A Sheaf-Theoretic Topos Model of the Physical "Continuum" and its Cohomological Observable Dynamics

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Abstract

The physical "continuum" is being modeled as a complex of events interconnected by the relation of extension and forming an abstract partially ordered structure. Operational physical procedures for discerning observable events assume their existence and validity locally, by coordinatizing the informational content of those observable events in terms of real-valued local observables. The localization process is effectuated in terms of topological covering systems on the events "continuum", that do not presuppose an underlying structure of points on the real line, and moreover, respect only the fundamental relation of extension between events. In this sense, the physical "continuum" is represented by means of a fibered topos-theoretic structure, modeled as a sheaf of algebras of continuous real-valued functions. Finally, the dynamics of information propagation, formulated in terms of continuous real-valued observables, is described in the context of an appropriate sheaf-cohomological framework corresponding to the localization process.

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1 Prologue

The semantics of the physical "continuum" in the standard interpretation of physical systems theories is associated with the codomain of valuation of physical attributes (Butterfield and Isham (2000)). Usually the notion of "continuum" is tied with the attribute of position, serving as the range of values characterizing this particular attribution. The model adopted to represent these values is the real line $\mathbb{R}$ and its powers, specified as a set theoretical structure of points that are independent and possess the property of infinite distinguishability with absolute precision. The adoption of the set-theoretic real line model is usually justified on the basis of operational arguments. Physical attributes are associated with the conception of observables, that is, physical quantities which, in principle, can be measured. Furthermore, physical systems theories stipulate that quantities admissible as measured results must be real numbers, since, it is accepted that the resort to real numbers has the advantage of making our empirical access secure. Thus, the crucial assumption underlying the employment of the real line and its powers for the modeling of the physical "continuum" is that real number representability constitutes our form of observation.

In this context, the geometrization of classical field theories as fiber bundles of some kind over a background spacetime points manifold, that is locally a power of the real line, is necessitated by the requirement of conferring numerical identity to the corresponding field events, conceived as being localized on charts of the spacetime manifold, being homeomorphic to a power of the real line. Closely related to the conceptualization of these geometrical models is the issue of localization in the physical "continuum". Operational procedures accompanying the development of physical systems theories can be understood as providing the means of probing the physical "continuum", via appropriate processes of localization in the "continuum", referring to localized events in terms of real-valued observable quantities.

Thus, physical observation presupposes, at the fundamental level, the development of localization processes in the "continuum" that accomplish the task of discerning observable events from it, and subsequently, assigning an individuality to them. It is important to notice however, that ascribing individuality to an event that has been observed by means of a localization scheme is not always equivalent to conferring a numerical identity to it, by means of a real value corresponding to a physical attribute. This is exactly the crucial assumption underlying the almost undisputed employment of the set theoretical model of the real line and its powers as models of the physical "continuum". The consequences of this common assumption, overlooked mainly because of the successful integration of the techniques of real analysis and classical differential geometry of smooth manifolds, in the argumentation and predictive power of physical theories, has posed enormous technical and interpretational problems related mainly with the appearance of singularities.

In this work we will attempt a transition in the semantics of the events "continuum" from a set-theoretic to a sheaf-theoretic one. The transition will be effectuated by using the syntax and technical machinery provided by category and topos theory (Artin, Grothendieck and Verdier (1972), Mac Lane and Moerdijk (1992), Mac Lane (1971), Borceaux (1994), Kelly (1971), Bell (2001), (1986), (1982) and (1988)). Conceptually, the proposed semantic transition is implemented and necessitated by the introduction of the following basic requirements admitting a sound physical basis of reasoning:

Requirement I: The primary conception of the physical "continuum" constitutes an inexhaustible complex of overlapping and non-overlapping events. The consideration of the notion of event as a primary concept immediately poses the following question: How are events being related to each other? If continuity is to be ascribed in the relations among events, then the fundamental relation is extension. The relata in the relation of extension are the events, such that each event is part of a larger whole and each event encompasses smaller events. Extension is also inextricably tied with the assumption of divisibility of events signifying a part-whole type of relation. In this sense, the physical "continuum" should constitute a representation of events ontology respecting the fundamental relation of extension. A natural working hypothesis in this sense, would be the modeling of the physical "continuum" by a partial order of events.

Requirement II: The notion of a "continuum" of events should not be necessarily based on the existence of an underlying structure of points on the real line. This equivalently means that localization processes for the individuation of events from the physical "continuum" should not depend on the existence of points. In this sense, ascribing individuality to an event that has been observed by means of a localization scheme should not be tautosemous to conferring a numerical identity to it, by means of a real value corresponding to a physical attribute, but only a limiting case of the localization process.

In order to construct a sheaf-theoretic model of the physical "continuum" based on the above physical requirements, and thus, accomplish the announced semantic transition, we further assume that, the localization process is being effectuated operationally in terms of suitable topological covering systems, which, do not presuppose an underlying structure of points on the real line, and moreover, respect only the fundamental relation of extension between events. In this sense, it will become apparent that the physical "continuum" can be precisely represented by means of a generalized fibered structure, such that, the partial order of events fibers over a base category of varying reference loci, corresponding to the open sets of a topological measurement space, ordered by inclusion. Moreover, we will explicitly show that, this fibered structure is being modeled as a sheaf of algebras of continuous real-valued functions, corresponding to observables. Finally, we are going to demonstrate that a sheaf-theoretic fibered construct of the physical continuum, as briefly described above, permits the formulation of the dynamical aspects of information propagation, in terms of a purely algebraic cohomological framework.

Generally speaking, the concept of sheaf expresses essentially gluing conditions, or equivalently, it formalizes the requirements needed for collating local observable information into global ones. The notion of local is characterized mathematically by means of a topological covering system, which, is the referent of topological closure conditions on the collection of covers, instantiating a localization process in the "continuum". It is important to emphasize that, the transition from locally defined observable information into global ones, elucidated by the concept of sheaf, takes place via a globally compatible family of localized information elements over a topological covering system of the "continuum". For a general mathematical and philosophical discussion of sheaves, variable sets, and related structures, the interested reader should consult (Lawvere (1975), Zafiris (2005)). Technical expositions of sheaf theory, being of particular interest in relation to the focus of the present work on topological localization processes, are provided by (Mac Lane and Moerdijk (1992), Bell (1986), Mallios (2004), and Bredon (1997)). Various applications of sheaf-theoretic fibered structures, based on the development of suitable localization schemes referring to the modeling and interpretation of quantum and complex systems, have been communicated, both conceptually and technically by the author, in the literature (Zafiris (2000), (2001), (2004), (2005), (2006) and (2007)).

2 Postulates on Observable Structures

The behavior of physical systems is adequately described by the collection of all observed data determined by the functioning of measurement devices in suitably specified experimental environments. Observables are precisely associated with physical quantities that, in principle, can be measured. The mathematical formalization of this procedure relies on the idea of expressing the observables by functions corresponding to measuring devices. Moreover, the usual underlying assumption on the basis of physical theories postulates that our global form of observation is represented by real-valued coefficients, and subsequently, global observables are modeled by real-valued functions corresponding to measuring devices.

At a further stage of development of this notion, two fundamental requirements are being postulated on the structure of observables:

Postulate I: The first postulate specifies the algebraic nature of the set of all observables, by assuming the structure of a commutative unital algebra $\mathbb{A}$ over the real numbers. The basic intuition behind this requirement is related with the fact that we can legitimately associate to any commutative algebra with unit a geometric object, called the spectrum of the algebra, such that the elements of the algebra, viz. the observables, can be considered as functions on the spectrum. The implemented principle is that the geometric structure of a measurement space can be completely recovered from the commutative algebra of observables defined on it. From a mathematical perspective, this principle has been well demonstrated in a variety of different contexts, known as Stone-Gel'fand duality in a functional analytic setting, or Grothendieck duality in an algebraic geometrical setting. Thus, to any commutative unital algebra of observables over the real numbers $\mathbb{R}$, we can associate a measurement space, namely its real spectrum, such that, each element of the algebra becomes a real-valued function on the spectrum. For instance, a real differential manifold M can be recovered completely from the $\mathbb{R}$-algebra $\mathcal{C}^{\infty}(M)$ of smooth real-valued functions on it, and in particular, the points of M may be recovered from the algebra $\mathcal{C}^{\infty}(M)$ as the $\mathbb{R}$-algebra morphisms (evaluations) $\mathcal{C}^{\infty}(M) \to \mathbb{R}$.

From a general systemic perspective, it is generally assumed that, real-number representability constitutes the universal form of observation instantiated in terms of the readings of measuring devices. Consequently, the set of all $\mathbb{R}$-algebra morphisms $\mathbb{A} \to \mathbb{R}$, assigning to each observable in $\mathbb{A}$, the reading of a measuring device in $\mathbb{R}$, encapsulates all the information collected about a physical system in measurement situations in terms of algebras of real-valued observables. Mathematically, the set of all $\mathbb{R}$-algebra morphisms $\mathbb{A} \to \mathbb{R}$ is identified as the $\mathbb{R}$-spectrum of the unital commutative algebra of observables $\mathbb{A}$. The physical semantics of this connotation denotes the set that can be $\mathbb{R}$-observed by means of this algebra.

Postulate II: The second postulate referring to the conceptualization of physical observables is related with the issue of localization. Usually, the operational specification of measurement environments assumes their existence locally, and the underlying assumption is that, the information gathered about local observables in different measurement situations can be collated together by appropriate means. The notion of local requires the specification of a topology on an assumed underlying measurement space over which algebras of observables may be localized. The net effect of this localization procedure of algebras of observables, together with the requirement of compatible information collation along localizations, are formalized by the notion of sheaf, as it will become clear in the sequel. A structure sheaf of commutative unital $\mathbb{R}$-algebras of observables incorporates exactly the conditions for the transition from locally collected observable data to globally defined ones. Moreover, the assumed underlying topological space acquires its characteristic features from the structure sheaf of $\mathbb{R}$-valued coefficients, standing for the observables.

Apart from the above pair of fundamental requirements, it is common practice that an extra smoothness assumption is postulated on the specification of observables. The underlying reason for the qualification of observables as real-valued smooth functions has to do with the fact that it is desirable to consider derivatives of observables and effectively set-up a dynamical framework of description in terms of differential equations defined over smooth differential manifolds. The notion of a smooth $\mathbb{R}$-algebra of observables on a differential manifold M, denoted by $\mathcal{C}^\infty(M)$, means that, locally, $\mathcal{C}^\infty(M)$ is like the $\mathbb{R}$-algebra $\mathcal{C}^\infty(\mathbb{R}^n)$ of infinitely differentiable functions on $\mathbb{R}^n$. The physical adaptation of the differential geometric mechanism of smooth manifolds has built-in the assumption that real-number representability constitutes our form of observation in terms of the readings of measuring devices. In this sense, $\mathbb{R}$-algebra morphisms (evaluations) $\mathcal{C}^\infty(M) \rightarrow \mathbb{R}$ can be legitimately identified with the points of a space ($\mathbb{R}$-spectrum) which can be observed by means of $\mathcal{C}^\infty(M)$, namely the points of a differential manifold M.

From the perspective of the present work, the adoption of the smoothness assumption on the structure of the algebra of observables, as a unique universal means of implementing a dynamical framework of information propagation, cannot be uncritically endorsed. More concretely, although the requirement of continuity is essential for the implementation of a topological localization process in the physical "continuum", the qualification of observables as, not only continuous, but, also smooth, real-valued functions should be seriously questioned. Clear indications about the restricted functionality of the smoothness assumption come from various sources, including the nature of observables in the quantum regime, as well as, the collapse of the smooth dynamical mechanism on singularities. In this sense, it seems reasonable to assume that the smoothness requirement on the algebra of observables, besides continuity, constitutes just a special case of observable coefficients, and most importantly, by no means is universally necessary, or even not substitutable, for coordinatizing the mechanism of information propagation in the physical "continuum". Of course, the validity of the inessentiality of the smoothness assumption can be verified, and thus operationally proved, in case that the development of a theory of dynamics of observables, can be formulated purely algebraically and independently of a smooth background manifold construct. In the sequel, we will show that this is indeed the case, by formulating dynamics in terms of the algebraic topological methodology of complexes and sheaf cohomology.

On the mathematical state of affairs, there exists a recently developed framework of algebraic differential geometry suited to overcome the smoothness restriction, called Abstract (Modern) Differential Geometry (ADG) (Mallios (1998), (2002), (2004), (2005), (2006), (2007); see also Mallios and Rosinger (1999), (2001), Mallios and Zafiris (2007), Zafiris (2007)). (ADG) generalizes mathematically the differential geometric mechanism of smooth manifolds, by explicitly demonstrating that most of the usual differential geometric constructions can be carried out by purely algebraic means without any use of any sort of C^{∞}-smoothness or any of the conventional calculus that goes with it.

Thus, on the physical state of affairs, the possibility of construction of a differential geometric framework along algebraic lines, like the paradigmatic case of (ADG), permits the formulation of information dynamics independently of any smoothness requirement on the structure of observables. This conclusion is important from a systemic viewpoint, since it allows the legitimate use of any appropriate $\mathbb{R}$-algebra sheaf of observables suited to a topological localization process, even singular algebra sheaves of generalized functions, without loosing the differential dynamical mechanism, prior believed to be solely associated with smooth manifolds. The above-stated conclusion can be properly formulated in terms of a third postulate referring on the dynamical structure of observables, which, signifies a relativity principle of an algebraic-topological origin. The latter concerns the covariance of the dynamical mechanism with respect to generalized algebras of continuous observables, which comply with appropriate well-defined cohomological conditions. These algebras of continuous observables may be conceived as topos-theoretic reference frames in the category of all sheaves of algebras of observables.

Postulate III: The global dynamical mechanism of information propagation should be independent of the particularities of various localization methodologies, which, besides continuity, are being locally employed for the extraction and subsequent coordinatization of the information content in terms of observables. Thus, algebra sheaves of smooth real-valued functions, together with, their associated, by measurement, manifold $\mathbb{R}$-spectrums, are, by no means, unique coordinatizations of the universal physical mechanism of qualitative information propagation in the physical "continuum" via observables. On the contrary, they constitute a particular instantiation of information coordinatization, or equivalently, a special topos-theoretic reference frame for the formation of the localization process. As an important consequence of the covariance property of the global dynamical mechanism, with respect to cohomologically proper sheaves of algebras of generalized observables, effectuating a localization process, we can now conceptually and technically disentagle the ascription of individuality to an event, that has been observed by means of that localization process, with the conferement of a numerical identity to it, by means of a real value, as has been undisputedly the case until now.

3 Localization in the Physical "Continuum"

We have initially hypothesized that the physical "continuum" may be considered as an abstract partial order of events with respect to the fundamental relation of extension. We assume that it exists as an object in a category of such abstract partial orders, with structure preserving morphisms as arrows, denoted in the sequel by $\mathcal{E}$. This category is required to be small, by construction, such that, the families of its objects and morphisms form genuine sets. The only technical requirement imposed is that $\mathcal{E}$ has all finite or arbitrary small colimits, thus it is a cocomplete small category.

The general purpose of a localization scheme amounts to filtering the information contained in an abstract partial order of events, representing an extensive event "continuum" in $\mathcal{E}$, through a concretely specified categorical environment, which is determined by an operational physical procedure. The latter specifies the kind of loci of variation that are used for individuation of events in the physical "continuum", such that, reference to concrete events of the specified kind can be made possible with respect to them. The kind of loci of variation signifies precisely the categorical environment employed operationally, for instance, the category of open sets, ordered by inclusion, in a topological measurement space, which, will essentially be the subject of our exposition in this work. Consequently, the abstract event "continuum" can obtain a concrete meaning in terms of localized events, represented appropriately by local real-valued observables, by referring to the categorical environment substantiated by the base open loci. In this sense, a localization scheme can be precisely conceived as a generalization of the notion of functional dependence. In the trivial case, the only locus is a point serving as a unique idealized measure of localization, and moreover, the only kind of reference frame is the one attached to a point. Pictorially, the instantiation of a localization scheme in the physical "continuum" can be represented as a fibered structure, such that the abstract partial order of events fibers over the base category of reference loci of the kind specified by operational means, identified in our present modeling attempt with the open sets of a topological measurement space.

Before proceeding in the technical exposition of the mathematical structures involved, it is instructive to discuss briefly a localization scheme of such an abstract partial order of events, over a base categorical environment $\mathcal{O}(\mathcal{X})$, consisting of open sets of a topological measurement space X, the arrows between them being inclusions. In other words, the reference loci in this operational environment are all the open sets of X, partially ordered by inclusion, or equivalently, the open sets $U \subseteq X$ are considered as varying base open loci, over which the partial order of the event "continuum" E fibers. This means that individuation of events in E from the perspective of the associated fibered structure of the induced localization scheme has meaning only with reference to the base open loci. Furthermore, it is essential to understand the shift in the semantics of continuous functions $f : X \to \mathbb{R}$, from the viewpoint of the fibered structure corresponding to the present localization scheme. If we consider local observers associated with the varying reference loci of the base category, then each of them in an measurement situation taking place over his reference locus U, individuates events by means of local real-valued observables, being continuous maps $U \to \mathbb{R}$. Thus, the local observers do not have a global perception of continuous functions $f : X \to \mathbb{R}$, but rather perceive events localized over their reference loci in terms of local observables. Of course, appropriate conditions are further needed for pasting their findings together, which, as we are going to explain in the sequel, constitute the necessary and sufficient conditions for a sheaf-theoretic construct. Intuitively, at this stage, we notice that in the present fibered structure of the events "continuum" , the viewpoint offered by a reference locus is not that of a real valued continuous function, but that of a continuously variable real number over the open locus, named local observable section and interpreted simply as a localized and accordingly individuated event. According to the interpretation put forward, we obtain a well-defined notion of localized events in the physical "continuum", varying over a multiplicity of domains of the kind determined by the topological categorical environment they share. Equivalently, there is constituted a resolution of the physical "continuum" as a variable multi-form structure of local observables over domains, characterized as reference loci, that admit a precise operational specification as open sets of a topological measurement space.

From a physical systemic perspective, the meaningful representation of an event "continuum" by means of a fibered structure, of the form described above, should incorporate the important notion of uniformity. The notion of uniformity in the physical "continuum", as filtered through the categorical environment of the open reference domains in a localization scheme, requires that, for any two events observed by means of local real-valued observables over the same open domain of measurement, the structure of all open reference loci that relate to the first cannot be distinguished, in any possible way, from the structure of open loci relating to the second. In this sense, all the localized events, within any particular reference locus, should be uniformly equivalent to each other. Moreover, the coherence of the localization scheme with respect to uniformity is secured, if the partial order of the event "continuum", represented as a fibered construct, is properly induced by lifting the partial order structure of open sets inclusions, form the base category $\mathcal{O}(\mathcal{X})$ to the fibers. The satisfaction of the above requirements, concerning a uniform fibered structure representation of an abstract partial order of events over a base categorical environment $\mathcal{O}(\mathcal{X})$, consisting of open sets of a topological measurement space X, lies at the basis of the sheaf theoretical conceptualization of the physical "continuum", which is going to be formalized categorically in the sequel.

4 Sheaf-Theoretic Fibered Structures

For the category of open sets $\mathcal{O}(\mathcal{X})$ on a topological measurement space X, ordered by inclusion, we will be considering the category of presheaves $\mathbf{Sets}^{\mathcal{O}(\mathcal{X})^{op}}$ of all contravariant functors from $\mathcal{O}(\mathcal{X})$ to $\mathbf{Sets}$ and all natural transformations between these functors. A functor $\mathbf{P}$ is a structure-preserving morphism of these categories, that is, it preserves composition and identities. A functor in the category $\mathbf{Sets}^{\mathcal{O}(\mathcal{X})^{op}}$ can be thought of, as constructing an image of $\mathcal{O}(\mathcal{X})$ in $\mathbf{Sets}$ contravariantly, or equivalently, as a contravariant translation of the language of $\mathcal{O}(\mathcal{X})$ into that of $\mathbf{Sets}$. Given another such translation (contravariant functor) $\mathbf{Q}$ of $\mathcal{O}(\mathcal{X})$ into $\mathbf{Sets}$ we need to compare them. This can be done by giving, for each object U in $\mathcal{O}(\mathcal{X})$ a transformation $T_U : \mathbf{P}(U) \to \mathbf{Q}(U)$ which compares the two images of the open set U. Not any morphism will do, however, as we would like the construction to be parametric in U, rather than ad hoc. Since U is an object in $\mathcal{O}(\mathcal{X})$ while $\mathbf{P}(U)$ is in $\mathbf{Sets}$ we cannot link them by a morphism. Rather the goal is that the transformation should respect the morphisms of $\mathcal{O}(\mathcal{X})$, or in other words the interpretations of $f : V \to U$ by $\mathbf{P}$ and $\mathbf{Q}$ should be compatible with the transformation under T. Then T is a natural transformation in the category of presheaves $\mathbf{Sets}^{\mathcal{O}(\mathcal{X})^{op}}$.

A presheaf $\mathbf{P}$ of $\mathbf{Sets}^{\mathcal{O}(\mathcal{X})^{op}}$ may be understood as a right action of $\mathcal{O}(\mathcal{X})$ on a set, which is partitioned into sorts, parameterized by the objects of $\mathcal{O}(\mathcal{X})$, and such that, whenever $F : V \to U$ is an inclusion arrow in $\mathcal{O}(\mathcal{X})$ and p is an element of $\mathbf{P}$ of sort U, then $p \circ F$ is specified as an element of $\mathbf{P}$ of sort V. Such an action $\mathbf{P}$ is referred as a $\mathcal{O}(\mathcal{X})$-variable set.

Thus, if $\mathcal{O}(\mathcal{X})^{op}$ is the opposite category of $\mathcal{O}(\mathcal{X})$, then $\mathbf{Sets}^{\mathcal{O}(\mathcal{X})^{op}}$ denotes the functor category of presheaves on varying reference loci U, being open sets of a topological measurement space X, partially ordered by inclusion, with objects all functors $\mathbf{P} : \mathcal{O}(\mathcal{X})^{op} \to \mathbf{Sets}$, and morphisms all natural transformations between such functors. Each object $\mathbf{P}$ in this category is a contravariant set-valued functor on $\mathcal{O}(\mathcal{X})$, called a presheaf of sets on $\mathcal{O}(\mathcal{X})$.

For each base open set U of $\mathcal{O}(\mathcal{X})$, $\mathbf{P}$(U) is a set, and for each arrow $F : V \to U$, $\mathbf{P}(f) : \mathbf{P}(U) \to \mathbf{P}$(V) is a set function. If $\mathbf{P}$ is a presheaf on $\mathcal{O}(\mathcal{X})$ and $p \in \mathbf{P}$(U), the value $\mathbf{P}(F)(p)$ for an arrow $F : V \to U$ in $\mathcal{O}(\mathcal{X})$ is called the restriction of p along F and is denoted by $\mathbf{P}(f)(p) = p \cdot F$.

Each base reference locus U of $\mathcal{O}(\mathcal{X})$ gives rise to a contravariant Hom-functor $\mathbf{y}[U] := Hom_{\mathcal{O}(\mathcal{X})}(-, U)$. This functor defines a presheaf on $\mathcal{O}(\mathcal{X})$. Its action on an object V of $\mathcal{O}(\mathcal{X})$ is given by

$$\mathbf{y}[U](V) := Hom_{\mathcal{O}(\mathcal{X})}(V, U)$$

whereas, its action on a morphism $x : W \to V$, for $v : V \to U$ is given by

$$\mathbf{y}[U](x) : Hom_{\mathcal{O}(\mathcal{X})}(V, U) \to Hom_{\mathcal{O}(\mathcal{X})}(W, U)$$

$$\mathbf{y}[U](x)(v) = v \circ x$$

Furthermore $\mathbf{y}$ can be made into a functor from $\mathcal{O}(\mathcal{X})$ to the contravariant functors on $\mathcal{O}(\mathcal{X})$:

$$\mathbf{y} : \mathcal{O}(\mathcal{X}) \to \mathbf{Sets}^{\mathcal{O}(\mathcal{X})^{op}}$$

$$U \mapsto Hom_{\mathcal{O}(\mathcal{X})}(-, U)$$

This is called the Yoneda embedding and it is a full and faithful functor.

The category of elements of a presheaf $\mathbf{P}$, denoted by $\int(\mathbf{P}, \mathcal{O}(X))$, admits the following objects-arrows description: The objects of $\int(\mathbf{P}, \mathcal{O}(X))$ are all pairs (U, p), with U in $\mathcal{O}(X)$ and $p \in \mathbf{P}$(U). The arrows of $\int(\mathbf{P}, \mathcal{O}(X))$, that is, $(\acute{U}, \acute{p}) \to (U, p)$, are those morphisms $Z : \acute{U} \to U$ in $\mathcal{O}(X)$, such that $\acute{p} = \mathbf{P}(Z)(p) := p \cdot Z$. Notice that the arrows in $\int(\mathbf{P}, \mathcal{O}(X))$ are those morphisms $Z : \acute{U} \to U$ in the base category $\mathcal{O}(X)$, that pull a chosen element $p \in \mathbf{P}$(U) back into $\acute{p} \in \mathbf{P}(\acute{U})$.

The category of elements $\int(\mathbf{P}, \mathcal{O}(X))$ of a presheaf $\mathbf{P}$, together with, the projection functor $\int_{\mathbf{P}} : \int(\mathbf{P}, \mathcal{O}(X)) \to \mathcal{O}(X)$ is called the split discrete fibration induced by $\mathbf{P}$, where $\mathcal{O}(X)$ is the base category of the fibration. We note that the fibers are categories in which the only arrows are identity arrows. If U is a open reference locus of $\mathcal{O}(X)$, the inverse image under $\int_{\mathbf{P}}$ of U is simply the set $\mathbf{P}(\mathrm{U})$, although its elements are written as pairs so as to form a disjoint union. The construction of the fibration induced by $\mathbf{P}$, is an instance of the general Grothendieck construction.

The relevance of the Grothendieck construction for the implementation of a localization process in the physical "continuum", according to the requirements of Section 3, has to do with the realization, that, the split discrete fibration induced by $\mathbf{P}$, where $\mathcal{O}(X)$ is the base category of the fibration, provides a well-defined notion of a uniform homologous fibered structure, in the following sense: Firstly, by the arrows specification defined in the category of elements of $\mathbf{P}$, any element p, determined over the reference locus U, is homologously related with any other element $\acute{p}$ over the reference locus $\acute{U}$, and so on, by variation over all the reference loci of the base category. Secondly, all the elements p of $\mathbf{P}$, of the same sort U, viz. determined over the same reference locus U, are uniformly equivalent to each other, since all the arrows in $\int(\mathbf{P}, \mathcal{O}(X))$ are induced by lifting arrows from the base $\mathcal{O}(X)$.

From a physical viewpoint, the purpose of introducing the notion of a presheaf $\mathbf{P}$ on $\mathcal{O}(\mathcal{X})$, in the environment of the functor category $\mathbf{Sets}^{\mathcal{O}(\mathcal{X})^{op}}$, amounts to the identification of an element of $\mathbf{P}$ of sort U, that is $p \in \mathbf{P}(\mathrm{U})$, with an event observed by means of a physical procedure over the reference locus U, being an open set of a topological measurement space X, such that, the interrelations of observed events over all reference domains of the base category $\mathcal{O}(\mathcal{X})$, fulfill the requirement of a uniform homologous fibered structure, explained in detail previously. The next crucial step of the construction, aims to the satisfaction of the following essential physical requirement: Since the operational specification of measurement environments assumed their existence locally, the information gathered about local events in different measurement situations should be collated together by appropriate means. Mathematically, this requirement is implemented by the methodology of sheafification or localization of the presheaf $\mathbf{P}$. In our context of enquiry, sheafification represents the process of conversion of the category of element-events of the presheaf $\mathbf{P}$ into a category of continuous real-valued functions, that is local observables, identified with the local sections of the corresponding sheaf.

A sheaf is an arbitrary presheaf $\mathbf{P}$ that satisfies the following condition: If $U = \bigcup_A U_a$, $U_a \in \mathcal{O}(\mathcal{X})$ and elements $p_a \in \mathbf{P}(U_a)$, $a \in I$, are such that for arbitrary a, $b \in I$ it holds:

$$p_a \mid U_{ab} = p_b \mid U_{ab}$$

where $U_{ab} := U_a \bigcap U_b$, and the symbol $\mid$ denotes the operation of restriction on the corresponding open domain, then there exists a unique element $p \in \mathbf{P}(U)$, such that $p \mid U_a = p_a$ for each a in I. Then an element of $\mathbf{P}(U)$ is called a section of the sheaf $\mathbf{P}$ over the open domain U. The sheaf condition means that sections can be glued together over the reference loci of the base category $\mathcal{O}(\mathcal{X})$.

We will show that if $\mathbf{A}$ is the contravariant functor that assigns to each open set $U \subset X$, the set of all real-valued continuous functions on U, then $\mathbf{A}$ is actually a sheaf. First of all, it is instructive to clarify that the specification of a topology on a measurement space X is solely used for the definition of the continuous functions on X; in the present case the continuous functions from any open set U in X to the real numbers $\mathbb{R}$. We notice that the continuity of each function $f : U \rightarrow \mathbb{R}$ can be determined locally. This property means that continuity respects the operation of restriction to open sets, and moreover that, continuous functions can be collated in a unique manner, as it is required for the satisfaction of the sheaf condition.

More concretely, if $f : U \rightarrow \mathbb{R}$ is a continuous function and $V \subset U$ is an open set in the topology, then the function f restricted to V is also continuous. The operation of restriction $f \mapsto f \mid V$, corresponds to a morphism of sets $\mathbf{A}(U) \to \mathbf{A}(V)$. Moreover, if $W \subset V \subset U$ stand for three nested open sets in the topology partially ordered by inclusion, the operation of restriction is transitive. Thus, the assignments;

$$U \mapsto \mathbf{A}(U)$$

$$\{V \hookrightarrow U\} \mapsto \{\mathbf{A}(U) \to \mathbf{A}(V) \quad by \quad f \mapsto f \mid V\}$$

amount to the definition of a presheaf functor $\mathbf{A}$ on $\mathcal{O}(\mathcal{X})$, in the category $\mathbf{Sets}^{\mathcal{O}(\mathcal{X})^{op}}$. Furthermore, if we consider that U is covered by open sets U_a, such that $U = \bigcup_A U_a$, $U_a \in \mathcal{O}(\mathcal{X})$, and also that, the I-indexed family of functions $f_a : U_a \to \mathbb{R}$ consists of continuous functions for all a in I, due to local determination of continuity, there is at most one continuous real-valued function $f : U \to \mathbb{R}$, with restrictions $f \mid U_a := f_a$ for all a in the index set I. Nevertheless, such a continuous function $f : U \to \mathbb{R}$ exists, if and only if, the f_a can be collated together on all the overlapping domains $U_a \bigcap U_b := U_{ab}$, such that:

$$f_a \mid U_{ab} = f_b \mid U_{ab}$$

Consequently, the presheaf of sets $\mathbf{A}$ of continuous real-valued functions on $\mathcal{O}(\mathcal{X})$, satisfies the sheaf condition, permitting in this sense, the characterization of events over the reference loci $U \subset X$ as local observables represented by real-valued continuous functions, the latter being local sections of the sheaf of sets $\mathbf{A}$. Actually, $\mathbf{A}$ is a sheaf of algebras over the field of the reals $\mathbb{R}$, because it is obvious that each set of sort U, $\mathbf{A}(U)$, is an $\mathbb{R}$-algebra under pointwise sum, product, and scalar multiple; whereas the morphisms $\mathbf{A}(U) \to \mathbf{A}(V)$ stand for $\mathbb{R}$-linear morphism of rings. In this algebraic setting, the sheaf condition means that the following sequence of $\mathbb{R}$-algebras of local observables is left exact;

$$0 \to \mathbf{A}(U) \to \prod_a \mathbf{A}(U_a) \to \prod_{a,b} \mathbf{A}(U_{ab})$$

Furthermore, we can define the inductive limit of $\mathbb{R}$-algebras $\mathbf{A}(U)$, denoted by $colim[\mathbf{A}(U)]$ as follows:

Let us consider that x is a point of the topological measurement space X. Moreover, let B be a set consisting of open subsets of X, containing x, such that the following condition holds: For any two open reference domains U, V, containing x, there exists an open set $W \in B$, contained in the intersection domain $U \bigcap V$. We may say that B constitutes a basis for the system of open reference domains around x. We form the disjoint union of all $\mathbf{A}(U)$, denoted by;

$$\mathbf{D}(x) := \coprod_{U \in B} \mathbf{A}(U)$$

Then we can define an equivalence relation in $\mathbf{D}(x)$, by requiring that $f \sim g$ for $f \in \mathbf{A}(U)$, $g \in \mathbf{A}(V)$, provided that they have the same restriction to a smaller open set contained in B. Then we define;

$$colim_B[\mathbf{A}(U)] := \mathbf{D}(x)/\sim_B$$

Furthermore, if we denote, generally, the inclusion mapping of V into U by;

$$i_{V,U} : V \hookrightarrow U$$

and also, the restriction morphism from U to V by;

$$\varrho_{U,V} : \mathbf{A}(U) \to \mathbf{A}(V)$$

we can introduce well-defined notions of addition and scalar multiplication on the set $colim_B[\mathbf{A}(U)]$, making it into an $\mathbb{R}$-module, or even an $\mathbb{R}$-algebra, as follows:

$$[f_U] + [g_V] := [\varrho_{U,W}(f_U) + \varrho_{V,W}(g_V)]$$

$$\lambda[g_V] := [\lambda g_V]$$

where f_U and g_V are elements in $\mathbf{A}(U)$ and $\mathbf{A}(V)$, that is real-valued continuous functions defined over the open domains U, V respectively, and $\lambda \in \mathbb{R}$. Now, if we consider that B and C are two bases for the system of open sets domains around $x \in X$, we can show that there are canonical isomorphisms between $colim_B[\mathbf{A}(U)]$ and $colim_C[\mathbf{A}(U)]$. In particular, we may take all the open subsets of X containing x: Indeed, we consider first the case when B is arbitrary and C is the set of all open subsets containing x. Then $C \supset B$ induces a morphism

$$colim_B[\mathbf{A}(U)] \rightarrow colim_C[\mathbf{A}(U)]$$

which is an isomorphism, since whenever V is an open subset containing x, there exists an open subset U in B contained in V. Since we can repeat that procedure for all bases of the system of open sets domains around $x \in X$, the initial claim follows immediately.

The inductive limit defined above, is denoted by $\mathbf{A}_x$, and referred as the stalk of $\mathbf{A}$ at the point $x \in X$. For an open reference domain W containing the point x, we obtain an $\mathbb{R}$-homomorphism of $\mathbf{A}(W)$ into the stalk at the point x;

$$i_{W,x} : \mathbf{A}(W) \rightarrow \mathbf{A}_x$$

For an element $f \in \mathbf{A}(W)$ its image $i_{W,x}(f) := f_x$ is called the germ of f at the point x.

The fibered structure that corresponds to the sheaf of real-valued continuous functions on a topological measurement space X is a bundle defined by the continuous mapping $\varphi : A \rightarrow X$, where;

$$\varphi^{-1}(x) = \mathbf{A}_x = colim_{\{x \in U\}}[\mathbf{A}(U)]$$

The mapping φ is locally a homeomorphism of topological spaces. The topology in A is defined as follows: for each $f \in \mathbf{A}(U)$, the set $\{f_x, x \in U\}$ is open, and moreover, an arbitrary open set is a union of sets of this form.

In the physical state of affairs, we remind that we have identified an element of $\mathbf{A}$ of sort U, that is a local section of $\mathbf{A}$, with an event f observed by means of a continuous physical procedure over the reference locus U.

Then the equivalence relation, used in the definition of the stalk $\mathbf{A}_x$ at the point $x \in X$ is interpreted as follows: Two events $f \in \mathbf{A}(U)$, $g \in \mathbf{A}(V)$, induce the same contextual information at x in X, provided that, they have the same restriction to a smaller open locus contained in the basis K. Then, the stalk $\mathbf{A}_x$ is the set containing all contextual information at x, that is the set of all equivalence classes. Moreover, the image in the stalk $\mathbf{A}_x$ of an event $f \in \mathbf{A}(U)$, that is the equivalence class of this event f, is precisely the germ of f at the point x.

The sheaf of real-valued continuous functions on a topological measurement space X is an object in the functor category of sheaves $\mathbf{Sh}(X)$ on varying reference loci U, being open sets of a topological measurement space X partially ordered by inclusion. The morphisms in $\mathbf{Sh}(X)$ are all natural transformations between such sheaf functors. It is instructive to notice that a sheaf makes sense only if the base category of reference loci is specified, which is equivalent in our context to the determination of a topology on an underlying measurement space X. Once this is accomplished, a sheaf can be thought of as measuring the space X. The functor categories of both, presheaves $\mathbf{Sets}^{\mathcal{O}(\mathcal{X})^{op}}$, and sheaves $\mathbf{Sh}(X)$, provide exemplary cases of categories, characterized as topoi. A topos can be conceived as a local mathematical framework, corresponding to a generalized model of set theory, or as a generalized algebraic space, corresponding to a categorical universe of variable information sets over the multiplicity of the reference loci of the base category. We recall that, formally a topos is a category, which has a terminal object, pullbacks, exponentials, and a subobject classifier, which is understood as an object of generalized truth values. The particular significance of the sheaf of real-valued continuous functions on X, that we have used as a uniform fibered structure of local observables for modeling an event "continuum", according to the physical requirements posed previously, is due to the following isomorphism: The sheaf of continuous real-valued functions on X, is isomorphic to the object of Dedekind real numbers in the topos of sheaves $\mathbf{Sh}(X)$, denoted in the sequel by $\mathbb{A}$. The aforementioned isomorphism validates the physical intuition of considering a local observable as a continuously variable real number over its locus of definition.

5 Topological Covering Systems

Until now, it has become evident that a sheaf-theoretic fibered model of the physical "continuum" is not based on an underlying structure of points. On the contrary, the fundamental entities are the base reference loci and their transformations, for instance, the open sets of a topological measurement space X, partially ordered by inclusion. The basic intuition behind their functioning is related with the expectation that, the reference domains of the base category in that fibered construct, serve the purpose of generalizing the notion of localization of events. In this sense, the unique measure of localization of the set-theoretical model, being a point, is substituted by a variety of localization measures, instantiated, for example, by the open sets of the base category ordered by inclusion. In the latter context, a point-localization measure, is identified precisely with the ultrafilter of open set domains containing the point. This identification permits the conception of other topological filters, being formed by the base reference loci, as generalized measures of localization. The meaningful association of filters with generalized localization measures in the physical "continuum" has to meet certain requirements, that, remarkably, have a sound physical basis, as it will become clear in the sequel, and leads to the notion of topological covering systems. It is significant, that, once the notion of a topological covering system has been crystallized, the sheaf-theoretic fibered model of an event "continuum" can be defined explicitly in its descriptive terms.

Topological covering systems are being effectuated by means of systems of covering devices on the base category of reference loci, called, in categorical terminology, covering sieves. Firstly, we shall explain the general notion of sieves, and afterwards, we shall specialize our exposition to the notion of covering sieves, showing that their applicability meets the physical requirements necessary for a conception of a points-free event "continuum".

A U-sieve with respect to a locus U in $\mathcal{O}(\mathcal{X})$, is defined as a family S of $\mathcal{O}(\mathcal{X})$-morphisms with codomain U, such that if $V \rightarrow U$ belongs to S and $D \rightarrow V$ is any $\mathcal{O}(\mathcal{X})$-morphism, then the composite $D \rightarrow V \rightarrow U$ belongs to S. We may think of a U-sieve as a right U-ideal, or equivalently, since $\mathcal{O}(\mathcal{X})$-morphisms are inclusions, as a downwards closed U-subdomain.

If we consider the contravariant representable functor of U in $\mathcal{O}(\mathcal{X})$, denoted by $\mathbf{y}[U] := Hom_{\mathcal{O}(\mathcal{X})}(-, U)$, then it is easy to realize that a U-sieve is equivalent to a subfunctor $\mathbf{S} \hookrightarrow \mathbf{y}[U]$ in $\mathbf{Sets}^{\mathcal{O}(\mathcal{X})^{op}}$.

In detail, given a U-sieve S, we define:

$$\mathbf{S}(V) = \{g/g : V \rightarrow U, g \in S\} \subseteq \mathbf{y}[U](V)$$

This definition yields a functor $\mathbf{S}$ in $\mathbf{Sets}^{\mathcal{O}(\mathcal{X})^{op}}$, which is obviously a subfunctor of $\mathbf{y}[U]$. Conversely, given a subfunctor $\mathbf{S} \hookrightarrow \mathbf{y}[U]$ in $\mathbf{Sets}^{\mathcal{O}(\mathcal{X})^{op}}$, the set:

$$S = \{g/g : V \rightarrow U, g \in \mathbf{S}(V)\}$$

for some locus V in $\mathcal{O}(\mathcal{X})$, is a U-sieve. Thus, epigramatically, we state:

$\langle$ U-sieve: S $\rangle$=$\langle$ Subfunctor of $\mathbf{y}[U]$: $\mathbf{S} \hookrightarrow \mathbf{y}[U]$ $\rangle$

We notice that if S is a U-sieve and $h : V \rightarrow U$ is any arrow to the locus U, then:

$$h^*(S) = \{f/cod(f) = V, (h \circ f) \in S\}$$

is a V-sieve, called the pullback of S along h. Consequently, we may define a presheaf functor $\mathbf{\Omega}$ in $\mathbf{Sets}^{\mathcal{O}(\mathcal{X})^{op}}$, such that its action on locoi U in $\mathcal{O}(\mathcal{X})$, is given by:

$$\mathbf{\Omega}(U) = \{S/S : U - sieve\}$$

and on arrows $h : V \to U$, by $h^*(-) : \mathbf{\Omega}(U) \to \mathbf{\Omega}(V)$, given by:

$$h^*(S) = \{f/cod(f) = V, (h \circ f) \in S\}$$

We notice that for a locus U in $\mathcal{O}(\mathcal{X})$, the set of all arrows into U, called the maximal sieve on U, and denoted by $t(U) := t_U$, is a U-sieve.

The natural question that arises in our context of enquiry is the following: How is it possible to restrict $\mathbf{\Omega}(U)$, that is the set of U-sieves for each locus U in $\mathcal{O}(\mathcal{X})$, such that each U-sieve of the restricted set can acquire the interpretation of a covering U-sieve with respect to a topological covering system. Equivalently stated, we wish to impose the satisfaction of appropriate conditions on the set of U-sieves for each locus U in $\mathcal{O}(\mathcal{X})$, such that the subset of U-sieves obtained, denoted by $\mathbf{\Omega}_{\mathcal{X}}(U)$, implement the relation of extension between events in the physical "continuum". In this sense, the U-sieves of $\mathbf{\Omega}_{\mathcal{X}}(U)$, for each locus U in $\mathcal{O}(\mathcal{X})$, to be thought as topological covering U-sieves, can be legitimately used for the definition of a localization scheme in the physical "continuum". The appropriate physical requirements for our purpose are the following:

[1]. The relation of extension among events in the physical "continuum" should be implemented by an appropriate relational property of open reference domains U in the base category $\mathcal{O}(\mathcal{X})$. In this sense, an arrow $V \to U$, such that V, U in $\mathcal{O}(\mathcal{X})$, is interpreted as a figure of U, and thus U, is interpreted as an extension of V in $\mathcal{O}(\mathcal{X})$. It is a natural requirement that the set of all figures of U should belong in $\mathbf{\Omega}_{\mathcal{X}}(U)$ for each locus U in $O(X)$.

[2]. The covering sieves should be stable under pullback operations, and most importantly, the stability conditions should be expressed functorially. This requirement means in particular that the intersection of covering sieves should also be a covering sieve for each open reference domain U in the base category $\mathcal{O}(\mathcal{X})$.

[3]. Finally, it would be desirable to impose: (i) a transitivity requirement on the specification of the covering sieves, such that intuitively stated,

covering sieves of figures of a locus in covering sieves of this locus, should be covering sieves of the locus themselves, and (ii) a requirement of common refinement of covering sieves.

If we take into account the above requirements we can define a topological covering system in the environment of $\mathcal{O}(\mathcal{X})$ as follows:

A topological covering system on $\mathcal{O}(\mathcal{X})$ is an operation J, which assigns to each open reference domain U in $\mathcal{O}(\mathcal{X})$, a collection $J(U)$ of U-sieves, called topological covering U-sieves, such that the following three conditions are satisfied:

[1]. For every open reference domain U in $\mathcal{O}(\mathcal{X})$ the maximal sieve $\{g : cod(g) = U\}$ belongs to $J(U)$ (maximality condition).

[2]. If S belongs to $J(U)$ and $h : V \to U$ is a figure of U, then $h^*(S) = \{f : V \to U, (h \circ f) \in S\}$ belongs to $J(V)$ (stability condition).

[3]. If S belongs to $J(U)$, and if for each figure $h : V_h \to U$ in S there is a sieve R_h belonging to $J(V_h)$, then the set of all composites $h \circ g$, with $h \in S$, and $g \in R_h$, belongs to $J(U)$ (transitivity condition).

As a consequence of the conditions above, we can check that any two U-covering sieves have a common refinement, that is: if S, R belong to $J(U)$, then $S \cap R$ belongs to $J(U)$.

If we consider the partially ordered set of open subsets of a topological measurement space X, viewed as the category of base reference loci $\mathcal{O}(\mathcal{X})$, then we specify that S is a covering U-sieve if and only if U is contained in the union of open sets in S. The above specification fulfills the requirements of topological covering sieves posed above, and consequently, defines a topological covering system on $\mathcal{O}(\mathcal{X})$.

Obviously a topological covering system J exists as a presheaf functor $\mathbf{\Omega}_{\mathcal{X}}$ in $\mathbf{Sets}^{\mathcal{O}(\mathcal{X})^{op}}$, such that: by acting on loci U in $\mathcal{O}(\mathcal{X})$, J gives the set of all covering U-sieves, denoted by $\mathbf{\Omega}_{\mathcal{X}}(U)$, whereas by acting on figures $h : V \to U$, it gives a morphism $h^*(-) : \mathbf{\Omega}_{\mathcal{X}}(U) \to \mathbf{\Omega}_{\mathcal{X}}(V)$, expressed as:

$h^*(S) = \{f/cod(f) = V, (h \circ f) \in S\}$, for $S \in \mathbf{\Omega}_{\mathcal{X}}(U)$.

Having introduced the notion of a topological covering system on the base category $\mathcal{O}(\mathcal{X})$, we can re-express the definition of a sheaf for that covering system on $\mathcal{O}(\mathcal{X})$, entirely in terms of covering sieves as follows:

A presheaf $\mathbf{Q}$ is a sheaf if and only if, for every covering U-sieve S, the inclusion morphism $\mathbf{S} \hookrightarrow \mathbf{y}[U]$ induces an isomorphism;

$$Hom(\mathbf{S}, \mathbf{Q}) \cong Hom(\mathbf{y}[U], \mathbf{Q})$$

The theoretical advantage of the above relies on the fact that it provides a description of sheaves entirely in terms of objects of the category of presheaves.

From a physical perspective, the consideration of covering sieves as generalized measures of localization of events in the physical "continuum", together with the requirements posed for the formation of topological covering systems, elucidates the sheaf-theoretic fibered model of local real-valued observables established previously, and moreover, justifies conceptually its relevance for the comprehension of a points-free events "continuum" that respects only the fundamental relation of extension between events.

6 Information Dynamics of Observables

The development of physical theories within the context of sheaf-theoretic fibered constructs of the physical "continuum", would require mechanisms of expressing the dynamics of information propagation in terms of continuous local observables, analogous to the ones afforded by the usual differential geometry of smooth manifolds.

For this purpose, we consider a sheaf of algebras of continuous real-valued functions on X, denoted by $\mathbb{A}$, which, represents a homologous and uniform fibered construct of the physical "continuum" in terms of scalar coefficients, characterized as local observables. In order to avoid unnecessary excess wording we refer to $\mathbb{A}$, simply, as an algebra of observables, interpreted inside the topos of sheaves $\mathbf{Sh}(X)$. In physical terminology, the introduction of new local attributes, related with the variation of the obsevables included in $\mathbb{A}$, is conceived as the result of local interactions caused by the presence of a field.

Algebraically, the process of extending the local form of observation with respect to the algebra of observables $\mathbb{A}$, due to field interactions, is described by means of a fibering, defined as an injective morphism of $\mathbb{R}$-algebras $\iota : \mathbb{A} \hookrightarrow \mathbb{B}$. Thus, the $\mathbb{R}$-algebra $\mathbb{B}$ is considered as a module over the algebra $\mathbb{A}$. A section of the fibering $\iota : \mathbb{A} \hookrightarrow \mathbb{B}$, is represented by a morphism of $\mathbb{R}$-algebras $s : \mathbb{B} \to \mathbb{A}$, left inverse to ι, that is $\iota \circ s = id_{\mathbb{B}}$. The fundamental extension of scalars of the $\mathbb{R}$-algebra $\mathbb{A}$ is obtained by tensoring $\mathbb{A}$ with itself over the distinguished subalgebra of the reals, that is $\iota : \mathbb{A} \hookrightarrow \mathbb{A} \bigotimes_{\mathbb{R}} \mathbb{A}$. Trivial cases of scalars extensions, in fact isomorphic to $\mathbb{A}$, induced by the fundamental one, are obtained by tensoring $\mathbb{A}$ with $\mathbb{R}$ from both sides, that is, $\iota_1 : \mathbb{A} \hookrightarrow \mathbb{A} \bigotimes_{\mathbb{R}} \mathbb{R}$, $\iota_2 : \mathbb{A} \hookrightarrow \mathbb{R} \bigotimes_{\mathbb{R}} \mathbb{A}$. In the present context of enquiry, the sought algebraic fibering should refer to an extension of the local form of observation, suitable for the description of local observables' infinitesimal variations, caused by local field interactions. The physical underpinning of that local fibering is based on the conception that, variable geometric configurations should be generated infinitesimally.

Consequently, if we follow the above algebraic line of reasoning, variable spectrum geometry, generated infinitesimally as a result of interactions, requires the extension of scalars of the algebra $\mathbb{A}$ by infinitesimal quantities, defined as a fibration:

$$d_* : \mathbb{A} \hookrightarrow \mathbb{A} \oplus \mathbf{M} \cdot \epsilon$$

$$f \mapsto f + d(f) \cdot \epsilon$$

where $d_*(f) =: df$ is considered as the infinitesimal part of the extended scalar, and ϵ the infinitesimal unit obeying $\epsilon^2 = 0$. The algebra of infinitesimally extended scalars, viz. $\mathbb{A} \oplus \mathbf{M} \cdot \epsilon$, is called the algebra of dual numbers over $\mathbb{A}$ with coefficients in the $\mathbb{A}$-module $\mathbf{M}$. It is immediate to see that the algebra $\mathbb{A} \oplus \mathbf{M} \cdot \epsilon$, as an abelian group is just the direct sum $\mathbb{A} \oplus \mathbf{M}$, whereas the multiplication is defined by:

$$(f + df \cdot \epsilon) \bullet (\acute{f} + d\acute{f} \cdot \epsilon) = f \cdot \acute{f} + (f \cdot d\acute{f} + \acute{f} \cdot df) \cdot \epsilon$$

Note that we further require that the composition of the augmentation $\mathbb{A} \oplus \mathbf{M} \cdot \epsilon \to \mathbb{A}$, with d_* is the identity. Equivalently, the above fibration, viz., the homomorphism of algebras $d_* : \mathbb{A} \hookrightarrow \mathbb{A} \oplus \mathbf{M} \cdot \epsilon$, can be formulated as a derivation, that is, in terms of an additive $\mathbb{R}$-linear morphism:

$$d : \mathbb{A} \to \mathbf{M}$$

$$f \mapsto df$$

that, moreover, satisfies the Leibniz rule:

$$d(f \cdot g) = f \cdot dg + g \cdot df$$

Since the formal symbols of differentials $\{df, f \in \mathbb{A}\}$, are reserved for the universal derivation, the $\mathbb{A}$-module $\mathbf{M}$ is identified as the free $\mathbb{A}$-module $\boldsymbol{\Omega}$ of 1-forms generated by these formal symbols, modulo the Leibniz constraint.

The crucial fact, regarding the algebraic construction above, has to do with the observation that, the locally free $\mathbb{A}$-module $\boldsymbol{\Omega}$ can be constructed explicitly from the fundamental form of scalars extension of $\mathbb{A}$, namely, $\iota : \mathbb{A} \hookrightarrow \mathbb{A} \bigotimes_{\mathbb{R}} \mathbb{A}$ by considering the morphism:

$$\delta : \mathbb{A} \bigotimes_{\mathbb{R}} \mathbb{A} \to \mathbb{A}$$

$$f_1 \otimes f_2 \mapsto f_1 \cdot f_2$$

Then by taking the kernel of this morphism of algebras, that is the ideal:

$$\mathbf{I} = \{f_1 \otimes f_2 \in \mathbb{A} \bigotimes_{\mathbb{R}} \mathbb{A} : \delta(f_1 \otimes f_2) = 0\} \subset \mathbb{A} \bigotimes_{\mathbb{R}} \mathbb{A}$$

it can be shown that the morphism of $\mathbb{A}$-modules:

$$\Sigma : \mathbf{\Omega} \to \frac{\mathbf{I}}{\mathbf{I}^2}$$

$$df \mapsto 1 \otimes f - f \otimes 1$$

is an isomorphism.

We can prove the above isomorphism as follows: The fractional object $\frac{\mathbf{I}}{\mathbf{I}^2}$ has an $\mathbb{A}$-module structure defined by:

$$f \cdot (f_1 \otimes f_2) = (f \cdot f_1) \otimes f_2 = f_1 \otimes (f \cdot f_2)$$

for $f_1 \otimes f_2 \in \mathbf{I}$, $f \in \mathbb{A}$. We can check that the second equality is true by proving that the difference of $(f \cdot f_1) \otimes f_2$ and $f_1 \otimes (f \cdot f_2)$ belonging to $\mathbf{I}$, is actually an elememt of $\mathbf{I}^2$, viz., the equality is true modulo $\mathbf{I}^2$. So we have:

$$(f \cdot f_1) \otimes f_2 - f_1 \otimes (f \cdot f_2) = (f_1 \otimes f_2) \cdot (f \otimes 1 - 1 \otimes f)$$

The first factor of the above product of elements belongs to $\mathbf{I}$, by assumption, whereas, the second factor also belongs to $\mathbf{I}$, since we have that:

$$\delta(f \otimes 1 - 1 \otimes f) = 0$$

Hence the product of elements above belongs to $\mathbf{I} \cdot \mathbf{I} = \mathbf{I}^2$. Consequently, we can define a morphism of $\mathbb{A}$-modules:

$$\Sigma : \mathbf{\Omega} \to \frac{\mathbf{I}}{\mathbf{I}^2}$$

$$df \mapsto 1 \otimes f - f \otimes 1$$

Now, we construct the inverse of that morphism as follows: The $\mathbb{A}$-module $\mathbf{\Omega}$ can be made an ideal in the algebra of dual numbers over $\mathbb{A}$, viz., $\mathbb{A} \oplus \mathbf{\Omega} \cdot \epsilon$. Moreover, we can define the morphism of algebras:

$$\mathbb{A} \times \mathbb{A} \to \mathbb{A} \oplus \mathbf{\Omega} \cdot \epsilon$$

$$(f_1, f_2) \mapsto f_1 \cdot f_2 + f_1 \cdot df_2 \epsilon$$

This is an $\mathbb{R}$-bilinear morphism of algebras, and thus, it gives rise to a morphism of algebras:

$$\Theta : \mathbb{A} \otimes_{\mathbb{R}} \mathbb{A} \to \mathbb{A} \oplus \mathbf{\Omega} \cdot \epsilon$$

Then, by definition we have that $\Theta(\mathbf{I}) \subset \mathbf{\Omega}$, and also, $\Theta(\mathbf{I}^2) = 0$. Hence, there is obviously induced a morphism of $\mathbb{A}$-modules:

$$\mathbf{\Omega} \leftarrow \frac{\mathbf{I}}{\mathbf{I}^2}$$

which is the inverse of Σ. Consequently, we conclude that:

$$\mathbf{\Omega} \cong \frac{\mathbf{I}}{\mathbf{I}^2}$$

Thus the free $\mathbb{A}$-module $\mathbf{\Omega}$ of 1-forms is isomorphic with the free $\mathbb{A}$-module $\frac{\mathbf{I}}{\mathbf{I}^2}$ of Kähler differentials of the algebra of scalars $\mathbb{A}$ over $\mathbb{R}$, conceived as a distinguished ideal in the algebra of infinitesimally extended scalars $\mathbb{A} \oplus \mathbf{\Omega} \cdot \epsilon$, due to interactions, according to the following split short exact sequence:

$$\mathbf{\Omega} \hookrightarrow \mathbb{A} \oplus \mathbf{\Omega} \cdot \epsilon \twoheadrightarrow \mathbb{A}$$

or equivalently formulated as:

$$0 \to \mathbf{\Omega}_{\mathbb{A}} \to \mathbb{A} \bigotimes_{\mathbb{R}} \mathbb{A} \to \mathbb{A}$$

By dualizing, we obtain the dual $\mathbb{A}$-module of $\mathbf{\Omega}$, that is $\mathbf{\Xi} := Hom(\mathbf{\Omega}, \mathbb{A})$. Thus we have at our disposal, expressed in terms of infinitesimal scalars extension of the algebra of observables $\mathbb{A}$, semantically intertwined with the generation of variable geometry as a result of local interaction, new types of observables related with the incorporation of differentials and their duals, called codifferentials or vectors.

Before proceeding further, it is instructive at this point to clarify the meaning of a universal derivation, playing a paradigmatic role in the construction of extended algebras of scalars, as above, in appropriate category-theoretic terms as follows: The covariant functor of left $\mathbb{A}$-modules valued derivations of $\mathbb{A}$:

$$\overleftarrow{\nabla}_{\mathbb{A}}(-) : \mathcal{M}^{(\mathbb{A})} \to \mathcal{M}^{(\mathbb{A})}$$

is being representable by the left $\mathbb{A}$-module of 1-forms $\mathbf{\Omega}^1(\mathbb{A})$ in the category of left $\mathbb{A}$-modules $\mathcal{M}^{(\mathbb{A})}$, according to the isomorphism:

$$\overleftarrow{\nabla}_{\mathbb{A}}(\mathbf{N}) \cong Hom_{\mathbb{A}}(\mathbf{\Omega}^1(\mathbb{A}), \mathbf{N})$$

Thus, $\mathbf{\Omega}^1(\mathbb{A})$ is characterized categorically as a universal object in $\mathcal{M}^{(\mathbb{A})}$, and the derivation:

$$d : \mathbb{A} \to \mathbf{\Omega}^1(\mathbb{A})$$

as the universal derivation. Furthermore, we can define algebraically, for each $n \in N$, $n \geq 2$, the n-fold exterior product:

$$\mathbf{\Omega}^n(\mathbb{A}) = \bigwedge^n \mathbf{\Omega}^1(\mathbb{A})$$

where $\mathbf{\Omega}(\mathbb{A}) := \mathbf{\Omega}^1(\mathbb{A})$, $\mathbb{A} := \mathbf{\Omega}^{\mathbf{0}}(\mathbb{A})$, and finally show analogously that the left $\mathbb{A}$-modules of n-forms $\mathbf{\Omega}^n(\mathbb{A})$ in $\mathcal{M}^{(\mathbb{A})}$ are representable objects in $\mathcal{M}^{(\mathbb{A})}$ of the covariant functor of left $\mathbb{A}$-modules valued n-derivations of $\mathbb{A}$, denoted by $\overleftarrow{\nabla}^n{}_{\mathbb{A}}(-) : \mathcal{M}^{(\mathbb{A})} \to \mathcal{M}^{(\mathbb{A})}$. We conclude that, all infinitesimally extended algebras of scalars, which, have been constructed from $\mathbb{A}$ by fibrations, presented equivalently, as derivations, are representable as left $\mathbb{A}$-modules of n-forms $\mathbf{\Omega}^n(\mathbb{A})$ in the category of left $\mathbb{A}$-modules $\mathcal{M}^{(\mathbb{A})}$.

We emphasize that the intelligibility of the algebraic modeling framework of dynamics, giving rise to variable geometric spectra, is based on the conception that infinitesimal variations in the observables of $\mathbb{A}$, are caused by interactions, meaning that they are being effectuated by the presence of a physical field. Thus, it is necessary to establish a purely algebraic representation of the notion of a physical field, as the causal agent of local interactions, and moreover, explain the functional role it assumes for the interpretation of the theory. The key idea for this purpose amounts to expressing the process of scalars extension, due to local interactions, covariantly, viz. in suitable functorial terms, and subsequently, identify that functor of infinitesimal scalars extension, induced by local interactions, with the functioning of a physical field that causes it. Regarding the first step of this strategy, we clarify that the general process of scalars extension from an algebra $\mathbb{W}$ to an algebra $\mathbb{T}$ is represented functorially by means of the functor of scalars extension, from $\mathbb{W}$ to $\mathbb{T}$ as follows:

$$\mathbf{F} : \mathcal{M}^{(\mathbb{W})} \to \mathcal{M}^{(\mathbb{T})}$$

$$\mathbf{E} \mapsto \mathbb{T} \bigotimes_{\mathbb{W}} \mathbf{E}$$

The second step involves the application of the functorial algebraic procedure for the case admitting the identifications:

$$\mathbb{W} = \mathbb{A}$$

$$\mathbb{T} = [\mathbb{A} \oplus \mathbf{\Omega}^1(\mathbb{A}) \cdot \epsilon]$$

corresponding to infinitesimal scalars extension. Consequently, the functionality of the notion of a physical field, as the causal agent of local interactions, admits a purely algebraic description as the functor of infinitesimal scalars extension, called a connection-inducing functor:

$$\widehat{\nabla} : \mathcal{M}^{(\mathbb{A})} \to \mathcal{M}^{(\mathbb{A} \oplus \mathbf{\Omega}^1(\mathbb{A}) \cdot \epsilon)}$$

$$\mathbf{E} \mapsto [\mathbb{A} \oplus \mathbf{\Omega}^1(\mathbb{A}) \cdot \epsilon] \bigotimes_{\mathbb{A}} \mathbf{E}$$

In this sense, the vectors of the left $\mathbb{A}$-module $\mathbf{E}$, are being infinitesimally extended into vectors of the left $(\mathbb{A} \oplus \mathbf{\Omega}^1(\mathbb{A}) \cdot \epsilon)$-module $[\mathbb{A} \oplus \mathbf{\Omega}^1(\mathbb{A}) \cdot \epsilon] \bigotimes_{\mathbb{A}} \mathbf{E}$. It is significant to notice that, these two kinds of vectors are being defined over different algebras. Hence, in order to compare them, we have to pull the infinitesimally extended ones back to the initial algebra of scalars, viz., the $\mathbb{R}$-algebra $\mathbb{A}$. Algebraically, this process is implemented by restricting the left $(\mathbb{A} \oplus \mathbf{\Omega}^1(\mathbb{A}) \cdot \epsilon)$-module $[\mathbb{A} \oplus \mathbf{\Omega}^1(\mathbb{A}) \cdot \epsilon] \bigotimes_{\mathbb{A}} \mathbf{E}$ to the $\mathbb{R}$-algebra $\mathbb{A}$.

If we perform this base algebra change, we obtain the left $\mathbb{A}$-module $\mathbf{E} \oplus [\boldsymbol{\Omega}^1(\mathbb{A})\bigotimes_{\mathbb{A}}\mathbf{E}] \cdot \epsilon$. Thus, the effect of the action of the physical field on the vectors of the left $\mathbb{A}$-module $\mathbf{E}$ can be expressed by means of the following comparison morphism of left $\mathbb{A}$-modules:

$$\nabla^{\star}{}_{\mathbf{E}} : \mathbf{E} \to \mathbf{E} \oplus [\boldsymbol{\Omega}^1(\mathbb{A})\bigotimes_{\mathbb{A}}\mathbf{E}] \cdot \epsilon$$

Equivalently, the irreducible amount of information incorporated in the comparison morphism, can be now expressed as a connection on $\mathbf{E}$, viz., as an $\mathbb{R}$-linear morphism of $\mathbb{A}$-modules:

$$\nabla_{\mathbf{E}} : \mathbf{E} \to \boldsymbol{\Omega}^1(\mathbb{A})\bigotimes_{\mathbb{A}}\mathbf{E} = \mathbf{E}\bigotimes_{\mathbb{A}}\boldsymbol{\Omega}^1(\mathbb{A}) := \boldsymbol{\Omega}^1(\mathbf{E})$$

such that, the following Leibniz type constraint is satisfied:

$$\nabla_{\mathbf{E}}(f \cdot v) = f \cdot \nabla_{\mathbf{E}}(v) + df \otimes v$$

for all $f \in \mathbb{A}$, $v \in \mathbf{E}$. Consequently, after having expressed the process of scalars extension in functorial algebraic terms, we can identify the functor of infinitesimal scalars extension, due to interactions, with the functional dependence induced by a physical field causing it. Thus, a local causal agent of a variable interaction geometry, viz., a physical field acting locally and causing infinitesimal variations of local observables, can be faithfully represented by means of a pair $(\mathbf{E}, \nabla_{\mathbf{E}})$, consisting of a left $\mathbb{A}$-module $\mathbf{E}$ and a connection $\nabla_{\mathbf{E}}$ on $\mathbf{E}$. We conclude, by emphasizing that, the functorial modeling of the universal mechanism of encoding physical interactions, by means of causal agents, as above, namely, physical fields effectuating infinitesimal scalars extension, is covariant with the algebra-theoretic specification of the structure of observables. Equivalently stated, the only actual requirement for the intelligibility of functoriality of interactions, by means of physical fields, rests on the algebra-theoretic specification of what we characterize structures of observables.

The next stage of development of a genuine functorial mechanism of dynamics, understood in the sense of local interactions caused by a physical field, involves the satisfaction of appropriate global constraints, that impose consistency requirements referring to the transition from the infinitesimal to the global. For this purpose it is necessary to employ the methodology of homological algebra. We start by reminding the algebraic construction, for each $n \in N$, $n \geq 2$, of the n-fold exterior product as follows: $\mathbf{\Omega}^n(\mathbb{A}) = \bigwedge^n \mathbf{\Omega}^1(\mathbb{A})$ where $\mathbf{\Omega}(\mathbb{A}) := \mathbf{\Omega}^1(\mathbb{A})$, $\mathbb{A} := \mathbf{\Omega}^{\mathbf{0}}(\mathbb{A})$. We notice that there exists an $\mathbb{R}$-linear morphism:

$$d^n : \mathbf{\Omega}^n(\mathbb{A}) \rightarrow \mathbf{\Omega}^{n+1}(\mathbb{A})$$

for all $n \geq 0$, such that $d^0 = d$. Let $\omega \in \mathbf{\Omega}^n(\mathcal{A})$, then ω has the form:

$$\omega = \sum f_i (dl_{i1} \bigwedge \ldots \bigwedge dl_{in})$$

with f_i, l_{ij}, $\in \mathbb{A}$ for all integers i, j. Further, we define:

$$d^n(\omega) = \sum df_i \bigwedge dl_{i1} \bigwedge \ldots \bigwedge dl_{in}$$

Then, we can easily see that the resulting sequence of $\mathbb{R}$-linear morphisms;

$$\mathbb{A} \rightarrow \mathbf{\Omega}^1(\mathbb{A}) \rightarrow \ldots \rightarrow \mathbf{\Omega}^n(\mathbb{A}) \rightarrow \ldots$$

is a complex of $\mathbb{R}$-modules, called the algebraic de Rham complex of $\mathbb{A}$, denoted consisely by $\langle \mathbf{\Omega}(\mathbb{A}) \rangle$. The notion of complex means that the composition of two consequtive $\mathbb{R}$-linear morphisms vanishes, that is $d^{n+1} \circ d^n = 0$, simplified symbolically as:

$$d^{\mathbf{2}} = 0$$

If we assume that $(\mathbf{E}, \nabla_{\mathbf{E}})$ is an interaction field, defined by a connection $\nabla_{\mathbf{E}}$ on the $\mathbb{A}$-module $\mathbf{E}$, then $\nabla_{\mathbf{E}}$ induces a sequence of $\mathbb{R}$-linear morphisms:

$$\mathbf{E} \rightarrow \mathbf{\Omega}^1(\mathbb{A}) \bigotimes_{\mathbb{A}} \mathbf{E} \rightarrow \ldots \rightarrow \mathbf{\Omega}^n(\mathbb{A}) \bigotimes_{\mathbb{A}} \mathbf{E} \rightarrow \ldots$$

or equivalently:

$$\mathbf{E} \to \mathbf{\Omega}^1(\mathbf{E}) \to \ldots \to \mathbf{\Omega}^n(\mathbf{E}) \to \ldots$$

where the morphism:

$$\nabla^n : \mathbf{\Omega}^n(\mathbb{A}) \bigotimes_{\mathbb{A}} \mathbf{E} \to \mathbf{\Omega}^{n+1}(\mathbb{A}) \bigotimes_{\mathbb{A}} \mathbf{E}$$

is given by the formula:

$$\nabla^n(\omega \otimes v) = d^n(\omega) \otimes v + (-1)^n \omega \wedge \nabla(v)$$

for all $\omega \in \mathbf{\Omega}^n(\mathbb{A})$, $v \in \mathbf{E}$. It is immediate to see that $\nabla^0 = \nabla_{\mathbf{E}}$. Let us denote by:

$$\mathbf{R}_{\nabla} : \mathbf{E} \to \mathbf{\Omega}^2(\mathbb{A}) \bigotimes_{\mathbb{A}} \mathbf{E} := \mathbf{\Omega}^2(\mathbf{E})$$

the composition $\nabla^1 \circ \nabla^0$. We see that $\mathbf{R}_{\nabla}$ is actually an $\mathbb{A}$-linear morphism, that is $\mathbb{A}$-covariant, and is called the curvature of the connection $\nabla_{\mathbf{E}}$. We notice that, the latter sequence of $\mathbb{R}$-linear morphisms, is actually a complex of $\mathbb{R}$-modules if and only if: $\mathbf{R}_{\nabla} = 0$. We say that the connection $\nabla_{\mathbf{E}}$ is integrable if $\mathbf{R}_{\nabla} = 0$, and we refer to the above complex as the algebraic de Rham complex of the integrable connection $\nabla_{\mathbf{E}}$ on $\mathbf{E}$ in that case. It is also usual to call a connection $\nabla_{\mathbf{E}}$ flat if $\mathbf{R}_{\nabla} = 0$. A flat connection defines a maximally undisturbed process of dynamical variation caused by the corresponding physical field. In this sense, a non-vanishing curvature signifies the existence of disturbances from the maximally symmetric state of that variation. These disturbances can be cohomologically identified as obstructions to deformation caused by physical sources. In that case, the algebraic de Rham complex of the algebra of scalars $\mathbb{A}$ is not acyclic, viz. it has non-trivial cohomology groups. These groups measure the obstructions caused by physical sources and are responsible for a non-vanishing curvature of the connection. Therefore, the field equations in the absence of physical sources simply read:

$$\mathbf{R}_{\nabla} = 0$$

It is essential to emphasize, that the algebraic cohomological framework of formulation of dynamical notions referring to the physical "continuum", which, is modeled by a sheaf-theoretic fibered structure of real-valued continuous observables, is based for its conceptualization and operative efficacy, neither, on the methodology of real Analysis, nor, on the restrictive assumption of smoothness of observables, but only, on the functorial expression of the process of infinitesimal scalars extensions. Nevertheless, it is instructive, to apply this algebraic framework for the case of smooth observables, in order to reproduce the smooth differential geometric mechanism of smooth manifolds geometric spectra. For this purpose, we consider that $\mathbb{A}$ stands for the sheaf of algebras of $\mathbb{R}$-valued smooth functions on X, denoted by $\mathcal{C}^{\infty}$, whereas, $\mathbf{\Omega}^{n}(\mathbb{A})$ stand for the locally free sheaves of $\mathcal{C}^{\infty}$-modules of differential n-forms on X. In this case, the algebraic de Rham complex of $\mathbb{A}$, gives rise to the corresponding differential de Rham complex of $\mathcal{C}^{\infty}$, as follows:

$$\mathcal{C}^{\infty} \to \mathbf{\Omega}^{1}(\mathcal{C}^{\infty}) \to \ldots \to \mathbf{\Omega}^{n}(\mathcal{C}^{\infty}) \to \ldots$$

The crucial mathematical observation concerning this complex, refers to the fact that, the augmented differential de Rham complex

$$\mathbf{0} \to \mathbb{R} \to \mathcal{C}^{\infty} \to \mathbf{\Omega}^{1}(\mathcal{C}^{\infty}) \to \ldots \to \mathbf{\Omega}^{n}(\mathcal{C}^{\infty}) \to \ldots$$

is actually exact. The exactness of the augmented differential de Rham complex, as above, constitutes an expression of the lemma of Poincaré, according to which, every closed $\mathcal{C}^{\infty}$-form on X is exact at least locally in X. Thus, the well-definability of the differential geometric dynamical mechanism of smooth manifolds is precisely due to the exactness of the augmented differential de Rham complex. This mathematical observation for the case of smooth observable coefficients, raises the issue of enrichment of the general functorial mechanism of infinitesimal scalars extensions, by the requirement of exactness of the respective augmented algebraic de Rham complex, securing in this sense, the well-definability of the dynamical mechanism for the general case, and reproducing the corresponding differential geometric mechanism of smooth manifolds faithfully, as well.

A positive settlement of this subtle issue comes from the mathematical theory of Abstract Differential Geometry (ADG). Actually, the axiomatic development of (ADG) *à* la Mallios in a fully-fledged mathematical theory, has been based on the exploitation of the consequences of the above-stated mathematical observation for the case of smooth observable coefficients. In this sense, the operational machinery of (ADG) is essentially implemented by the imposition of the exactness requirement of the following abstract de Rham complex, interpreted inside the topos of sheaves $\mathbf{Shv}(X)$:

$$\mathbf{0} \rightarrow \mathbb{R} \rightarrow \mathbb{A} \rightarrow \mathbf{\Omega}^1(\mathbb{A}) \rightarrow \ldots \rightarrow \mathbf{\Omega}^n(\mathbb{A}) \rightarrow \ldots$$

(ADG)'s power of abstracting and generalizing the classical calculus on smooth manifolds basically lies in the possibility of assuming other more general coordinate sheaves $\mathbb{A}$, while, at the same time retaining, via the exactness of the algebraic augmented de Rham complex, as above, the mechanism of differentials, instantiated paradigmatically, in the first place, in the case of classical differential geometry on smooth manifolds.

For our physical purposes, we conclude any cohomologically appropriate sheaf of algebras $\mathcal{A}$, characterized by the exactness property posed previously, can be legitimately regarded as a sheaf of local observables, capable of providing a well-defined dynamical mechanism, independently of any smooth manifold background, analogous, however, to the one supported by smooth manifolds.

Conclusively, it is instructive to recapitulate and add some further remarks on the physical semantics associated with the preceding algebraic cohomological dynamical framework by invoking the sheaf-theoretic terminology explicitly. The basic mathematical objects involved in the development of that framework consists of a sheaf of commutative unitary algebras $\mathbb{A}$, identified with the sheaf of algebras of real-valued local observables, a sheaf of locally free $\mathbb{A}$-modules $\mathbf{E}$ of rank n, as well as, the sheaf of locally free $\mathbb{A}$-modules of universal 1-forms $\mathbf{\Omega}$ of rank n. We assume that these sheaves have a common base space, over which they are localized, namely, an arbitrary topological measurement space X. A topological covering system of X is defined simply by an open covering $\mathcal{U} = \{U \subseteq X : \ U \text{ open in } X\}$ of X such that, any locally free $\mathbb{A}$-module sheaf $\mathbf{N}$ splits locally, by definition, that is, with respect to every U in $\mathcal{U}$, into a finite n-fold Whitney sum $\mathbb{A}^n$ of $\mathbb{A}$ with itself as $\mathbf{N}|_U = \mathbb{A}^n|_U$. For this reason, a topological covering system $\mathcal{U}$ of X may be called a coordinatizing open cover of $\mathbf{N}$. Hence, the local sections of the structure $\mathbb{R}$-algebra sheaf $\mathbb{A}$ relative to the coordinatizing open cover $\mathcal{U}$ obtain the meaning of local coordinates, while $\mathbb{A}$ itself may be called 'the coefficient' or 'continuously variable real number coordinate sheaf' of $\mathbf{N}$. Furthermore, a pair $(\mathbf{E}, \nabla_{\mathbf{E}})$, consisting of a left $\mathbb{A}$-module sheaf $\mathbf{E}$ and a connection $\nabla_{\mathbf{E}}$ on $\mathbf{E}$, represents a local causal agent of a variable interaction geometry, viz., a physical field acting locally and causing infinitesimal variations of coordinates, standing for local observables. In this sense, the local sections of $\mathbb{A}$-module sheaf $\mathbf{E}$, relative to the open cover $\mathcal{U}$, coordinatize the states of the corresponding physical field. The connection $\nabla_{\mathbf{E}}$ on $\mathbf{E}$, is given by an $\mathbb{R}$-linear morphism of $\mathbb{A}$-modules sheaves:

$$\nabla_{\mathbf{E}} : \mathbf{E} \to \mathbf{\Omega}^1(\mathbb{A}) \bigotimes_{\mathbb{A}} \mathbf{E} = \mathbf{E} \bigotimes_{\mathbb{A}} \mathbf{\Omega}^1(\mathbb{A}) := \mathbf{\Omega}^1(\mathbf{E})$$

such that, the following Leibniz condition holds:

$$\nabla_{\mathbf{E}}(f \cdot v) = f \cdot \nabla_{\mathbf{E}}(v) + df \otimes v$$

for all $f \in \mathbb{A}$, $v \in \mathbf{E}$. Notice that, by definition, the connection $\nabla_{\mathbf{E}}$ is only an $\mathbb{R}$-linear morphism of $\mathbb{A}$-modules sheaves. Hence, although it is $\mathbb{R}$-covariant, it is not $\mathbb{A}$-covariant as well. The connection $\nabla_{\mathbf{E}}$ on $\mathbf{E}$ contains the irreducible amount of information encoded in the process of infinitesimal scalars extension caused by local interactions, induced by the corresponding field.

A significant observation has to do with the fact that if $\mathbf{E} = \mathbb{A}$, considered as an $\mathbb{A}$-module over itself, then, the $\mathbb{R}$-linear morphism of sheaves of $\mathbb{A}$-modules

$$d : \mathbf{\Omega}^0(\mathbb{A}) := \mathbb{A} \to \mathbf{\Omega}(\mathbb{A})^1 := \mathbf{\Omega}(\mathbb{A})$$

is a natural connection, which is also integrable, or flat, since, $\langle \mathbf{\Omega}(\mathbb{A}) \rangle$ is actually a complex, namely the algebraic de Rham complex of $\mathbb{A}$.

If we consider a coordinatizing open cover $e^U \equiv \{U;\ (e_i)_{0 \leq i \leq n-1}\}$ of the $\mathbb{A}$-module sheaf $\mathbf{E}$ of rank n, every continuous local section $s \in \mathbf{E}(U)$, where, $U \in \mathcal{U}$, can be expressed uniquely as a superposition

$$s = \sum_{i=1}^{n} s_i e_i$$

with coefficients s_i in $\mathbb{A}(U)$. The action of $\nabla_{\mathbf{E}}$ on these sections of $\mathbf{E}$ is expressed as follows:

$$\nabla_{\mathbf{E}}(s) = \sum_{i=1}^{n} (s_i \nabla_{\mathbf{E}}(e_i) + e_i \otimes d(s_i))$$

where,

$$\nabla_{\mathbf{E}}(e_i) = \sum_{i=1}^{n} e_i \otimes \omega_{ij},\ 1 \leq i, j \leq n$$

where, $\omega = (\omega_{ij})$ denotes an $n \times n$ matrix of sections of local 1-forms. Consequently we have;

$$\nabla_{\mathbf{E}}(s) = \sum_{i=1}^{n} e_i \otimes (d(s_i) + \sum_{j=1}^{n} s_j \omega_{ij}) \equiv (d + \omega)(s)$$

Thus, every connection $\nabla_{\mathbf{E}}$, where, $\mathbf{E}$ is a locally free finite rank-n sheaf of modules $\mathbf{E}$ on X, can be decomposed locally as follows:

$$\nabla_{\mathbf{E}} = d + \omega$$

In this context, $\nabla_{\mathbf{E}}$ is identified as a covariant derivative, being decomposed locally as a sum consisting of a flat part tautosemous with d, and a generally non-flat part ω, called the gauge potential (vector potential), signifying a measure of deviation from the maximally undisturbed process of dynamical variation (represented by the flat part), caused by the corresponding physical field. The behavior of the gauge potential part ω of $\nabla_{\mathbf{E}}$ under local gauge transformations constitutes the 'transformation law of vector potentials' and is established in the following manner: Let $e^U \equiv \{U;\ e_{i=1\cdots n}\}$ and $h^V \equiv \{V;\ h_{i=1\cdots n}\}$ be two coordinatizing open covers of $\mathbf{E}$ over the open sets U and V of X, such that $U \bigcap V \neq \emptyset$. Let us denote by $g = (g_{ij})$ the following change of local gauge matrix:

$$h_j = \sum_{i=1}^{n} g_{ij} e_i$$

Under such a local gauge transformation (g_{ij}), the gauge potential part ω of $\nabla_{\mathbf{E}}$ transforms as follows:

$$\omega' = g^{-1}\omega g + g^{-1} dg$$

Furthermore, it is instructive to find the local form of the curvature $\mathbf{R}_{\nabla}$ of a connection $\nabla_{\mathbf{E}}$, where, $\mathbf{E}$ is a locally free finite rank-n sheaf of modules $\mathbf{E}$ on X, defined by the following $\mathbb{A}$-linear morphism of sheaves:

$$\mathbf{R}_{\nabla} := \nabla^1 \circ \nabla^0 : \mathbf{E} \to \mathbf{\Omega}^2(\mathbb{A}) \bigotimes_{\mathbb{A}} \mathbf{E} := \mathbf{\Omega}^2(\mathbf{E})$$

Due to its property of $\mathbb{A}$-covariance, a non-vanishing curvature represents in this context, the $\mathbb{A}$-covariant, and thus, observable (by $\mathbb{A}$-scalars) disturbance from the maximally symmetric state of the variation caused by the corresponding physical field. In this sense, it may be accurately characterized physically as 'gauge field strength'. Moreover, since the curvature $\mathbf{R}_{\nabla}$ is an $\mathbb{A}$-linear morphism of sheaves of $\mathbb{A}$-modules, $\mathbf{R}_{\nabla}$ may be thought of as an element of $\mathbf{End}(\mathbf{E}) \bigotimes_{\mathbb{A}} \mathbf{\Omega}^2(\mathbb{A}) := \mathbf{\Omega}^2(\mathbf{End}(\mathbf{E}))$, that is:

$$\mathbf{R}_{\nabla} \in \mathbf{\Omega}^2(\mathbf{End}(\mathbf{E}))$$

Hence, the local form of the curvature $\mathbf{R}_\nabla$ of a connection $\nabla_\mathbf{E}$, consists of local $n \times n$ matrices having for entries local 2-forms on X.

The behavior of the curvature $\mathbf{R}_\nabla$ of a connection $\nabla_\mathbf{E}$ under local gauge transformations constitutes the 'transformation law of gauge field strengths'. If we agree that $g = (g_{ij})$ denotes the change of gauge matrix, we have previously considered in the discussion of the transformation law of gauge potentials, we deduce the following local transformation law of gauge field strengths:

$$\mathbf{R}_\nabla \overset{g}{\mapsto} \mathbf{R}_\nabla{}' = g^{-1}(\mathbf{R}_\nabla)g$$

7 Epilogue

The basic nucleus of ideas at the core of this article, aiming at a conceptual and technical replacement of the axiomatic set-theoretic model of the physical "continuum" by a constructive fibered sheaf-theoretic topos model, taking into account, the realistic operational systemic procedures of localization processes for discerning and coordinatizing observable events, resolves around four crucial physical issues of paramount importance.

The first of them refers to the enunciation of the meaning of an appropriate structure of observables, used for the coordinatization of events in the physical "continuum". The term appropriate is being qualified precisely by a twofold determination of observable structures. The first component concerns their algebraic nature, by stipulating the structure of a commutative unital algebra $\mathbb{A}$ over the real numbers. The significance of this stipulation lies on the fact that the categorical dual of a commutative unital algebra is understood as a geometric space, called the spectrum of the algebra, such that, the elements of the algebra, viz. the observables, can be considered as functions on the spectrum. This consideration can be properly actualized, by employing the second component of determination of appropriate observable structures, concerning their topological nature. The latter is responsible for the localization of the information contained in the algebras of observables, with respect to a category of reference loci, being amenable to an operational specification. The net effect of the algebraic and topological organization of information, realized by means of a vertical (on the fibers) and horizontal (on the reference base) conceptual dimension respectively, can be formalized by the notion of a sheaf of commutative unital algebras of continuous real-valued observables, functioning as a homologous uniform and coherent fibered construct of the physical "continuum".

The second physical issue concerns a novel conception of the notion of systemic localization precesses. More concretely, in classical theories localization has been conceived by means of metrical properties on a pre-existing smooth set-theoretic spacetime manifold. In contradistinction, we have argued that general localization schemes, in agreement with realistic operational measurement procedures, should be understood in terms of topological covering systems of the physical "continuum". These covering systems elucidate the primary functionality of a localization process, being constituted by the properties of covariance with respect to pullback operations of covering sieves and transitivity. Furthermore, they invoke suitable criteria for collating local observables into global ones. Notice that, the notion of functional dependence introduced by localization schemes, is formalized exclusively in functorial algebraic terms of relational information content with respect to the category of base reference loci, without any supporting notion of a smooth metrical set-theoretic backround manifold. In this sense, the resolution focus in the physical "continuum" has been shifted from point-set to topological localization models, that effectively, induce a transition in the semantics of observables from a set-theoretic to a sheaf-theoretic one. Subsequently, that semantic transition effectuates the conceptual replacement of the classical metrical ruler of localization on a smooth background manifold, with a multiplicity of sheaf-cohomological rulers of algebraic topological localization in the respective spectra of the topos of all sheaves of algebras of continuous observables.

The third physical issue concerns the functoriality, or equivalently stated, covariance of the dynamical mechanism of information propagation, with respect to generalized algebras of continuous observables, complying with the cohomological conditions for the formation of exact complexes. The significance of this covariance property, invoking the formulation of functorial dynamics in the physical "continuum", lies on the fact that, algebra sheaves of smooth real-valued functions, together with, their associated, by measurement, manifold $\mathbb{R}$-spectrums do not constitute unique coordinatizations of the universal physical mechanism of qualitative information propagation via observables. Thus, the smoothness assumption on the structure of observables is not necessary for the formulation of the dynamical mechanism. In this perspective, the assumed collapse of dynamics at singularities of a base manifold is actually only phenomenal, meaning that, it appears to break down exclusively due to the employment of inappropriate coefficients (smooth functions) for the coordinatization of these loci in the physical "continuum".

Finally, the fourth physical issue concerns the functorial modeling of the notion of a physical field, which, is endowed with the semantics of a causal agent inducing local interactions, implemented by the algebraic process of infinitesimal scalar extensions of the algebra sheaf of local observables, coordinatizing events in the physical "continuum". In the context of this conceptualization, admitting a well-defined functorial algebraic formulation, the dynamics of information propagation being caused by the presence of some interaction field, as a particular application of the general functorial mechanism, can be adequately generated by the connection morphism. The flat instantiation of the connection defines a maximally undisturbed process of dynamical variation caused by that field. In general, a connection can be decomposed into the sum of a flat component and a non-flat one. The non-flat part, called gauge potential, constitutes a measure of deviation from the maximally symmetric state of dynamical variation, defined respectively by the flat or integrable part. The disturbance from that maximally symmetric state of dynamical variation, caused by the non-integrability of the connection, becomes observable geometrically, meaning that, it becomes covariantly represented with respect to the local sections of the algebra sheaf of observables, by means of a non-vanishing curvature. Thus, the latter is legitimately endowed with the physical semantics of gauge field strength. Finally, disturbances of the previous form and functionality may be cohomologically identified as obstructions to topological deformation caused by physical sources.

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