The composition results in a new coequalizer diagram in $\mathcal{A}_C$ of the following form:

$$\coprod_{A_C} A_C \underset{y_2}{\overset{y_1}{\rightrightarrows}} \coprod_s D_C \xrightarrow{G} C_C$$

where the first coproduct is indexed by all A_C in the commutative diagrams in $\mathcal{A}_C$, representing elements of epimorphic families.

6.3 The J-Sheaf $\mathbf{R}(A_Q)$

For each quantum observables algebra A_Q in $\mathcal{A}_Q$, we consider the contravariant Hom-functor $\mathbf{R}(A_Q) = Hom_{\mathcal{A}_Q}(-, A_Q)$ in $\mathbf{Sets}^{\mathcal{A}_C{}^{op}}$. If we apply this representable functor to the latter coequalizer diagram we obtain an equalizer diagram in **Sets** as follows:

$$\prod_{A_C} Hom_{\mathcal{A}_Q}(A_C, A_Q) \leftleftarrows \prod_{s \in S} Hom_{\mathcal{A}_Q}(D_C, A_Q) \longleftarrow Hom_{\mathcal{A}_Q}(C_C, A_Q)$$

where the first product is indexed by all A_C in the commutative diagrams in $\mathcal{A}_C$, representing elements of epimorphic families.

The equalizer in **Sets**, thus obtained, says explicitly that the contravariant Hom-functor $\mathbf{R}(A_Q) = Hom_{\mathcal{A}_Q}(-, A_Q)$ in $\mathbf{Sets}^{\mathcal{A}_C{}^{op}}$, satisfies the sheaf condition for the covering sieve S. Moreover, the equalizer condition holds, for every covering sieve in the Grothendieck topology of epimorphic families.

By rephrasing the above, we conclude that the representable Hom-functor $\mathbf{R}(A_Q)$ is a sheaf for the Grothendieck topology of epimorphic families on the category of commutative observables algebras.

7 EQUIVALENCE OF THE TOPOS $\mathbf{Sh}(\mathcal{A}_C, \mathbf{J})$ WITH $\mathcal{A}_Q$

We claim, that if the functor $\mathbf{R}$ from $\mathcal{A}_Q$ to presheaves

$$\mathbf{R}(A_Q) : A_C \mapsto Hom_{\mathcal{A}_Q}(\mathbf{M}(A_C), A_Q)$$

sends quantum observables algebras A_Q in $\mathcal{A}_Q$ not just into presheaves, but into sheaves for the Grothendieck topology of epimorphic families, $\mathbf{J}$, on the category of commutative observables algebras $\mathcal{A}_C$, the fundamental adjunction restricts to an equivalence of categories $\mathbf{Sh}(\mathcal{A}_C, \mathbf{J}) \cong \mathcal{A}_Q$. Thus, $\mathcal{A}_Q$

is, in effect, a Grothendieck topos. Hence, in an epigrammatic manner we can assert that; appropriately sheafifying in the Grothendieck sense is equivalent to quantizing, equivalently, quantizing means, in effect, sheafifying à la Grothendieck!.

7.1 Covering Sieves on Quantum Observables Algebras

If we consider a quantum observables algebra A_Q, and all quantum algebraic morphisms of the form $\psi : E_C \to A_Q$, with domains E_C, in the generating subcategory of commutative observables algebras $\mathcal{A}_C$, then the family of all these maps ψ, constitute an epimorphism:

$$T : \coprod_{(E_C \in \mathcal{A}_\mathcal{C}, \psi : E_C \to A_Q)} E_C \to A_Q$$

We notice that the quantum algebraic epimorphism T is actually the same as the map,

$$T : \coprod_{(E_C \in \mathcal{A}_\mathcal{C}, \psi : \mathbf{M}(E_C) \to A_Q)} \mathbf{M}(E_C) \to A_Q$$

since the coordinatization functor $\mathbf{M}$ is, by the fact that $\mathcal{A}_C$ is a full subcategory of $\mathcal{A}_Q$, just the inclusion functor $\mathbf{M} : \mathcal{A}_C \hookrightarrow \mathcal{A}_Q$.

Subsequently, we may use the same arguments, as in the discussion of the Grothendieck topology of epimorphic families of the previous section, in order to assert that the epimorphism T can be presented as a coequalizer diagram of the form [DI] in $\mathcal{A}_Q$ as follows:

$$\coprod_\nu A_C \overset{y_1}{\underset{y_2}{\rightrightarrows}} \coprod_{(E_C \in \mathcal{A}_C, \psi : E_C \to A_Q)} E_C \xrightarrow{T} A_Q$$

where the first coproduct is indexed by all ν, representing commutative diagrams in $\mathcal{A}_Q$, of the form:

$$\begin{array}{ccc} A_C & \xrightarrow{l} & E_C \\ \downarrow{k} & & \downarrow{\psi} \\ \acute{E_C} & \xrightarrow{\acute{\psi}} & A_Q \end{array}$$

where A_C, E_C, $\acute{E_C}$ are objects in the generating subcategory $\mathcal{A}_C$ of $\mathcal{A}_Q$.

We say that a sieve on a quantum observables algebra defines a covering sieve by objects of its generating subcategory $\mathcal{A}_C$, when the quantum algebraic morphisms belonging to the sieve define an epimorphism

$$T : \coprod_{(E_C \in \mathcal{A}_\mathcal{C}, \psi : \mathbf{A}(E_C) \to A_Q)} \mathbf{M}(E_C) \to A_Q$$

In this case the epimorphic families of quantum algebraic morphisms constituting covering sieves of quantum observables algebras fit into coequalizer diagrams of the latter form [DI].

From the physical point of view covering sieves of the form defined above, are equivalent with commutative algebras localization systems of quantum observables algebras. These localization systems filter the information of quantum observabes algebras, through commutative algebras domains, associated with procedures of measurement of observables. We will discuss localizations systems in detail, in order to unravel the physical meaning of the requirements underlying the notion of Grothendieck topology, and subsequently, the notion of covering sieves defined previously. It is instructive to begin with the notion of a system of prelocalizations for a quantum observables algebra.

A **system of prelocalizations** for a quantum observables algebra A_Q in $\mathcal{A}_Q$ is a *subfunctor of the Hom-functor* $\mathbf{R}(A_Q)$ of the form $\mathbf{S} : \mathcal{A_C}^{op} \to \mathbf{Sets}$, namely, for all A_C in $\mathcal{A_C}$ it satisfies $\mathbf{S}(A_C) \subseteq [\mathbf{R}(A_Q)](A_C)$. Hence, a system of prelocalizations for a quantum observables algebra A_Q in $\mathcal{A}_Q$ is an *ideal* $\mathbf{S}(A_C)$ *of quantum algebraic morphisms* from commutative algebras domains of the form

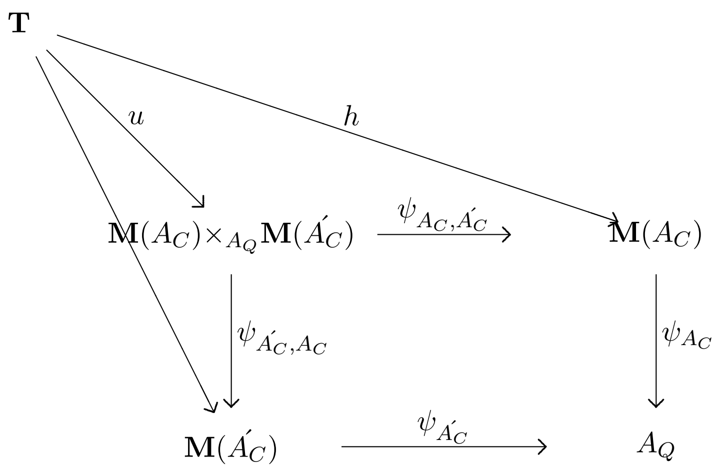
$$\psi_{A_C} : \mathbf{M}(A_C) \longrightarrow A_Q, \qquad A_C \in \mathcal{A_C}$$

such that $\{\psi_{A_C} : \mathbf{M}(A_C) \longrightarrow A_Q$ in $\mathbf{S}(A_C)$, and $\mathbf{M}(v) : \mathbf{M}(\acute{A_C}) \to \mathbf{M}(A_C)$ in $\mathcal{A}_Q$ for $v : \acute{A_C} \to A_C$ in $\mathcal{A_C}$, implies $\psi_{A_C} \circ \mathbf{M}(v) : \mathbf{M}(\acute{A_C}) \longrightarrow \mathcal{A}_Q$ in $\mathbf{S}(A_C)\}$.

The introduction of the notion of a system of prelocalizations of a quantum observabes algebra has a sound operational physical basis: In every concrete *experimental context*, the set of observables that can be observed in this context forms a *unital commutative algebra*. The above remark is equivalent to the statement that a *measurement-induced commutative algebra of observables* serves as a *local reference frame, in a topos-theoretical environment*, relative to which a measurement result is being coordinatized. Adopting the aforementioned perspective on quantum observables algebras, the operation of the Hom-functor $\mathbf{R}(A_Q)$ is equivalent to depicting an ideal of algebraic morphisms which are to play the role of local coverings of a quantum observables algebra, by coordinatizing commutative algebras related with measurement situations. From a geometrical viewpoint, we may thus characterize the maps $\psi_{A_C} : \mathbf{M}(A_C) \longrightarrow A_Q, \quad A_C \in \mathcal{A_C}$, in a system of prelocalizations for a quantum observables algebra A_Q, as a cover of A_Q by an algebra of commutative observables.

Under these intuitive identifications, we say that a family of commutative domains covers $\psi_{A_C} : \mathbf{M}(A_C) \longrightarrow A_Q, \quad A_C \in \mathcal{A_C}$, is the generator of the system of prelocalization $\mathbf{S}$, iff this system is the smallest among all that contains that family. It is evident that a quantum observables algebra can have many systems of measurement prelocalizations, that, remarkably, form an ordered structure. More specifically, systems of prelocalizations constitute a partially ordered set, under inclusion. Furthermore, the intersection of any number of systems of prelocalization is again a system of prelocalization.We emphasize that the minimal system is the empty one, namely $\mathbf{S}(A_C) = \emptyset$ for all $A_C \in \mathcal{A_C}$, whereas the maximal system is the Hom-functor $\mathbf{R}(A_Q)$ itself, or equivalently, all quantum algebraic morphisms $\psi_{A_C} : \mathbf{M}(A_C) \longrightarrow A_Q$, that is the set $Hom_{\mathcal{A}_Q}(\mathbf{M}(A_C), A_Q)$.

The transition from a system of prelocalizations to a system of localizations for a quantum observables algebra, can be effected under the restriction that, certain compatibility conditions have to be satisfied on the overlap of the coordinatizing commutative domain covers. In order to accomplish this, we use a *pullback* diagram in $\mathcal{A}_Q$ as follows:



The pullback of the commutative domains covers $\psi_{A_C} : \mathbf{M}(A_C) \longrightarrow A_Q, A_C \in \mathcal{A}_{\mathcal{C}}$ and $\psi_{A'_C} : \mathbf{M}(A'_C) \longrightarrow A_Q, A'_C \in \mathcal{A}_{\mathcal{C}}$ with common codomain the quantum observables algebra A_Q, consists of the object $\mathbf{M}(A_C)\times_{A_Q}\mathbf{M}(A'_C)$ and two arrows $\psi_{A_C A'_C}$ and $\psi_{A'_C A_C}$, called projections, as shown in the above diagram. The square commutes and, for any object T and arrows h, g that make the outer square commute, there is a unique $u : T \longrightarrow \mathbf{M}(A_C)\times_{A_Q}\mathbf{M}(A'_C)$ that makes the whole diagram commutative. Hence, we obtain the condition:

$$\psi_{A'_C} \circ g = \psi_{A_C} \circ h$$

We notice that if ψ_{A_C} and $\psi_{A'_C}$ are 1-1, then the pullback is isomorphic with the intersection $\mathbf{M}(A_C) \cap \mathbf{M}(A'_C)$. Then, we can define the pasting map, which is an isomorphism, as follows:

$$W_{A_C,A'_C} : \psi_{A'_C A_C}(\mathbf{M}(A_C)\times_{A_Q}\mathbf{M}(A'_C)) \longrightarrow \psi_{A_C A'_C}(\mathbf{M}(A_C)\times_{A_Q}\mathbf{M}(A'_C))$$

by putting

$$W_{A_C,A'_C} = \psi_{A_C A'_C} \circ {\psi_{A'_C A_C}}^{-1}$$

Then we have the following conditions: ("pull-back compatibility")

$$W_{A_C,A_C} = 1_{A_C} \qquad with \qquad 1_{A_C} := id_{A_C}$$

$$W_{A_C,A'_C} \circ W_{A'_C,A''_C} = W_{A_C,A''_C} \qquad if \quad \mathbf{M}(A_C) \cap \mathbf{M}(A'_C) \cap \mathbf{M}(A''_C) \neq 0$$

$$W_{A_C,A'_C} = {W^{-1}}_{A'_C,A_C} \qquad if \quad \mathbf{M}(A_C) \cap \mathbf{M}(A'_C) \neq 0$$

The pasting map provides the means to guarantee that $\psi_{\acute{A_C}A_C}(\mathbf{M}(A_C)\times_{A_Q}\mathbf{M}(\acute{A_C}))$ and $\psi_{A_C\acute{A_C}}(\mathbf{M}(A_C)\times_{A_Q}\mathbf{M}(\acute{A_C}))$ are going to cover the same part of a quantum observables algebra in a compatible way.

Given a system of measurement prelocalizations for a quantum observables algebra $A_Q \in \mathcal{A}_Q$, we call it a **system of localizations** *iff the above compatibility conditions are being satisfied.*

The compatibility conditions established, provide the necessary relations for understanding a system of measurement localizations for a quantum observables algebra as a structure sheaf or sheaf of coefficients from local commutative covering domains of observables algebras. This is related to the fact that systems of measurement localizations are actually subfunctors of the representable Hom-functor $\mathbf{R}(A_Q)$ of the form $\mathbf{S} : \mathcal{A_C}^{op} \rightarrow \mathbf{Sets}$, namely, all A_C in $\mathcal{A_C}$ satisfy $\mathbf{S}(A_C) \subseteq [\mathbf{R}(A_Q)](A_C)$. In this sense the pullback compatibility conditions express gluing relations on overlaps of commutative domains covers and convert a presheaf subfunctor of the Hom-functor (system of prelocalizations) into a sheaf for the Grothendieck topology specified.

7.2 Unit and Counit of the Adjunction

We focus again our attention in the fundamental adjunction and investigate the unit and the counit of it. For any presheaf $\mathbf{P} \in \mathbf{Sets}^{\mathcal{A}_C{}^{op}}$, we deduce that the unit $\delta_{\mathbf{P}} : \mathbf{P} \longrightarrow Hom_{\mathcal{A_Q}}(\mathbf{M}(\_), \mathbf{P} \otimes_{\mathcal{A}_C} \mathbf{M})$ has components:

$$\delta_{\mathbf{P}}(A_C) : \mathbf{P}(A_C) \longrightarrow Hom_{\mathcal{A_Q}}(\mathbf{M}(A_C), \mathbf{P} \otimes_{\mathcal{A}_C} \mathbf{M})$$

for each commutative algebra object A_C of $\mathcal{A}_C$. If we make use of the representable presheaf $y[A_C]$, we obtain:

$$\delta_{\mathbf{y}[A_C]} : \mathbf{y}[A_C] \rightarrow Hom_{\mathcal{A_Q}}(\mathbf{M}(\_), \mathbf{y}[A_C] \otimes_{\mathcal{A}_C} \mathbf{M})$$

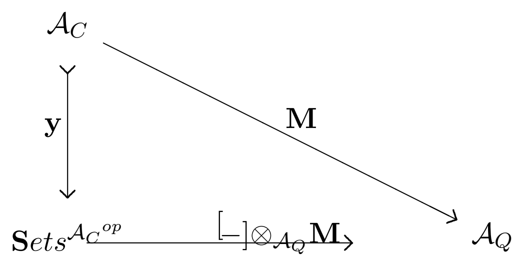
Hence, for each object A_C of $\mathcal{A}_C$ the unit, in the case considered, corresponds to a map,

$$\mathbf{M}(A_C) \rightarrow \mathbf{y}[A_C] \otimes_{\mathcal{A}_C} \mathbf{M}$$

But, since

$$\mathbf{y}[A_C] \otimes_{\mathcal{A}_C} \mathbf{M} \cong \mathbf{M} \circ \mathbf{G}_{\mathbf{y}[A_C]}(A_C, 1_{A_C}) = \mathbf{M}(A_C)$$

the unit for the representable presheaf of commutative algebras, which is a sheaf for the Grothendieck topology of epimorphic families, is clearly an isomorphism. By the preceding discussion we can see that the diagram commutes

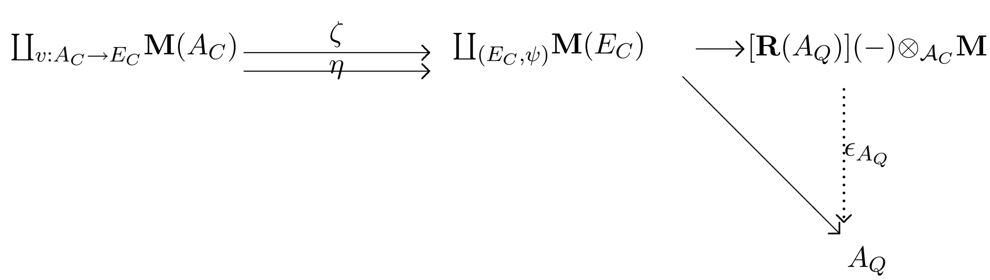


Thus, the unit of the fundamental adjunction, referring to the representable sheaf $\mathbf{y}[A_C]$ of the category of commutative observables algebras, provides a map (quantum algebraic morphism) $\mathbf{M}(A_C) \longrightarrow \mathbf{y}[A_C]\otimes_{\mathcal{A}_C}\mathbf{M}$, which is an isomorphism.

On the other side, for each quantum observables algebra object A_Q of $\mathcal{A}_Q$, the counit is

$$\epsilon_{A_Q} : Hom_{\mathcal{A}_Q}(\mathbf{M}(\_), A_Q)\otimes_{\mathcal{A}_C}\mathbf{M} \longrightarrow A_Q$$

The counit corresponds to the vertical map in the following coequalizer diagram [DII]:



where the first coproduct is indexed by all arrows $v : A_C \to E_C$, with A_C, E_C objects of $\mathcal{A}_C$, whereas the second coproduct is indexed by all objects A_C in $\mathcal{A}_C$ and arrows $\psi : \mathbf{M}(E_C) \to A_Q$, belonging to a covering sieve of A_Q by objects of its generating subcategory.

It is important to notice the similarity in form of diagrams [DI] and [DII]. Based on this observation, it is possible to prove that if the domain of the counit of the adjunction is restricted to sheaves for the Grothendieck topology of epimorphic families on $\mathcal{A}_C$, then the counit defines a quantum algebraic isomorphism;

$$\epsilon_{A_Q} : Hom_{\mathcal{A}_Q}(\mathbf{M}(\_), A_Q) \otimes_{\mathcal{A}_C} \mathbf{M} \simeq A_Q$$

In order to substantiate our thesis, we inspect diagrams [DI] and [DII], observing that it is enough to prove that the pairs of arrows (ζ, η) and (y_1, y_2) have isomorphic coequalizers, since, then, the counit is obviously an isomorphism. Thus, we wish to show that a covering sieve of a quantum event algebra

$$T : \coprod_{(E_C \in \mathcal{A}_C, \psi : \mathbf{M}(E_C) \to A_Q)} \mathbf{M}(E_C) \to A_Q$$

is the coequalizer of (y_1, y_2) iff it is the coequalizer of (ζ, η). In the following discussion, we may omit the explicit presence of the inclusion functor $\mathbf{M}$, for the same reasons stated previously.

We consider a covering sieve of a quantum observables algebra A_Q, consisting of quantum algebraic morhisms $T_{(E_C, \psi)}$, that together constitute an epimorphic family in $\mathcal{A}_Q$. We observe that the condition $T \cdot y_1 = T \cdot y_2$ is equivalent to the condition [CI] as follows:

$$T_{(E_C, \psi)} \cdot l = T_{(\acute{E_C}, \acute{\psi})} \cdot k$$

for each commutative square ν. Furthermore, the condition $T \cdot \zeta = T \cdot \eta$ is equivalent to the condition [CII] as follows:

$$T_{(E_C, \psi)} \cdot u = T_{(\acute{E_C}, \psi \cdot u)}$$

for every commutative algebras morphism $u : \acute{E_C} \to E_C$, with A_C, E_C objects of $\mathcal{A}_C$ and $\psi : E_C \to A_Q$, belonging to a covering sieve of A_Q by objects of its generating subcategory. Therefore, our thesis is proved if we show that $[CI] \Leftrightarrow [CII]$.

On the one hand, $T \cdot \zeta = T \cdot \eta$, implies for every commutative diagram of the form ν:

$$
\begin{array}{ccc}
A_C & \xrightarrow{l} & E_C \\
\Big\downarrow k & & \Big\downarrow \psi \\
\acute{E_C} & \xrightarrow{\acute{\psi}} & A_Q
\end{array}
$$

the following relations:

$$T_{(E_C,\psi)} \cdot l = T_{(A_C,\psi \cdot l)} = T_{(A_C,\acute{\psi} \cdot k)} = T_{(\acute{E_C},\acute{\psi})} \cdot k$$

Thus $[CI] \Rightarrow [CII]$

On the other hand, $T \cdot y_1 = T \cdot y_2$, implies that for every commutative algebras morphism $u : \acute{E_C} \to E_C$, with A_C, E_C objects of $\mathcal{A}_C$ and $\psi : E_C \to A_Q$, the diagram of the form ν

$$
\begin{array}{ccc}
\acute{E_C} & \xrightarrow{u} & E_C \\
id \Big\downarrow & & \Big\downarrow \psi \\
\acute{E_C} & \xrightarrow{\psi \cdot u} & A_Q
\end{array}
$$

commutes and provides the condition

$$T_{(E_C,\psi)} \cdot u = T_{(\acute{E_C},\psi \cdot u)}$$

Thus $[CI] \Leftarrow [CII]$.

Consequently, the pairs of arrows (ζ, η) and (y_1, y_2) have isomorphic coequalizers, proving that *the counit of the fundamental adjunction restricted to sheaves for the Grothendieck topology of epimorphic families on $\mathcal{A}_C$ is an isomorphism.*

8 ABSTRACT DIFFERENTIAL GEOMETRY IN THE QUANTUM REGIME

8.1 The Quantum Quotient Algebra Sheaf of Coefficients

Having at our disposal the *sheaf theoretical representation of a quantum observables algebra* through the counit isomorphism established above, for the Grothendieck topology of epimorphic families of covers from local commutative algebras domains of observables, we may attempt to apply the methodology of Abstract (alias, Modern) Differential Geometry (ADG), in order to set up a differential geometric mechanism suited to the quantum regime of observables structures.

First of all we notice that the transition from the classical to the quantum case is expressed in terms of the relevant arithmetics used, as a transition from a commutative algebra of observables presented as a sheaf, if localized over a measurement topological space, to a globally non-commutative algebra of observables, presented correspondingly as a sheaf of locally commutative algebras of coefficients for the Grothendieck topology specified over the category of commutative subalgebras of the former. It is instructive to remind that the latter sheaf theoretical representation is established according to the counit isomorphism by

$$\epsilon_{A_Q} : \mathbf{R}(A_Q) \otimes_{\mathcal{A}_C} \mathbf{M} \simeq A_Q$$

Furthermore, we may give an explicit form of the elements of $\mathbf{R}(A_Q) \otimes_{\mathcal{A}_C} \mathbf{M}$ according to the coequalizer of coproduct definition of the above tensor product

$$\coprod_{v:\acute{A_C} \to A_C} \mathbf{M}(\acute{A_C}) \underset{\eta}{\overset{\zeta}{\rightrightarrows}} \coprod_{(A_C, \psi_{A_C})} \mathbf{M}(A_C) \xrightarrow{\chi} \quad \mathbf{R}(A_Q) \otimes_{\mathcal{A}_C} \mathbf{M}$$

According to the diagram above for elements $\psi_{A_C} \in \mathbf{R}(A_Q)(A_C)$, $v : \acute{A_C} \to A_C$ and $\acute{\xi} \in \mathbf{M}(\acute{A_C})$, the following equations hold:

$$\zeta(\psi_{A_C}, v, \acute{\xi}) = (\psi_{A_C} v, \acute{\xi}), \qquad \eta(\psi, v, \acute{\xi}) = (\psi_{A_C}, v\acute{\xi})$$

symmetric in $\mathbf{R}(A_Q)$ and $\mathbf{M}$. Hence the elements of $\mathbf{R}(A_Q) \otimes_{\mathcal{A}_C} \mathbf{M}$ are all of the form $\chi(\psi_{A_C}, \xi)$. This element can be written as

$$\chi(\psi_{A_C}, \xi) = \psi_{A_C} \otimes \xi, \quad \psi_{A_C} \in \mathbf{R}(A_Q)(A_C), \xi \in \mathbf{M}(A_C)$$

Thus if we take into account the definitions of ζ and η above, we obtain

$$\psi_{A_C} v \otimes \acute{\xi} = \psi_{A_C} \otimes v\acute{\xi}$$

We conclude that $\mathbf{R}(A_Q)\otimes_{\mathcal{A}_C}\mathbf{M}$ is actually the quotient of $\coprod_{(A_C,\psi_{A_C})}\mathbf{M}(A_C)$ by the smallest equivalence relation generated by the above equations. Moreover, if there exists A_D in $\mathcal{A}_C$ and homomorphisms $w : A_D \rightarrow A_C$, $v : A_D \rightarrow \acute{A}_C$, such that: $w\bar{\xi} = \xi$, $v\bar{\xi} = \acute{\xi}$, and $\psi_{A_C} w = \psi_{\acute{A_C}}$, $\xi \in \mathbf{M}(A_C), \acute{\xi} \in \mathbf{M}(\acute{A}_C), \bar{\xi} \in \mathbf{M}(A_D), \psi_{A_C} \in \mathbf{R}(A_Q)(A_C), \acute{\psi}_{A_C} \in \mathbf{R}(A_Q)(\acute{A}_C)$ then the identification equations take the form

$$\psi_{A_C} \otimes \xi = \psi_{\acute{A_C}} \otimes \acute{\xi}$$

If we denote by $l_Q(A_C)$ the ideal generated by the equivalence relation, corresponding to the above identification equations, for each A_C in $\mathcal{A}_C$, we conclude that locally, in the Grothendieck topology defined, an element of $\mathbf{R}(A_Q)\otimes_{\mathcal{A}_C}\mathbf{M}$ can be written in the form:

$$\psi_{A_C} \otimes \xi = (\psi_{A_C}, \xi) + l_Q(A_C) \equiv [\psi_{A_C} \otimes \xi]$$

Subsequently a quantum observables algebra admits a sheaf theoretical representation in terms of an algebra sheaf that, locally, that is, over a particular cover, has the quotient form;

$$\mathbf{R}(A_Q)(A_C)\otimes_{\mathcal{A}_C}\mathbf{M}(A_C) = [\mathbf{R}(A_Q)(A_C) \times \mathbf{M}(A_C)]/l_Q(A_C)$$

In this sense, the quantum arithmetics can be described locally, that is, over a particular cover in a localization system of a quantum observables algebra, as an algebra $\mathbf{K}(A_C) := [\mathbf{R}(A_Q)(A_C) \times \mathbf{M}(A_C)]/l_Q(A_C)$. The latter can be further localized over a "topological measurement space", categorically dual to the commutative observables algebra A_C, that serves as the *algebra sheaf of a differential triad* $\Delta = (\mathbf{K}(A_C), \delta_{\mathbf{K}(A_C)}, \Omega(\mathbf{K}(A_C)))$, attached to this particular cover. The appropriate specification of the $\mathbf{K}(A_C)$-module $\Omega(\mathbf{K}(A_C))$ is going to be the subject of a detailed discussion in what follows: From a physical viewpoint a reasonable choice would be the identification of $\Omega(\mathbf{K}(A_C))$ with the $\mathbf{K}(A_C)$-module $\Phi(\mathbf{K}(A_C))$ of all localized quotient commutative algebra of observables sheaf endomorphisms $\nabla_{\mathbf{K}(A_C)} : \mathbf{K}(A_C) \rightarrow \mathbf{K}(A_C)$, which are $\mathcal{R}$-linear and satisfy the Leibniz rule ("derivations"). Thus the differential structure on a local commutative domain cover, $\psi_{A_C} : \mathbf{M}(A_C) \hookrightarrow A_Q$, A_C in $\mathcal{A_C}$, being an inclusion, would be naturally defined in the following manner:

$$(\psi_{A_C}, \xi) + l_Q(A_C) \mapsto (\psi_{A_C}, \nabla_{A_C}\xi) + l_Q(A_C)$$

where $\nabla_{A_C} := \nabla : A_C \to A_C$ is an A_C-valued derivation of A_C, which we call differential variation of first-order, or equivalently differential 1-variation, applied to the observable ξ. In the sequel, we will specify the necessary conditions required for the existence of ∇ for a general commutative algebra of observables A_C.

In this sense, we may form the conclusion that locally in the Grothendieck topology specified, there exists a naturally defined differential operator, that has the following form over a particular cover for each A_C in $\mathcal{A}_C$:

$$\delta_{\mathbf{K}(A_C)}(\psi_{A_C} \otimes \xi) := (\psi_{A_C}, \nabla_{A_C}\xi) + l_Q(A_C)$$

At this point we remind that a covering sieve, or equivalently, localization system of a quantum observables algebra contains epimorphic families from local commutative domain covers, such that each element associated with a covering sieve is represented by a commutative diagram of the form

$$\begin{array}{ccc} A_C & \xrightarrow{\;l\;} & E_C \\ \Big\downarrow k & & \Big\downarrow \psi \\ \acute{E}_C & \xrightarrow{\;\acute{\psi}\;} & A_Q \end{array}$$

where A_C, E_C, $\acute{E}_C$ are objects in the generating subcategory $\mathcal{A}_C$ of $\mathcal{A}_Q$.

Moreover they fit all together in a coequalizer diagram

$$\coprod_{\nu} A_C \quad \underset{y_2}{\overset{y_1}{\rightrightarrows}} \quad \coprod_{(E_C \in \mathcal{A}_C, \psi : E_C \to A_Q)} E_C \xrightarrow{\;T\;} A_Q$$

where the first coproduct is indexed by all ν, representing commutative diagrams in $\mathcal{A}_Q$ of the form above.

Thus, having specified a differential triad $\Delta = (\mathbf{K}(A_C), \delta_{\mathbf{K}(A_C)}, \Omega(\mathbf{K}(A_C)))$ attached to each particular cover, we may specify a diagram of differential triads that, in turn corresponds to an element associated with an epimorphic covering sieve in the Grothendieck topology defined on $\mathcal{A}_C$. This diagram of differential triads, together with the corresponding coequalizer of coproduct diagram, contain all the information necessary for the set-up of the differential geometric mechanism suited to the quantum regime. *Hence, the transition from the classical to the quantum case amounts to a change of perspective from a single differential triad to a diagram of differential triads interlocking in such a way that information related to observation in different covering domains is compatible on their overlaps.*

8.2 Differential 1-variations

A derivation ∇ of a commutative observables algebra A_C is an R-linear endomorphism of the R-commutative arithmetic A_C, denoted by $\nabla : A_C \to A_C$, that satisfies the Leibniz rule:

$$\nabla(\zeta\xi) = \zeta\nabla(\xi) + \xi\nabla(\zeta)$$

for all ζ, ξ belonging to A_C.

We also define the set of all derivations of A_C, denoted by $G(A_C)$. It is obvious that $G(A_C)$ is a left A_C-module. Remarkably, $G(A_C)$ can be also endowed with a Lie algebra structure if we define an R-linear skew-symmetric operator, called commutator of derivations in $G(A_C)$ as follows:

For any two derivations ∇_1, $\nabla_2 \in G(A_C)$ their commutator, denoted as $[\nabla_1, \nabla_2]$, is given by;

$$[\nabla_1, \nabla_2] = \nabla_1 \circ \nabla_2 - \nabla_2 \circ \nabla_1$$

We can easily check that the commutator $[\nabla_1, \nabla_2]$ is skew-symmetric, and also, it is a derivation belonging to $G(A_C)$. Furthermore, the commutator derivation satisfies the Jacobi identity as follows;

$$[\nabla_1, [\nabla_2, \nabla_3]] = [[\nabla_1, \nabla_2], \nabla_3] + [\nabla_2, [\nabla_1, \nabla_3]]$$

Actually if we consider $\nabla \in G(A_C)$, and also, $\zeta, \xi \in A_C$, then we define;

$$(\overleftarrow{\zeta}\nabla)(\xi) := \zeta(\nabla(\xi))$$

It is clear that $\overleftarrow{\zeta}\nabla \in G(A_C)$, thus $G(A_C)$ is a left A_C-module. We notice that we can also define a right A_C-module structure on $G(A_C)$ according to the rule;

$$(\overrightarrow{\zeta}\nabla)(\xi) := \nabla(\zeta\xi)$$

Now, we may define a commutator as follows;

$$[\hat{\zeta}, \nabla](\xi) := (\overrightarrow{\zeta}\nabla - \overleftarrow{\zeta}\nabla)(\xi) = (\nabla(\zeta))\xi$$

Thus, for any $\zeta \in A_C$, we can define the Lie derivative operator;

$$L_\zeta : G(A_C) \to G(A_C)$$

$$L_\zeta(\nabla) := [\hat{\zeta}, \nabla]$$

Moreover, if we consider operators L_ζ, L_η, we can easily show that they commute, and furthermore, the identity below is being satisfied;

$$(L_\eta \circ L_\zeta)(\nabla) = 0$$

for every ζ, $\eta \in A_C$. Thus we can state the following:

If we consider a commutative observables algebra A_C, then an R*-linear morphism $\nabla \in G(A_C)$ is called a differential* 1*-variation if for all η, $\zeta \in A_C$, and corresponding commutator operators L_η, L_ζ, the following identity holds:*

$$(L_\eta \circ L_\zeta)(\nabla) = 0$$

In the case that the classical commutative arithmetic A_C represents $\mathcal{C}^\infty(X, R)$, then the above identity is satisfied and differential 1-variations are tautosemous with the usual fist-order linear differential operators of the form $\nabla = \kappa^i \partial_i + \lambda$, where $\kappa^i, \lambda \in \mathcal{C}^\infty(X, R)$.

The fact that the set of all derivations of A_C, say $G(A_C)$, has an A_C-module structure, motivates the definition of an M-valued derivation of an observables algebra A_C, for an arbitrary A_C-module M as follows:

An M-valued derivation ∇_M of an observables algebra A_C is an R-linear morphism, denoted by $\nabla_M : A_C \to M$, that satisfies the Leibniz rule:

$$\nabla_M(\zeta\xi) = \zeta\nabla_M(\xi) + \xi\nabla_M(\zeta)$$

for all ζ, ξ belonging to A_C. If we consider $\nabla_M \in G(M)$, and also, $\zeta, \xi \in A_C$, then we have;

$$(\overleftarrow{\zeta}\nabla_M)(\xi) := \zeta(\nabla_M(\xi))$$

It is clear that $\overleftarrow{\zeta}\nabla_M \in G(M)$, thus $G(M)$ is a left A_C-module. We also notice that we can define a right A_C-module structure on $G(M)$ as follows;

$$(\overrightarrow{\zeta}\nabla_M)(\xi) := \nabla_M(\zeta\xi)$$

Now, we can define a commutator according to;

$$[\hat{\zeta}, \nabla_M](\xi) := (\overrightarrow{\zeta}\nabla_M - \overleftarrow{\zeta}\nabla_M)(\xi) = (\nabla_M(\zeta))\xi$$

Thus, for any $\zeta \in A_C$, we can again define the Lie derivative operator, as follows;

$$L_\zeta : G(M) \to G(M)$$

$$L_\zeta(\nabla_M) := [\hat{\zeta}, \nabla_M]$$

Furthermore, if we consider operators L_ζ, L_η, for every ζ, $\eta \in A_C$, they commute, and also, satisfy the identity;

$$(L_\eta \circ L_\zeta)(\nabla_M) = 0$$

In this sense, we can state the criterion of identification of M-valued derivations, in an analogous manner as in 8.2, as follows:

If we consider the set $S(M)$, consisting of R*-linear morphisms of a commutative observables algebra A_C into an arbitrary A_C-module M, then the elements of $S(M)$ are identified as M-valued derivations, ∇_M, of the algebra A_C, if for any ζ, $\eta \in A_C$, and corresponding Lie derivative operators L_ζ, L_η from $S(M)$ to itself, the following identity holds:*

$$(L_\eta \circ L_\zeta)(\nabla_M) = 0$$

Furthermore, it is instructive to notice that, for an R-linear morphism θ of A_C-modules, M, $N \in S(M,N)$; where $S(M,N)$ denotes the bimodule of all R-linear morphisms of A_C-modules M and N, we can analogously define a commutator operator $\hat{L}_\zeta$, for every $\zeta \in A_C$, according to;

$$\hat{L}_\zeta : S(M,N) \to S(M,N)$$

such that;

$$\hat{L}_\zeta(\theta) := [\hat{\zeta}, \theta] = (\overrightarrow{\zeta}\theta - \overleftarrow{\zeta}\theta)$$

Thus, we can consider commutator operators $\hat{L}_\eta$, $\hat{L}_\zeta$, for η, $\zeta \in A_C$, and also, take their composition, denoted by $\hat{L}_\eta \circ \hat{L}_\zeta$. Then we can give the following definition:

We consider an observables algebra A_C, and let M, N be A_C-modules. An R*-linear morphism $\theta \in S(M, N)$ is called a differential* 1*-variation induced by the action of M on N if for all η, $\zeta \in A_C$, and corresponding commutator operators $\hat{L}_\eta$, $\hat{L}_\zeta$, the following identity holds:*

$$(\hat{L}_\eta \circ \hat{L}_\zeta)(\theta) = 0$$

Let us denote the set of all differential 1-variations induced by the action of M on N, by $V^1{}_{A_C}(M, N)$. The set $V^1{}_{A_C}(M, N)$ can be endowed with a bimodule structure, where multiplication from the left by elements ζ of A_C is denoted by $\overleftarrow{\zeta}\theta$, whereas multiplication from the right is denoted by $\overrightarrow{\zeta}\theta$, according to;

$$(\overleftarrow{\zeta}\theta)(m) := \zeta \cdot \theta(m)$$

$$(\overrightarrow{\zeta}\theta)(m) := (\theta \circ \zeta)(m)$$

for every $\zeta \in A_C$. We also denote by $\hat{V}^1_{A_C}(M, N)$ the bimodule structure, whereas by $\overleftarrow{V}^1{}_{A_C}(M, N)$ and $\overrightarrow{V}^1{}_{A_C}(M, N)$, the left and right A_C-module structures, respectively. Moreover, the bimodule of all differential 1-variations, induced by the action of A_C, being an A_C-module over itself, on N, is denoted by $\hat{V}^1_{A_C}(N)$. We also denote by $\overleftarrow{V}^1{}_{A_C}(N)$ the left A_C-module structure, whereas by $\overrightarrow{V}^1{}_{A_C}(N)$ the right A_C-module structure.

8.3 Left Modules of 1-Forms

From now on, we shall focus our attention to the left module structure alone. The correspondence $N \mapsto \overleftarrow{V}^1{}_{A_C}(M, N)$ if applied to all objects and arrows of the category of A_C-modules $\mathcal{M}^{(A_C)}$, specifies a covariant functor from the category of A_C-modules to themselves;

$$\overleftarrow{\mathbf{V}}^1{}_{A_C}(M, -) : \mathcal{M}^{(A_C)} \to \mathcal{M}^{(A_C)}$$

Furthermore, we define the R-linear map;

$$l : M \to (A_C) \otimes_{\mathrm{R}} M$$

by setting;

$$l(m) = 1 \otimes m$$

where $m \in M$. The codomain of the map l is called the tensor product of the left A_C-modules A_C and M, and most significantly it is an A_C-module itself, where the left multiplication is specified by;

$$\overleftarrow{\zeta}(\xi \otimes m) := (\zeta\xi) \otimes m$$

where ζ, $\xi \in A_C$, and $m \in M$. The tensor product $(A_C)\otimes_{\mathrm{R}} M$, can be further endowed with a right A_C-module structure defined by

$$\overrightarrow{\zeta}(\xi \otimes m) := \xi \otimes (\zeta m)$$

Thus, we may form a commutator operator for every $\zeta \in A_C$ defined as follows;

$$\hat{L}_\zeta : (A_C)\otimes_{\mathrm{R}} M \to (A_C)\otimes_{\mathrm{R}} M$$

$$\hat{L}_\zeta(l(m)) := [\hat{\zeta}, l(m)] = (\overrightarrow{\zeta} l(m) - \overleftarrow{\zeta} l(m))$$

Subsequently, we can consider for η, $\zeta \in A_C$, the corresponding commutator operators $\hat{L}_\eta$, $\hat{L}_\zeta$, and also, take their composition. Consequently, the elements $((\hat{L}_\eta \circ \hat{L}_\zeta)(l))(m)$ generate a submodule of the tensor product $(A_C)\otimes_{\mathrm{R}} M$, denoted by $\underline{M}$. Moreover, we may form a quotient A_C-module corresponding to each A_C-module M, defined as follows;

$$\pi(M) := ((A_C)\otimes_{\mathrm{R}} M)/\underline{M}$$

It is straightforward to see, if we take into account the definition of the quotient A_C-module $\underline{M}$, that the map ;

$$\Pi : M \to \pi(M)$$

defined by the assignment

$$m \mapsto \Pi(m) := (l(m)) mod(\underline{M}) := [l(m)]$$

for each A_C-module M, is a differential 1-variation.

Moreover, the above map for each A_C-module M, gives rise to a covariant functor

$$\mathbf{\Pi} : \mathcal{M}^{(A_C)} \rightarrow \mathcal{M}^{(A_C)}$$

[i]. Its action on a A_C-module in $\mathcal{M}^{(A_C)}$ is given by;

$$\mathbf{\Pi}(M) := \pi(M)$$

[ii]. Its action on a morphism of A_C-modules $\alpha : M \rightarrow N$, for $[l(m)] \in \pi(M)$ is given by;

$$\mathbf{\Pi}(\alpha) : \mathbf{\Pi}(M) \rightarrow \mathbf{\Pi}(N)$$

$$\mathbf{\Pi}(\alpha)([l(m)]) = \alpha \circ [l(m)]$$

Now, we consider that θ is a differential 1-variation, that is $\theta \in \overleftarrow{\mathbf{V}}_{A_C}(M,N)$. Obviously, $\overleftarrow{\mathbf{V}}_{A_C}(M,N) \subset Hom_{\mathrm{R}}(M,N)$, so we may further consider the morphism;

$$\chi : Hom_{A_C}(((A_C)\otimes_{\mathrm{R}}M),N) \rightarrow Hom_{\mathrm{R}}(M,N)$$

defined by the relation;

$$\chi(\tau) = \tau \circ l$$

Next, we apply the commutator operator $\hat{L}_\zeta$ on $\chi(\tau)$, taking into account that τ is an A_C-morphism, as follows;

$$\hat{L}_\zeta(\chi(\tau)) = \hat{L}_\zeta(\tau \circ l) = \tau \circ \hat{L}_\zeta(l) = \chi(\hat{L}_\zeta(\tau))$$

Consequently, $\chi(\tau) \in Hom_{\mathrm{R}}(M,N)$, is a differential 1-variation, iff

$$\hat{L}_\zeta(\chi(\tau)) = 0$$

or equivalently, iff;

$$\tau(\underline{M}) = 0$$

Thus, by restricting the codomain of χ to elements being qualified as differential 1-variations, we obtain the following isomorphism;

$$\iota : Hom_{A_C}(\mathbf{\Pi}(M),N) \rightarrow \overleftarrow{\mathbf{V}}_{A_C}(M,N)$$

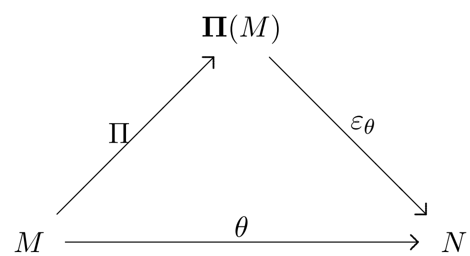
Its inverse is denoted by ε and is subsequently defined as;

$$\varepsilon : \overleftarrow{\boldsymbol{\nabla}}_{A_C}(M, N) \to Hom_{A_C}(\mathbf{\Pi}(M), N)$$

$$\theta \mapsto \varepsilon_\theta$$

$$\theta = \varepsilon_\theta \circ \Pi$$

according to the diagram below;



Hence, we can draw the conclusion that the covariant functor corresponding to a left A_C-module M in $\mathcal{M}^{(A_C)}$;

$$\overleftarrow{\boldsymbol{\nabla}}_{A_C}(M, -) : \mathcal{M}^{(A_C)} \to \mathcal{M}^{(A_C)}$$

is being representable by the left A_C-module in $\mathcal{M}^{(A_C)}$;

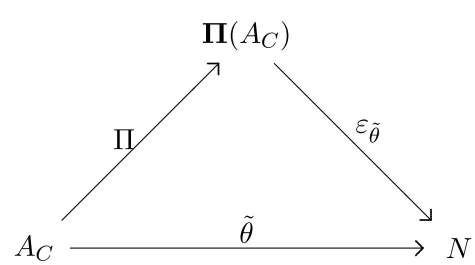
$$\mathbf{\Pi}(M) := ((A_C) \otimes_{\mathrm{R}} M)/\underline{M}$$

according to the established isomorphism;

$$\overleftarrow{\boldsymbol{\nabla}}_{A_C}(M, N) \cong Hom_{A_C}(\mathbf{\Pi}(M), N)$$

As a consequence, if we consider the case $M = A_C$, we obtain;

$$\overleftarrow{\boldsymbol{\nabla}}_{A_C}(N) \cong Hom_{A_C}(\mathbf{\Pi}(A_C), N)$$



where the map,

$$\Pi : A_C \to \mathbf{\Pi}(A_C)$$

is defined by the assignment

$$\zeta \mapsto \Pi(\zeta) = [l(\zeta)] = [1 \otimes \zeta]$$

Now, we may form the quotient left A_C-module $\Omega^1(A_C)$ defined as follows;

$$\Omega^1(A_C) := \mathbf{\Pi}(A_C)/Im(\Pi)$$

where $Im(\Pi)$ denotes the submodule of $\mathbf{\Pi}(A_C)$ depicted by the image of the morphism Π. There exists a natural projection mapping defined by;

$$pr : \mathbf{\Pi}(A_C) \to \Omega^1(A_C)$$

So, we may form the composition;

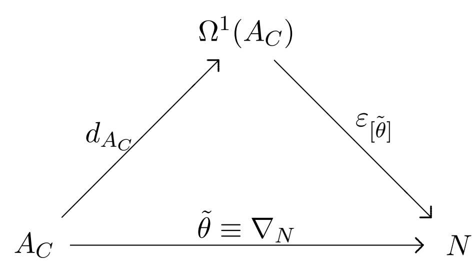
$$d_{A_C} : A_C \to \Omega^1(A_C)$$

$$d_{A_C} := pr \circ \mathbf{\Pi}$$

Then, d_{A_C} is clearly an $\Omega^1(A_C)$-valued derivation of A_C.

In a suggestive terminology, d_{A_C} *is called a first order differential of the observables algebra* A_C*, whereas the left* A_C*-module* $\Omega^1(A_C)$ *is characterized as the module of* 1*-forms of* A_C*. In this sense, a differential* 1*-variation is tautosemous with a first order differential of* A_C*, evaluated on* 1*-forms in* $\Omega^1(A_C)$*.*

Consequently, we may further consider the following commutative diagram;



We conclude that for any N-valued derivation $\tilde{\theta} \equiv \nabla_N$ of A_C, there exists a uniquely defined morphism $\varepsilon_{[\tilde{\theta}]} : \Omega^1(A_C) \to N$ making the diagram above commutative.

In functorial language the statement above means that the covariant functor of left A_C-modules valued derivations of A_C;

$$\overleftarrow{\nabla}_{A_C} : \mathcal{M}^{(A_C)} \to \mathcal{M}^{(A_C)}$$

is being representable by the left A_C-module of 1-forms in $\mathcal{M}^{(A_C)}$;

$$\Omega^1(A_C) := \mathbf{\Pi}(A_C)/Im(\Pi)$$

for every commutative arithmetic A_C, according to the isomorphism;

$$\overleftarrow{\nabla}_{A_C}(N) \cong Hom_{A_C}(\Omega^1(A_C), N)$$

Consequently, the conclusion stated above resolves completely the issue related with the appropriate specification of the $\mathbf{K}(A_C)$-module $\Omega(\mathbf{K}(A_C))$ in 8.1. If we remind the relevant discussion, it has been initially conjectured that from a physical viewpoint, a reasonable choice would be the identification of $\Omega(\mathbf{K}(A_C))$ with the $\mathbf{K}(A_C)$-module $\Phi(\mathbf{K}(A_C))$ of all derivations, that is localized arithmetics endomorphisms $\nabla_{\mathbf{K}(A_C)} : \mathbf{K}(A_C) \to \mathbf{K}(A_C)$, which are $\mathcal{R}$-linear and satisfy the Leibniz rule. From the isomorphism established above, the covariant functor of $\mathbf{K}(A_C)$-modules valued derivations of $\mathbf{K}(A_C)$ is being representable by the $\mathbf{K}(A_C)$-module of 1-forms in $\mathcal{M}^{\mathbf{K}(A_C)}$; Hence, we finally identify the $\mathbf{K}(A_C)$-module $\Omega(\mathbf{K}(A_C))$ in 8.1 with the $\mathbf{K}(A_C)$-module of 1-forms $\Omega^1(\mathbf{K}(A_C))$ and from now on we use them interchangeably.

Summarizing and recapitulating, we state that the differential structure on a local commutative domain cover, $\psi_{A_C} : \mathbf{M}(A_C) \hookrightarrow A_Q$, A_C in $\mathcal{A_C}$, being an inclusion, is defined as follows:

$$(\psi_{A_C}, \xi) + l_Q(A_C) \mapsto (\psi_{A_C}, d_{A_C}\xi) + l_Q(A_C) \equiv [(\psi_{A_C}, d_{A_C}\xi)]$$

Hence, locally in the Grothendieck topology specified, there exists a naturally defined differential operator, that has the following form over a particular cover for each A_C in $\mathcal{A}_C$;

$$d_{\mathbf{K}(A_C)}(\psi_{A_C} \otimes \xi) := (\psi_{A_C}, d_{A_C}\xi) + l_Q(A_C) \equiv [(\psi_{A_C}, d_{A_C}\xi)]$$

8.4 Non-Local Information Encoded in Ideals

If we focus our attention to a localization system of compatible overlapping commutative domain covers, we can specify accurately the information encoded in the ideal $l_Q(A_C)$ in $\mathbf{K}(A_C)$, where $\psi_{A_C} : \mathbf{M}(A_C) \hookrightarrow A_Q$, A_C in $\mathcal{A}_\mathcal{C}$, stands for a local cover belonging to this system. More concretely the ideal $l_Q(A_C)$ contains information about all the other covers in the localization system that are compatible in pullback diagrams over A_Q with the specified one. This is evident if we inspect the isomorphism pasting map $W_{A_C,\acute{A_C}} = \psi_{A_C\acute{A_C}} \circ \psi_{\acute{A_C}A_C}{}^{-1}$ and noticing that its existence guarantees the satisfaction of the relations needed, as has been explained previously, for the establishment of the identification equations $\psi_{A_C} \otimes \xi = \psi_{\acute{A_C}} \otimes \acute{\xi}$ in the localization system. Thus, essentially the information encoded in the ideal $l_Q(A_C)$ refers to all other local covers that are compatible with the specified one in the localization system. This is a unique peculiar characteristic of the quantum arithmetics as substantiated in the form of the algebras $\mathbf{K}(A_C)$. Remarkably, each one of them in a covering sieve contains information about all the others in the same sieve that can be made compatible, and explicitly, the content of this information is encoded in the structure of an ideal. This is a crucial observation and pertains to discussions of non-locality characterizing the behavior of quantum systems. In our perspective the assumed paradoxical behaviour of quantum systems exhibiting non-local correlations stems from two factors: The first factor has to do with the employment of supposedly unrelated classical arithmetics, while the second stems from the identification of the general notion of localization in the sense of Grothendieck with the restricted notion of spatial localization. These two factors, of course are intimately connected, since if somebody sticks blindly to the notion of spatial localization, that works nicely for a space of points but is completely inadequate to function in a category of points, is not possible to think of a correlation of arithmetics that are spatially employed far apart from each other for the description of the observables of the same quantum system, that can even be the whole universe itself. This is only possible if the notion of localization is detached from its spatial connotation, as it is the case with Grothendieck localization in categories. We have seen in detail how the functioning of covering sieves permits the conception of localization systems in a generalized topological sense and subsequently the natural appearance of commutative local arithmetics correlated by means of compatible information content.

8.5 The Abstract De Rham Complex

We consider the differential triad $\Delta = (\mathbf{K}(A_C), d_{\mathbf{K}(A_C)}, \Omega(\mathbf{K}(A_C)))$ that has been attached to each particular cover in a localization system of a quantum observables algebra. We further localize over a topological measurement space X, that we may consider as a nonvoid open subset in R^n, or an n-dimensional manifold. In this setting we assume that the classical commutative arithmetic A_C represents $\mathcal{C}^\infty(X, R)$, whereas its corresponding module of variations is the respective set of 1-forms.

Now, given the differential triad $\Delta = (\mathbf{K}(A_C), d_{\mathbf{K}(A_C)}, \Omega(\mathbf{K}(A_C)))$ localized sheaf-theoretically over a finite open covering $U = (U_a)$ of X as above, we define algebraically, for each $n \in N$, $n \geq 2$ the n-fold exterior product $\Omega^n(\mathbf{K}(A_C)) = \bigwedge^n \Omega^1(\mathbf{K}(A_C))$, where $\Omega(\mathbf{K}(A_C)) := \Omega^1(\mathbf{K}(A_C))$.

Furthermore, we assume the existence of an R-linear sheaf morphism $d^1 : \Omega^1(\mathbf{K}(A_C)) \to \Omega^2(\mathbf{K}(A_C))$, satisfying the Leibniz rule as follows:

$$d^1(ft) = f d^1(t) + \vartheta(f) \wedge t$$

for every $f \in \mathbf{K}(A_C)(U)$, $t \in \Omega^1(\mathbf{K}(A_C))(U)$, $U \subseteq X$. Moreover, we require that $d^1 \circ d^0 = 0$, where $d^0 := d_{\mathbf{K}(A_C)}$.

Based on the above, we can now further construct the R-linear sheaf morphism $d^2 : \Omega^2(\mathbf{K}(A_C)) \to \Omega^3(\mathbf{K}(A_C))$, satisfying:

$$d^2(t \wedge r) = t \wedge d^1(r) + d^1(t) \wedge r$$

where $t, r \in \Omega^1(A_C)(U)$, $U \subseteq X$. Finally, we may assume that d^2 satisfies: $d^2 \circ d^1 = 0$.

Thus, by iteration, for each $n \in N$, $n \geq 3$ we can construct the R-linear sheaf morphism $d^n : \Omega^n(\mathbf{K}(A_C)) \to \Omega^{n+1}(\mathbf{K}(A_C))$, satisfying:

$$d^n(t \wedge r) = (-1)^{n-1} t \wedge d^1(r) + d^{n-1}(t) \wedge r$$

where $t \in \Omega^{n-1}(\mathbf{K}(A_C))(U)$, $r \in \Omega^1(\mathbf{K}(A_C))(U)$, $U \subseteq X$.

In the above framework we obtain the following relations:

$$d^3 \circ d^2 = d^4 \circ d^3 = \ldots = d^{n+1} \circ d^n = \ldots = 0$$

where $n \in N$, $n \geq 2$. This fact allows the construction of the de Rham complex in our case, as a complex of R-linear sheaf morphisms as follows:

$$0 \to R \to \mathbf{K}(A_C) \to \Omega^1(\mathbf{K}(A_C)) \to \Omega^2(\mathbf{K}(A_C)) \to \ldots$$

Now, if we remind that $\mathbf{K}(A_C)$ consists of elements of the form $\psi_{A_C} \otimes \xi = (\psi_{A_C}, \xi) + l_Q(A_C)$, as well as, that the differential structure is defined by means of $(\psi_{A_C}, \xi) + l_Q(A_C) \mapsto (\psi_{A_C}, d_{A_C}\xi) + l_Q(A_C)$, where $d_{A_C}\xi$ corresponds to the usual differential of a smooth observable ξ, in the case considered, it can be checked the exactness of the de Rham complex above, by reduction to the well-known classical case of smooth functions. In this sense, there can be obtained a version of the Poincare Lemma corresponding to $\mathbf{K}(A_C)$.

8.6 Functoriality of the Differential Geometric Mechanism

At this subtle point of the present discussion the major conceptual innovation of ADG consists of the realization that the *differential geometric mechanism* as it is explicated by the functioning of differential triads *is not dependent on both, the arithmetics employed, and the localization methodology adopted*. Put differently, *the form of the mechanism describing the propagation of information is universally the same*, irrespectively of the arithmetics employed for encoding and decoding its content, as well as, the localization contexts deviced for its qualification through observation. This essentially means that *the nature of the differential geometric mechanism is functorial*; therefore, the differential equations based on it, as well [37].

In order to explain the claim presented above in the context of our inquiry related with the transition from the classical to quantum regime of observable structure we will make use of a topos-theoretic argument. The argument is based on the observation that in the functorial environment of the topos of presheaves over the category of commutative arithmetics the difference between classical and quantum observable behaviour is expressed as a switch on the representable functors of the corresponding arithmetics from $\mathbf{y}[A_C] \otimes_{\mathcal{A}_C} \mathbf{M}$ to $\mathbf{R}(A_Q) \otimes_{\mathcal{A}_C} \mathbf{M}$. In the classical case,

$$\mathbf{y}[A_C] \otimes_{\mathcal{A}_C} \mathbf{M} \cong \mathbf{M} \circ \mathbf{G}_{\mathbf{y}[A_C]}(A_C, 1_{A_C}) = \mathbf{M}(A_C)$$

and the modelling functor $\mathbf{M}$ is assumed to be the identity functor. Under this identification in the classical case, we may equivalently assume that the category of commutative arithmetics may be endowed with a discrete Grothendieck topology, such that, the representable presheaves of commutative arithmetics $\mathbf{y}[A_C]$ are being transformed into sheaves for this topology. In the quantum case, respectively,

$$\mathbf{R}(A_Q) \otimes_{\mathcal{A}_C} \mathbf{M} \cong A_Q$$

by virtue of the counit isomorphism, and moreover, $\mathbf{R}(A_Q)$ becomes a sheaf for the Grothendieck topology of epimorphic families from commutative domain arithmetics. Furthermore, inspecting the unit of the established adjunction as applied to the representable functors $\mathbf{y}[A_C]$ and $\mathbf{R}(A_Q)$ we obtain the corresponding isomorphisms:

$$\delta_{\mathbf{y}[A_C]} : \mathbf{y}[A_C] \to Hom_{\mathcal{A}_Q}(\mathbf{M}(\_), \mathbf{y}[A_C] \otimes_{\mathcal{A}_C} \mathbf{M})$$

$$\delta_{\mathbf{R}(A_Q)} : \mathbf{R}(A_Q) \to Hom_{\mathcal{A}_Q}(\mathbf{M}(\_), \mathbf{R}(A_Q) \otimes_{\mathcal{A}_C} \mathbf{M})$$

At this instance, if we remind the construction of differential triads, we realize the following: In the classical case, a differential triad is specified locally by the triple $\Delta_C = (A_C, d_{A_C}, \Omega(A_C))$, whereas in the quantum case, a differential triad is specified locally by the triple $\Delta_Q = (\mathbf{K}(A_C), d_{\mathbf{K}(A_C)}, \Omega(\mathbf{K}(A_C)))$. Of course, the notion of local is solely determined with respect to the imposed Grothendieck topology in each case correspondingly. Thus, formally the differential geometric mechanism is expressed in the classical case by a category of differential triads attached to the category of commutative arithmetics equipped with the discrete Grothendieck topology, which is equivalent to considering disjoint differential triads globally, whereas in the quantum case, by a category of differential triads attached to the category of commutative arithmetics equipped with the Grothendieck topology of epimorphic families, which is equivalent, correspondingly, to considering diagrams of interconnected differential triads globally. Consequently, the mechanism expressed universally as a morphism from the employed arithmetics to the modules of variation of these arithmetics is functorial with respect to its domain and codomain instantiations in each case.

Conclusively, whereas the "mechanism of differentials" can be relativized with respect to different arithmetics and different modules of variation, the form it assumes is covariant, and simply expressed as an $\mathcal{R}$-linear Leibniz morphism from the arithmetics to their corresponding modules of variation, provided that the same localization procedure is respectively employed in both the domain and the codomain of this morphism within the categorical environments specified.

8.7 The Notions of Connection and Curvature

Interesting things start to happen from a differential geometric point of view when the assumed localization of the domain categorical environment is different from the one that is actually applicable in the codomain. This is exactly the case when a single classical commutative arithmetic A_C, in the environment of $\mathcal{A_C}$ equipped with the discrete Grothendieck topology, attempts to describe a quantum system whose actual variation is described by the module $\Omega^{A_Q}(A_C)$, by setting up a mechanism of propagation of information. Although we have not defined strictly $\Omega^{A_Q}(A_C)$ yet, for the heuristic purposes of the intuitive discussion of this section, we may assume that it exists and denotes the A_Q-module of differentials on A_C corresponding to the arrow $A_C \rightarrow A_Q$. We will see in the next section, where $\Omega^{A_Q}(A_C)$ is strictly defined, that it actually stands for an abelian group object of differentials in the comma category $\mathcal{A_Q}/A_Q$.

It is instructive to consider two observers using classical commutative arithmetic A_C and $\acute{A_C}$ respectively. The observers may localize their arithmetics over a topological measurement space X and thus have at their disposal the corresponding algebra sheaves over X. In the terminology of ADG a locally finite open covering $U = (U_a)$ of X constitutes a local frame. From the perspective of the arithmetics of the observers, within their operational categorical environment, viz., $\mathcal{A_C}$ equipped with the discrete Grothendieck topology, quantum observable behaviour is being inferred and uniquely determined, up to isomorphism in the same categorical environment, by the cocycle $W_{\acute{A_C},A_C}$, provided that $\acute{A_C} \bigcap A_C \neq 0$, using the suggestive notation of Section 7.1. Thus, essentially each observer is equipped with an arrow $\acute{A_C} \bigcap A_C \rightarrow \acute{A_C}$ and $\acute{A_C} \bigcap A_C \rightarrow A_C$ respectively, that provides information about quantum observable behaviour. Let us now restrict our attention to the observer using the commutative arithmetic A_C, and pose the following question: How should the observer A_C set up a differential geometric mechanism of information propagation related with quantum observable behaviour? First of all, it is obvious that the expression of the mechanism should be constrained by the existence of the arrow $\acute{A_C} \bigcap A_C \rightarrow A_C$ in the environment of $\mathcal{A_C}$. This means that the observer should relativize the mechanism with respect to information contained in $\acute{A_C} \bigcap A_C$. For this purpose the observer restricts the arithmetic A_C at the image of the cocycle in A_C, viz., restricts the scalars from A_C to $\acute{A_C} \bigcap A_C$. Thus, the observer becomes capable of expressing the mechanism in terms of the A_C-module sheaf $E(A_C)$, written suggestively as $E(A_C) := [Res]^{A_C}{}_{\acute{A_C} \bigcap A_C} A_C$, meaning that $\acute{A_C} \bigcap A_C$ is understood as the A_C-module sheaf $E(A_C)$. Furthermore, from the perspective of the arithmetic A_C the observer perceives variation of information regarding quantum behaviour by relativizing the A_C-module sheaf of differentials $\Omega(A_C)$ with respect to the arrow $\acute{A_C} \bigcap A_C \rightarrow A_C$. Let us denote this relativization by $\Omega^{A_C}{}_{\acute{A_C} \bigcap A_C} = \Omega(E(A_C))$, meaning the A_C-module of differentials on $\acute{A_C} \bigcap A_C$. Thus, the observer A_C should be able to set up a differential geometric mechanism of information propagation related with quantum observable behaviour, by means of the following $\mathcal{R}$-linear Leibniz sheaf morphism:

$$D_{A_C} : E(A_C) \rightarrow \Omega(E(A_C))$$

We will now explain that the sheaf morphism D_{A_C} is actually a connection on the A_C-module sheaf $E(A_C)$, introduced by the observer A_C in order to express the relativization of the differential mechanism with respect to the arrow $\acute{A_C} \bigcap A_C \rightarrow A_C$ that induces information about quantum behaviour in the categorical environment $\mathcal{A_C}$. For this purpose, we initially notice that to give a derivation $d_{A_C} : A_C \rightarrow \Omega(A_C)$ is equivalent to giving a

$\mathcal{R}$-linear sheaf morphism of $\mathcal{R}$-algebras

$$\tilde{d}_{A_C} : A_C \to A_C \bigoplus \Omega(A_C) \cdot \epsilon$$

$$a \mapsto a + da \cdot \epsilon$$

where $A_C \bigoplus \Omega(A_C) \cdot \epsilon$, with $\epsilon^2 = 0$, is the ring of dual numbers over A_C with coefficients in $\Omega(A_C)$. We note that as an abelian group $A_C \bigoplus \Omega(A_C) \cdot \epsilon$ is the direct sum $A_C \bigoplus \Omega(A_C)$, and the multiplication law is defined by

$$(a + da \cdot \epsilon) \bullet (\acute{a} + d\acute{a} \cdot \epsilon) = (a \cdot \acute{a} + (a \cdot d\acute{a} + \acute{a} \cdot da) \cdot \epsilon)$$

We further require that the composition of the augmentation

$$A_C \bigoplus \Omega(A_C) \cdot \epsilon \to A_C$$

with $\tilde{d}_{A_C}$ is the identity.

At a next stage, if we use the functor of scalars extension, referring to the sheaf morphism of $\mathcal{R}$-algebras $\tilde{d}_{A_C} : A_C \to A_C \bigoplus \Omega(A_C) \cdot \epsilon$, we obtain:

$$E(A_C) \mapsto E(A_C) \bigotimes_{A_C} [A_C \bigoplus \Omega(A_C) \cdot \epsilon]$$

Notice that $E(A_C) \bigotimes_{A_C} [A_C \bigoplus \Omega(A_C) \cdot \epsilon]$ is an $A_C \bigoplus \Omega(A_C) \cdot \epsilon$-module. Hence, by restricting it to A_C, denoted obviously by the same symbol, we obtain a comparison morphism of A_C-modules as follows:

$$\tilde{D}_{A_C} : E(A_C) \to E(A_C) \bigotimes_{A_C} [A_C \bigoplus \Omega(A_C) \cdot \epsilon]$$

Thus, the information incorporated in the comparison morphism can be now expressed as a connection on $E(A_C)$, viz., as an $\mathcal{R}$-linear Leibniz sheaf morphism:

$$D_{A_C} : E(A_C) \to E(A_C) \bigotimes_{A_C} \Omega(A_C)$$

Hence the A_C-module of differentials on $\acute{A_C} \bigcap A_C$, i.e. $\Omega(E(A_C))$, is identified with the tensor product of A_C-modules $E(A_C) \bigotimes_{A_C} \Omega(A_C)$, that is:

$$\Omega(E(A_C)) \equiv E(A_C) \bigotimes_{A_C} \Omega(A_C)$$

Thus, the differential geometric mechanism of information propagation, related with quantum observable behaviour, that the observer A_C sets up for this purpose, which is expressed in terms of the $\mathcal{R}$-linear Leibniz sheaf morphism, $D_{A_C} : E(A_C) \to \Omega(E(A_C))$, is equivalent with the introduction of a connection on the A_C-module $E(A_C)$ in order to account for that observable behaviour, defined by means of the R-linear Leibniz sheaf morphism, $D_{A_C} : E(A_C) \to \Omega(A_C) \otimes E(A_C)$. We conclude this discussion by realizing the following;

Whereas an observer using quantum arithmetics, expressed locally in the Grothendieck topology of epimorphic families of its categorical environment by means of the algebra $\mathbf{K}(A_C) := [\mathbf{R}(A_Q)(A_C) \times \mathbf{M}(A_C)]/l_Q(A_C)$, *formulates the differential geometric mechanism locally in terms of the Leibniz morphism* ${d^0}_{\mathbf{K}(A_C)} : \mathbf{K}(A_C) \to \Omega(\mathbf{K}(A_C))$, *an observer using classical arithmetics, expressed locally in the atomic Grothendieck topology of its corresponding categorical environment by means of the algebra* A_C *has to device the notion of connection in order to express the same mechanism.*

In the latter case, from the viewpoint of a classical observer using a commutative arithmetic, in a discretely topologized categorical environment, not respecting the localization properties holding in the codomain of variations of his observations, and by virtue of invariance of the mechanism under relativizations, the only way that the formed discrepancy can be compensated is through the introduction of the notion of connection. Furthermore, using a local frame of $E(A_C)$, it can be readily shown that the A_C-connection D_{A_C} can be locally expressed in the form

$$D_{A_C} = {d^0}_{A_C} + \omega_{A_C}$$

Hence, D_{A_C} is locally determined uniquely by ω_{A_C}, called the local A_C-connection matrix of D_{A_C}. This means that the A_C-connection D_{A_C} of an observer using a commutative arithmetic plays locally the role of *potential*.

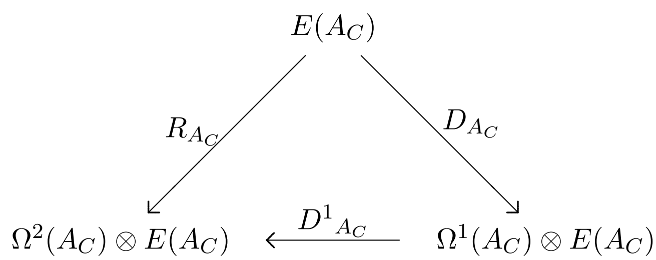
The notion of A_C-connection is always accompanied by the *notion of curvature*, that in the context of ADG is expressed as another appropriately defined sheaf morphism, however, now, respecting the arithmetic used, in contradistinction with what happens with D_{A_C}. More concretely, algebraically is possible to define the various exterior powers of the module of variations $\Omega(A_C) = \Omega^1(A_C)$, and furthermore, assume the existence of a second $\mathcal{R}$-linear morphism

$$d^1{}_{A_C} : \Omega^1(A_C) \to \Omega^2(A_C) := \Omega^1(A_C) \wedge \Omega^1(A_C)$$

such that $d^1{}_{A_C} \circ d^0{}_{A_C} = 0$, where $d^1{}_{A_C}$ is called the first exterior derivation. Moreover, it is possible to define the 1st prolongation of D_{A_C} by

$$D^1{}_{A_C} : \Omega^1(A_C) \otimes E(A_C) \to \Omega^2(A_C) \otimes E(A_C)$$

Finally, we can define the curvature of the given A_C-connection by the following commutative diagram:



where $R_{A_C} := D^1{}_{A_C} \circ D_{A_C}$.

It is readily seen that the curvature R_{A_C} of the given A_C-connection D_{A_C} is an A_C-morphism of the A_C-modules involved, that is

$$R_{A_C} \in Hom(E(A_C), \Omega^2(A_C) \otimes E(A_C))$$

The physical meaning of the curvature R_{A_C} refers to the detectable effect or *strength of the potential* represented by the connection D_{A_C}. From our prism of interpretation, we emphasize that the curvature R_{A_C} is the effect detected by an observer employing a commutative arithmetic in a discretely topologized categorical environment, in the attempt to understand the quantum localization properties in the codomain of variations of his observations, after having introduced a potential in order to reproduce the differential geometric mechanism.

9 QUANTUM FUNCTORIAL DIFFERENTIAL GEOMETRIC MECHANISM

9.1 Relativization and Abelian group Objects

In the previous Section we have noticed that in the functorial environment of the topos of presheaves over the category of commutative arithmetics the difference between classical and quantum observable behaviour is expressed as a switch on the representable functors of the corresponding arithmetics from $\mathbf{y}[A_C]\otimes_{\mathcal{A}_C}\mathbf{M}$ to $\mathbf{R}(A_Q)\otimes_{\mathcal{A}_C}\mathbf{M}$. The problem of establishing a well defined functorial differential geometric mechanism suitable for quantum observables algebras, based on the adjunction

$$\mathbf{L} : \mathbf{Sets}^{\mathcal{A}_C{}^{op}} \underset{\longrightarrow}{\longleftarrow} \mathcal{A}_Q : \mathbf{R}$$

necessitates the construction of a cohomological scheme of interpretation of these algebras. For this purpose, it is indispensable to have well defined notions of cohomology modules and derivations in the category $\mathcal{A}_Q$, as it is actually the case in the category $\mathcal{A}_C$. In order to accomplish this task we adopt the following strategy: Firstly, we unfold the notions of modules and derivations in the paradigmatic case of the category $\mathcal{A}_C$ using the method of relativization, and secondly, we adapt appropriately the definition of these notions in the category $\mathcal{A}_Q$.

The categorical method of relativization involves the passage to comma categories. The initial problem that is posed in this context of inquiry has to do with the possibility of representing the information contained in an A_C-module, where A_C is a commutative arithmetic in $\mathcal{A}_C$, with a suitable object of the relativization of $\mathcal{A}_C$ with respect to A_C, viz., with an object of the comma category $\mathcal{A}_C/A_C$. For this purpose, we define the split extension of the commutative arithmetic A_C, considered as a commutative ring, by an A_C-module M, denoted by $A_C \bigoplus M$, as follows: The underlying set of $A_C \bigoplus M$ is the cartesian product $A_C \times M$, where the group and ring theoretic operations are defined respectively as;

$$(a, m) + (b, n) := (a + b, m + n)$$

$$(a, m) \bullet (b, n) := (ab, a \cdot n + b \cdot m)$$

Notice that the identity element of $A_C \bigoplus M$ is $(1_{A_C}, 0_M)$, and also that, the split extension $A_C \bigoplus M$ contains an ideal $0_{A_C} \times M := \langle M \rangle$, that corresponds naturally to the A_C-module M. Thus, given a commutative arithmetic A_C in $\mathcal{A}_\mathcal{C}$, the information of an A_C-module M, consists of an object $\langle M \rangle$ (ideal in $A_C \bigoplus M$), together with a split short exact sequence in $\mathcal{A}_\mathcal{C}$;

$$\langle M \rangle \hookrightarrow A_C \bigoplus M \to A_C$$

We infer that the ideal $\langle M \rangle$ is identified with the kernel of the epimorphism $A_C \bigoplus M \to A_C$, viz.,

$$\langle M \rangle = Ker(A_C \bigoplus M \to A_C)$$

From now on we focus our attention to the comma category $\mathcal{A}_\mathcal{C}/A_C$, noticing that $id_{A_C} : A_C \to A_C$ is the terminal object in this category. If we consider the split extension of the commutative arithmetic A_C, by an A_C-module M, that is $A_C \bigoplus M$, then the morphism:

$$\lambda : A_C \bigoplus M \to A_C$$

$$(a, m) \mapsto a$$

is obviously an object of $\mathcal{A}_\mathcal{C}/A_C$. It is a matter of simple algebra to realize that it is actually an abelian group object in the comma category $\mathcal{A}_\mathcal{C}/A_C$. This equivalently means that for every object ξ in $\mathcal{A}_\mathcal{C}/A_C$ the set of morphisms $Hom_{\mathcal{A}_\mathcal{C}/A_C}(\xi, \lambda)$ is an abelian group in **Sets**. Moreover, the arrow $\gamma : \kappa \to \lambda$ is a morphism of abelian groups in $\mathcal{A}_\mathcal{C}/A_C$ if and only if for every ξ in $\mathcal{A}_\mathcal{C}/A_C$ the morphism;

$$\hat{\gamma}_\xi : Hom_{\mathcal{A}_\mathcal{C}/A_C}(\xi, \kappa) \to Hom_{\mathcal{A}_\mathcal{C}/A_C}(\xi, \lambda)$$

is a morphism of abelian groups in **Sets**. We denote the category of abelian group objects in $\mathcal{A}_\mathcal{C}/A_C$ by the suggestive symbol $[\mathcal{A}_\mathcal{C}/A_C]_{\mathbf{Ab}}$. Based on our previous remarks it is straightforward to show that the category of abelian group objects in $\mathcal{A}_\mathcal{C}/A_C$ is equivalent with the category of A_C-modules, viz.:

$$[\mathcal{A}_\mathcal{C}/A_C]_{\mathbf{Ab}} \cong \mathcal{M}^{(A_C)}$$

Thus, we have managed to characterize intrinsically A_C-modules as abelian group objects in the relativization of the category of commutative arithmetics $\mathcal{A}_\mathcal{C}$ with respect to A_C, and moreover, we have concretely identified them as kernels of split extensions of A_C.

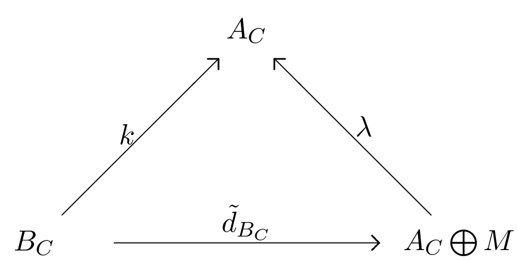
This characterization is particularly useful if we consider an A_C-module M as a cohomology module, or equivalently, as a codomain for derivations of objects of $\mathcal{A}_\mathcal{C}/A_C$. For this purpose, let us initially notice that if $k : B_C \to A_C$ is an arbitrary object in $\mathcal{A}_\mathcal{C}/A_C$, then any A_C-module M is also a B_C-module via the map k. We define a derivations functor from the comma category $\mathcal{A}_\mathcal{C}/A_C$ to the category of abelian groups **Ab**:

$$\mathbf{Der}(-, M) : \mathcal{A}_\mathcal{C}/A_C \to \mathbf{Ab}$$

Then if we evaluate the derivations functor at the commutative arithmetic B_C we obtain:

$$\mathbf{Der}(B_C, M) \cong Hom_{\mathcal{A}_\mathcal{C}/A_C}(B_C, A_C \bigoplus M)$$

This means that, given an object $k : B_C \to A_C$ in $\mathcal{A}_\mathcal{C}/A_C$, then a derivation $d_{B_C} : B_C \to M$ is the same as the following morphism in $\mathcal{A}_\mathcal{C}/A_C$:



Now we notice that the morphism: $\lambda : A_C \bigoplus M \to A_C$ is actually an object in $[\mathcal{A}_\mathcal{C}/A_C]_{\mathbf{Ab}}$. Hence, we consider it as an object of $[\mathcal{A}_\mathcal{C}/A_C]$ via the action of an inclusion functor:

$$\Upsilon_{A_C} : [\mathcal{A}_\mathcal{C}/A_C]_{\mathbf{Ab}} \hookrightarrow [\mathcal{A}_\mathcal{C}/A_C]$$

$$[\lambda : A_C \bigoplus M \to A_C] \mapsto [\Upsilon_{A_C}(\lambda) : \Upsilon_{A_C}(M) \to A_C]$$

Thus we obtain the isomorphism:

$$\mathbf{Der}(B_C, M) \cong Hom_{\mathcal{A}_\mathcal{C}/A_C}(B_C, \Upsilon_{A_C}(M))$$

The inclusion functor Υ_{A_C} has a left adjoint functor;

$$\mathbf{\Omega}^{A_C} : [\mathcal{A_C}/A_C] \to [\mathcal{A_C}/A_C]_{\mathbf{Ab}}$$

Consequently, if we further take into account the equivalence of categories $[\mathcal{A_C}/A_C]_{\mathbf{Ab}} \cong \mathcal{M}^{(A_C)}$, the isomorphism above takes the following final form:

$$\mathbf{Der}(B_C, M) \cong Hom_{\mathcal{M}^{(A_C)}}(\mathbf{\Omega}^{A_C}(B_C), M)$$

We conclude that the derivations functor $\mathbf{Der}(-, M) : \mathcal{A_C}/A_C \to \mathbf{Ab}$ is being represented by the abelianization functor $\mathbf{\Omega}^{A_C} : [\mathcal{A_C}/A_C] \to [\mathcal{A_C}/A_C]_{\mathbf{Ab}}$. Furthermore, the evaluation of the abelianization functor $\mathbf{\Omega}^{A_C}$ at an object $k : B_C \to A_C$ of $\mathcal{A_C}/A_C$, viz. $\mathbf{\Omega}^{A_C}(B_C)$, is interpreted as the A_C-module of differentials on B_C.

At this stage of development it is obvious that, for cohomological purposes, we can easily adapt the previously established notions of modules and derivations to the category of quantum observables algebras $\mathcal{A_Q}$. Firstly, we simply define the category of A_Q-modules as the category of abelian group objects in the comma category $\mathcal{A_Q}/A_Q$, viz.;

$$\mathcal{M}^{(A_Q)} := [\mathcal{A_Q}/A_Q]_{\mathbf{Ab}}$$

Secondly, we use the above definition in order to introduce the notion of an A_Q-module for derivations in the category $\mathcal{A_Q}$. For this purpose we define the derivations functor from the comma category $\mathcal{A_Q}/A_Q$ to the category of abelian groups $\mathbf{Ab}$:

$$\mathbf{Der}(-, N) : \mathcal{A_Q}/A_Q \to \mathbf{Ab}$$

where N is now an A_Q-module, or equivalently, an abelian group object in $\mathcal{A_Q}/A_Q$. Hence, if $K : B_Q \to A_Q$ denotes an object of $\mathcal{A_Q}/A_Q$ we obtain the isomorphism:

$$\mathbf{Der}(B_Q, N) \cong Hom_{\mathcal{A_Q}/A_Q}(B_Q, \Upsilon_{A_Q}(N))$$

where;

$$\Upsilon_{A_Q} : [\mathcal{A_Q}/A_Q]_{\mathbf{Ab}} \hookrightarrow [\mathcal{A_Q}/A_Q]$$

denotes the corresponding inclusion functor, having a left adjoint abelianization functor:

$$\mathbf{\Omega}^{A_Q} : [\mathcal{A}_{\mathcal{Q}}/A_Q] \to [\mathcal{A}_{\mathcal{Q}}/A_Q]_{\mathbf{Ab}}$$

Consequently we obtain again the following isomorphism:

$$\mathbf{Der}(B_Q, N) \cong Hom_{\mathcal{M}^{(A_Q)}}(\mathbf{\Omega}^{A_Q}(B_Q), N)$$

We conclude that the derivations functor $\mathbf{Der}(-, N) : \mathcal{A}_{\mathcal{Q}}/A_Q \to \mathbf{Ab}$ is being represented by the abelianization functor $\mathbf{\Omega}^{A_Q} : [\mathcal{A}_{\mathcal{Q}}/A_Q] \to [\mathcal{A}_{\mathcal{Q}}/A_Q]_{\mathbf{Ab}}$. Furthermore, the evaluation of the abelianization functor $\mathbf{\Omega}^{A_Q}$ at an object $K : B_Q \to A_Q$ of $\mathcal{A}_{\mathcal{Q}}/A_Q$, viz. $\mathbf{\Omega}^{A_Q}(B_Q)$, is interpreted correspondingly as the A_Q-module of differentials on B_Q.

9.2 Cohomology of Quantum Observables Algebras

The representation of quantum observables algebras A_Q in $\mathcal{A}_{\mathcal{Q}}$ in terms of sheaves over commutative arithmetics A_C in $\mathcal{A}_{\mathcal{C}}$ for the Grothendieck topology of epimorphic families on $\mathcal{A}_{\mathcal{C}}$, is based on the existence of the adjunctive correspondence $\mathbf{L} \dashv \mathbf{R}$ as follows:

$$\mathbf{L} : \mathbf{Sets}^{\mathcal{A}_{\mathcal{C}}{}^{op}} \underset{\longrightarrow}{\longleftarrow} \mathcal{A}_{\mathcal{Q}} : \mathbf{R}$$

which says that the functor of points $\mathbf{R}$ defined by

$$\mathbf{R}(A_Q) : A_C \mapsto Hom_{\mathcal{A}_{\mathcal{Q}}}(\mathbf{M}(A_C), A_Q)$$

has a left adjoint $\mathbf{L} : \mathbf{Sets}^{\mathcal{A}_{\mathcal{C}}{}^{op}} \to \mathcal{A}_{\mathcal{Q}}$, which is defined for each presheaf $\mathbf{P}$ in $\mathbf{Sets}^{\mathcal{A}_{\mathcal{C}}{}^{op}}$ as the colimit

$$\mathbf{L}(\mathbf{P}) = Colim\{\mathbf{G}(\mathbf{P}, \mathcal{A}_{\mathcal{C}}) \xrightarrow{\mathbf{G_P}} \mathcal{A}_{\mathcal{C}} \xrightarrow{\mathbf{M}} \mathcal{A}_{\mathcal{Q}}\}$$

Equivalently, there exists a bijection, natural in $\mathbf{P}$ and A_Q as follows:

$$Nat(\mathbf{P}, \mathbf{R}(A_Q)) \cong Hom_{\mathcal{A}_{\mathcal{Q}}}(\mathbf{LP}, A_Q)$$

The adjunction can be characterized in terms of the unit and the counit categorical constructions. For any presheaf $\mathbf{P} \in \mathbf{Sets}^{\mathcal{A}_{C}{}^{op}}$, the unit is defined as:

$$\delta_{\mathbf{P}} : \mathbf{P} \longrightarrow \mathbf{RLP}$$

On the other side, for each object A_Q of $\mathcal{A_Q}$ the counit is:

$$\epsilon_{A_Q} : \mathbf{LR}(A_Q) \longrightarrow A_Q$$

The composite endofunctor $\mathbf{G} := \mathbf{LR} : \mathcal{A_Q} \to \mathcal{A_Q}$, together with the natural transformations $\delta : \mathbf{G} \to \mathbf{G} \circ \mathbf{G}$, called comultiplication, and also, $\epsilon : \mathbf{G} \to \mathbf{I}$, called counit, where $\mathbf{I}$ is the identity functor on $\mathcal{A_Q}$, is defined as a comonad $(\mathbf{G}, \delta, \epsilon)$ on the category of quantum observables algebras $\mathcal{A_Q}$, provided that the diagrams below commute for each object A_Q of $\mathcal{A_Q}$;

$$\begin{array}{ccc}
\mathbf{G}A_Q & \xrightarrow{\delta_{A_Q}} & \mathbf{G}^2 A_Q \\
\Big\downarrow {\scriptstyle \delta_{A_Q}} & & \Big\downarrow {\scriptstyle \delta_{\mathbf{G}A_Q}} \\
\mathbf{G}^2 A_Q & \xrightarrow{\mathbf{G}\delta_{A_Q}} & \mathbf{G}^3 A_Q
\end{array}$$

$$\begin{array}{ccccc}
 & & \mathbf{G}A_Q & & \\
 & \swarrow\!\!= & \Big\downarrow {\scriptstyle \delta_{A_Q}} & =\!\!\searrow & \\
\mathbf{G}A_Q & \xleftarrow{\epsilon_{\mathbf{G}A_Q}} & \mathbf{G}^2 A_Q & \xrightarrow{\mathbf{G}\epsilon_{A_Q}} & \mathbf{G}A_Q
\end{array}$$

For a comonad $(\mathbf{G}, \delta, \epsilon)$ on $\mathcal{A_Q}$, a $\mathbf{G}$-coalgebra is an object A_Q of $\mathcal{A_Q}$, being equipped with a structural map $\kappa : A_Q \to \mathbf{G}A_Q$, such that the following conditions are satisfied;

$$1 = \epsilon_{A_Q} \circ \kappa : A_Q \to \mathbf{G}A_Q$$

$$\mathbf{G}\kappa \circ \kappa = \delta_{A_Q} \circ \kappa : A_Q \to \mathbf{G}^2 A_Q$$

With the above obvious notion of morphism, this gives a category $\mathcal{A_Q}_{\mathbf{G}}$ of all $\mathbf{G}$-coalgebras.

The counit of the comonad $(\mathbf{G}, \delta, \epsilon)$ on $\mathcal{A_Q}$, that is:

$$\epsilon_{A_Q} : \mathbf{G}A_Q := \mathbf{LR}(A_Q) = \mathbf{R}(A_Q) \otimes_{\mathcal{A_C}} \mathbf{M} \longrightarrow A_Q$$

is intuitively the first step of a functorial free resolution of an object A_Q in $\mathcal{A_Q}$. Thus, by iteration of $\mathbf{G}$, we may extend ϵ_{A_Q} to a free simplicial resolution of A_Q in $\mathcal{A_Q}$. Most importantly, we will consider the case of defining cohomology groups $\tilde{\mathbf{H}}^n(A_Q, X_Q)$, $n \geq 0$, of a quantum observables algebra A_Q in $\mathcal{A_Q}$ with coefficients in an A_Q-module X_Q, relative to the given underlying functor of points $\mathbf{R} : \mathcal{A_Q} \to \mathbf{Sets}^{\mathcal{A_C}^{op}}$, defined by $\mathbf{R}(A_Q) : A_C \mapsto Hom_{\mathcal{A_Q}}(\mathbf{M}(A_C), A_Q)$, having a left adjoint $\mathbf{L} : \mathbf{Sets}^{\mathcal{A_C}^{op}} \to \mathcal{A_Q}$.

Thus, let $(\mathbf{G}, \delta, \epsilon)$ be the comonad on $\mathcal{A_Q}$, that is induced by the adjoint pair of functors $\mathbf{L} : \mathbf{Sets}^{\mathcal{A_C}^{op}} \leftrightarrows \mathcal{A_Q} : \mathbf{R}$. The following simplicial object in $\mathcal{A_Q}$ is called the free simplicial comonadic resolution of a quantum observables algebra A_Q in $\mathcal{A_Q}$, denoted by $\mathbf{G}_\star A_Q \to A_Q$:

$$A_Q \xleftarrow{\epsilon_0 := \epsilon} \mathbf{G}A_Q \underset{\epsilon_1}{\overset{\epsilon_0}{\leftleftarrows}} \mathbf{G}^2 A_Q \xleftarrow{\epsilon_{0,1,2}} \cdots \xleftarrow{\epsilon_{0,1,\ldots,n-1}} \mathbf{G}^n A_Q \xleftarrow{\epsilon_{0,1,\ldots,n}} \mathbf{G}^{n+1} A_Q \quad \cdots$$

In the simplicial resolution above, $\epsilon_{0,1,2}$ denotes a triplet of arrows etc. Notice that, $\mathbf{G}^{n+1}$ is the term of degree n, whereas the face operator $\epsilon_i : \mathbf{G}^{n+1} \to \mathbf{G}^n$ is $\mathbf{G}^i \circ \epsilon \circ \mathbf{G}^{n-i}$, where $0 \leq i \leq n$. We can verify the following simplicial identities;

$$\epsilon_i \circ \epsilon_j = \epsilon_{j+1} \circ \epsilon_i$$

where $i \leq j$. The comonadic resolution $\mathbf{G}_\star A_Q \to A_Q$ induces clearly a comonadic resolution in the comma category $[\mathcal{A_Q}/A_Q]$, which we still denote by $\mathbf{G}_\star A_Q \to A_Q$.

An n-cochain of a quantum observables algebra A_Q with coefficients in an A_Q-module X_Q, where, by definition, X_Q is an object in $[\mathcal{A_Q}/A_Q]_{\mathbf{Ab}}$, is defined as a map $\mathbf{G}^{n+1} A_Q \to \mathbf{\Upsilon}_{A_Q}(X_Q)$ in the comma category $[\mathcal{A_Q}/A_Q]$. We remind that, since X_Q is an abelian group object in $[\mathcal{A_Q}/A_Q]$, the set $Hom_{\mathcal{A_Q}}(A_Q, X_Q)$ has an abelian group structure for every object A_Q in $\mathcal{A_Q}$, and moreover, for every arrow $\acute{A_Q} \to A_Q$ in $\mathcal{A_Q}$, the induced map of sets $Hom_{\mathcal{A_Q}}(A_Q, X_Q) \to Hom_{\mathcal{A_Q}}(\acute{A_Q}, X_Q)$ is an abelian groups map. Then, we can identify the set of n-cochains with the abelian group of derivations of $\mathbf{G}^{n+1} A_Q$ into the abelian group object X_Q in $[\mathcal{A_Q}/A_Q]_{\mathbf{Ab}}$. Hence, we consider an n-cochain as a derivation map $\mathbf{G}^{n+1}A_Q \to X_Q$.

Consequently, the face operators ϵ_i, induce abelian group maps;

$$Der(\epsilon_i A_Q, X_Q) : Der(\mathbf{G}^n A_Q, X_Q) \to Der(\mathbf{G}^{n+1} A_Q, X_Q)$$

Thus, the cohomology can be established by application of the contravariant functor $\mathbf{Der}(-, X_Q)$ on the free simplicial resolution of a quantum observables algebra A_Q in $\mathcal{A}_\mathcal{Q}$, obtaining the following cochain complex of abelian groups;

$$0 \xrightarrow{d^0} Der(\mathbf{G}A_Q, X_Q) \xrightarrow{d^1} Der(\mathbf{G}^2 A_Q, X_Q)$$

$$\xrightarrow{d^2} \cdots \xrightarrow{d^n} Der(\mathbf{G}^{n+1} A_Q, X_Q) \xrightarrow{d^{n+1}} \cdots$$

where, because of the aforementioned simplicial identities we have:

$$d^{n+1} = \sum_i (-1)^i Der(\epsilon_i A_Q, X_Q)$$

where $0 \leq i \leq n+1$, and also;

$$d^{n+1} \circ d^n = 0$$

written symbolically as;

$$d^{\mathbf{2}} = 0$$

Finally we may also make use of the following isomorphism:

$$Der(\mathbf{G}A_Q, X_Q) \simeq Hom(\mathbf{\Omega}^{A_Q}(\mathbf{G}A_Q), X_Q)$$

where the abelinazation functor $\mathbf{\Omega}^{A_Q} : [\mathcal{A}_\mathcal{Q}/A_Q] \to [\mathcal{A}_\mathcal{Q}/A_Q]_{\mathbf{Ab}}$ represents the derivations functor $\mathbf{Der}(-, X_Q) : \mathcal{A}_\mathcal{Q}/A_Q \to \mathbf{Ab}$. In this precise sense, $\mathbf{\Omega}^{A_Q}(\mathbf{G}A_Q) := \hat{\mathbf{\Omega}}(\mathbf{G}A_Q)$ is identified with the A_Q-module of first order differentials or 1-forms on $\mathbf{G}A_Q$. Thus, equivalently, we obtain the following cochain complex of abelian groups;

$$0 \xrightarrow{d^0} Hom(\hat{\mathbf{\Omega}}(\mathbf{G}A_Q), X_Q) \xrightarrow{d^1} Hom(\hat{\mathbf{\Omega}}(\mathbf{G}^2A_Q), X_Q)$$

$$\xrightarrow{d^2} \cdots \xrightarrow{d^n} Hom(\hat{\mathbf{\Omega}}(\mathbf{G}^{n+1}A_Q), X_Q) \xrightarrow{d^{n+1}} \cdots$$

Now, we define the cohomology groups $\tilde{\mathbf{H}}^n(A_Q, X_Q)$, $n \geq 0$, of a quantum observables algebra A_Q in $\mathcal{A}_\mathcal{Q}$ with coefficients in an A_Q-module X_Q as follows:

$$\tilde{\mathbf{H}}^n(A_Q, X_Q) := \mathbf{H}^n[\mathbf{Der}(\mathbf{G}_\star A_Q, X_Q)] = \frac{Ker(d^{n+1})}{Im(d^n)}$$

According to the above, a 1-cocycle is a derivation map $\omega : \mathbf{G}^2A_Q \to X_Q$, such that:

$$\epsilon_2\omega \circ \epsilon_0\omega = \epsilon_1 \circ \omega$$

where $\epsilon_i : \mathbf{G}^3A_Q \to \mathbf{G}^2A_Q$, $i = 0, 1, 2$. Correspondingly, a 1-coboundary is a derivation map $\upsilon : \mathbf{G}A_Q \to X_Q$, which can be presented as a mapping of 1-cocycles $\upsilon : \omega \to \acute{\omega}$ modulo the conditions:

$$\omega \circ \epsilon_0\upsilon = \epsilon_1\upsilon \circ \acute{\omega}$$

9.3 Functorial Connection and Curvature Equation

In this Section we are going to introduce the notion of a functorial quantum connection, together with, the associated curvature of that connection. The connection is intentionally termed functorial because it is precisely induced by the functor of points of a quantum observables algebra A_Q in $\mathcal{A}_\mathcal{Q}$, restricted as usual to commutative arithmetics. For this purpose it is necessary to define appropriately an A_Q-module, denoted by Ω_{A_Q}, that is going to play the role of a universal object of quantum differential 1-forms in analogy to the classical case. At this stage, it is instructive to remind briefly the analogous construction of the classical universal object of differential 1-forms Ω_{A_C}, corresponding to a commutative arithmetic A_C. According to Kahler, the free $\mathcal{A}$-module $\mathbf{\Omega}$ can be constructed explicitly form the fundamental form of scalars extension of A_C, that is $\iota : A_C \hookrightarrow A_C \bigotimes_\mathcal{R} A_C$ by considering the diagonal morphism:

$$\delta : A_C \bigotimes_{\mathcal{R}} A_C \to A_C$$

$$f_1 \otimes f_2 \mapsto f_1 \cdot f_2$$

where f_1, $f_2 \in A_C$. Then by taking the kernel of this morphism of algebras, that is the ideal;

$$I = \{f_1 \otimes f_2 \in A_C \bigotimes_{\mathcal{R}} A_C : \delta(f_1 \otimes f_2) = 0\} \subset A_C \bigotimes_{\mathcal{R}} A_C$$

it can be easily proved that the morphism of A_C-modules

$$\Sigma : \Omega_{A_C} \to \frac{I}{I^2}$$

$$df \mapsto 1 \otimes f - f \otimes 1$$

is an isomorphism. Thus the free A_C-module Ω_{A_C} of 1-forms is isomorphic with the free A_C-module $\frac{I}{I^2}$ of Kahler differentials of the commutative arithmetic A_C over $\mathcal{R}$, according to the following split short exact sequence:

$$\Omega_{A_C} \hookrightarrow A_C \oplus \Omega_{A_C} \cdot \epsilon \to A_C$$

where $\epsilon^2 = 0$, formulated equivalently as follows:

$$0 \to \Omega_{A_C} \to A_C \bigotimes_{\mathcal{R}} A_C \to A_C$$

In the quantum case, as we have explained in detain in Section 9.2, the counit of the adjunction $\epsilon_{A_Q} : \mathbf{LR}(A_Q) \to A_Q$, defined by the composite endofunctor $\mathbf{G} := \mathbf{LR} : \mathcal{A}_{\mathcal{Q}} \to \mathcal{A}_{\mathcal{Q}}$, constitutes the first step of a functorial free resolution of a quantum observables algebra A_Q in $\mathcal{A}_{\mathcal{Q}}$, generated by iterating the endofunctor $\mathbf{G}$. In this setting, and in analogy to the classical case, we define the A_Q-module Ω_{A_Q} of quantum differential 1-forms, by means of the following split short exact sequence:

$$0 \to J_{A_Q} \to \mathbf{R}(A_Q) \otimes_{\mathcal{A}_C} \mathbf{M} \to A_Q$$

According to the above, we obtain that $\Omega_{A_Q} = \frac{J_{A_Q}}{J^2{}_{A_Q}}$, where $J_{A_Q} = \mathbf{Ker}(\epsilon_{A_Q})$ denotes the kernel of the counit of the adjunction. Subsequently, we may apply the algebraic construction, for each $n \in N$, $n \geq 2$, of the n-fold exterior product ${\Omega^n}_{A_Q} = \bigwedge^n {\Omega^1}_{A_Q}$. Thus, we may now set up the algebraic de Rham complex of A_Q as follows:

$$A_Q \to \Omega_{A_Q} \to \ldots \to {\Omega^n}_{A_Q} \to \ldots$$

For the purpose of introducing the notion of a functorial quantum connection, the crucial idea comes from the realization that the functor of points of a quantum observables algebra restricted to commutative arithmetics, viz., $\mathbf{R}(A_Q)$, is a left exact functor, because it is the right adjoint functor of the established adjunction. Thus, it preserves the short exact sequence defining the object of quantum differential 1-forms, in the following form:

$$0 \to \mathbf{R}(\Omega_{A_Q}) \to \mathbf{R}(\mathbf{G}(A_Q)) \to \mathbf{R}(A_Q)$$

Hence, we immediately obtain that: $\mathbf{R}(\Omega_{A_Q}) = \frac{Z}{Z^2}$, where $Z = \mathbf{Ker}(\mathbf{R}(\epsilon_{A_Q}))$.

Then, in analogy to the paradigmatic classical algebraic situation, we define the notion of a functorial quantum connection, denoted by $\nabla_{\mathbf{R}(A_Q)}$, in terms of the following Leibniz natural transformation:

$$\nabla_{\mathbf{R}(A_Q)} : \mathbf{R}(A_Q) \to \mathbf{R}(\Omega_{A_Q})$$

Thus, the quantum connection $\nabla_{\mathbf{R}(A_Q)}$ induces a sequence of functorial morphisms, or equivalently, natural transformations as follows:

$$\mathbf{R}(A_Q) \to \mathbf{R}(\Omega_{A_Q}) \to \ldots \to \mathbf{R}({\Omega^n}_{A_Q}) \to \ldots$$

Let us denote by;

$$\mathbf{R}_\nabla : \mathbf{R}(A_Q) \to \mathbf{R}({\Omega^2}_{A_Q})$$

the composition $\nabla^1 \circ \nabla^0$ in the obvious notation, where $\nabla^0 := \nabla_{\mathbf{R}(A_Q)}$. The natural transformation $\mathbf{R}_\nabla$ is called the curvature of the functorial quantum connection $\nabla_{\mathbf{R}(A_Q)}$. Furthermore, the latter sequence of functorial morphisms, is actually a complex if and only if $\mathbf{R}_\nabla = 0$. We say that the quantum connection $\nabla_{\mathbf{R}(A_Q)}$ is integrable or flat if $\mathbf{R}_\nabla = 0$, referring to the above complex as the functorial de Rham complex of the integrable connection $\nabla_{\mathbf{R}(A_Q)}$ in that case. Thus we arrive at the following conclusion: The vanishing of the curvature of the functorial quantum connection, viz.:

$$\mathbf{R}_\nabla = 0$$

can be interpreted as the transposition of Einstein's equations in the quantum regime, that is inside the topos $\mathbf{Shv}(\mathcal{A}_{\mathcal{C}})$ of sheaves of algebras over the base category of commutative algebraic contexts, in the absence of cohomological obstructions. We may explain the curvature of the quantum connection as the effect of non-trivial interlocking of commutative arithmetics, in some underlying diagram of a quantum observables algebras being formed by such localizing commutative arithmetics. The non-trivial gluing of commutative arithmetics in localization systems of a quantum observables algebra is caused by topological obstructions. These obstructions can be associated with the elements of the non-trivial cohomology groups of a quantum observables algebra A_Q, in $\mathcal{A}_{\mathcal{Q}}$. From a physical viewpoint, these obstructions can be understood as geometric phases related with the monodromy of the quantum connection, being evaluated at points A_C of the functor of points of a quantum observables algebra A_Q restricted to commutative arithmetics, which, in turn, has been respectively interpreted as a prelocalization system of A_Q. Intuitively, a non-vanishing curvature may be understood as the non-local attribute detected by an observer employing a commutative arithmetic in a discretely topologized categorical environment, in the attempt to understand the quantum localization properties, after having introduced a potential, or equivalently, a connection, in order to account for these properties by means of a differential geometric mechanism. Thus, the physical meaning of curvature is associated with the apparent existence of non-local correlations from the restricted spatial perspective of disjoint classical commutative arithmetics A_C.

10 EPILOGUE

The representation of quantum observables algebras, A_Q in $\mathcal{A}_Q$, as sheaves, with respect to the Grothendieck topology of epimorphic families on $\mathcal{A}_C$, is of a remarkable physical significance. If we remind the discussion of the physical meaning of the adjunction, expressed in terms of the information content, communicated between commutative arithmetics and quantum arithmetics, we arrive to the following conclusion: the totality of the content of information, included in the quantum species of observables structure remains invariant, under commutative algebras decodings, corresponding to local arithmetics for measurement of observables, in covering sieves of quantum observables algebras, if, and only if, the counit of the fundamental adjunction is a quantum algebraic isomorphism. In this manner, the fundamental adjunction is being restricted to an equivalence of categories $\mathbf{Sh}(\mathcal{A}_C, \mathbf{J}) \cong \mathcal{A}_Q$; making thus, in effect, $\mathcal{A}_Q$ a Grothendieck topos, equivalent with the topos of sheaves on the site $(\mathcal{A}_C, \mathbf{J})$. The above correspondence, that can be understood as a topos-theoretic generalization of Bohr's correspondence principle, essentially shows that the process of quantization is categorically equivalent with the process of subcanonical localization and sheafification of information in commutative terms, appropriately formulated in a generalized topological fashion, *à la Grothendieck*.

We also claim that the sheaf-theoretic representation of a quantum observables algebra reveals that its deep conceptual significance is related not to its global non-commutative character, but, on the precise manner that distinct local contexts of observation, understood as commutative arithmetics, are being interconnected together, so as its informational content is preserved in the totality of its operational commutative decodings. By the latter, we precisely mean contextual operational procedures for probing the quantum regime of observable structure, which categorically give rise to covering sieves, substantiated as interconnected epimorphic families of the generalized elements of the sheafified functor of points of a quantum observables algebra $\mathbf{R}(A_Q)$. The sheaf-theoretic representation expresses exactly the compatibility of these commutative algebras of observables on their overlaps in such a way as to leave invariant the amount of information contained in a quantum system. We may adopt the term reference frames of commutative arithmetics for a geometric characterization of these local contexts of encoding the information related to a quantum system, emphasizing their prominent role in the organization of meaning associated with a quantum algebra of observables. Moreover this terminology signifies the intrinsic contextuality of algebras of quantum observables, as filtered through the base commutative localizing category, and is suggestive of the introduction of a relativity principle in the quantum level of observable structure, as a categorical extension of Takeuti's and Davis's research program [38, 39], related, in the present embodiment, with the invariance of the informational content with respect to commutative arithmetics reference frames contained in covering sieves of quantum observables algebras.

Furthermore, the sheaf-theoretic representation of quantum observables algebras makes possible the extension of the mechanism of Differential Geometry in the quantum regime by a proper adaptation of the methodology of ADG in a topos-theoretic environment. More concretely, in the terminology of ADG the differential geometric mechanism is incorporated in the functioning of differential triads consisting of commutative localized arithmetics, modules of variations of arithmetics and Leibniz sheaf morphisms from the domains of the former to the codomains of the latter, instantiating appropriate differentials. Most importantly the mechanism itself is functorial in nature, or equivalently, is always in force irrespectively of the relativizations pertaining the domains and codomains of the differentials introduced, provided that the same localization properties are respected in the corresponding categorical environments of the domains and codomains of differentials.

The differential geometric mechanism is expressed in the classical case by a category of differential triads attached to the category of commutative observable algebras equipped with the discrete Grothendieck topology, whereas in the quantum case, by a category of differential triads attached to the category of commutative observables algebras, being a generating subcategory of the category of quantum observables algebras, equipped with the Grothendieck topology of epimorphic families. As a consequence of the difference in the categorical localization properties classical arithmetics are different from quantum arithmetics. In a local cover belonging to a covering epimorphic sieve, a quantum arithmetic appears as a quotient of a commutative algebra over an ideal, incorporating information about all the other covers being compatible with it in a localization system of the former. Despite the difference in the corresponging arithmetics and modules of variations of them, the mechanism expressed universally as a morphism from the employed arithmetics to the modules of variations of these arithmetics is functorial with respect to its domain and codomain instantiations localized categorically in the same fashion in each case. Realization of this subtle point has subsequently forced us to argue that the real power of the abstract differential geometric mechanism, referring to propagation of information related to observation, is substantiated in cases where the categorical localization of the arithmetics used for observation is different from the categorical localization that is actually applicable in the modules containing variations of observations. As we have seen this is exactly the case when a disjoint classical commutative arithmetic A_C, in the environment of $\mathcal{A_C}$ equipped with the discrete Grothendieck topology, is used for description of a quantum system whose actual variation is described by the A_Q-module $\Omega^{A_Q}(A_C)$, denoting the A_Q-module of differentials on A_C corresponding to the arrow $A_C \to A_Q$. From the perspective of classical arithmetics in discretely topologized categorical environments the explication of the differential geometric mechanism necessitates the introduction of a connection, termed quantum potential, for the explanation of the -peculiar from their viewpoint- categorical localization rules respected by variations of observations in the quantum regime. The detectable effect emanating from the introduction of this potential for the description of the mechanism of information propagation in the resources offered by their arithmetics, is the appearance of the strength of the employed potential, expressed geometrically as the curvature of the associated connection. It is instructive to make clear that, in the present scheme, the geometric notion of curvature does not refer to an underlying background manifold, since such a structure has neither been postulated nor has it been required at all in the development of the differential geometric mechanism according to ADG. The physical meaning of curvature is associated with the apparent existence of non-local correlations from the perspective of disjoint classical commutative arithmetics A_C. Put differently, curvature is the detectable effect on a locus associated with a classical commutative arithmetic in a discretely topologized categorical environment, when observation of quantum behavior takes place, constituting the denotator of non-local correlations, stemming exclusively from the restricted sense of spatial locality that the locus shares in its discrete classical categorical environment. On the contrary, the form of quantum arithmetics, constructed by epimorphic covering families of sieves from interlocking commutative domain reference frames, incorporate a generalized notion of localization, not associated with its former restricted spatial connotation, but being defined only in the relational local terms of compatible information collation among those frames.

Considering the above scheme of interpretation seriously, we may assert, characteristically, that the transition in the meaning of generalized localization and its observable effects, as related with the formulation of a dynamical model that reflects the transition from the classical to the quantum regime, can be understood in terms of the observational locoi, corresponding to the respective arithmetics, as a conceptual transformation from a space of unrelated points endowed with a classically conceived point-localization structure, to a category of interconnected generalized points, being themselves localizing morphisms in covering sieves, and ultimately constituting a Grothendieck topos.

Thus, essentially the transition reflects a shift in the semantics of localization schemes, from a set-theoretic to a topos-theoretic one. Put differently, the notion of space of the classical theory is replaced by that one of a Grothendieck topos, equivalent with the topos of sheaves on the site $(\mathcal{A}_C, \mathbf{J})$, where the latter is simply understood as a generalized spatial framework of interrelation of experimentally gathered information, referring to quantum observable behaviour, being expressed in reference frames of interlocking commutative arithmetics. Remarkably, the algebraic sheaf-theoretic framework of ADG, conceived via the categorical and topos-theoretic adaptation attempted in this work, vindicates further the possibility of extending the "mechanism of differentials" in the quantum regime. The latter is being effectuated by the realization that the character of the mechanism is functorial with respect to the kind of arithmetics used for description of observable physical behaviour. Put differently, this means that the differential geometric mechanism of description of information propagation in physical terms, is covariant with respect to the arithmetics employed, denoting reference frames of a topos-theoretic nature, for decoding its content. Consequently, we are naturally directed towards a functorial formulation of the differential geometric mechanism, characterizing the dynamics of information propagation, through observable attributes, localized over commutative reference frames of variable form, thus, affording a language of dynamics suited to localization schemes of a topos theoretical nature, suitable for the transcription of dynamics from the classical to the quantum regime of observable behaviour.

More concretely, the process of gluing information along the loci of overlapping commutative arithmetics, interpreted in this generalized topos-theoretic localization environment, generates dynamics, involving the transition from the classical to the quantum regime, by means of the notion of a functorial connection and its associated curvature natural transformation, conceived in a precise cohomological manner. In this sense, the vanishing of the curvature of the functorial quantum connection, viz., $\mathbf{R}_{\nabla} = 0$, can be interpreted as the transposition of Einstein's equations in the quantum regime, that is inside the topos $\mathbf{Shv}(\mathcal{A}_{\mathcal{C}})$ of sheaves of algebras over the base category of commutative algebraic contexts, in the absence of cohomological obstructions.

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