Topos-Theoretic Classification of Quantum Events Structures In Terms of Boolean Reference Frames

ELIAS ZAFIRIS

University of Athens

Institute of Mathematics

Panepistimiopolis, 15784 Athens

Greece

Abstract

We construct a sheaf-theoretic representation of quantum events structures, in terms of Boolean localization systems. These covering systems are constructed as ideals of structure-preserving morphisms of quantum events algebras from varying Boolean domains, identified with physical contexts of measurement . The modeling sheaf-theoretic scheme is based on the existence of a categorical adjunction between presheaves of Boolean events algebras and quantum events algebras. On the basis of this adjoint correspondence, we also prove the existence of an object of truth values in the category of quantum logics, characterized as subobject classifier. This classifying object plays the equivalent role that the two-valued Boolean truth values object plays in classical events structures. We construct the object of quantum truth values explicitly, and furthemore, demonstrate its functioning for the valuation of propositions in a typical quantum measurement situation.

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[0] *E-mail* : **ezafiris@math.uoa.gr**

1 Prologue

The notion of the logic of a physical theory has been introduced in 1936 by von Neumann and G. Birkhoff in a paper entitled "*The Logic of Quantum Mechanics*". For classical theories the appropriate logic is a Boolean algebra; but for quantum theories a non-Boolean logical structure is necessary, which can be characterized as an orthocomplemented lattice, or a partial Boolean algebra, or some other structure of a related form. The logic of a physical theory reflects the structure of the propositions describing the behavior of a physical system in the domain of the corresponding theory.

Naturally, the typical mathematical structure associated with logic is an ordered structure. The original quantum logical formulation of quantum theory [1, 2] depends in an essential way on the identification of propositions with projection operators on a complex Hilbert space. A non-classical, non-Boolean logical structure is effectively induced which has its origins in quantum theory. More accurately the Hilbert space quantum logic is axiomatized as a complete, atomic, orthomodular lattice. Equivalently, it can be isomorphic to the partial Boolean algebra of closed subspaces of the Hilbert space associated with the quantum system, or alternatively, the partial Boolean algebra of projection operators of the system. On the contrary, the propositional logic of classical mechanics is Boolean logic, meaning that the class of models, over which validity and associated semantic notions are defined for the propositions of classical mechanics, is the class of Boolean logic structures.

In a previous work we have proposed a sheaf-theoretic scheme, that accommodates the formalization of quantum event and observable algebras as structured interlocking families of Boolean event algebras [3]. In the present work our purpose is the study of the truth values structures suited to represent accurately the quantum domain of discourse, according to the sheaf-theoretic representation established. We will argue that generalized classical logic structures, interconnected non-trivially, provide the building blocks of an appropriate conceptual environment by means of which it is possible to comprehend the complexity of the structures of quantum propositions. We hold the view that the logic of quantum propositions reflects literal ontological structures of the quantum domain of discourse, and the perspective offered by the proposed scheme, with respect to a logical truth values interpretation, reveals the relevant ontological aspects as well.

Traditionally, the vast majority of the attempts to explore the logical structures associated with quantum mechanical systems are based on a set theoretical language. We propose a transition in the syntax of the theory involved, which as will see effects a transition in the semantics of quantum logics. This transition hopefully clarifies the relationship between the ontological structures associated with the classical and quantum domains of discourse, as it is reflected on their logical intertransformability. The mathematical language which is best suited to fulfill our objectives is provided by category and topos theory [4-10]. This is due to the fact that, these theories provide the means to relate the form and meaning of non-Boolean quantum logical structure with suitable interlocking local Boolean contexts, and most importantly, this can be done in a universal way.

The development of the conceptual and technical machinery of localization systems for generating non-trivial global events structures, as it has been recently demonstrated in [11], effectuates a transition in the semantics of events from a set-theoretic to a sheaf-theoretic one. This is a crucial semantic difference that characterizes the present approach in comparison to the vast literature on quantum measurement and quantum logic. More precisely, quantum events algebras can be represented as sheaves for an appropriate covering system defined on the Boolean localizing category. This process is formalized categorically by the concept of localization systems, where, the specified maps from Boolean contexts induced by measurement situations of observables, play the role of covers of a quantum structure of events. In more detail, the notion of local is characterized by a categorical Grothendieck topology, the axioms of which express closure conditions on the collection of covers. In this sense, the information available about each map of the specified covering system may be used to determine completely a quantum events structure. In this paper, we will avoid to mention Grothendieck topologies explicitly, although the definition of a Boolean localization system, gives rise to a precise Grothendieck topology [11] .

The category of sheaves is a topos, and consequently, comes naturally equipped with an object of generalized truth values, called subobject classifier. This object of truth values, being remarkably a sheaf itself, namely an object of the topos, is the appropriate conceptual tool for the organization of the logical dimension of the information included in the category of quantum events algebras, as it is encoded in Boolean localization systems.

More concretely, we will show that the logic of propositions describing a quantum system can be comprehended via equivalence relations in the sheaf of coefficients defined over the category of Boolean logical structures for an appropriate covering system of the latter, defined as a Boolean localization system. We emphasize that the significance of the sheaf-theoretical conception of a quantum logical structure, lies on the fact that it is supported by the well defined underlying notion of multi-valued truth structure of a topos.

The fact that a quantum events algebra is actually a non-trivial global object is fully justified by Kochen-Specker theorem [12]. According to this, there are no two-valued homomorphisms on the algebra of quantum propositions. Consequently, a quantum logical algebra cannot be embedded into a Boolean one. We note that a two-valued homomorphism on a classical event algebra is a classical truth value assignment on the propositions of the physical theory, represented by the elements of the Boolean algebra, or a yes-no assignment on the corresponding properties represented by the elements of the algebra. In this work, we will show eventually that the categorical environment specifying a quantum event algebra in terms of Boolean localization systems, contains an object of truth values, or classifying object, that constitutes the appropriate tool for the definition of a quantum truth values assignment, corresponding to valuations of propositions describing the behavior of quantum systems.

Contextual topos theoretical approaches to quantum structures of truth values have been also considered, from a different viewpoint in [13, 14], and discussed in [15, 16]. Of particular relevance to the present work, regarding the specification of a quantum truth values object, although not based on category theory methods, seems to be the approach to the foundations of quantum logic by Takeuti and Davis [17, 18], according to whom, quantization of a proposition of classical physics is equivalent to interpreting it in a Boolean extension of a set theoretical universe, where B is a complete Boolean algebra of projection operators on a Hilbert space.

In Section 2, we recapitulate the categorical framework that leads to the sheaf-theoretic representation of quantum events algebras, by formulating the existence of a categorical adjunction between the categories of Boolean presheaves and quantum events algebras. Moreover we explain the notions of Boolean systems of localizations for quantum events algebras and formulate a representation theorem in terms of the counit of the Boolean-quantum adjunction. In Section 3, we introduce the notion of a subobject functor and specify the necessary and sufficient conditions for being representable by an object in the category of quantum logics, to be identified as a quantum truth values object. In Section 4, we construct the representation of the quantum truth values object in tensor product form, and moreover, we prove that it plays the role of subobject classifier in the category of quantum events algebras. Furthermore, we formulate explicitly, the relevant criterion of truth for a complete description of reality. In Section 5, we propose the use of quantum truth values as the proper range for valuations of propositions associated with the behavior of quantum systems and demonstrate their functioning. Finally we conclude in Section 6.

2 The Categorical Framework of Representation

2.1 Sheaf-Theoretic Modeling of Quantum Events Algebras

Definition: A **Quantum events structure** is a small cocomplete category, denoted by $\mathcal{L}$, which is called the category of quantum events algebras. The objects of $\mathcal{L}$ are quantum events algebras and the arrows are quantum algebraic homomorphisms.

Definition: **A quantum events algebra** L in $\mathcal{L}$, is defined as an orthomodular σ-orthoposet, that is, as a partially ordered set of quantum events, endowed with a maximal element 1, and with an operation of orthocomplementation $[-]^* : L \longrightarrow L$, which satisfy, for all $l \in L$, the following conditions: [a] $l \leq 1$, [b] $l^{**} = l$, [c] $l \vee l^* = 1$, [d] $l \leq \acute{l} \Rightarrow \acute{l}^* \leq l^*$, [e] $l \bot \acute{l} \Rightarrow l \vee \acute{l} \in L$, [f] for $l, \acute{l} \in L, l \leq \acute{l}$ implies that l and $\acute{l}$ are compatible, where $0 := 1^*$, $l \bot \acute{l} := l \leq \acute{l}^*$, and the operations of meet $\wedge$ and join $\vee$ are defined as usually.

Remark: We recall that $l, \acute{l} \in L$ are compatible if the sublattice generated by $\{l, l^*, \acute{l}, \acute{l}^*\}$ is a Boolean algebra, namely if it is a Boolean sublattice. The σ-completeness condition, namely that the join of countable families of pairwise orthogonal events must exist, is also required in order to have a well defined theory of observables over L.

Definition: A **quantum algebraic homomorphism** in $\mathcal{L}$ is a morphism $K \xrightarrow{H} L$, which satisfies, for all $k \in K$, the following conditions: [a] $H(1) = 1$, [b] $H(k^*) = [H(k)]^*$, [c] $k \leq \acute{k} \Rightarrow H(k) \leq H(\acute{k})$, [d] $k \perp \acute{k} \Rightarrow H(k \vee \acute{k}) \leq H(k) \vee H(\acute{k})$, [e] $H(\bigvee_n k_n) = \bigvee_n H(k_n)$, where $k_1, k_2, \ldots$ countable family of mutually orthogonal events.

Definition: A **Classical events structure** is a small category, denoted by $\mathcal{B}$, which is called the category of Boolean events algebras. The objects of $\mathcal{B}$ are σ-Boolean algebras of events and the arrows are the corresponding Boolean algebraic homomorphisms.

Definition: **A functor of local Boolean coefficients**, $\mathbf{A} : \mathcal{B} \longrightarrow \mathcal{L}$, assigns to Boolean events algebras in $\mathcal{B}$, that instantiates a model coordinatizing category, the underlying quantum events algebras from $\mathcal{L}$, and to Boolean homomorphisms the underlying quantum algebraic homomorphisms.

Remark: The functor $\mathbf{M}$ acts as a forgetful functor, forgetting the extra Boolean structure of $\mathcal{B}$.

Definition: The **functor category of presheaves on Boolean events algebras**, denoted by $\mathbf{Sets}^{\mathcal{B}^{op}}$, has objects all functors $\mathbf{P} : \mathcal{B}^{op} \longrightarrow \mathbf{Sets}$, and morphisms all natural transformations between such functors, where $\mathcal{B}^{op}$ is the opposite category of $\mathcal{B}$.

Definition: Each object $\mathbf{P}$ in $\mathbf{Sets}^{\mathcal{B}^{op}}$ is a contravariant set-valued functor on $\mathcal{B}$, called a **presheaf** on $\mathcal{B}$.

Remark: For each Boolean algebra B of $\mathcal{B}$, $\mathbf{P}$(B) is a set, and for each arrow $f : C \longrightarrow B$, $\mathbf{P}(f) : \mathbf{P}(B) \longrightarrow \mathbf{P}$(C) is a set function. If $\mathbf{P}$ is a presheaf on $\mathcal{B}$ and $p \in \mathbf{P}$(B), the value $\mathbf{P}(f)(p)$ for an arrow $f : C \longrightarrow B$ in $\mathcal{B}$ is called the restriction of x along f and is denoted by $\mathbf{P}(f)(p) = p \cdot f$.

Remark: Each object B of $\mathcal{B}$ gives rise to a contravariant Hom-functor $\mathbf{y}[B] := Hom_{\mathcal{B}}(-, B)$. This functor defines a presheaf on $\mathcal{B}$. Its action on an object C of $\mathcal{B}$ is given by

$$\mathbf{y}[B](C) := Hom_{\mathcal{B}}(C, B)$$

whereas its action on a morphism $D \xrightarrow{x} C$, for $v : C \longrightarrow B$ is given by

$$\mathbf{y}[B](x) : Hom_{\mathcal{B}}(C, B) \longrightarrow Hom_{\mathcal{B}}(D, B)$$

$$\mathbf{y}[B](x)(v) = v \circ x$$

Furthermore, $\mathbf{y}$ can be made into a functor from $\mathcal{B}$ to the contravariant functors on $\mathcal{B}$

$$\mathbf{y} : \mathcal{B} \longrightarrow \mathbf{Sets}^{\mathcal{B}^{op}}$$

such that $B \mapsto Hom_{\mathcal{B}}(-, B)$. This is called the Yoneda embedding and it is a full and faithful functor.

Remark: The functor category of presheaves on Boolean events algebras $\mathbf{Sets}^{\mathcal{B}^{op}}$, provides an instantiation of a structure known as topos. A topos exemplifies a well defined notion of a universe of variable sets. It can be conceived as a local mathematical framework corresponding to a generalized model of set theory or as a generalized space.

Definition : The category of elements of the presheaf P, denoted by $\int(\mathbf{P}, \mathcal{B})$ has objects all pairs (B, p), and morphisms $(\acute{B}, \acute{p}) \rightarrow (B, p)$ are those morphisms $u : \acute{B} \rightarrow B$ of $\mathcal{B}$ for which $p \cdot u = \acute{p}$.

Definition: Projection on the second coordinate of $\int(\mathbf{P}, \mathcal{B})$, defines a functor $\int_{\mathbf{P}} : \int(\mathbf{P}, \mathcal{B}) \rightarrow \mathcal{B}$. $\int(\mathbf{P}, \mathcal{B})$ together with the projection functor $\int_{\mathbf{P}}$ is defined as the **split discrete fibration** induced by $\mathbf{P}$, where $\mathcal{B}$ is the base category of the fibration as in the diagram below.

Remark: We note that the fibers are categories in which the only arrows are identity arrows. If B is an object of $\mathcal{B}$, the inverse image under $\int_{\mathbf{P}}$ of B is simply the set $\mathbf{P}$(B), although its elements are written as pairs so as to form a disjoint union. The construction of the fibration induced by $\mathbf{P}$, is an instance of the general Grothendieck construction [8].

$$\begin{array}{ccc} \int(\mathbf{P}, \mathcal{B}) & & \\ \Big\downarrow \int_{\mathbf{P}} & & \\ \mathcal{B} & \xrightarrow{\mathbf{P}} & \mathbf{Sets} \end{array}$$

Remark: The construction of the split discrete fibration induced by $\mathbf{P}$, where $\mathcal{B}$ is the base category of the fibration, incorporates the physically important requirement of uniformity [11]. The notion of uniformity, requires that for any two events observed over the same domain of measurement, the structure of all Boolean contexts that relate to the first cannot be distinguished in any possible way from the structure of Boolean contexts relating to the second. In this sense, all the observed events within any particular Boolean context should be uniformly equivalent to each other. It is easy to notice that the composition law in the category of elements of the presheaf $\mathbf{P}$, expresses precisely the above uniformity condition.

Definition : The **functor of generalized elements of a quantum events algebra** L in the environment of the category of presheaves on Boolean events algebras, $\mathbf{Sets}^{\mathcal{B}^{op}}$, or **functor of Boolean frames of a quantum event algebra** L, is defined by:

$$\mathbf{R} : \mathcal{L} \to \mathbf{Sets}^{\mathcal{B}^{op}}$$

from $\mathcal{L}$ to the category of presheaves of Boolean events algebras $\mathbf{Sets}^{\mathcal{B}^{op}}$, where, the action on an object B in $\mathcal{B}$ is given by

$$\mathbf{R}(L)(B) := \mathbf{R}_L(B) = Hom_{\mathcal{L}}(\mathbf{M}(B), L)$$

whereas, the action on a morphism $D \xrightarrow{x} B$ in $\mathcal{B}$, for $v : B \longrightarrow L$ is given by

$$\mathbf{R}(L)(x) : Hom_{\mathcal{L}}(\mathbf{M}(B), L) \longrightarrow Hom_{\mathcal{B}}(\mathbf{M}(D), L)$$

$$\mathbf{R}(L)(x)(v) = v \circ x$$

Theorem: There exists a pair of adjoint functors $\mathbf{L} \dashv \mathbf{R}$ as follows (for a proof see [3]):

$$\mathbf{L} : \mathbf{Sets}^{\mathcal{B}^{op}} \rightleftarrows \mathcal{L} : \mathbf{R}$$

The Boolean-quantum adjunction consists of the functors $\mathbf{L}$ and $\mathbf{R}$, called left and right adjoints with respect to each other respectively, as well as the natural bijection:

$$Nat(\mathbf{P}, \mathbf{R}(L)) \cong Hom_{\mathcal{L}}(\mathbf{LP}, L)$$

Remark: The established bijective correspondence, interpreted functorially, says that the functor $\mathbf{R}$ from $\mathcal{L}$ to presheaves given by

$$\mathbf{R}(L) : B \mapsto Hom_{\mathcal{L}}(\mathbf{M}(B), L)$$

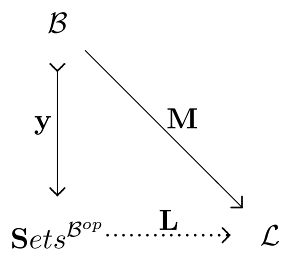
has a left adjoint $\mathbf{L} : \mathbf{Sets}^{\mathcal{B}^{op}} \to \mathcal{L}$, which is defined for each presheaf of Boolean algebras $\mathbf{P}$ in $\mathbf{Sets}^{\mathcal{B}^{op}}$ as the colimit

$$\mathbf{L}(\mathbf{P}) = Colim\{\int(\mathbf{P}, \mathcal{B}) \xrightarrow{\int_{\mathbf{P}}} \mathcal{B} \xrightarrow{\mathbf{M}} \mathcal{L}\}$$

Corollary: The Boolean coefficients functor $\mathbf{M}(B)$ is characterized as the colimit of the representable presheaf on the category of Boolean algebras (for a proof see [3]), as follows:

$$\mathbf{L}\mathbf{y}[B](B) \cong \mathbf{M} \circ \int_{\mathbf{y}[B]} (B, 1_B) = \mathbf{M}(B)$$

Remark: The following diagram (with the Yoneda embedding $\mathbf{y}$) commutes.



Corollary: The colimit in the category of elements of the functor of Boolean frames $\mathbf{L}(\mathbf{R}(L))$ is a quantum event algebra (for a proof see [3], [11]).

Definition: A **system of Boolean prelocalizations** for a quantum events algebra L in $\mathcal{L}$ is a subfunctor of the Hom-functor $\mathbf{R}(L)$ of the form $\mathbf{S} : \mathcal{B}^{op} \to \mathbf{Sets}$, that is for all B in $\mathcal{B}$, it satisfies $\mathbf{S}(B) \subseteq [\mathbf{R}(L)](B)$.

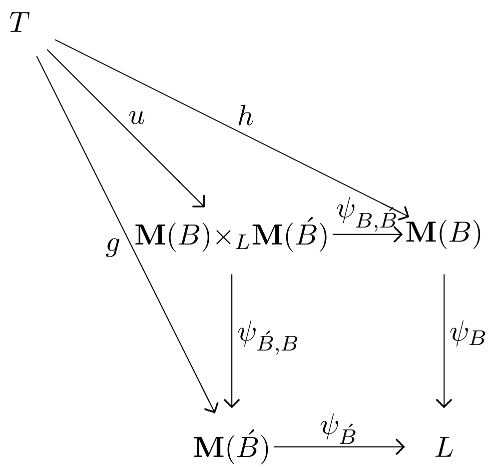
Remark: A system of Boolean prelocalizations for a quantum events algebra L in $\mathcal{L}$ is equivalent to a right ideal $\mathbf{S} \triangleright \mathbf{R}(L)$, defined by the requirement that, for each B in $\mathcal{B}$, $\mathbf{S}(B)$ is a set of quantum algebraic homomorphisms of the form $\psi_B : \mathbf{M}(B) \longrightarrow L$, satisfying the following property:

$\langle$ If $\psi_B : \mathbf{M}(B) \longrightarrow L \in \mathbf{S}(B)$, and $\mathbf{M}(v) : \mathbf{M}(\acute{B}) \longrightarrow \mathbf{M}(B)$ in $\mathcal{L}$, for $v : \acute{B} \longrightarrow B$ in $\mathcal{B}$, then $\psi_B \circ \mathbf{M}(v) : \mathbf{M}(\acute{B}) \longrightarrow \mathcal{L} \in \mathbf{S}(B)$ $\rangle$.

Definition: A family of Boolean covers $\psi_B : \mathbf{M}(B) \longrightarrow L$, B in $\mathcal{B}$, is the **generator of a system of Boolean prelocalizations S**, if and only if, this system is the smallest among all that contains that family.

Remark: The systems of Boolean prelocalizations constitute a partially ordered set under inclusion. The minimal system is the empty one, namely $\mathbf{S}(B) = \emptyset$ for all B in $\mathcal{B}$, whereas the maximal system is the Hom-functor $\mathbf{R}(L)$ itself.

Definition: The **pullback of the Boolean covers**: $\psi_B : \mathbf{M}(B) \longrightarrow L$, B in $\mathcal{B}$, and $\psi_{\acute{B}} : \mathbf{M}(\acute{B}) \longrightarrow L$, $\acute{B}$ in $\mathcal{B}$, with common codomain the quantum events algebra L, consists of the object $\mathbf{M}(B) \times_L \mathbf{M}(\acute{B})$ and two arrows $\psi_{B\acute{B}}$ and $\psi_{\acute{B}B}$, called projections, as shown in the following diagram. The square commutes and for any object T and arrows h and g that make the outer square commute, there is a unique $u : T \longrightarrow \mathbf{M}(B) \times_L \mathbf{M}(\acute{B})$ that makes the whole diagram commute.



Remark: If ψ_B and $\psi_{\acute{B}}$ are injective, then their pullback is isomorphic with the intersection $\mathbf{M}(B) \cap \mathbf{M}(\acute{B})$.

Definition: The **pasting isomorphism** of Boolean covers, is defined as follows:

$$\Omega_{B,\acute{B}} : \psi_{\acute{B}B}(\mathbf{M}(B)\bigcap_L \mathbf{M}(\acute{B})) \longrightarrow \psi_{B\acute{B}}(\mathbf{M}(B)\bigcap_L \mathbf{M}(\acute{B}))$$

$$\Omega_{B,\acute{B}} = \psi_{B\acute{B}} \circ \psi_{\acute{B}B}{}^{-1}$$

Theorem: The Boolean coordinatizing maps $\psi_{\acute{B}B}(\mathbf{M}(B)\times_L \mathbf{M}(\acute{B}))$ and $\psi_{B\acute{B}}(\mathbf{M}(B)\times_L \mathbf{M}(\acute{B}))$ cover the same part of a quantum events algebra in a compatible way.

Proof: An immediate consequence of the previous definition is the satisfaction of the following **Boolean coordinates cocycle conditions**:

$$\Omega_{B,B} = 1_B \qquad 1_B : identity \quad of \quad B$$

$$\Omega_{B,\acute{B}} \circ \Omega_{\acute{B},\acute{\acute{B}}} = \Omega_{B,\acute{\acute{B}}} \qquad if \quad \mathbf{M}(B) \cap \mathbf{M}(\acute{B}) \cap \mathbf{M}(\acute{\acute{B}}) \neq 0$$

$$\Omega_{B,\acute{B}} = \Omega^{-1}{}_{\acute{B},B} \qquad if \quad \mathbf{M}(B) \cap \mathbf{M}(\acute{B}) \neq 0$$

Thus, the pasting morhism assures that $\psi_{\acute{B}B}(\mathbf{M}(B)\times_L \mathbf{M}(\acute{B}))$ and $\psi_{B\acute{B}}(\mathbf{M}(B)\times_L \mathbf{M}(\acute{B}))$ cover the same part of a quantum events algebra in a compatible way.

Definition: Given a system of prelocalizations for a quantum events algebra $L \in \mathcal{L}$, we call it **a system of Boolean localizations**, or equivalently, a **structure sheaf of Boolean coefficients**, iff the Boolean coordinates cocycle conditions are satisfied, and moreover, the quantum algebraic events structure is preserved.

Definition: For any presheaf $\mathbf{P} \in \mathbf{Sets}^{\mathcal{B}^{op}}$, the unit is defined as

$$\delta_{\mathbf{P}} : \mathbf{P} \longrightarrow \mathbf{RLP}$$

On the other side, for each quantum event algebra L in $\mathcal{L}$ the counit is defined as

$$\epsilon_L : \mathbf{LR}(L) \longrightarrow L$$

Boolean Representation Theorem: The representation of a quantum events algebra L in $\mathcal{L}$, in terms of a coordinatization system of localizations, consisting of Boolean coefficients, is full and faithful, if and only if the counit of the Boolean-quantum adjunction, restricted to that system, is an isomorphism, that is, structure-preserving, injective and surjective.

Remark: The counit of the adjunction, restricted to a system of localizations is a quantum algebraic isomorphism, if and only if the right adjoint functor is full and faithful, or equivalently, if and only if the cocone from the functor $\mathbf{M} \circ \int_{\mathbf{R}(L)}$ to the quantum event algebra L is universal for each L in $\mathcal{L}$. In the latter case we characterize the coordinatization functor $\mathbf{M} : \mathcal{B} \longrightarrow \mathcal{L}$, a proper modeling functor.

2.2 Physical Semantics

The physical significance of this representation lies on the fact that the whole information content in a quantum events algebra is preserved by every covering Boolean system, qualified as a system of measurement localizations. The preservation property is established by the counit isomorphism. It is remarkable that the categorical notion of adjunction provides the appropriate formal tool for the formulation of invariant properties, giving rise to preservation principles of a physical character.

If we return to the intended representation, we realize that the surjective property of the counit guarantees that the Boolean domain covers, being themselves objects in the category of elements $\int(\mathbf{R}(L), B)$, cover entirely the quantum event algebra L, whereas its injective property guarantees that any two covers are compatible in a system of measurement localizations. Moreover, since the counit is also a homomorphism, it preserves the algebraic structure.

In the physical state of affairs, each cover corresponds to a set of Boolean events actualized locally in a measurement situation. The equivalence classes of Boolean domain covers represent quantum events in L through compatible coordinatizations by Boolean coefficients. Consequently, the structure of a quantum event algebra is being generated by the information that its structure preserving maps, encoded as Boolean covers in localization systems carry, as well as their compatibility relations. Most significantly, the same compatibility conditions provide the necessary relations for understanding a system of localizations for a quantum event algebra as a structure sheaf of Boolean coefficients associated with local contexts of measurement of observables.

Finally, the operational substantiation of the sheaf theoretical scheme of representation of quantum event algebras, is naturally provided by the application of Stone's representation theorem for Boolean algebras. According to this theorem, it is legitimate to replace Boolean algebras by fields of subsets of a space, playing the equivalent role of a local context for measurement. We note that in an equivalent topological interpretation, we could consider a local measurement space as a compact Hausdorff space, the compact open subsets of which are the maximal filters or the prime ideals of the underlying Boolean algebra. If we replace each Boolean algebra B in $\mathcal{B}$ by its set-theoretical representation $[\mathbf{\Sigma}, B_\Sigma]$, consisting of a local measurement space $\mathbf{\Sigma}$ and its local field of subsets B_Σ, it is possible to define local measurement space covers $(B_\Sigma, \psi_{B_\Sigma} : \mathbf{A}(B_\Sigma) \longrightarrow L)$ and corresponding space localization systems for a quantum event algebra L in $\mathcal{L}$. Again from local measurement space covers $(B_\Sigma, \psi_{B_\Sigma} : \mathbf{M}(B_\Sigma) \longrightarrow L)$ we may form their equivalence classes by using the colimits construction in the category of elements of $\mathbf{R}(L)$. Then by taking into account the conditions for compatibility on overlaps we can establish a full and faithful representation of quantum events in L by equivalence classes of local measurement space covers. Under these circumstances we may interpret these equivalence classes as the statistical experimental actualizations of the quantum events in $\mathcal{L}$. The pullback compatibility condition, which is in bijective correspondence with the one in $\mathcal{L}$ since it holds in a localization system, may be interpreted in the operational context as denoting that two local space representations of quantum events satisfy the compatibility condition on overlaps if and only if they support measurements of observables sharing the same experimental arrangement.

The above set-up could be the ideal starting point for the development of quantum probability, as a contextual probability theory on a structure sheaf of Boolean coefficients associated with local contexts of measurement of observables. Following this line of thought we may obtain important insights regarding probabilistic notions in quantum theory. In the prologue we have expressed the thesis that the logic of quantum propositions reflects literal ontological structures of the quantum domain of discourse. Of course the substantiation of this claim necessitates a thorough investigation of the truth values structures suited to express the quantum domain of discourse. In classical theories it is well known that the logic of events, or equivalently, propositions referring to the behavior of a classical system is characterized by valuations into the trivial Boolean two-valued truth values object $\mathbf{2} := \{0, 1\}$ stating that a proposition is true or false. Moreover the notion of probability has been designed as a superstructure on the truth values object **2**, expressing an ignorance of all the relevant details permitting a sharp true/false value assignment on the propositions of the theory. In this sense classical probabilities are not objective, but constitute a measure of ignorance. On the other side, in quantum theories a true/false value assignment is possible under the specification of a Boolean preparatory context of measurement and only after a measurement device provides a response as a result of its interaction with a quantum system. This state of affairs is at the heart of the problem of quantum measurement and makes necessary a careful re-examination of all the relevant assumptions concerning valuations of propositions that belong in quantum event structures. In this manner, if the truth values structures suited for valuations of quantum propositions prove to be different from the trivial two-valued classical ones, the notion of quantum probability acquires an objective meaning and its interpretation cannot be based on ignorance. Rather, it can be conceived as a measure of indistinguishability in the generalized topological sense of covering systems on categories, being in agreement at the same time, with the physical semantics of a sheaf theoretical interpretation. In the sequel, our objective will be exactly the investigation of the truth values structures suited to express valuations in quantum event algebras. Fortunately the categorical framework provides all the necessary means for this purpose.

3 The Subobject Functor

3.1 Existence of the Subobject Functor in $\mathcal{L}$

We have seen previously that the counit of the fundamental adjunction, restricted to localization systems of a quantum event algebra is a quantum algebraic isomorphism, iff the right adjoint functor is full and faithful. This fact is important, because it permits us to consider the category of quantum events algebras as a reflection of the category of presheaves of Boolean event algebras $\mathbf{Sets}^{B^{op}}$. It is methodologically appropriate to remind that the coordinatization functor, $\mathbf{M} : \mathcal{B} \longrightarrow \mathcal{L}$, is called a proper modeling functor iff the right adjoint functor of the established adjunction is full and faithful. In this sense, a proper modeling functor guarantees a full and faithful corresponding representation of quantum event algebras in terms of Boolean localization systems, such that the whole information content contained in a quantum structure of events is totally preserved by its covering systems of Boolean domain coordinatizations. Furthermore, the fact that $\mathcal{L}$ can be conceived as reflection of $\mathbf{Sets}^{B^{op}}$, secures that $\mathcal{L}$ is a complete category, as well as that, monic arrows are preserved by the right adjoint functor $\mathbf{R}$. Since $\mathcal{L}$ is a complete category, there is a terminal object for insertion of information related with the structure of events it represents, and also, there exist pullbacks securing the satisfaction of compatibility relations. In particular, since pullbacks of monic arrows also exist, there exists a subobject functor.

Definition: The **subobject functor** of the category of quantum event algebras is defined as:

$$\mathbf{Sub} : \mathcal{L}^{op} \to \mathbf{Sets}$$

This is, remarkably, a contravariant functor by pulling back. Composition of this functor with a proper modeling functor provides a presheaf functor in $\mathbf{Sets}^{B^{op}}$ as follows:

$$\mathbf{Sub} \circ \mathbf{M} : \mathcal{B}^{op} \to \mathcal{L}^{op} \to \mathbf{Sets}$$

In a compact notation we obtain:

$$\Theta(\mathbf{M}(-)) : \mathcal{B}^{op} \to \mathbf{Sets}$$

such that:

$$\mathcal{B}^{op} \ni B \mapsto [Dom(m) \hookrightarrow^{m} \mathbf{M}(B)] \in \mathbf{Sets}$$

where the range denotes an equivalence class of monic algebraic homomorphisms to $\mathbf{M}(B)$.

Definition: We say that $\Theta(\mathbf{M}(B))$, is the set of all **subobjects** of $\mathbf{M}(B)$ in the category of quantum events algebras $\mathcal{L}$.

Furthermore it is easy to verify that, $\Theta(\mathbf{M}(B))$ is a partially ordered set under inclusion of subobjects.

3.2 Representation of the Subobject Functor in $\mathcal{L}$

A natural question that arises in this context, is if it could be possible to represent the subobject functor by means of a quantum events algebra, Ω, that is an object of $\mathcal{L}$, which would play the role of a classifying object in $\mathcal{L}$. The representation of the subobject functor in $\mathcal{L}$, is significant from a physical perspective, since it would allow to associate the concrete classifying object Ω, with the functioning of a truth values object, in a sense similar to the role played by the two-valued Boolean object $\mathbf{2} := \{0, 1\}$, in characterization of the logic of propositions referring to the behavior of classical systems. In that case, subobjects of a quantum events algebra should be characterized in terms of characteristic functions, that take values, not in $\mathbf{2}$, but precisely, in the truth values object Ω in $\mathcal{L}$. Most importantly, in that case the category of quantum events algebras $\mathcal{L}$, is endowed with a subobject classifier, defined categorically as follows:

Definition: The **subobject classifier** of the category of quantum events algebras is a universal monic quantum homomorphism,

$$T := True : 1 \hookrightarrow \Omega$$

such that, to every monic arrow, $m : K \hookrightarrow L$ in $\mathcal{L}$, there is a unique characteristic arrow ϕ_m, which, with the given monic arrow m, forms a pullback diagram

$$\begin{array}{ccc} K & \xrightarrow{\ !\ } & 1 \\ \Big\downarrow m & & \Big\downarrow T \\ L & \xrightarrow{\ \phi_m\ } & \Omega \end{array}$$

This is equivalent to saying that every subobject of L in $\mathcal{L}$, is uniquely a pullback of the universal monic T.

Subobject Representation Theorem: The subobject functor can be represented in the category of quantum event algebras, $\mathcal{L}$, iff there exists a classifying object Ω in $\mathcal{L}$, that is, iff there exists an isomorphism for each Boolean domain object of the model category, as follows:

$$\Theta(\mathbf{M}(-)) \simeq \mathbf{R}(\Omega) := Hom_{\mathcal{L}}(\mathbf{M}(-), \Omega)$$

Proof: We have seen previously, that the counit of the adjunction, for each quantum event algebra object L of $\mathcal{L}$, is

$$\epsilon_L : \mathbf{LR}(L) \longrightarrow L$$

ϵ_L, being a a quantum algebraic isomorphism, guarantees a full and faithful representation of a quantum event algebra in terms of a covering or localization system consisting of Boolean domain coordinatizations via the action of a proper modeling functor. From the other side, we have seen that for any presheaf $\mathbf{P} \in \mathbf{Sets}^{B^{op}}$, the unit is defined as

$$\delta_{\mathbf{P}} : \mathbf{P} \longrightarrow \mathbf{RLP}$$

It is easy to see that if we consider as $\mathbf{P} \in \mathbf{Sets}^{B^{op}}$, the subobject functor $\Theta(\mathbf{M}(-))$ we obtain the following arrow:

$$\delta_{\Theta(\mathbf{M}(-))} : \Theta(\mathbf{M}(-)) \longrightarrow \mathbf{RL}\Theta(\mathbf{M}(-))$$

or equivalently:

$$\delta_{\Theta(\mathbf{M}(-))} : \Theta(\mathbf{M}(-)) \longrightarrow Hom_{\mathcal{L}}(\mathbf{M}(-), \mathbf{L}\Theta(\mathbf{M}(-))$$

Hence, by inspecting the unit of the adjunction arrow $\delta_{\Theta(\mathbf{M}(-))}$, we conclude that the subobject functor becomes representable in the category of quantum events algebras if we, equivalently, prove the following:

Subobject Unit Theorem: Given that the counit of the Boolean-quantum adjunction is an isomorphism, if the unit $\delta_{\Theta(\mathbf{M}(-))}$ is also an isomorphism, then the subobject functor becomes representable in $\mathcal{L}$, by the quantum events algebra classifying object Ω, characterized explicitly as, $\Omega := \mathbf{L}\Theta(\mathbf{M}(-))$, and thus, the category of quantum events algebras $\mathcal{L}$ is endowed with a subobject classifier, that functions as a quantum truth values object in $\mathcal{L}$. The inverse of the theorem also obviously holds.

Proof: Firstly, it is easy to notice that, in case, the unit $\delta_{\Theta(\mathbf{M}(-))}$ is an isomorphism, then the classifying quantum event algebra Ω, is constructed by application of the left adjoint functor, that is, as the colimit taken in the category of elements of the modeled subobject functor, according to:

$$\Omega := \mathbf{L}\Theta(\mathbf{M}(-))$$

We may verify the above immediately, by realizing that if the unit $\delta_{\Theta(\mathbf{M}(-))}$ is an isomorphism, then;

$$\Omega := \mathbf{L}\Theta(\mathbf{M}(-)) \simeq \mathbf{L}[\mathbf{R}\mathbf{L}\Theta(\mathbf{M}(-))] \simeq \mathbf{L}\mathbf{R}\Omega$$

is precisely an expression of the counit isomorphism for the quantum event algebra Ω.

Next, we will show explicitly that, if the unit $\delta_{\Theta(\mathbf{M}(-))}$ is an isomorphism, the category of quantum events algebras $\mathcal{L}$ is endowed with a subobject classifier, that functions as a quantum truth values object in $\mathcal{L}$. For this purpose, we consider a monic quantum homomorphism $l : K \hookrightarrow L$, denoting a subobject of L, in $\mathcal{L}$, and subsequently, we define a natural transformation in $\mathbf{Sets}^{\mathcal{B}^{op}}$:

$$\Phi_l : \mathbf{R}(L) \to \Theta(\mathbf{M}(-))$$

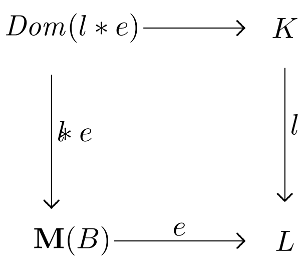
specified for each Boolean event algebra B, in $\mathcal{B}$ by:

$$[\Phi_l]_B : \mathbf{R}(L)(B) \to \Theta(\mathbf{M}(B))$$

such that for an element e in $\mathbf{R}(L)(B)$, we have:

$$[\Phi_l]_B(e) := l * e$$

where the monic arrow $l * e$, denotes the pullback of l along e in $\mathcal{L}$, as in the following diagram:



Furthermore, if we take into account the subobjects of the terminal object 1 in $\mathcal{L}$, denoted by the uniquely defined monic quantum algebraic homomorphisms $\kappa : K \hookrightarrow 1$, we may define a natural transformation in $\mathbf{Sets}^{\mathcal{B}^{op}}$:

$$\Upsilon : \mathbf{R}(1) \to \Theta(\mathbf{M}(-))$$

specified for each Boolean event algebra B, in $\mathcal{B}$ by:

$$[\Upsilon]_B : \mathbf{R}(1)(B) \to \Theta(\mathbf{M}(B))$$

such that for the unique element $\alpha(B)$ in $\mathbf{R}(1)(B)$, we have:

$$[\Upsilon]_B(\alpha(B)) := id_{\mathbf{M}(B)}$$

At a next stage, we may combine the natural transformations, defined previously, in order to obtain, for each monic quantum algebraic homomorphism $l : K \hookrightarrow L$, the following commutative diagram in $\mathbf{Sets}^{B^{op}}$, that by construction is a pullback as it can be easily seen.

$$\begin{array}{ccc} \mathbf{R}(K) & \longrightarrow & \mathbf{R}(1) \\ \Big\downarrow \mathbf{R}(l) & & \Big\downarrow \Upsilon \\ \mathbf{R}(L) & \xrightarrow{\Phi_l} & \Theta(\mathbf{M}(-)) \end{array}$$

Moreover we consider the arrows obtained by composing, the arrows $[\Phi_l]$ and $[\Upsilon]$, with the unit isomorphism $\delta_{\Theta(\mathbf{M}(-))}$ as follows:

$$\delta_{\Theta(\mathbf{M}(-))} \circ \Phi_l : \mathbf{R}(L) \to \Theta(\mathbf{M}(-)) \to \mathbf{R}(\Omega)$$

$$\delta_{\Theta(\mathbf{M}(-))} \circ \Upsilon : \mathbf{R}(1) \to \Theta(\mathbf{M}(-)) \to \mathbf{R}(\Omega)$$

Concerning the latter composite arrow, we may define:

$$\mathbf{R}(T) := \delta_{\Theta(\mathbf{M}(-))} \circ \Upsilon : \mathbf{R}(1) \hookrightarrow \mathbf{R}(\Omega)$$

and using the fact that the right adjoint functor is full and faithful, by the counit isomorphism, we obtain a uniquely defined monic quantum homomorphism

$$T := true : 1 \hookrightarrow \Omega$$

The previous pullback diagram, together with the composite arrows $\delta_{\Theta(\mathbf{M}(-))} \circ \Phi_l$, $\delta_{\Theta(\mathbf{M}(-))} \circ \Upsilon$, facilitate the immediate verification of the claim, as follows: We wish to show that, if the unit of the adjunction $\delta_{\Theta(\mathbf{M}(-))}$ is an isomorphism, then

$$\mathbf{Sub}(L) \simeq Hom_L(L, \Omega)$$

such that that the category of quantum event algebras $\mathcal{L}$ is endowed with a subobject classifier. So we define a map

$$\varpi_L : \mathbf{Sub}(L) \to Hom_L(L, \Omega)$$

such that the element e of the range, defined by:

$$\mathbf{Sub}(L) \ni l \mapsto [e : L \to \Omega]$$

is specified by the requirement:

$$\mathbf{R}(e) = \delta_{\Theta(\mathbf{M}(-))} \circ \Phi_l : \mathbf{R}(L) \to \mathbf{R}(\Omega)$$

Hence, for the subobject l of L, in $\mathcal{L}$, and the element e of $Hom_L(L, \Omega)$, with $e = \varpi_L(l)$, we obtain the following pullback diagram in $\mathbf{Sets}^{B^{op}}$,

$$\begin{array}{ccc} \mathbf{R}(K) & \longrightarrow & \mathbf{R}(1) \\ \Big\downarrow \mathbf{R}(l) & & \Big\downarrow \mathbf{R}(T) \\ \mathbf{R}(L) & \xrightarrow{\mathbf{R}(e)} & \mathbf{R}(\Omega) \end{array}$$

Using again the argument of the counit isomorphism, that specifies the right adjoint as a full and faithful functor, we obtain a pullback diagram in $\mathcal{L}$:

$$\begin{array}{ccc} K & \xrightarrow{!} & 1 \\ \Big\downarrow l & & \Big\downarrow T \\ L & \xrightarrow{e} & \Omega \end{array}$$

Moreover, it is straightforward to show that, $\varpi_L : \mathbf{Sub}(L) \to Hom_L(L, \Omega)$ is 1-1 and epi. Thus, we have verified that, if the unit $\delta_{\Theta(\mathbf{M}(-))}$ is an isomorphism, then the category of quantum events algebras, $\mathcal{L}$, is endowed with a subobject classifier, according to the above pullback diagram. Precisely stated, the subobject classifier in $\mathcal{L}$, is specified by the monic quantum algebraic homomorphism $T := True : 1 \hookrightarrow \Omega$, such that, $\mathbf{R}(T) := \delta_{\Theta(\mathbf{M}(-))} \circ \Upsilon$. It is easy to notice that the inverse obviously holds. As a consequence we conclude that the diagram below;

$$
\begin{array}{ccc}
Dom(\lambda) & \xrightarrow{\quad !\quad} & \mathbf{LR}(1) \\
\Big\downarrow \lambda & & \Big\downarrow \mathbf{L}\Upsilon = T \\
\mathbf{M}(B) & \xrightarrow{\varpi_{\mathbf{M}(B)}(\lambda) = \zeta} & \mathbf{L}\Theta(\mathbf{M}(-))
\end{array}
$$

is a pullback square for each quantum algebraic homomorphism

$$\varpi_{\mathbf{M}(B)}(\lambda) = \zeta : \mathbf{M}(B) \to \mathbf{L}\Theta(\mathbf{M}(-))$$

from a Boolean domain modelled object, such that λ is a subobject of $\mathbf{M}(B)$. It is important to notice, that in this case:

$$\varpi_{\mathbf{M}(B)}(\lambda) = \delta_{\Theta(\mathbf{M}(B))}(\lambda)$$

and since this holds for arbitrary subobject λ of $\mathbf{M}(B)$, we have

$$\varpi_{\mathbf{M}(B)} = \delta_{\Theta(\mathbf{M}(B))}$$

The above completes the proof of the theorem.

It is instructive to remark the significance of the subobject representation theorem, in relationship with the notion of **quantum sets**. This notion can acquire a precise meaning in the present framework, if we remind the analogy with classical sets. We notice that classical sets are specified by the rule which states that the subsets of any set are represented as characteristic functions into **2**. By analogy, we may say that quantum sets admit a specification by the rule according to which, the subsets of a quantum set are represented as characteristic arrows in the quantum truth values object Ω. We may easily associate a quantum set, specified as above, by the colimit in the category of elements of a presheaf of local spaces, where each local space is the representation of a Boolean event algebra using Stone's representation theorem, as has already been explained in Section 2.2.

4 Tensor Product Representation of the Quantum Subobject Classifier

4.1 Quantum Truth Values Representation

The quantum truth values object Ω, has been characterized as $\Omega = \mathbf{L}\Theta(\mathbf{M}(-))$, that is, as the colimit taken in the category of elements of the modeled subobject functor. In what follows, we are going to exploit the categorical construction of the colimit defined above, as a coequalizer of a coproduct. For this purpose, it is necessary to consider the category of elements of $\Theta(\mathbf{M}(-))$, denoted by $\int(\boldsymbol{\Theta}(\mathbf{M}(-)), \mathcal{B})$. Its objects are all pairs $(B, \varphi_{\mathbf{M}(B)})$, where $\varphi_{\mathbf{M}(B)}$ is a subobject of $\mathbf{M}(B)$, that is, a monic quantum homomorphism in $\mathbf{M}(B)$. The morphisms of the category of elements of $\Theta(\mathbf{M}(-))$ are given by the arrows $(\acute{B}, \phi_{\mathbf{M}(\acute{B})}) \longrightarrow (B, \varphi_{\mathbf{M}(B)})$, namely they are those morphisms $u : \acute{B} \longrightarrow B$ of $\mathcal{B}$ for which $\varphi_{\mathbf{M}(B)} * u = \phi_{\mathbf{M}(\acute{B})}$, where $\varphi_{\mathbf{M}(B)} * u$ denotes the pullback of the subobject of $\mathbf{M}(\acute{B})$, $\varphi_{\mathbf{M}(B)}$, along u.

Tensor Product Representation Theorem: The quantum truth values object Ω, given by the colimit in the category of elements of the modeled subobject functor, $\Omega = \mathbf{L}\Theta(\mathbf{M}(-))$, admits the following coequalizer representation in tensor product form:

$$\coprod_{v:\acute{B}\to B}\mathbf{M}(\acute{B}) \underset{\eta}{\overset{\zeta}{\rightrightarrows}} \coprod_{(B,\varphi_{\mathbf{M}(B)}))}\mathbf{M}(B) \xrightarrow{\chi} \Theta(\mathbf{M}(-))\otimes_{\mathcal{B}}\mathbf{M}$$

In the diagram above the second coproduct is over all the objects $(B, \varphi_{\mathbf{M}(B)})$ with $\varphi_{\mathbf{M}(B)} \in \Theta(\mathbf{M}(B))$ of the category of elements, while the first coproduct is over all the maps $v : (\acute{B}, \phi_{\mathbf{M}(\acute{B})}) \longrightarrow (B, \varphi_{\mathbf{M}(B)})$ of that category, so that $v : \acute{B} \longrightarrow B$ and the condition $\varphi_{\mathbf{M}(B)} * u = \phi_{\mathbf{M}(\acute{B})}$ is satisfied.

The proof of the theorem above, makes use of standard category theoretical arguments, regarding the representation of a colimit as a coequalizer of coproduct [5]. For physical purposes, it is essential to describe the truth values of $\Omega = \Theta(\mathbf{M}(-))\otimes_{\mathcal{B}}\mathbf{M}$ explicitly, and then, demonstrate how they can be used for valuations of propositions in typical quantum measurement situations.

Quantum Truth Values Theorem: The quantum truth values in Ω admit a tensor product representation in the following form:

$$[\delta_{\Theta(\mathbf{M}(B))}]^{\varphi_{\mathbf{M}(B)}}(b) = \varphi_{\mathbf{M}(B)} \otimes b$$

where

$$[\varphi_{\mathbf{M}(B)} * v] \otimes \acute{b} = \varphi_{\mathbf{M}(B)} \otimes v(\acute{b}), \quad \varphi_{\mathbf{M}(B)} \in \Theta(\mathbf{M}(B), \acute{b} \in \mathbf{M}(\acute{B}), v : \acute{B} \rightarrow B, v(\acute{b}) = b$$

and a Boolean cover of the truth values object in a localization system, using the unit of the adjunction, is expressed as:

$$\delta_{\Theta(\mathbf{M}(B))}(\varphi_{\mathbf{M}(B)}) = [\delta_{\Theta(\mathbf{M}(B))}]^{\varphi_{\mathbf{M}(B)}}$$

Proof: First of all, it is essential to prove that, the set $\Theta(\mathbf{M}(-)) \otimes_{\mathcal{B}} \mathbf{M}$ endowed with the relevant structure, is actually a quantum event algebra, for every B in $\mathcal{B}$, so that, its elements can be interpreted as quantum truth values.

According to the coequalizer representation of the colimit established previously, if we consider its interpretation in the category of **Sets**, the coproduct $\amalg_{\varphi_{\mathbf{M}(B)}} \mathbf{M}(B)$ is a coproduct of sets, which is equivalent to the product $\Theta(\mathbf{M}(B)) \times \mathbf{M}(B)$ for $B \in \mathcal{B}$. The coequalizer is thus the definition of the tensor product $\Theta(\mathbf{M}(-)) \otimes \mathcal{A}$ of the set valued functors:

$$\Theta(\mathbf{M}(-)) : \mathcal{B}^{op} \longrightarrow \mathbf{Sets}, \qquad \mathbf{M} : \mathcal{B} \longrightarrow \mathbf{Sets}$$

$$\coprod_{B,\acute{B}} \Theta(\mathbf{M}(B)) \times Hom(\acute{B}, B) \times \mathbf{M}(\acute{B}) \quad \underset{\eta}{\overset{\zeta}{\rightrightarrows}}$$

$$\underset{\eta}{\overset{\zeta}{\rightrightarrows}} \quad \coprod_{B} \Theta(\mathbf{M}(B)) \times \mathbf{M}(B) \overset{\chi}{\rightarrow} \quad \Theta(\mathbf{M}(-)) \otimes \mathbf{M}(B)$$

According to the preceding diagram for elements $\varphi_{\mathbf{M}(B)} \in \Theta(\mathbf{M}(B))$, $v : \acute{B} \rightarrow B$ and $\acute{q} \in \mathbf{M}(\acute{B})$ the following equations hold:

$$\zeta(\varphi_{\mathbf{M}(B)}, v, \acute{q}) = (\varphi_{\mathbf{M}(B)} * v, \acute{q}), \qquad \eta(\varphi_{\mathbf{M}(B)}, v, \acute{q}) = (\varphi_{\mathbf{M}(B)}, v(\acute{q}))$$

symmetric in $\Theta(\mathbf{M}(B))$ and $\mathbf{M}$. Hence the elements of the set $\Theta(\mathbf{M}(B)) \otimes_{\mathcal{B}} \mathbf{M}$ are all of the form $\chi(\varphi_{\mathbf{M}(B)}), q)$. This element can be written as

$$\chi(\varphi_{\mathbf{M}(B)}), q) = \varphi_{\mathbf{M}(B)} \otimes q, \quad \varphi_{\mathbf{M}(B)} \in \Theta(\mathbf{M}(B)), q \in \mathbf{M}(B)$$

Thus if we take into account the definitions of ζ and η above, we obtain

$$[\varphi_{\mathbf{M}(B)}*v]\otimes\acute{q} = \varphi_{\mathbf{M}(B)}\otimes v(\acute{q}), \quad \varphi_{\mathbf{M}(B)} \in \Theta(\mathbf{M}(B), \acute{q} \in \mathbf{M}(\acute{B}), v : \acute{B} \longrightarrow B$$

We conclude that the set $\Theta(\mathbf{M}(B))\otimes_{\mathcal{B}}\mathbf{M}$ is actually the quotient of the set $\amalg_B \Theta(\mathbf{M}(B)) \times \mathbf{M}(B)$ by the equivalence relation generated by the above equations. Furthermore, if we define:

$$[\varphi_{\mathbf{M}(B)} * v] = \phi_{\mathbf{M}(\acute{B})}$$

$$v(\acute{q}) = q$$

where $\phi_{\mathbf{M}(\acute{B})}$ is a subobject of $\mathbf{M}(\acute{B})$ and $q \in \mathbf{M}(B)$, we obtain the equations:

$$\phi_{\mathbf{M}(\acute{B})} \otimes \acute{q} = \varphi_{\mathbf{M}(B)} \otimes q$$

At a next stage, since pullbacks exist in $\mathcal{L}$, we may consider the arrows $h : \mathbf{M}(D) \to \mathbf{M}(B)$ and $\acute{h} : \mathbf{M}(D) \to \mathbf{M}(\acute{B})$ and the following pullback diagram in $\mathcal{L}$:

$$\begin{array}{ccc} \mathbf{M}(D) & \xrightarrow{h} & \mathbf{M}(B) \\ \downarrow{\scriptstyle \acute{h}} & & \downarrow \\ \mathbf{M}(\acute{B}) & \longrightarrow & L \end{array}$$

such that the relations that follow are satisfied: $h(d) = q$, $\acute{h}(d) = \acute{q}$ and $\varphi_{\mathbf{M}(B)} \otimes h = \phi_{\mathbf{M}(\acute{B})} \otimes \acute{h}$. Then we obtain:

$$\varphi_{\mathbf{M}(B)}\otimes q = \varphi_{\mathbf{M}(B)}\otimes h(d) = [\varphi_{\mathbf{M}(B)}*h]\otimes d = [\phi_{\mathbf{M}(\acute{B})}*\acute{h}]\otimes d = \phi_{\mathbf{M}(\acute{B})}\otimes\acute{h}(d) = \phi_{\mathbf{M}(\acute{B})}\otimes\acute{q}$$

We may further define:

$$\varphi_{\mathbf{M}(B)} * h = \phi_{\mathbf{M}(\acute{B})} * \acute{h} = \varepsilon_{\mathbf{M}(D)}$$

Then, it is obvious that:

$$\varphi_{\mathbf{M}(B)} \otimes q = \varepsilon_{\mathbf{M}(D)} \otimes d$$

$$\phi_{\mathbf{M}(\acute{B})} \otimes \acute{q} = \varepsilon_{\mathbf{M}(D)} \otimes d$$

It is then evident that we may define a partial order on the set $\Theta(\mathbf{M}(B)) \otimes_{\mathcal{B}} \mathbf{M}$ as follows:

$$\varphi_{\mathbf{M}(B)} \otimes b \leq \varrho_{\mathbf{M}(C)} \otimes c$$

iff there exist quantum algebraic homomorphisms $\beta : \mathbf{M}(D) \to \mathbf{M}(B)$ and $\gamma : \mathbf{M}(D) \to \mathbf{M}(C)$, and some d_1, d_2 in $\mathbf{M}(D)$, such that: $\beta(d_1) = b$, $\gamma(d_2) = c$, and $\varphi_{\mathbf{M}(B)} * \beta = \varrho_{\mathbf{M}(C)} * \gamma = \varepsilon_{\mathbf{M}(D)}$. Thus we obtain:

$$\varphi_{\mathbf{M}(B)} \otimes b = \varepsilon_{\mathbf{M}(D)} \otimes d_1$$

$$\varrho_{\mathbf{M}(C)} \otimes c = \varepsilon_{\mathbf{M}(D)} \otimes d_2$$

We conclude that:

$$\varphi_{\mathbf{M}(B)} \otimes b \leq \varrho_{\mathbf{M}(C)} \otimes c$$

iff

$$\varepsilon_{\mathbf{M}(D)} \otimes d_1 \leq \varepsilon_{\mathbf{M}(D)} \otimes d_2 \Longleftrightarrow d_1 \leq d_2$$

The set $\Theta(\mathbf{M}(B)) \otimes_{\mathcal{B}} \mathbf{M}$ may be further endowed with a maximal element which admits the following presentations:

$$\mathbf{1} = \varepsilon_{\mathbf{M}(Z)} \otimes 1 \qquad \forall \varepsilon_{\mathbf{M}(Z)} \in \Theta(\mathbf{M}(Z))$$

$$\mathbf{1} = id_{\mathbf{M}(B)} \otimes b \qquad \forall b \in \mathbf{M}(B)$$

and an orthocomplementation operator:

$$[\varepsilon_{\mathbf{M}(Z)} \otimes z]^{\star} = \varepsilon_{\mathbf{M}(Z)} \otimes z^{\star}$$

Then it is easy to verify that the set $\Theta(\mathbf{M}(-)) \otimes_{\mathcal{B}} \mathbf{M}$ endowed with the prescribed operations is actually a quantum event algebra, for every B in $\mathcal{B}$. Consequently, the truth values in Ω are represented in the form

$$[\delta_{\Theta(\mathbf{M}(B))}]^{\varphi_{\mathbf{M}(B)}}(b) = \varphi_{\mathbf{M}(B)} \otimes b$$

where

$$[\varphi_{\mathbf{M}(B)} * v] \otimes \acute{b} = \varphi_{\mathbf{M}(B)} \otimes v(\acute{b}), \quad \varphi_{\mathbf{M}(B)} \in \Theta(\mathbf{M}(B), \acute{b} \in \mathbf{M}(\acute{B}), v : \acute{B} \to B, v(\acute{b}) = b$$

and a Boolean cover of the truth values object in a localization system, using the unit of the adjunction, is expressed as:

$$\delta_{\Theta(\mathbf{M}(B))}(\varphi_{\mathbf{M}(B)}) = [\delta_{\Theta(\mathbf{M}(B))}]^{\varphi_{\mathbf{M}(B)}}$$

4.2 Criterion of Truth in the Quantum Regime

In order to understand the functioning of the quantum truth values object Ω in the category of quantum event algebras, we need to establish a criterion of truth, that can be used for valuations of propositions describing quantum events.

First of all, it is necessary to provide a definition of the value *true*. For this purpose we remind the following:

$$\mathit{True} : \mathbf{L}R(1) \hookrightarrow \mathbf{L}\Theta(\mathbf{M}(-))$$

$$\Upsilon : \mathbf{R}(1) \to \Theta(\mathbf{M}(-))$$

specified for each Boolean event algebra B, in $\mathcal{B}$ by:

$$[\Upsilon]_B : \mathbf{R}(1)(B) \to \Theta(\mathbf{M}(B))$$

such that for the unique element $\alpha(B)$ in $\mathbf{R}(1)(B)$, we have:

$$[\Upsilon]_B(\alpha(B)) := id_{\mathbf{M}(B)}$$

Then by the commutativity of the diagram below

$$\begin{array}{ccc} & \mathbf{M}(B) & \\ {\scriptstyle [\delta_{\mathbf{R}(1)}]^{\alpha(B)}} \swarrow & & \searrow {\scriptstyle [\delta_{\Theta(\mathbf{M}(B))}]^{id_{\mathbf{M}(B)}}} \\ \mathbf{LR}(1) & \xrightarrow{\mathit{True}} & \mathbf{L}\Theta(\mathbf{M}(-)) \end{array}$$

we may easily conclude that

$$\mathbf{1} = id_{\mathbf{M}(B)} \otimes b = \mathit{True}(([\delta_{\mathbf{R}(1)}]^{\alpha(B)})(b)) := \mathit{true}$$

Having specified the value *true* of the quantum truth values object Ω, we define the notion of truth with respect to the category of quantum event algebras as follows:

Criterion of Truth: The notion of truth with respect to the category of quantum events algebras is specified as follows:

$$[\delta_{\Theta(\mathbf{M}(B))}]^{\varphi_{\mathbf{M}(B)}}(b) = \varphi_{\mathbf{M}(B)} \otimes b = true \qquad \textbf{iff} \qquad b \in Dom(\varphi_{\mathbf{M}(B)})$$

where

$$[\varphi_{\mathbf{M}(B)} * v] \otimes \acute{b} = \varphi_{\mathbf{M}(B)} \otimes v(\acute{b}), \quad \varphi_{\mathbf{M}(B)} \in \Theta(\mathbf{M}(B), \acute{b} \in \mathbf{M}(\acute{B}), v : \acute{B} \to B, v(\acute{b}) = b$$

and a Boolean cover of the truth values object in a localization system, using the unit of the adjunction, is expressed as:

$$\delta_{\Theta(\mathbf{M}(B))}(\varphi_{\mathbf{M}(B)}) = [\delta_{\Theta(\mathbf{M}(B))}]^{\varphi_{\mathbf{M}(B)}}$$

Furthermore according to the pullback diagram below, $\varphi_{\mathbf{A}(B)} = l \star e$, for a subobject of a quantum event algebra $l : K \hookrightarrow L$, and a Boolean domain cover $e : \mathbf{A}(B) \to L$.

$$\begin{array}{ccc} Dom(l * e) & \longrightarrow & K \\ \Big\downarrow{\scriptstyle l * e} & & \Big\downarrow{\scriptstyle l} \\ \mathbf{A}(B) & \xrightarrow{\ e\ } & L \end{array}$$

We conclude that, the characteristic function of a subobject of a quantum event algebra $l : K \hookrightarrow L$, is specified as an equivalence class of pullbacks of the subobject along its restrictions on a localization system of compatible Boolean domain covers. In particular, if the Boolean covers are monic morphisms, each pullback is expressed as the intersection of the subobject with the corresponding cover in the Boolean localization system. Moreover, the value $\mathbf{1} = true$ in Ω is assigned to all those b that belong in $Dom(\varphi_{\mathbf{M}(B)})$ according to the pullback diagram above, or equivalently, to all those b, that belong to the restrictions of a subobject of a quantum event algebra along the covers of a localization system of the latter. We emphasize that the specification of the classifying object Ω, characterized by the truth values $[\delta_{\Theta(\mathbf{M}(B))}]^{\varphi_{\mathbf{M}(B)}}(b) = \varphi_{\mathbf{M}(B)} \otimes b$, implies a localization of truth in the quantum regime, with respect to Boolean points belonging to a covering system of a quantum events algebra.

Furthermore, we may identify the value $\mathbf{1} = \mathit{true}$ in Ω, with the maximal element of Ω, as follows:

$$\mathbf{1} = \varepsilon_{\mathbf{M}(Z)} \otimes 1 \qquad \forall \varepsilon_{\mathbf{M}(Z)} \in \Theta(\mathbf{M}(Z))$$

The identity above implies that $\mathbf{1}$ belongs in $\varepsilon_{\mathbf{M}(Z)}$, for all $\varepsilon_{\mathbf{M}(Z)} \in \Theta(\mathbf{M}(Z))$. Thus, the quantum truth values can be characterized as equivalence classes of filters of covers in a Boolean localization system, and the maximal value true corresponds to an equivalence class of ultrafilters. Using this observation, it is straightforward to state the truth value criterion, in case of monic covers in a Boolean localization system of a quantum events algebra as follows:

Criterion of Truth for Monic Boolean Covers:

$$[\delta_{\Theta(\mathbf{M}(C))}]^{\varphi_{\mathbf{M}(C)}}(c) = \varphi_{\mathbf{M}(C)} \otimes c = \mathit{true} \qquad \textbf{iff} \qquad c = \Omega_{B,C}(1)$$

that is, if and only if, c is in the image of the maximal element in $\mathbf{M}(B)$, via the isomorphism pasting map $\Omega_{B,C}$, where;

$$\Omega_{B,\acute{B}} : \rho_{\mathbf{M}(B)}(\mathbf{M}(B) \bigcap \mathbf{M}(\acute{B})) \longrightarrow \varrho_{\mathbf{M}(\acute{B})}(\mathbf{M}(B) \bigcap \mathbf{M}(\acute{B}))$$

according to the pullback diagram below:

$$\begin{array}{ccc} \mathbf{M}(B)\bigcap\mathbf{M}(\acute{B}) & \xrightarrow{\varrho_{\mathbf{M}(\acute{B})}} & \mathbf{M}(B) \\ \downarrow{\scriptstyle \rho_{\mathbf{M}(B)}} & & \downarrow{\scriptstyle \psi_B} \\ \mathbf{M}(\acute{B}) & \xrightarrow{\psi_{\acute{B}}} & L \end{array}$$

is defined by;

$$\Omega_{B,\acute{B}} = \rho_{\mathbf{M}(B)} \circ \varrho_{\mathbf{M}(\acute{B})}{}^{-1}$$

5 Application in Quantum Measurement

The use of the quantum truth values object Ω, in conjunction with the language of Boolean reference frames, for valuations of propositions related with the behavior of quantum systems, provide a powerful formal mechanism capable of resolving the problems associated with the quantum regime of description of reality. The provocative claim that follows from the topos-theoretic logical framework developed in the previous Sections, has to do with the realization that the quantum measurement problem, as well as its associated problem of quantum state reduction, are not related with any actual physical mechanism, but on the contrary, are logical consequences of the use of an inappropriate classifying object in the category of quantum event algebras. More concretely, the two elements Boolean algebra **2**, used for valuations of propositions related with the behavior of a classical system, cannot be also used for valuations of propositions of quantum systems, because it cannot play the role of a subobject classifier in the category of quantum event algebras, as in the classical case. Thus, the conceptual essence of existence of a quantum truth values object Ω in the category of quantum event algebras, as specified concretely in the previous Section, is associated with the fact that Ω constitutes the appropriate quantum algebra or quantum logic for valuations of propositions describing the behavior of a quantum system, in correspondence with the classical case, where the two elements Boolean algebra **2** is properly used.

In this sense, propositions associated with the description of the behavior of a quantum system in various contexts of observation, identified by Boolean covers in localization systems of a quantum event algebra, are being properly assigned truth values in Ω, by means of:

$$[\delta_{\Theta(\mathbf{M}(B))}]^{\varphi_{\mathbf{M}(B)}}(b) = \varphi_{\mathbf{M}(B)} \otimes b$$

where

$$[\varphi_{\mathbf{M}(B)} * v] \otimes \acute{b} = \varphi_{\mathbf{M}(B)} \otimes v(\acute{b}), \quad \varphi_{\mathbf{M}(B)} \in \Theta(\mathbf{M}(B), \acute{b} \in \mathbf{M}(\acute{B}), v : \acute{B} \to B, v(\acute{b}) = b$$

and furthermore b may be though as representing the element (for instance projection operator) that identifies a proposition p in the context of $\mathbf{M}(B)$. It is instructive to notice that the description of reality in the quantum regime, by means of the quantum truth values in Ω, is relativized and localized with respect to Boolean reference frames.

More specifically, a **complete description of reality** is characterized by the requirement that:

$$\mathit{true} = \mathbf{1} = \varepsilon_{\mathbf{M}(Z)} \otimes 1 \qquad \forall \varepsilon_{\mathbf{M}(Z)} \in \Theta(\mathbf{M}(Z))$$

$$true = \mathbf{1} = id_{\mathbf{M}(B)} \otimes b \qquad \forall b \in \mathbf{M}(B)$$

Now, we are in a position to interpret from our perspective, a typical measurement situation referring to a quantum system prepared to pass through a slit, where a counter has been put to record by clicking, the passage through the slit. If we denote a Boolean domain preparation context, that contains both the measuring apparatus as well as the system observed, by $\mathbf{M}(B)$, then we may form the propositions: $\langle p \rangle :=$*counter clicks*, $\langle q \rangle :=$*system passes through the slit*, as well as, the composite proposition $\langle$ *Counter clicks* $\Rightarrow$ *system passes through the slit* $\rangle := \langle\ p \rightarrow q\ \rangle$

The proposition $\langle\ p \rightarrow q\ \rangle$ is assigned the value *true* in Ω, expressing a complete description of the state of affairs. Moreover, in every Boolean cover of a localization system, the maximal element corresponds to $\langle p \rightarrow q \rangle = \langle \neg p \bigvee q \rangle$. We notice that, the above is not enough to infer that $\langle q \rangle$ is true. In order to infer the above, we need to use the Boolean reference frame that contains only the measuring apparatus, being obviously a subobject of the preparatory Boolean frame $\mathbf{M}(B)$. If we denote by $\varepsilon_{\mathbf{M}(B)}$, the monic that corresponds to the specified subobject, we easily deduce that

$$\varepsilon_{\mathbf{M}(B)} \otimes p = true$$

since obviously p is contained in $Dom(\varepsilon_{\mathbf{M}(B)})$, and for notational convenience we have identified the proposition $\langle p \rangle$ with its corresponding element p in $\mathbf{M}(B)$. Now, it is evident that, with respect to the Boolean frame containing only the apparatus, we can say that the proposition $\langle q \rangle :=$*system passes through the slit* is true. In this perspective, the existence of a measuring apparatus plays the role of an ultrafilter in the preparatory context $\mathbf{M}(B)$, transforming truth with respect to Ω, into two-valued truth with respect to $\mathbf{2}$. This is effectuated by the fact that the monic subobject of $\mathbf{M}(B)$, containing only the measuring apparatus, is equivalent to a classical valuation map $\mathbf{A}(B) \rightarrow \mathbf{2}$, as can be easily seen from the ultrafilter characterization.

Thus, the role of the measuring apparatus in a typical measurement situation as the above, provides precisely the means for the transformation of the quantum truth values object Ω, into the classical object $\mathbf{2}$. In this sense, we conclude that the physics of the apparatus specifies a Boolean frame, in which a unique decomposition of the proposition $p \rightarrow q$ is possible, such that the proposition q is legitimately assigned the value *true*, only with respect to that frame, namely an ultrafilter in $\mathbf{M}(B)$. From that perspective, Kochen-Specker theorem is an expression of the fact that a unique apparatus cannot reduce all propositions in a quantum event algebra to classical two-valued truth, or equivalently, that truth/false assignments cannot be performed in a global quantum logic with respect to a unique Boolean cover. The latter realization justifies again, *a posteriori*, the use of variable Boolean contexts interlocking non-trivially in localization systems of a quantum algebra of events, interpreted as Boolean reference frames in a sheaf-theoretic environment. We may further argue that, the variation of the base Boolean events algebra is actually arising from any operational procedure aiming to fix the state of a quantum system, and corresponds in this sense, to the variation of all possible Boolean preparatory contexts for measurement. In this setting, the notion of truth is adjacent to equivalent classes of compatible filters, instantiating subobjects of preparatory contexts for measurement, whereas the value *true*, that provides a complete description of reality, is prescribed by the rule $true = \mathbf{1} = \varepsilon_{\mathbf{M}(B)} \otimes 1 \qquad \forall \varepsilon_{\mathbf{M}(B)} \in \Theta(\mathbf{M}(B))$.

6 Epilogue

In this paper, we have proposed a category-theoretic framework for the interpretation of quantum events structures and their logical semantics. The scheme of interpretation is based on the existence of the Boolean-quantum adjunction. From that adjunction, characterized by means of the counit and unit natural transformations, we have constructed a sheaf-theoretic representation of quantum events algebras in terms of Boolean localization systems, as well as, a quantum subobject classifier, that plays the role of a classifying object in the quantum universe of discourse. In this sense, the Boolean-quantum adjunction incorporates both, the semantics of representation of quantum logics as sheaves of local Boolean coefficients, and, the semantics of truth values encoded in the specification of a classifying object in the category of quantum logics.

Thus, from a physical viewpoint, the Boolean-quantum adjunction stands as a theoretical platform for decoding the global structural information contained in quantum algebras of events via processes of localization in Boolean reference frames, realized as physical contexts for measurement of observables, and subsequent processes of information classification in terms of truth values. The functioning of this platform is based on the establishment of a bidirectional dependence between the Boolean and quantum structural levels of events in local or partial congruence. Most significantly, the dependence takes place through the topos-theoretic universe of sheaves of sets over the Boolean points of quantum events algebras, where these generalized points play the role of local Boolean covers, effectuating in this sense the idea of partial congruence mentioned above. Of course, the sheaf-theoretic requirements secure the compatibility of the Booleanized information in the overlapping regions of physical measurement contexts.

Additionally the global closure of this bidirectional dependence, is necessary to be constrained to obey certain conditions, such that its total constitutive information content, unfolded in the multitude of local Boolean reference frames, is both, preserved and, coherently organized in a logical manner. Remarkably, the necessary and sufficient conditions for both of these requirements, that is:

[i] preservation of the quantum information content in Boolean localization systems, and,

[ii] logical classification of quantum information by means of truth values, is supplied by the Boolean-quantum adjunctive correspondence itself, via the counit and unit natural transformations respectively.

More concretely, regarding the first condition, we conclude that it is satisfied if the counit of the adjunction is an isomorphism for each quantum events algebra. In this case, there exists a full and faithful sheaf-theoretic representation of quantum events algebras in the descriptive terms of Boolean covering systems, characterized as Boolean localization systems of measurement. Regarding the second condition, we conclude that it is satisfied, given the validity of the first condition, if the unit of the adjunction is an isomorphism for the subobject functor. In this case respectively, the subobject functor is representable in the category of quantum events algebras by a classifying object in that category, called subobject classifier. The classifying object plays the role of a quantum truth values object and may be legitimately used for valuations of quantum propositions, in exact correspondence with the use of the two-valued Boolean object, used for valuations of classical propositions. Moreover, the quantum subobject classifier provides the key logical device for the analysis of typical quantum measurement situations, providing a criterion of truth for a complete description of reality in the quantum regime.

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