CATEGORY-THEORETIC ANALYSIS OF THE NOTION OF COMPLEMENTARITY FOR QUANTUM SYSTEMS

ELIAS ZAFIRIS

University of Athens

Department of Mathematics

Panepistimiopolis, 15784 Athens

Greece

e.mail: ezafiris@math.uoa.gr

Abstract

In this paper we adopt a category-theoretic viewpoint in order to analyze the semantics of complementarity for quantum systems. Based on the existence of a pair of adjoint functors between the topos of presheaves of the Boolean kind of structure and the category of the quantum kind of structure, we establish a twofold complementarity scheme which constitutes an instance of the concept of adjunction. It is further argued, that the established scheme is inextricably connected with a realistic philosophical attitude, although substantially different from the classical one.

Keywords: Complementarity, Quantum Logic, Category Theory, Adjunction, Topos Theory.

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1 Prologue

In the interpretation of physical theories, our empirical access to the world is being objectified through the concept of observables. Observables denote physical quantities, that in principle, can be measured in the context of appropriate experimental arrangements. In any experiment performed by an observer, the propositions that can be made concerning a physical quantity, are of the type which asserts, that the value of the physical quantity lies in some measurable set of the real numbers. A proposition of this kind, corresponds to an event, as it is apprehended by an observer using his measuring device. We may claim that the real line endowed with its measurable structure acts as a modelling object, that schematizes the space of events of an observed system, by projecting into it its specific structure.

In this work we will attempt to argue that the concept of complementarity, describing the behaviour of quantum systems (Bohr (1958), Feyerabend (1958), Folse (1985)), is being formulated mathematically through the categorical notion of adjunction (Kan (1958), Marquis (2002)), that has been recently proved to exist between the category of quantum event algebras and and the topos of presheaves of Boolean event algebras (Zafiris (2001)). We will demonstrate that the notion of categorical adjunction embodies precisely the semantics of complementarity, and furthermore, explicates its functioning through a twofold scheme consisting of a horizontal and a vertical conceptual dimension.

The typical mathematical structure associated with the conception of events is a structure endowed with an ordering relation. In the Hilbert space formalism of Quantum theory events are considered as closed subspaces of a separable, complex Hilbert space corresponding to a physical system (Birkhoff and von Neumann (1936), Varadarajan (1968), Bub (1997), Rawling and Selesnick (2000)). Then, the quantum event structure is identified with the lattice of closed subspaces of the Hilbert space, ordered by inclusion and carrying an orthocomplementation operation that is provided by the orthogonal complement of the closed subspaces. More accurately, the Hilbert space quantum event structure is a complete, atomic, orthomodular lattice. Equivalently, it may be conceived as being isomorphic to the partial Boolean algebra of closed subspaces of the Hilbert space of the system, or alternatively the partial Boolean algebra of projection operators suited to the description of the properies of the system. It represents the event structure of a quantum mechanical system, just as the event structure of a classical system is a Boolean algebra isomorphic to the Boolean algebra of Borel subsets of the phase space of the system.

In logic-oriented approaches the equivalence of events and propositions is made literal. Generally the logical structure of a theory is reflected on the algebraic structure of its propositional calculus. If we consider the set of all feasible contingent propositions in the universe of discourse, then while the propositional system of classical mechanics is isomorphic to a Boolean lattice, on the contrary, the logical structure of a quantum system is, neither Boolean, nor possible to be embedded, into a Boolean lattice. In a Boolean propositional structure all propositions are mutually compatible in the sense that they are simultaneously decidable. Due to the fact that, in quantum theory not all propositions are compatible, incompatible propositions are allowed in the universe of discourse. The quantum logical formulation of Quantum theory depends in an essential way on the identification of propositions with projection operators on a complex Hilbert space. Furthermore, the order relations and the lattice operations of the lattice of quantum propositions are associated with the logical implication relation and the logical operations of conjunction, disjunction and negation of propositions. In effect a non-classical, non-Boolean logical structure is induced which has its origins in Quantum theory.

The complementarity scheme we will develop in the sequel, is based on the notion that observables in Boolean domain contexts, can be conceived as providing a coordinatization of the Quantum world by establishing a relativity principle in a topos-theoretical environment. An intuitive flavor of this insight is due to the validity of Kochen-Specker theorem (Kochen-Specker

(1967)), understood as expressing the impossibility of probing the entire manifestation of a quantum mechanical system with the use of a single system of Boolean devices. On the other side, in every concrete experimental context, the set of events that have been actualized in this context forms a Boolean algebra. Hence it is reasonable to claim that an observable picks a specific Boolean algebra, which can be considered as a Boolean subalgebra of the quantum algebra of events. Stated more precisely, an observable schematizes a quantum structure of events by correlating its Boolean subalgebras picked by measurements, with the smallest Boolean algebra containing all the clopen sets of the real line. In this sense, Boolean domain observables play the role of coordinatizing objects in the process of probing the Quantum world, setting the conceptual ground for the development of the ideas of complementarity. Equivalently we may assert, that a Boolean algebra in the lattice of quantum events picked by an observable, serves as a reference frame, relative to which the measurement result is being coordinatized. This perspective essentially suggests the identification of Boolean covers in systems of measurement localization for quantum event algebras with reference frames, relative to which the results of measurements are actually coordinatized, such that, every cover in a system of Boolean localizations for a quantum algebra of events will correspond to a set of classical Boolean events that become realizable in the experimental context of it.

2 The Conceptual Setting of the Categorical Framework

The Syntactical Path : Our investigation regarding quantum complementarity will be based on the existence of partial structural congruences between the quantum and Boolean kinds of event structure. The mathematical language which is at best suited to fulfill our objectives is category theory. Category theory provides a general theoretical framework for dealing with systems formalized through appropriate mathematical structures putting the emphasis on their mutual relations and transformations (MacLane (1971), Kelly (1971), Bell (1982), Borceaux (1994), Lawvere and Shanuel (1997)) . The basic categorical principles that we adopt in our exposition are summarized as follows:

[i] To each kind of mathematical structure used to model a system, there corresponds a **category** whose objects have that structure, and whose morphisms preserve it.

[ii] To any natural construction on structures of one kind, yielding structures of another kind, there corresponds a **functor** from the category of the first specified kind to the category of the second. The implementation of this principle is associated with the fact that a construction is not merely a function from objects of one kind to objects of another kind, but must preserve

the essential relationships among objects.

The adoption of the categorical syntax incorporates silently a conceptual shift in the way that one is likely to have previously thought about the mathematical structures considered. In particular, mappings with structure, called arrows, rather than, sets with structure, called objects, are to be regarded as primary (Bell (1986) and (2002)).

The Epistemological Path : The central axis of our epistemological path relies on the observation that the set theoretical axiomatizations of quantum event structures hides the intrinsic significance of Boolean localizing contexts in the formation of these structures. Moreover, the operational procedures followed in quantum measurement are based explicitly in the employment of appropriate Boolean environments. The construction of these contexts of observation are related with certain abstractions and can be metaphorically considered as pattern recognition mechanisms. In this way, we may argue that, the real significance of a quantum observable structure proves to be, not at the level of events, but at the level of specific interlocking of distinct or overlapping Boolean localization contexts of observation together, forming an intelligible coherent whole. The categorical analysis of quantum complementarity will be based, on the qualification of this idea in the position of a leading epistemological principle.

In particular, the epistemological path to be followed, implements the intuitively clear idea of probing the structure of a quantum algebra of events in terms of localizing Boolean environments, admitting an unquestionable operational interpretation. The process suggested may be decomposed in three levels:

The first level is constitutive of the introduction of a covering scheme, according to which, local Boolean domain objects cover entirely a quantum algebra of events by coordinatizing structure preserving morphisms. These morphisms from the Boolean domain localizing objects, capture in essence separately, complementary features of the quantum system of enquiry, and provide a structured decomposition of a quantum event algebra in the language of local Boolean covers.

The second level is constitutive of the establishment of an appropriate environment of compatibility between overlapping covers. This is necessary, since it guarantees an efficient pasting code between different coordinatizations of a quantum algebra of events.

The third level, finally, is constitutive of the integration of the qualitative content incorporated in local Boolean contexts of observation, effectuated by the establishment of an isomorphism between the structure of a quantum event algebra, and, the totality of local morphisms applied upon it in a covering system, consisting of a family of Boolean domain objects, in conjunction with, the pasting information between them.

The fact that a quantum event algebra is actually an object admitting a multitude of different localizations, is motivated again by Kochen-Specker theorem. According to this there are no two-valued homomorphisms on the algebra of quantum events. Subsequently, a quantum event algebra cannot be embedded into a Boolean one. We note parenthetically, that a two-valued homomorphism on a classical event algebra is a classical truth value assignment on the events, or equivalently, propositions in the logic-oriented approach, represented by the elements in the Boolean algebra, or a true-false assignment on the corresponding properties represented by the elements of the algebra.

3 Preliminaries

Categories : A category $\mathcal{C}$ is a class of objects and morphisms of objects such that the following properties are satisfied:

[1]. For any objects X, Y all morphisms $f : X \to Y$ form a set denoted $Hom_{\mathcal{C}}(X, Y)$;

[2]. For any object X an element $id_X \in Hom_{\mathcal{C}}(X, X)$ is distinguished; it is called the identity morphism;

[3]. For arbitrary objects X, Y, Z the set mapping is defined

$$Hom_{\mathcal{C}}(X,Y) \times Hom_{\mathcal{C}}(Y,Z) \to Hom_{\mathcal{C}}(X,Z)$$

For morphisms $g \in Hom_{\mathcal{C}}(X,Y)$, $h \in Hom_{\mathcal{C}}(Y,Z)$ the image of the pair (g,h) is called the composition; it is denoted $h \circ g$. The composition operation is associative.

[4]. For any $f \in Hom_{\mathcal{C}}(X,Y)$ we have $id_Y \circ f = f \circ id_X = f$.

For an arbitrary category $\mathcal{C}$ the opposite category $\mathcal{C}^{op}$ is defined in the following way: the objects are the same, but $Hom_{\mathcal{C}^{op}}(X,Y) = Hom_{\mathcal{C}}(Y,X)$, namely all arrows are inverted. A category $\mathcal{C}$ is called small if the classes of its objects and morphisms form genuine sets respectively.

Functors : Let $\mathcal{C}$, $\mathcal{D}$ be categories; a covariant functor $\mathbf{F} : \mathcal{C} \to \mathcal{D}$ is a class mapping that transforms objects to objects and morphisms to morphisms preserving compositions and identity morphisms:

$$\mathbf{F}(id_X) = id_{\mathbf{F}(X)}; \mathbf{F}(g \circ f) = \mathbf{F}(g) \circ \mathbf{F}(f)$$

A contravariant functor $\mathbf{F} : \mathcal{C} \to \mathcal{D}$ is, by definition, a covariant functor $\mathbf{F} : \mathcal{C} \to \mathcal{D}^{op}$.

Natural Transformations : Let $\mathcal{C}$, $\mathcal{D}$ be categories, and let further $\mathbf{F}$, $\mathbf{G}$, be functors from the category $\mathcal{C}$ to the category $\mathcal{D}$. A natural transformation

τ from $\mathbf{F}$ to $\mathbf{G}$ is a mapping assigning to each object A in $\mathcal{C}$ a morphism τ_A from $\mathbf{F}(A)$ to $\mathbf{G}(A)$ in $\mathcal{D}$, such that for every arrow $f : A \to B$ in $\mathcal{C}$ the following diagram in $\mathcal{D}$ commutes;

$$\begin{tikzcd}
\mathbf{F}(A) \arrow[r, "\tau_A"] \arrow[d, "\mathbf{F}(f)"'] & \mathbf{G}(A) \arrow[d, "\mathbf{G}(f)"] \\
\mathbf{F}(B) \arrow[r, "\tau_B"] & \mathbf{G}(B)
\end{tikzcd}$$

That is, for every arrow $f : A \to B$ in $\mathcal{C}$ we have:

$$\mathbf{G}(f) \circ \tau_A = \tau_B \circ \mathbf{F}(f)$$

Natural Isomorphisms : A natural transformation $\tau : \mathbf{F} \to \mathbf{G}$ is called a natural isomorphism if every component τ_A is invertible.

Adjoint Functors : Let $\mathbf{F} : \mathcal{C} \to \mathcal{D}$ and $\mathbf{G} : \mathcal{D} \to \mathcal{C}$ be functors. We say that $\mathbf{F}$ is left adjoint to $\mathbf{G}$, if there exists a bijective correspondence between the arrows $\mathbf{F}(C) \to D$ in $\mathcal{D}$ and $C \to \mathbf{G}(D)$ in $\mathcal{C}$.

$$\mathbf{F} : \mathcal{C} \rightleftarrows \mathcal{D} : \mathbf{G}$$

Pictorially we have;

$$
\begin{tikzcd}
\mathbf{F}(C) \arrow[d] & C \arrow[d] \\
D & \mathbf{G}(D)
\end{tikzcd}
$$

where the left part is in $\mathcal{D}$ and the right in $\mathcal{C}$.

Diagrams : A diagram $\mathbf{X} = (\{X_i\}_{i\in I}, \{F_{ij}\}_{i,j\in I})$ in a category $\mathcal{C}$ is defined as an indexed family of objects $\{X_i\}_{i\in I}$ and a family of morphisms sets $\{F_{ij}\}_{i,j\in I} \subseteq Hom_{\mathcal{C}}(X_i, X_j)$.

Cocones : A cocone of the diagram $\mathbf{X} = (\{X_i\}_{i\in I}, \{F_{ij}\}_{i,j\in I})$ in a category $\mathcal{C}$, consists of an object X in $\mathcal{C}$, and for every $i \in I$, a morphism $f_i : X_i \to X$, such that $f_i = f_j \circ f_{ij}$ for all $j \in I$, that is, such that for every i, $j \in I$, and for every $f_{ij} \in F_{ij}$ the diagram below commutes

$$
\begin{tikzcd}
& X & \\
X_i \arrow[ur, "f_i"] \arrow[rr, "f_{ij}"] & & X_j \arrow[ul, "f_j"']
\end{tikzcd}
$$

Colimits : A colimit of the diagram $\mathbf{X} = (\{X_i\}_{i\in I}, \{F_{ij}\}_{i,j\in I})$ is a cocone with the property that for every other cocone given by morphisms $f_{\acute{i}} : X_i \to \acute{X}$, there exists exactly one morphism $f : X \to \acute{X}$, such that $f_{\acute{i}} = f \circ f_i$, for all $i \in I$ (universality property).

Reversing the arrows in the above definitions of cocone and colimit of a diagram $\mathbf{X} = (\{X_i\}_{i\in I}, \{F_{ij}\}_{i,j\in I})$ in a category $\mathcal{C}$, results in the dual notions called cone and limit of $\mathbf{X}$ respectively. Moreover, starting with a diagram $\mathbf{X} = (\{X_i\}_{i\in I}, \{F_{ij}\}_{i,j\in I})$ in a category $\mathcal{C}$, that consists only of the objects X_i, $i \in I$, as nodes but without morphisms, that is all $F_{ij} = \emptyset$, we obtain the notion of the categorical *coproduct*, $\coprod_{i\in I} X_i$ (as a special colimit) and *product*, $\prod_{i\in I} X_i$ (as a special limit) respectively. The morphisms f_i in the corresponding definitions are called canonical injections of the coproduct and canonical projections of the product, respectively. We emphasize that we can derive special notions of limits and colimits, corresponding to the shape of the base diagram $\mathbf{X}$. In this sense we obtain the following; an *initial object* is the colimit of the diagram consisting of the empty set. A *coequalizer* is the colimit of a diagram consisting of two parallel arrows $A \rightrightarrows B$. A *pushout* is the colimit of a diagram of the form:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd}$$

The dual notions are the following; a *terminal object* is the limit of the diagram consisting of the empty set. An *equalizer* is the limit of a diagram consisting of two parallel arrows $A \rightrightarrows B$. A *pullback* is the limit of a diagram of the form:

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[r, "g"] & A \end{tikzcd}$$

4 Categories of Boolean and Quantum Kinds of Structure

4.1 Categories of Event Algebras

A Classical event structure is a small category, denoted by $\mathcal{B}$, which is called the category of Boolean event algebras. Its objects are Boolean algebras of events, and its arrows are Boolean algebraic homomorphisms.

A Quantum event structure is a small category, denoted by $\mathcal{Q}$, which is called the category of Quantum event algebras.

Its objects are Quantum algebras of events, that is, partially ordered sets of Quantum events, endowed with a maximal element 1, and with an operation of orthocomplementation $[-]^* : Q \longrightarrow Q$, which satisfy, for all $e \in Q$ the following conditions: [a] $e \leq 1$, [b] $e^{**} = e$, [c] $e \vee e^* = 1$, [d] $e \leq \acute{e} \Rightarrow \acute{e}^* \leq e^*$, [e] $e \perp \acute{e} \Rightarrow e \vee \acute{e} \in Q$, [g] $e \vee \acute{e} = 1, e \wedge \acute{e} = 0 \Rightarrow e = \acute{e}^*$, where

$0 := 1^*$, $e \perp \acute{e} := e \leq \acute{e}^*$, and the operations of meet $\wedge$ and join $\vee$ are defined as usually.

Its arrows are Quantum algebraic homomorphisms, that is maps $Q \xrightarrow{H} K$, which satisfy, for all $k \in K$ the following conditions: [a] $H(1) = 1$, [b] $H(k^*) = [H(k)]^*$, [c] $k \leq \acute{k} \Rightarrow H(k) \leq H(\acute{k})$, [d] $k \perp \acute{k} \Rightarrow H(k \vee \acute{k}) \leq H(k) \vee H(\acute{k})$.

We can check the following:

[1]. In the Hilbert space formalism of Quantum theory events are considered as closed subspaces of a seperable, complex Hilbert space corresponding to a physical system. Then the quantum event structure is identified with the lattice of closed subspaces of the Hilbert space, ordered by inclusion and carrying an orthocomplementation operation which is given by the orthogonal complement of the closed subspaces. For a seperable complex Hilbert space of dimension at least three, the lattice is also a quantum event algebra (the Hilbert space quantum event algebra).

[2]. Obviously every Boolean event algebra is also a quantum event algebra.

[3]. The Lindenbaum algebra corresponding to propositions describing the behavior of a quantum system is also a quantum event algebra.

4.2 The topos of Boolean-variable sets of events

For the category of Boolean event algebras $\mathcal{B}$ we will be considering the category of presheaves $\mathbf{Sets}^{\mathcal{B}^{op}}$ of all contravariant functors from $\mathcal{B}$ to $\mathbf{Sets}$ and all natural transformations between these (Bell (1988), MacLane and Moerdijk (1992)). A functor $\mathbf{P}$ is a structure-preserving morphisms of these categories, that is it preserves composition and identities. A functor in the category $\mathbf{Sets}^{\mathcal{B}^{op}}$ can be thought of as constructing an image of $\mathcal{B}$ in $\mathbf{Sets}$ contravariantly, or as a contravariant translation of the language of $\mathcal{B}$ into that of $\mathbf{Sets}$. Given another such translation (contravariant functor) $\mathbf{N}$ of $\mathcal{B}$ into $\mathbf{Sets}$ we need to compare them. This can be done by giving, for each Boolean object B in $\mathcal{B}$ a transformation $\tau_B : \mathbf{P}(B) \longrightarrow \mathbf{N}(B)$ which compares the two images of the Boolean object B. Not any morphism will do, however, as we would like the construction to be parametric in B, rather than ad hoc. Since B is an object in $\mathcal{B}$ while $\mathbf{P}(B)$ is in $\mathbf{Sets}$ we cannot link them by a morphism. Rather the goal is that the transformation should respect the morphisms of $\mathcal{B}$, or in other words the interpretations of $v : B \longrightarrow C$ by $\mathbf{P}$ and $\mathbf{N}$ should be compatible with the transformation under τ. Then τ is a natural transformation in the category of presheaves $\mathbf{Sets}^{\mathcal{B}^{op}}$.

The category of presheaves $\mathbf{Sets}^{\mathcal{B}^{op}}$ of all contravariant functors from $\mathcal{B}$ to $\mathbf{Sets}$ and all natural transformations between them is a topos (Artin, Grothendieck and Verdier (1972), Lawvere (1975), Bell (1986)). A topos exemplifies a well defined notion of a universe of variable sets. It can be conceived as a local mathematical framework corresponding to a generalized model of set theory or as a generalized space. Moreover, it provides a natural example of a many-valued truth structure, which remarkably is not ad hoc, but reflects genuine constraints of the surrounding universe. Applications of topos theory in quantum logic have been also considered from a different viewpoint in Butterfield and Isham (1999 and 2000).

Each presheaf functor may be equivalently characterized as a Boolean-variable set of events as follows:

An object $\mathbf{P}$ of $\mathbf{Sets}^{\mathcal{B}^{op}}$ may be understood as a right action of $\mathcal{B}$ on a set which is partitioned into sorts parameterized by the objects of $\mathcal{B}$ and such that whenever $v : C \longrightarrow B$ is a Boolean event algebras homomorphism and p is an element of $\mathbf{P}$ of sort B, then pv is specified as an element of $\mathbf{P}$ of sort C, such that the following conditions are satisfied, in Lawvere's notation (1975);

$$p1_B = p, \quad p(vw) = (pv)w, \quad wv : D \longrightarrow C \longrightarrow B$$

Such an action $\mathbf{P}$ is referred as a [Boolean algebras]-variable set or briefly $\mathcal{B}$-set. The fact that any morphism $\tau : \mathbf{P} \longrightarrow \mathbf{N}$ in the category of presheaves

$\mathbf{Sets}^{\mathcal{B}^{op}}$ is a natural transformation is expressed by the condition

$$\tau(p, v) = \tau(p)(v)$$

where the first action of v is the one given by $\mathbf{P}$ and the second by $\mathbf{N}$.

Of fundamental importance is the embedding functor $\mathbf{y}_{\mathcal{B}} : \mathcal{B} \longrightarrow \mathbf{Sets}^{\mathcal{B}^{op}}$. This functor associates to each Boolean algebra A of $\mathcal{B}$ the $\mathcal{B}$-set $\mathbf{y}_{\mathcal{B}}(A) = Hom_{\mathcal{B}}(-, A) := \mathcal{B}(-, A)$, whose B-th sort is the set $\mathcal{B}(B, A)$ of $\mathcal{B}$ morphisms $B \longrightarrow A$, with action by composition: $xv : C \longrightarrow B \longrightarrow A$. This is a functor because for any Boolean homomorphism $A \longrightarrow D$ we obtain a $\mathcal{B}$-map $\mathcal{B}(-, A) \longrightarrow \mathcal{B}(-, D)$, which has a functorial behavior under composition $A \longrightarrow D \longrightarrow E$, due to the associativity of composition in $\mathcal{B}$. Due to the embedding $\mathbf{y}_{\mathcal{B}} : \mathcal{B} \longrightarrow \mathbf{Sets}^{\mathcal{B}^{op}}$ it is useful to think of A as $\mathbf{y}_{\mathcal{B}}(A)$ in $\mathbf{Sets}^{\mathcal{B}^{op}}$. Furthermore for any $\mathcal{B}$-set and for any Boolean event algebra A of $\mathcal{B}$, the set of elements of $\mathbf{P}$ of sort A is identified naturally with the set of $\mathbf{Sets}^{\mathcal{B}^{op}}$-morphisms from $\mathbf{y}_{\mathcal{B}}(A) \longrightarrow \mathbf{P}$. Thus it is also useful to think of the elements of $\mathbf{P}$ of sort A as morphisms $A \longrightarrow \mathbf{P}$ in $\mathbf{Sets}^{\mathcal{B}^{op}}$.

By the term points of $\mathbf{Sets}^{\mathcal{B}^{op}}$ we mean morphisms $\mathbf{1} \longrightarrow \mathbf{P}$, where $\mathbf{1}$ is the terminal object of $\mathbf{Sets}^{\mathcal{B}^{op}}$. It is defined as the $\mathcal{B}$-set which for any Boolean algebra B in $\mathcal{B}$, has exactly one element of sort B, namely $B : B \longrightarrow \mathbf{1}$. Then it follows that for any $\mathbf{P}$ there is exactly one morphism $\mathbf{P} \longrightarrow \mathbf{1}$, denoted by $\mathbf{P}$ as well. A morphism $\mathbf{1} \longrightarrow \mathbf{P}$ operates at each sort, picking an element p_B of each sort B, such that the condition $p_C = p_B v$ is satisfied for all $v : C \longrightarrow B$ in $\mathcal{B}$.

The morphisms $p_C \longrightarrow p_B$, satisfying the condition $p_C = p_B v$ for all $v : C \longrightarrow B$ in $\mathcal{B}$ can become actual arrows in a category, called the category of elements of the presheaf of Boolean event algebras $\mathbf{P}$.

4.3 Category of Elements of a Boolean-Variable Set of Events

Because $\mathcal{B}$ is by construction a small category, there is a set consisting of all the elements of all the sets $\mathbf{P}(\mathrm{B})$, and similarly there is a set consisting of all the functions $\mathbf{P}(f)$. We may exploit this observation about $\mathbf{P} : \mathcal{B}^{op} \longrightarrow \mathbf{Sets}$ by taking the disjoint union of all the sets of the form $\mathbf{P}(\mathrm{B})$ for all Boolean objects B of $\mathcal{B}$. The elements of this disjoint union can be represented as pairs (B, p) for all objects B of $\mathcal{B}$ and elements $p \in \mathbf{P}(\mathrm{B})$. We can say that we construct the disjoint union of sets by labelling the elements. Now we can construct a category whose set of objects is the disjoint union just mentioned. This structure is called the category of elements of $\mathbf{P}$, denoted by $\mathbf{G}(\mathbf{P}, \mathcal{B})$. Its objects are all pairs (B, p), and its morphisms $(\acute{B}, \acute{p}) \longrightarrow (B, p)$ are those morphisms $u : \acute{B} \longrightarrow B$ of $\mathcal{B}$ for which $pu = \acute{p}$. Projection on the second coordinate of $\mathbf{G}(\mathbf{P}, \mathcal{B})$, defines a functor $\mathbf{G}(\mathbf{P}) : \mathbf{G}(\mathbf{P}, \mathcal{B}) \longrightarrow \mathcal{B}$. $\mathbf{G}(\mathbf{P}, \mathcal{B})$

together with the projection functor $\mathbf{G}(\mathbf{P})$ is called the split discrete fibration induced by $\mathbf{P}$, and $\mathcal{B}$ is the base category of the fibration. The discreteness pertains to the fact that the fibers are categories in which the only arrows are identity arrows. If B is an object of $\mathcal{B}$, the inverse image under $\mathbf{G}(\mathbf{P})$ of B is simply the set $\mathbf{P}(\mathrm{B})$, although its elements are written as pairs so as to form a disjoint union.

5 The Fundamental Adjunction underlying the Complementarity Scheme

The significance of the previous constructions for the explication of the complementarity concept as it may be formalized through the structural interrelations between Boolean and quantum event algebras is revealed by the introduction of appropriate functors. This is consistent with the categorical principle that to any natural construction on structures of one kind, yielding structures of another kind, there corresponds a suitable functor from the category of the first kind to the category of the second.

The trinity of functors introduced for our purposes consists of:

[i :] A local coefficients coordinatizing functor $\mathbf{A} : \mathcal{B} \longrightarrow \mathcal{Q}$.

[ii :] A functor $\mathbf{R}$ from $\mathcal{Q}$ to presheaves given by

$$\mathbf{R}(Q) : B \mapsto Hom_{\mathcal{Q}}(\mathbf{A}(B), Q)$$

[iii :] The introduction of the notion of a covering system, or system of localizations of quantum event algebra Q in $\mathcal{Q}$. This amounts to the consideration that $\mathbf{P}$ is a subfunctor of the Hom-functor $\mathbf{R}(Q)$ of the form $\mathbf{S} : \mathcal{B}^{op} \longrightarrow \mathbf{Sets}$, namely for all B in $\mathcal{B}$ it satisfies $\mathbf{S}(B) \subseteq [\mathbf{R}(Q)](B)$.

The functor $\mathbf{R}(Q)$ is the key for the establishment of a categorical adjunction expressed by the bijection natural in $\mathbf{P}$ and Q. A detailed presentation of the adjunction appears in Zafiris (2001):

$$Nat(\mathbf{P}, \mathbf{R}(Q)) \cong Hom_{\mathcal{Q}}(\mathbf{LP}, Q)$$

where the left adjoint $\mathbf{L} : \mathbf{Sets}^{\mathcal{B}^{op}} \longrightarrow \mathcal{Q}$, is defined for each presheaf of Boolean algebras $\mathbf{P}$ in $\mathbf{Sets}^{\mathcal{B}^{op}}$ as the colimit taken in the category of elements of $\mathbf{P}$

$$\mathbf{L}(\mathbf{P}) = Colim\{\mathbf{G}(\mathbf{P}, \mathcal{B}) \longrightarrow \mathcal{B} \longrightarrow \mathcal{Q}\}$$

Furthermore it has been shown that the categorical construction of this colimit as a coequalizer of a coproduct reveals the fact that this left adjoint is like the tensor product $-\otimes_{\mathcal{B}}\mathbf{A}$.

Consequently there is a pair of adjoint functors $\mathbf{L} \dashv \mathbf{R}$ as follows:

$$\mathbf{L} : \mathbf{Sets}^{\mathcal{B}^{op}} \underset{\longrightarrow}{\longleftarrow} \mathcal{Q} : \mathbf{R}$$

The main thesis of this paper is that the above adjunction formalizes and sets the conceptual ground for the understanding of the complementarity concept in quantum theory.

The adjunction established between the topos of presheaves of Boolean event algebras and the category of quantum event algebras is subsequently employed in order to provide a representation of quantum event algebras in terms of Boolean covering systems. This is accomplished through a functorial construction, which introduces the notion of a system of localizations of quantum event algebra Q in $\mathcal{Q}$. This notion is equivalent with the requirement that $\mathbf{P}$ is a subfunctor of the Hom-functor $\mathbf{R}(Q)$ of the form $\mathbf{S} : \mathcal{B}^{op} \longrightarrow \mathbf{Sets}$. Equivalently it may be described as a right ideal $\mathbf{S}(B)$ of structure preserving morphisms of the form

$$\psi_B : \mathbf{A}(B) \longrightarrow Q, \qquad B \in \mathcal{B}$$

such that $\{\psi_B : \mathbf{A}(B) \longrightarrow Q$ in $\mathbf{S}(B)$, and $\mathbf{A}(v) : \mathbf{A}(\acute{B}) \rightarrow \mathbf{A}(B)$ in $\mathcal{Q}$ for $v : \acute{B} \rightarrow B$ in $\mathcal{B}$, implies $\psi_B \circ \mathbf{A}(v) : \mathbf{A}(\acute{B}) \longrightarrow \mathcal{Q}$ in $\mathbf{S}(B)\}$.

The functioning of the notion of a system of localizations for quantum event algebra Q in $\mathcal{Q}$, is not adequate without the satisfaction of appropriate compatibility relations among distinctive components in this system. Compatibility relations are formulated in a categorical language through the pullback construction as follows:

The pullback of the maps: $\psi_B : \mathbf{A}(B) \longrightarrow Q$, $B \in \mathcal{B}$, and $\psi_{\acute{B}} : \mathbf{A}(\acute{B}) \longrightarrow Q$, $\acute{B} \in \mathcal{B}$, with common codomain the quantum event algebra Q, consists of the object $\mathbf{A}(B) \times_Q \mathbf{A}(\acute{B})$ and two arrows $\psi_{B\acute{B}}$ and $\psi_{\acute{B}B}$, called projections, as shown in the following diagram. The square commutes and for any object T and arrows h and g that make the outer square commute, there is a unique $u : T \longrightarrow \mathbf{A}(B) \times_Q \mathbf{A}(\acute{B})$ that makes the whole diagram commute. Hence we obtain the compatibility condition:

$$\psi_{\acute{B}} \circ g = \psi_B \circ h$$

$$\begin{tikzcd}
T \arrow[dr, "u"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "g"'] & & \\
& \mathbf{A}(B) \times_Q \mathbf{A}(\acute{B}) \arrow[r, "\psi_{B,\acute{B}}"] \arrow[d, "\psi_{\acute{B},B}"] & \mathbf{A}(B) \arrow[d, "\psi_B"] \\
& \mathbf{A}(\acute{B}) \arrow[r, "\psi_{\acute{B}}"] & Q
\end{tikzcd}$$

In essence the subfunctors of the Hom-functor $\mathbf{R}(Q)$ supply an ideal of algebraic homomorphisms which fulfill the task of covering a quantum event algebra by local modelling objects entirely.

The coordinatizing Boolean domain mappings $\psi_B : \mathbf{A}(B) \longrightarrow Q, \quad B \in \mathcal{B}$, in a system of localizations for quantum event algebra Q are characterized as Boolean covers, whereas their domains B play the role of local Boolean coefficients domains, the elements of B the role of Boolean coefficients, and finally, the Boolean homomorphisms $v : B \longrightarrow \acute{B}$ in $\mathcal{B}$ may be characterized as pasting maps.

Finally, we may talk about Q and a system of compatible Boolean localizations generating Q, in an equivalent fashion, if and only if, the counit of the fundamental adjunction restricted to subfunctors of the Hom-functor $\mathbf{R}(Q)$, qualified as Boolean covering systems, is an isomorphism, namely structure-preserving, injective and surjective. If we focus our attention to a Boolean covering system for quantum event algebra Q, we observe that the objects of the category of elements $\mathbf{G}(\mathbf{R}(Q), B)$ are precisely the local coordinatizing Boolean covers and its maps are the transition functions. It is instructive to remind that the objects of the category of elements $\mathbf{G}(\mathbf{R}(Q), B)$ are pairs $(B, \psi_B : \mathbf{A}(B) \longrightarrow Q)$, with B in $\mathcal{B}$ and ψ_B an arrow in $\mathcal{Q}$, namely a quantum algebraic homomorphism; a morphism $(\acute{B}, \psi_{\acute{B}}) \longrightarrow (B, \psi_B)$ in the category of elements is an arrow $v : \acute{B} \longrightarrow B$ in $\mathcal{B}$, namely a Boolean homomorphism, with the property that $\psi_{\acute{B}} = \psi_B \circ \mathbf{A}(v) : \mathbf{A}(\acute{B}) \longrightarrow Q$; in other words, v must take the chosen Boolean chart ψ_B in $\mathbf{G}(\mathbf{R}(Q), B)$ back into $\psi_{\acute{B}}$ in $\mathbf{G}(\mathbf{R}(Q), \acute{B})$. These morphisms are composed by composing the underlying arrows v of $\mathcal{B}$.

6 Complementarity as an Instance of the Categorical Adjunction Concept

6.1 Critical remarks on the correspondence

The representation of a quantum event algebra as an interlocking system of Boolean domain localization maps acquires a concrete physical meaning by a subsequent association of a Boolean cover with a reference frame, or equivalently a physical context, relative to which a measurement result admits a coordinatization. Such a conceptual viewpoint has been also suggested from a non-category theoretic perspective in Davis (1977) and Takeuti (1978). This association is justified by the fact that in every concrete experimental context, the set of events that have been actualized in this context forms a Boolean algebra. Furthermore, as has already been mentioned, it is unambiguously suggested by the physical interpretation of Kochen-Specker theorem, according to which it not possible to understand completely a quantum mechanical system with the use of a single system of Boolean devices. The proposed association suggests that a Boolean cover $(B, \psi_B : \mathbf{A}(B) \longrightarrow Q)$ in a system of localizations for quantum event algebra, corresponds to a set of Boolean classical events that become actualized in the experimental context of B.

Associating a Boolean cover in the mathematical descriptive language, with a concrete physical context in the physical one, takes into account two fundamental distinctions inextricably connected with the quantum theoretical formalism:

The first of them refers to a distinction being made between an observable event and the physical context that constitutes a set of necessary and sufficient constraints for the occurrence of an event of the observed kind. To the event, there corresponds a formal descriptive proposition language. To the physical context there corresponds a context-description in a formal descriptive language assuming existence at the level of covers, followed by an appropriate terminology providing names for the characterization of the language of events occurring in that context. These latter descriptions can be said that belong to the constitutive level of the Boolean localization systems.

The second refers to a distinction being made between possessed physical quantities, as those found in classical physics, and dispositional ones, as those found in quantum physics. The dispositional character of quantum observables is associated with the fact that they may only be specified via the measurement process, and more precisely, as relationally appearing with respect to theoretical or actual Boolean preparatory environments. In the mathematical descriptive terminology this distinction is encoded in a transition from globally Boolean event structures to globally non-Boolean event structures, being covered by a multitude of Boolean domain localization charts. In this sense, creating a preparatory Boolean environment for a system to interact with a measuring device, does not determine which event will take place, but it does determine the kind of event that will take place. It forces the outcome, whatever it is, to belong to a certain definite Boolean chart of events for which the standard measurement conditions are invariant. Such a set of standard conditions for a definite kind of measurement is named a physical context and is reflected to a Boolean reference frame in the mathematical descriptive language.

In the sense of these fundamental distinctions, observables schematize the quantum event structure by correlating Boolean charts picked by measurements with the smallest Boolean algebra of all the clopen sets of the real line, playing the precise role of coordinatizing objects in the process of probing the Quantum world.

6.2 Substantiation of a Twofold Complementarity Scheme

The previous critical remarks have prepared the conceptual ground for the implementation of the main thesis of this paper, namely that the interpretative concept of complementarity in Quantum theory is being formalized mathematically, and moreover clarified and substantiated, through the categorical notion of adjunction between the categories of quantum event algebras and the topos of presheaves of Boolean event algebras.

As a concise prologue, it is of value to remark that the notion of complementarity as a property of language, being forced as a tool of interpretation by virtue of the context dependence of quantum events is taken by Bohr. In the original conception of this notion, Bohr's argumentation consisted of two basic premises:

[i] Every quantum event is an observer-related event.

[ii] The study of complementary phenomena demand the use of mutually exclusive experimental arrangements.

With respect to premise [i], we claim that this is precisely the functioning of a Boolean reference frame. But, with respect to premise [ii], we may argue that the restriction of complementary phenomena to those only corresponding to mutually exclusive measurement contexts imposes a strict adherence to the two-valued logical machinery of classical logic, since does not permit truth-value assignments, like the multi-valued ones of topoi, expressing in the spirit of our discussion, partial compatibility of overlapping Boolean domain environments. Thus, in order to accommodate these logical possibilities, it is necessary to adopt a weaker version of premise [ii], that permits partially or locally compatible physical contexts in covering systems of the quantum event structure. An analogues stance towards the meaning of the complementarity concept has first been advocated by Heelan (1970). We will return to this subtle issue later in this work.

Our main objective at the present stage is the defence of the claim that the adjunction construction embodies the semantics of the notion complementarity and, moreover, it explicates its functioning via a twofold scheme consisting of a horizontal and a vertical dimension. For this purpose, we consider the natural bijection

$$Nat(\mathbf{P}, \mathbf{R}(Q)) \cong Hom_{\mathcal{Q}}(\mathbf{LP}, Q)$$

We notice that the functors $\mathbf{R}$ and $\mathbf{L}$ are not inverses, since we can see that neither $\mathbf{RL}$ nor $\mathbf{LR}$ need be isomorphic to an identity functor. One way of thinking about this is to recall the analogy between functors and translations and make it literal.

If we consider that $\mathbf{Sets}^{\mathcal{B}^{op}}$ is the universe of [Boolean event algebras]-variable sets, and $\mathcal{Q}$ that of quantum event algebras, then the functor $\mathbf{L}$: $\mathbf{Sets}^{\mathcal{B}^{op}} \longrightarrow \mathcal{Q}$ can be understood as a translational code from variable sets of Boolean localization domains, standing as physical contexts of measurement, to the algebra of events describing globally the behavior of a quantum system. On the other side, the functor $\mathbf{R} : \mathcal{Q} \longrightarrow \mathbf{Sets}^{\mathcal{B}^{op}}$ can be conceived as a translational code in the inverse direction. In general, the content of the information is not possible to remain completely invariant translating from one language to another and back, that is by encoding and decoding a message. However, there remain two ways for a [Boolean event algebras]-variable set $\mathbf{P}$, or else multiple filters structured window, to communicate a message to a quantum event algebra Q. Either the information is specified in quantum descriptive terms with $\mathbf{P}$ translating, which we can represent as the quantum homomorphism $\mathbf{LP} \rightarrow Q$, or the information is given in Boolean descriptive terms with Q translating, represented as the natural transformation $\mathbf{P} \rightarrow \mathbf{R}(Q)$. In the first case, Q thinks that is questioned in its own quantum descriptive terms, while in the second $\mathbf{P}$ thinks that it poses a question in Boolean terms. The natural bijection then corresponds to the assertion that these two distinct ways of communication, objectified as interactions via the channels of measuring devices, are equivalent.

Thus, the adjunctive correspondence is precisely constitutive of the meaning embodied in the process of relating relations arising from the partial congruences of two different globally descriptive levels of event language in communication. Most importantly, it engulfs all the necessary and sufficient conditions for the formulation of a two-directional dependent variation regulated simultaneously by the Boolean and quantum structural levels in local congruence vertically displayed. This process is actualized operationally in any preparatory context of a measurement situation with the purpose of extracting information semantically associated with the behavior of a quantum system via observable quantities. In turn, the repetition of this process in distinct or overlapping localizing environments for measurement, such that the totality of all similar or different manifestations of a quantum system's behavior is exhausted, gives rise to a variation of the observable characteristics, which is not always compatible according to the descriptive ideals of classical physics. This fact is not really paradoxical, from the contextual viewpoint of Boolean reference frames. It is just a consequence of the partial compatibility of overlapping observation contexts with intentionally specified Boolean descriptive languages at preparatory environments. Remarkably, the uncertainty principle can be understood in this setting as expressing exactly the measure of partial compatibility of diverse observable characteristics of the same quantum system in one and the same observational context, or equivalently, Boolean reference frame. Of course, the global closure of this, essentially, communicative process, realized as Boolean morphogenetic information filtration in the vertical direction, and as transitory information circulation in the horizontal, is constrained to obey certain conditions, such that its total constitutive information content, unfolded in the multitude of Boolean domain coordinatized instantiations of observables, is preserved and coherently organized. At this stage, the adjunctive correspondence itself, via the counit characterizing it, guides to the conclusion that a full and faithful representation of the structure of events of a globally non-Boolean quantum algebra, in terms of families of coordinatizing Boolean domain homomorphisms, being qualified as covering or localization systems, is guaranteed if and only if the counit is a quantum isomorphism. This conclusion, subsequently, is the referent of the invariance property pertaining the preservation of the total qualitative information content embodied in a quantum algebra of events through the process of unfolding in Boolean reference frames of covering systems and then enfolding back.

We may systematize the discussion above by arguing that the established adjunction manifests complementarity in a twofold holistic scheme, consisting of two interdependent interpretational directions.

Vertical complementarity : Vertical complementarity acquires a meaning by conceptually relating two hierarchically different but partially congruent descriptive levels, namely the levels of quantum event structures and those of Boolean event structures modelled functorially in Sets. It relates them by the pair of adjoint functors $\mathbf{L} \dashv \mathbf{R}$ as follows:

$$\mathbf{L} : \mathbf{Sets}^{\mathcal{B}^{op}} \underset{\longrightarrow}{\longleftarrow} \mathcal{Q} : \mathbf{R}$$

and the natural bijection

$$Nat(\mathbf{P}, \mathbf{R}(Q)) \cong Hom_{\mathcal{Q}}(\mathbf{LP}, Q)$$

which establishes an equivalence of translational codes between the language of a system of multiple Boolean filtering windows, and the language of a quantum structure.

Horizontal Complementarity : Horizontal complementarity acquires a meaning by conceptually relating different Boolean reference frames, used as Boolean domain localization contexts for the observation of manifested behavior of a quantum system, and subsequently, integrating them in a coherently organized covering system.

By virtue of, both, the horizontal and the vertical dimension of the complementarity notion, the quantum kind of structure is being objectified via isomorphic classes of the Boolean kind localization systems, each containing interconnected, partially or locally compatible, measurement environments. The twofold scheme of complementarity contains a significant interpretative power as formalized in the categorical adjunction construction and the subsequent Boolean manifold representation of quantum event algebras as we will further analyze.

The Boolean covers interlocking in localization systems, and providing reference frames for the local description of the quantum event structure in terms of contextual events, function like complementary pattern recognition mechanisms, such that complementary manifestations of the same quantum system are being formed by the deliberate abstractions associated with the preparation of a physical context. These abstractions inextricably connected with every Boolean domain filtering mechanism project the globally non-Boolean quantum event structure into Boolean local models endowed with a sharp or fuzzy Boolean logic, and, furthermore, make possible the manifestation of qualitatively named observables in the corresponding Boolean environments of discourse.

If we consider a set of standard conditions for a specified kind of measurement, to this there corresponds a Boolean chart of events, and a propositional language in which the events can be described. Each event language correlated with some standard measurement conditions constituting a physical context, can be considered as a single element, corresponding to a local Boolean cover in the category of elements $\mathbf{G}(\mathbf{R}(Q), B)$. Then, compatible event languages in measurement localization systems are interconnected through the colimit construction and can be pasted together, in order to form a Boolean manifold event language representation of the quantum structure. The language in which correlations between distinct or overlapping event languages are expressed, corresponding to concrete physical contexts, is the language admissible at the level of covering systems, that by virtue of the adjunction counit isomorphism, may provide a full and faithful representation of the quantum event language globally. Its resources are the names of the various descriptive event languages in the category of elements, and predicates describing the corresponding Boolean reference frames. Statements in the Boolean manifold language can be conceived as linguistic pictures -not, however, of events, but of a variety of physical contexts for different charts of events. Thus, the meaning of nonclassical logic characterizing the quantum propositional calculus is at the level in which Boolean domain covers, or Boolean reference frames, are related to one another, and not at the level of single quantum mechanical events.

It is appropriate to remark that each quantum event algebra can have many covering systems, or systems of localizations, which form a partial ordered set under inclusion. We note that the minimal system is the empty one, namely $\mathbf{S}(B) = \emptyset$ for all $B \in \mathcal{B}$, whereas the maximal system is the Hom-functor $\mathbf{R}(Q)$ itself. Moreover intersection of any number of systems of localization is again a system of localization. Furthermore, we say that a family of Boolean charts $\psi_B : \mathbf{A}(B) \longrightarrow Q, \quad B \in \mathcal{B}$ generates the system of localizations $\mathbf{S}$ if and only if this system is the smallest among all that contains this family. From the logical viewpoint, we can assert that the quantum event language accommodates a partial ordering of different localization systems consisting of physical measurement contexts. Hence it is legitimate to think of quantum logic, as the logic appropriate at the level of correlations of distinct or overlapping Boolean contexts for measurement, presupposing the validity of sharp or fuzzy Boolean logic in the domain of discourse of distinct experimental situations.

The covering process leads naturally to a contextual description of quantum events, with respect to Boolean reference frames of measurement, and finally to a representation of them as equivalence classes of Boolean events. The latter term is justified by the fact that, in case, Q signifies a truth-value structure, each cover can be interpreted as a fuzzy Boolean algebra of events, corresponding to measurement of an observable. More concretely, since covers are maps $[\psi_B] : \mathbf{A}(B) \longrightarrow Q$, each Boolean event realized in the domain B, besides its true or false truth value assignment in a measurement context, related to the outcome of an experiment that has taken place, may be also legitimately assigned a truth value, representing its relational information content for the comprehension of the coherence of the whole quantum structure, measured by the degrees in the poset Q, or equivalently by the degrees assigned to its poset structure of localization systems. This observation forms a logical manifestation of vertical complementarity and is deduced as a consequence of the established adjunction.

At the level of horizontal complementarity we observe that since two different local Boolean charts, each corresponding to a specified measurement context, is possible to overlap, experimental arrangements fulfil the task of probing the quantum event structure from different Boolean reference frames, objectified by the use of qualitatively different pattern recognition mechanisms. But, by virtue of the equivalence and compatibility relations, encoded in the form of the established adjunction through the colimit and pullback categorical constructions, these different observational frames can be conceived as being equivalent, and moreover, that is possible to establish the same quantum event, although via complementary observable manifestations.

We may sum up, by asserting that, in the twofold complementarity scheme, although every quantum event is an event for a particular kind of a Boolean domain observation chart materialized in a measurement context, or else it is an observer-related event, the correlations of these Boolean reference frames, in the Boolean manifold representation of the quantum event structure, permits a conception of quantum events as equivalence classes of Boolean fuzzy events. Adopting this viewpoint, we may claim that the meaning of the uncertainty relation in Quantum theory is a characterization of the limitations imposed by the theory, and reflected in the quantum event structure on an individual system in one and the same physical measurement context, or Boolean reference frame. In practice, since ideally precise measurement contexts with sharp classical logic are only theoretically possible, simultaneously imprecise measurement contexts with fuzzy Boolean logic, constitute the appropriate reference frames for the probation of the quantum event structure.

Schematically, we may argue that the quantum event structure is being unfolded through abstractions related with Boolean domain measurement environments, and the twofold complementarity scheme, formalized by the categorical adjunction construction, guarantees the correct communicability, both at the horizontal direction of Boolean reference frames at the same level, and the vertical direction linking the Boolean with the Quantum hierarchical levels by the manifold scheme.

Finally, it is of particular importance in the conceptual manifestation of the complementarity scheme, through the established pair of adjoint functors, to emphasize the clear geometric intuitions underlying this notion. We may notice that any Boolean event algebra B of $\mathcal{B}$, may be viewed as a generic figure in $\mathbf{Sets}^{\mathcal{B}^{op}}$ and any $p : B \longrightarrow \mathbf{P}$ as a particular figure, not necessarily monomorphic, of sort B. If $\mathbf{P}$ is any subfunctor of the Hom-functor $\mathbf{R}(Q)$, consisting a system of localizations for quantum algebra Q, then the counit isomorphism informs us that the Boolean charts in a system of localizations, representing measurement physical contexts, provide singular figures of shape B in Q, used to probe the quantum object Q by means of maps $\psi_B : \mathbf{A}(B) \longrightarrow Q$ from localizing Boolean domains. In this perspective it is clear that when a Boolean reference frame is considered as a figure of shape B in Q, we think of Q as a fixed object and of $\mathbf{A}(B)$ as variable, so as to give all possible shapes of figures in Q. Furthermore, the compatibility relations that the Boolean charts obey, determine to what extent the corresponding figures overlap, and what the structure of this overlap is. In this geometrical perspective, we view each Boolean reference frame for observation of the quantum event structure, by virtue of the adjoint situation, as a B-parameterized family, or equivalently as a varying element, in that if we evaluate it at various stages, we will vary it through various points of Q. Thus the Boolean reference frames effect a naming or coordinatization of elements of Q by B, emphasizing the fact that each map $\psi_B : \mathbf{A}(B) \longrightarrow Q$ produces a structure in Q. It is then clear that quantum events manifest themselves only in relation to Boolean reference frames, which in turn, justify entirely their characterization as pattern recognition mechanisms, effecting a coordinatization of quantum event structure, in terms of observable quantities in Boolean domain environments. At a higher level, the categorical construction of colimit, results in the interconnection of the Boolean reference frames in the category of elements, in order to capture all complementary manifestations of a quantum system via contextual observables, and remarkably, this is done in a coherent way respecting partial compatibility on overlaps. Then, the nature of the quantum structural kind is revealed: [i] through the concept of localization systems, represented functorially as subfunctors of the representable functor $\mathbf{R}(Q)$, that precisely concretize the interconnecting machinery of the colimit in the category of their elements, and, [ii] through the adjunctive correspondence, instantiating the twofold complementarity scheme, which being a universal construction, establishes a translational code between the kind of Boolean structure modelled in Sets, and the kind of quantum structure. Finally, the counit of this adjunction, restricted to Boolean localization systems, being an isomorphism, is the referent of a closure condition, signifying the invariance property pertaining the preservation of the total qualitative information content embodied in a quantum structure through the process of unfolding in Boolean reference frames and then enfolding back.

7 Philosophical Reflections on the Twofold Complementarity Scheme

The idea of a reality admitting objective existence is the starting point of all scientific investigations and the conceptual basis pertaining the realistic interpretation of classical theories of physics. We will attempt to demonstrate that the twofold complementarity scheme, as appearing as an instance of the categorical adjunction concept, provides a realistic perspective on quantum structure, although conceptually different from the classical one.

The classical view of reality presupposes that the objects of our observation are the entities in the world. Moreover an entity may be specified, both qualitatively and quantitatively in a definite manner, independently of procedures of observation, being capable of assuming individuality in isolation. According to the interpretative standards of classical realism, an objective description is one that determines the properties possessed by an independently real physical object, standing for an entity, by adopting a representation of that object as a physical system isolated from any observational interaction. Quantum theory does not conform to the descriptive ideals of the classical realism. Thus, the complementarity scheme appearing in the Boolean manifold representation of quantum event structures, should not be judged by the descriptive ideals of the classical realism position, but instead, should be considered as generating a generalization of the classical framework, mainly by initiating a revision of the classical realism assumptions, concerning in particular, the descriptive concepts involved in association with the objects they are used to describe.

We claim that the descriptive framework of the twofold complementarity scheme makes sense only on the basis of a revised realistic interpretation. Indeed, we hold the view that the propositions of a theory relate to real being and moreover that the referents of the theory are objects admitting individuality. Put differently, we claim that the quantum event structure reflects an objective physical reality having existence independently of some mind perceiving it. Evidently, such an interpretational viewpoint is compatible with the formalism of quantum theory if and only if a non-Boolean event structure is being manifested globally.

Let us initially notice, that we describe our observations using notions of validity adhering to sharp or fuzzy Boolean logic, as a consequence of the preparation of Boolean environments of measurement. This is due to the fact that, only in such environments it is possible to separate sharply or fuzzily elements and conceive them as existing in isolation from the rest of the world. Indeed, the Boolean specification of environments of measurement, engulfs the silent assumption of an almost atomic underlying topology. Of course, such an assumption, in general, can be used as a methodological tool of enquiry, only locally or partially, and never be claimed to assume global validity in the name of empirical findings, being themselves amenable exactly to the specification of these environments. The appropriate border in the domain of validity of such a Boolean description is decided by the relevant abstractions of a measurement situation, conceived precisely as a Boolean localizing or coordinatizing environment. It is instructive at this point, to remind Bohr's definition of the word phenomenon, to refer exclusively to observations obtained under specific circumstances, including the account of the whole experiment. Thus, for methodological purposes, we may adapt Bohr's concept of phenomenon as a referent of the assignment of an observable quantity to a system, legitimately thought as approximately isolated, strictly in the context of a Boolean domain local environment of measurement. Let us notice that this consideration is naturally forced just on the basis of a globally non-Boolean event structure, necessitating a multitude of, locally considered, isolated manifestations in Boolean reference frames. In this sense, every Boolean filtering window corresponds to a particular phenomenon and every proposition of the universe of discourse belongs to at least one Boolean frame, thus every proposition, is in principle, descriptive of a classically conceived observable, only under the choice of a local physical context.

It is also important to notice that the physical claim of complementary descriptions embodies in an essential manner the claim that the description of a phenomenon as above, necessitates a theoretical conception of the phenomenon as the referent of a two-levelled interactive process. The first level of this process is constitutive of the generation of a localizing environment, as a reality probing filter, endowed with an intentionally prepared mechanism of abstraction determined explicitly by the qualitative nature of the specific Boolean environment. The second level of this process, is in turn, constitutive of the actualization of the phenomenon only after interaction with the relevant measuring apparatus attached as a binary code to the Boolean frame. Most remarkably this process, considered in conjunction with a naturally induced variability of phenomena in the global perspective of the Boolean manifold synthesis, is suggestive of the fact that, the interacting parts actually form an individual whole, the latter being just expressed, as a variable division along different abstraction lines, between, only such, isolated local manifestations and measuring devices, and precisely characterized by the constraint of satisfying collectively the closure condition of the counit isomorphism. Thus, the conception of phenomena in this sense, enforces reference to quantum systems as actually existing objects behind the phenomena, the latter being only their local or partial manifestations in Boolean environments designed for that purpose. Evidently, each separate phenomenon cannot be regarded as a representation of a quantum system. Only at the level of Boolean localization systems, constituting structured interconnected multiplicities of phenomena, an isomorphic representation of the global behavior of such an entity becomes possible.

In the light of the above analysis, it is legitimate to say that the twofold complementarity scheme associated with the meaning of the categorical adjunction concept, replaces the classical static monolithic realist view, with a form of realism, admitting multiplicities of a really existing object, as an expression of qualitative structured variation or fuzziness in the observable universe of discourse, which are simultaneously, strictly constrained to obey collectively and globally a closure condition, constitutive of the preservation of the meaning engulfed, in the so conceived, really existing quantum object, being thus, comprehended as a sheaf a of local or partial phenomenal manifestations. Hence, it is explicitly ruled out any interpretation in the classical realist sense of a one-to-one correspondence between the concepts used to describe a phenomenal object and the presumed properties of an independent reality. Still, a realistic understanding of the description of quantum objects is retained by the interpretative power of twofold complementarity in terms of the categorical adjunction concept and the associated closure condition. Consequently, by virtue of this scheme of interpretation, the classical realist assumption that knowledge of an object is achieved by forming a representation of that object as a substance possessing properties is rejected, and subsequently, replaced by the possibility of formulating local or partial contextual theoretical structures allowing different or overlapping phenomenal descriptions, grounded on the same actually existing object, where the sameness is precisely determined by the preservation of meaning closure condition.

8 Epilogue

In this work we have explicated a twofold complementarity scheme for the comprehension of the quantum kind of observable structure, being substantiated as an instance of a categorical adjunction between the topos of presheaves of the Boolean kind of structure and the category of the quantum kind of structure. The key element of interpretation, that this scheme reveals, is that an object behaving in terms of the quantum kind of structure, is possible to be communicated in its entirety, only through isomorphisms from interconnected families of covers organized in Boolean localization systems, that by virtue of the adjunctive correspondence established, have the potential of unfolding its meaning, and simultaneously, preserving it consistently, providing, in this sense, a complementarity-based conception of the process of quantum becoming. We may conclusively, argue that the quantum level of reality can be conceived only through a relational perspective. It seems that relations occupy a substantially remarkable territory in our description of nature, one which was once occupied exclusively by properties. An actually existing quantum object is not described through isolated properties, but only through its relations with localizing physical contexts, that, when interconnected according to the specifications of partial compatibility and closure, totally reproduce its meaning as a real entity. Significantly, this fact is expressed in the most fundamental form in the language of category theory as an instance of the adjunction concept. The categorical framework reveals precisely that the essence of the quantum structure of reality is to be sought not in its internal constitution as a set-theoretical entity endowed with qualities, but rather, in the form of its relationship with the Boolean kind of structure through the established network of adjoint functors between the topos of [Boolean event algebras]-variable sets and the category of quantum event algebras.

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