E. VASSILIOU - M. PAPATRIANTAFYLLOU

First Steps to Smooth Manifolds.
An Undergraduate Course

ΔΕΙΓΜΑ ΠΡΩΤΗΣ ΓΡΑΦΗΣ ΒΙΒΛΙΟΥ
ΥΠΟ ΠΡΟΕΤΟΙΜΑΣΙΑ
ΥΠΟΚΕΙΤΑΙ ΣΕ ΑΝΑΘΕΩΡΗΣΗ ΚΑΙ ΔΙΟΡΘΩΣΕΙΣ

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ΠΡΩΤΟ ΣΚΑΛΙ

....

Κι αν είσαι στο σκαλί το πρώτο, πρέπει
νάσαι υπερήφανος κ' ευτυχισμένος.
Εδώ που έφτασες, λίγο δεν είναι
τόσο που έκανες, μεγάλη δόξα.
Κι αυτό ακόμη το σκαλί το πρώτο
πολύ από τον κοινό τον κόσμο απέχει.

....

Κ.Π. Καβάφης (1863–1933)

*THE FIRST STEP**

....

Just to be on the first step
should make you happy and proud.
To have come this far is no small achievement:
what you have done is a glorious thing.
Even this first step
is a long way above the ordinary world.

....

C.P. Cavafy (1863–1933)

*C. P. Kavafy, Collected Poems. Translated by Edmund Keeley and Philip Sherrard, Princeton University Press, Princeton N. J., 1992.

Preface

A manifold is a sophisticated concept even for mathematicians. For example, a great mathematician such as Jacques Hadamard "felt insuperable difficulty . . . in maintaining more than a rather elementary and superficial knowledge of the theory of Lie groups", a notion based on that of a manifold.

S.S. Chern [16, p. 344]

What this book is about. It is a text intended as a rigorous and very detailed introduction to some basic notions of the *elementary* theory of differential manifolds, for the benefit of undergraduate and beginning graduate students of mathematics and physics. It is designed to serve as a preparatory step to more advanced subjects, treated in the differential geometry of smooth manifolds and fiber bundles, global analysis, symplectic geometry, gauge theory and much more. These subjects are usually taught in graduate courses, and many excellent books deal with them. However, advanced undergraduate and beginning graduate students feel a kind of "shock" (according to some students of ours) when they first get in touch with the theory of manifolds. Our purpose is to help them to overcome this difficulty, by working at a leisurely pace, and by giving details which are usually omitted in a regular graduate course. This may demystify the difficulties implicit in Chern's epigraph.

The origins of the book. We both have taught the material included here, for many years. Part of this material was the core of a compulsory undergraduate course given, alongside with general topology, functional analysis, and measure theory, at our Department. In the 2000's the previous courses became optional, a policy naturally leading to analogous changes in graduate studies curricula.

Now, more than a decade after this "reform", many beginning graduate students in pure mathematics, in physics, and applied fields such as statistics, control theory, PDE's, economics, computer graphics and the like, where notions of the theory of smooth manifolds are needed, they regret for not having attended an introductory course at an earlier stage so that they would be adequately prepared to meet the harder demands of graduate studies. Urged by such reactions and the favorable reception of our lecture notes (in Greek), we decided to present them to a wider audience, in a new, thoroughly revised and expanded version.

Other motives. Beside the above reasons, the book owes its existence to some thoughts of ours about the present day teaching of differential geometry: In most Departments of Mathematics, differential geometry focuses, at best, on a brief presentation of curves and surfaces. Regrettably, there are students having not even heard the word manifold.

However, after Riemann's revolutionary vision of geometry and the explosion of modern physical theories, differential geometry has revealed a universe of unprecedented unity and beauty. Therefore, without abandoning Euler, Gauss and many other glorious ancestors, who paved the way to the present evolution of ideas, we believe that the student of mathematics and physics should be familiar, as early as possible, with the elementary aspects of the theory of (smooth) manifolds. The latter, conceived by Riemann, nowadays should be considered as classical as the theory of smooth curves and surfaces in $\mathbb{R}^3$. Students should understand that the space-time of General Relativity is not a (vulgar) "4-dimensional space", but "a 4-dimensional manifold" (equipped with an appropriate metric etc.) Even those destined to teach in secondary education, they will be helped to aspire future generations in many ways.

The contents in brief. With the previous thoughts in mind, we have selected a few introductory topics organized as follows:

Chapter 1 is dealing with smooth (alias differentiable) manifolds. They are sets which locally look like Euclidean spaces. We do not require any topological structure beforehand on them Instead, by means of *charts* and *atlases*, we build up a differential structure inducing, in a natural way, a topological structure. If the manifold is already equipped with a topology, a simple criterion ensures whether the latter coincides or not with the topology induced by the differential structure. We find this approach more convenient for the beginner. Later on, to avoid unnecessary repetitions, we impose certain restrictions on the topology of the manifolds. Another important notion here is that of the smoothness of maps between manifolds for which we provide many useful criteria.

Chapter 2 is a detailed study of the tangent spaces and the derived tangent bundle of a smooth manifold. Tangent vectors are described as equivalence classes of suitable smooth curves, and, later, as derivations of the algebra of (germs of) $\mathbb{R}$-valued smooth functions on the manifold. The latter approach, often used as the primary definition of a tangent vector, seems unmotivated to the novice, whence our choice to start with the first, more geometrical, (kinematic) approach. With a little work, the reader will also master derivations which are indispensable in later chapters. Special attention is given to various useful computations. Furthermore, given a smooth map between manifolds, we define its differential (or tangent map) by which one extends the ordinary derivative (differential) to the case of manifolds. We illustrate the calculus thus obtained by proving the analogs of some classical results in the new context.

In Chapter 3 we study smooth vector fields as sections of the tangent bundle and describe their differentiability in many useful ways. We explain their relationship with differential equations and flows. The latter are treated in detail, though some proofs may be omitted on a first reading. The set of vector fields of a smooth manifold is an example of a Lie algebra. This is an important algebraic structure, probably met by the reader for the first time. The Lie algebra of a Lie group (studied in Chapter 4) is an indispensable tool for the study of the latter groups. Chapter 3 closes with the Lie derivative of vector field.

Chapter 4 is a brief introduction to Lie groups. They provide an important example of the intertwining of two different structures; namely, the algebraic structure of a group with the differential structure of a manifold. The Lie algebra, one-parameter subgroups, the exponential map, and the adjoint representation of a Lie group are elaborated in full details. The elementary material included in this chapter is a direct application of the main ideas developed in the previous chapters, in particular those referring to vector fields and differentials.

Chapter 5 starts with regular submanifolds. Unlike the case of topological spaces, an arbitrary subset of a manifold is not necessarily a manifold too, unless additional conditions are satisfied. Immersions and submersions are the next subject. They are smooth maps that locally look like the natural inclusions and projections, respectively. Although they are elegantly defined by simple properties of their differentials, their local descriptions need some extra work.

Chapter 6 touches upon the Riemannian structure and linear connections on smooth manifolds. Roughly speaking, these notions introduce the reader to the realm of Riemannian geometry, as well as to the theory of connections, one of the most important tools of the modern geometry of fiber bundles, with important applications in geometry and physics. From here, one can follow a variety of directions, according to her/his needs and taste.

Chapter 7 is dealing with differential forms. We approach them in an elementary way, avoiding the use of tensor fields and exterior algebras. We start with local forms in Euclidean spaces and then we proceed to the general case of forms on manifolds. Central notions such as the exterior product, exterior differentiation, the pull-back and Lie derivatives of differential forms are carefully explained. We also prove the exactness of the local de Rham complex.

We would like to note that, although we are dealing with finite-dimensional manifolds, we often try to give coordinate-free proofs. These suggest ways to extend various results to the (infinite-dimensional) Banach framework, of course under appropriate modifications.

As we see from the preceding description and the table of contents, we have included (for the sake of completeness) more material than that normally addressed to undergraduate students. Complicated topics or proofs that can be skipped on a first reading are included between two black triangles ▼ and ▲.

The exercises. Each section is followed by exercises. For most of them there are hints or detailed solutions (depending on their difficulty) given at the end of the book. The reader is advised to try to deal with them before seeking help in the solutions. Very few exercises complement the theory and are used in subsequent parts of the book.

Prerequisites. Since we have tried to write a self-contained text, we have included two appendices reviewing what is needed here from the general topology and multivariate calculus. In the case of calculus, the linearity of the derivative is systematically used, since this property is a key to further developments. In this respect, the book also helps the reader to systematize the classical differential calculus and to see how it is applied to a more abstract context. Naturally, the Jacobian matrix is indispensable for finite-dimensional *computations*. On the other hand, although very little of the general (point-set) topology is needed, application of it to the first chapter gives the reader a useful perspective from the familiar topology of $\mathbb{R}^n$ to that of an abstract space. Needless to add that basic notions of linear algebra, set theory and functions surely belong to the elementary background of every student of mathematics and physics, that is why there is no particular mention of them.

The audience. As already explained, we have in mind (advanced) undergraduate and beginning graduate students of mathematics and physics wishing to be acquainted, as early as possible, with the material described above. With this background, they can easily approach more advanced topics such as vector and principal bundles, connection theory, Riemannian geometry, characteristic classes, symplectic geometry, gauge theory and so on. We note that students with some knowledge of surface theory will trace in it the origin of many notions of modern differential geometry. However, the former theory is not required to access our exposition. With a little effort (from the part of teachers and students) the back-bone of this book can be covered in one semester, possibly with the omission of certain sections.

About the bibliography. In the last 40 years or so, the literature on the geometry of manifolds and related topics has been enriched with a considerable collection of excellent textbooks and specialized treatises, as well as a myriad of research papers. From this enormous list, we have included here, apart from the sources we specifically cite in the text, and those we have used as students, teachers and writers, and –to our opinion– they better serve the purposes of oursthe present approach. Omissions by no means reflect any disdain. On the contrary, we are deeply indebted to all the authors whose works have influenced the development of our discipline, as well as our personal mathematical formation.

Acknowledgments. We are grateful to our teacher and mentor, the late Professor

A. Mallios (1932–2014), who first introduced smooth manifolds in the undergraduate curriculum of our Department. This was an exciting moment for all of us. We also thank our numerous students, whose questions, comments and suggestions helped us to give the present form to our preliminary notes (in Greek). Similarly, thanks are due to many colleagues for discussions about the need of such a book and its contents. Finally, we are indebted to the Publishers for including our work in this series.

We will be happy if we succeed to convey to our readers the same excitement we felt when we first came across the theory of smooth manifolds and modern differential geometry.

E. V – M. P. Athens, June 2016.

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Chapter 1

Differential Structures

It is hardly an exaggeration to say that manifolds are the most important spaces in mathematics.[...] They are important almost everywhere in mathematics from calculus on.

W.M. Boothby [8, p. 145]

Motivated by the central idea of cartography, we define the structure of a differential or smooth manifold, based on the abstraction of (geographical) charts and atlases. The existence of a differential structure on a set M supplies the latter with a canonical topology whose basic properties are exhibited. The next step is to define a differentiability notion for maps between smooth manifolds. This is a half way towards the development of a differential calculus on an abstract space such as a smooth manifold. This development is completed in Chapter 2, when the necessary linear spaces, supporting a notion of derivative, are provided by the tangent spaces of the manifold. The final section of this chapter is dealing with bump functions and smooth partitions of unity on a manifold M. The former allow to extend local differentiable functions to ones with domain M. On the other hand, local objects, defined on an open cover of the manifold, can be glued together by means of a partition of unity to determine a global object on M.

1.1 Motivation

Many terms encountered in the first sections come from cartography, the art of making charts. The problem of drawing charts is very old. Ancient astronomers, like Hipparhus (190–125 B.C) and Ptolemy (138–180 A.D), used the method of *stereographic projection* (as it was named around the 1600's) to draw celestial charts. Much later, in medieval times, the same method was used to make primitive geographical charts of the then known world.

In making charts, one essentially tries to *project*, that is, to represent parts of the Earth into pieces of the plane. Let us describe first the ***stereographic projection from the north pole*** shown in the next figure.

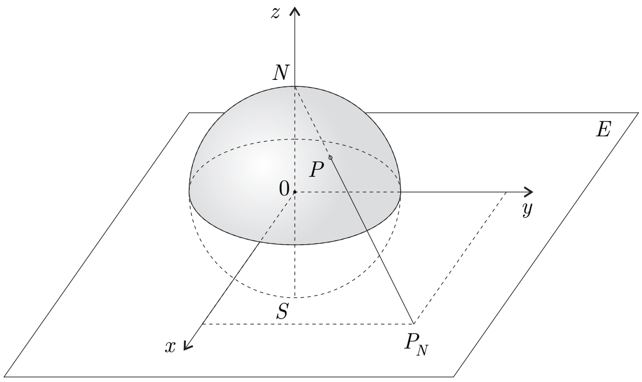

Figure 1.1 Stereographic projection from the north pole

Idealizing the situation, we imagine that the Globe is represented by the unit sphere S^2 centered at the origin 0 of the cartesian system of coordinates in $\mathbb{R}^3$. We also imagine that a (very large) sheet of paper is unfolded on the plane $x0y$, identified with the plane E of the equator. Then the straight line from the north pole N to a point P on the sphere meets E at a point P_N. The latter is the stereographic projection of P from the north pole. Conversely, joining any point of E with N, we obtain a point on S^2.

A little thought may convince the reader that, in this way, all the points of the sphere but one, namely the north pole, are in 1–1 correspondence to the points of the whole plane E; in other words, $S^2 \setminus N = S^2 - \{N\}$ is projected bijectively onto $E \equiv \mathbb{R}^2$. This process yields a very primitive *chart*, a useless one indeed, since points close to N are projected very far away, thus the image of regions near to the noth pole are monstrously distorted. Nevertheless, the essence of chart making lies in this process.

Since the image of the north pole does not appear in our chart, we make a new one containing the image of N by applying, in a similar way, the ***stereographic projection from the south pole*** S. Then $S^2 \setminus S$ is bijectively projected onto E. Clearly, the new chart does not contain the image of S. However, binding together the previous two charts we get a (geographical) *atlas*.

For the same sphere we may obtain better charts (only of theoretical value, of course) by projecting various hemispheres to appropriate planes. More precisely, we consider the ***north hemisphere*** (without the equator)

$$S_z^+ := \{(x, y, z) \in S^2 \ : \ z > 0\}, \tag{1.1.1}$$

and we project it onto the plane E of the equator, as in Figure 1.2. Clearly, a point (x, y, z) of the hemisphere is mapped to the point (x, y) belonging to the ***unit disc***

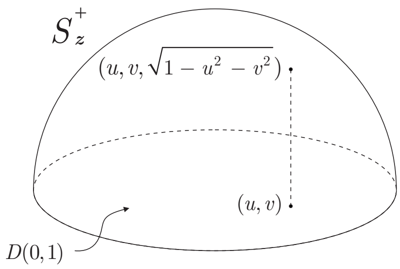

Figure 1.2 The north hemisphere

$$D_z := \left\{(a,b) \in \mathbb{R}^2 \;:\; a^2 + b^2 < 1\right\} \tag{1.1.2}$$

(the index z indicates that this disc is perpendicular to the z-axis). Conversely, any $(a, b) \in D_z$ is the projection of $(a, b, +\sqrt{1-a^2-b^2}) \in S_z^+$. Thus, a bijective correspondence between the noth hemisphere and the disc D_z is established, providing a chart. Note that D_z is an *open* subset of $\mathbb{R}^2$, a fact which will be important importance in the definition of smooth manifolds.

Analogously, we consider the south hemisphere S_z^- (along the z-axis), the west and east hemispheres (along the x-axis), and the two hemispheres along the y-axis. In this way we get an atlas with six charts. All the aforementioned hemispheres are shown in the standard picture of Figure 1.3

The details of the preceding charts, obtained by the stereographic projections and the hemispheres, will be elaborated in Examples 1.2.12 (D1,D2), pp. 14, 15.

Of course, in modern cartography more complicated projections are used (see, for instance, [52, Chapter 8^{bis}]), and very sophisticated devices, mounted on satellites, produce highly accurate charts. Further details on this matter are beyond the scope of this book.

The idea to correspond parts of the sphere to open subsets of $\mathbb{R}^2$ had been applied to arbitrary surfaces in 3-dimensional space. Thus the mathematical tools of the Euclidean space can be transferred, so to speak, to a surface. The development of the modern differential geometry of surfaces bears the seal of C.F. Gauss's ingenuity. In the study of these objects, the fact that the latter are embedded in $\mathbb{R}^3$ plays a crucial role. However, the theorem of Gauss known as *Theorema Egregium* revealed the existence of an *intrinsic geometry*. This means that the geometry of the surface can be recovered by means of measurements on the surface itself, without recourse to any ambient space. This led G.F.B. Riemann to conceive the idea of an abstract space, not necessarily embedded in some Euclidean space, that he called *manifold*.

Riemann's revolutionary ideas first appeared in his famous Inaugural Lecture *Über die Hypothesen, welche der Geometrie zu Grunde liegen* (: On the hypotheses

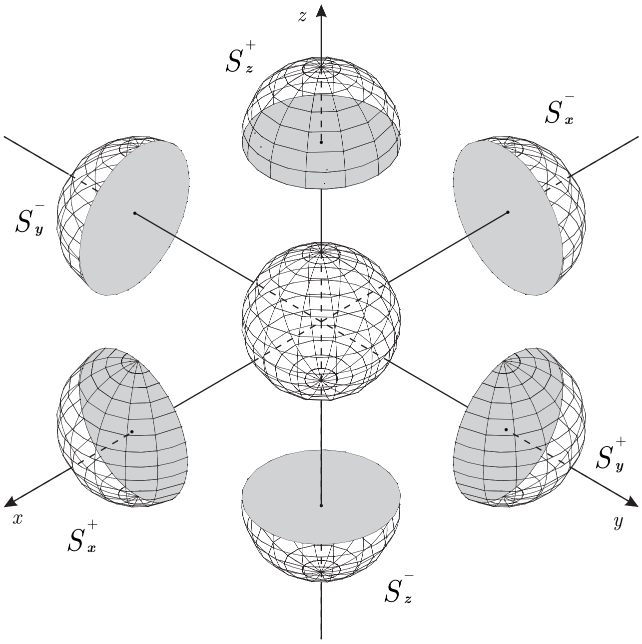

Figure 1.3 Six typical hemispheres

which lie at the foundations of geometry)*, delivered at the University of Göttingen in 1854, as part of the requirements for the position of 'Privatdozent'. For a long time they remained at the possession of a few mathematicians until A. Einstein applied Riemann's geometrical ideas to describe the mathematical background of the theory of General Relativity. This had a tremendous impact on the development of the theory of manifolds and led to an explosion of modern ideas, methods and related disciplines.

1.2 Charts and atlases: formal definitions

Formalizing the process of chart making described in § 1.1, here we provide an arbitrary set with charts and atlases.

Definition 1.2.1. Let M be a nonempty set. An (m-***dimensional***) ***chart*** on M is a pair (U, ϕ), where $U \subseteq M$ and

$$\phi \colon U \longrightarrow \phi(U) \subseteq \mathbb{R}^m$$

is an 1–1 map (onto $\phi(U)$), with $\phi(U)$ *open* in $\mathbb{R}^m$ (see also Figure 1.4).

*For a translation of this lecture and many useful comments see [68, Vol. II]

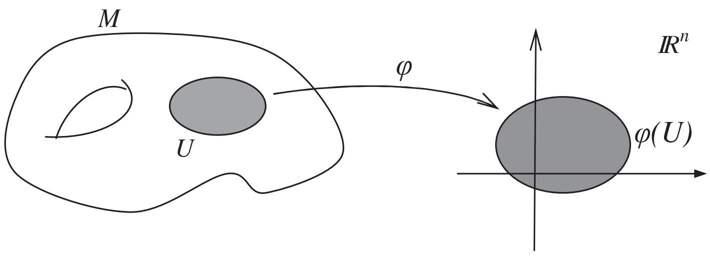

Figure 1.4 A (coordinate) chart

The following terminology is of frequent use: If U is a proper subset of M, the pair (U, ϕ) is called a ***local*** chart, whereas $U = M$ characterizes a ***global*** chart. A ***chart at*** (or ***about***) $x \in M$ is a chart (U, ϕ) on M such that $x \in U$. The Euclidean space $\mathbb{R}^m$ is said to be the ***model*** of the chart.

Global charts are generally rare. We describe a few of them in Examples 1.2.12. If there is a global chart (M, ϕ), then M is identified with the open subset $\phi(M)$ of $\mathbb{R}^m$; therefore, M may be considered as a topological space on which one defines differentiable mappings in the usual sense.

A local chart (U, ϕ) is also called a ***coordinate chart*** because it induces the ***local coordinate functions*** or simply ***local coordinates***

$$x^i := \mathrm{pr}_i \circ \phi; \qquad i = 1, \ldots, m,$$

where $\mathrm{pr}_i \colon \mathbb{R}^m \to \mathbb{R}$ is the projection to the i-th factor. The same projections coincide with the ***ordinary coordinate functions*** (u^i) of $\mathbb{R}^m$. Thus, with respect to (U, ϕ), the coordinates of a point $x \in U$ are the numbers

$$x^i(x) = \mathrm{pr}_i(\phi(x)) = u^i(\phi(x)), \qquad i = 1, \ldots, m.$$

To specify the coordinate functions, we also write $(U, \phi) = (U, x^1, \ldots, x^m)$. In this respect, U is the ***coordinate domain*** and ϕ the ***coordinate function*** Occasionally, instead of $x^i(x)$, we simply write x^i, omitting x. The context will clarify whether we are referring to the coordinate functions on U or the coordinates of the point $x \in U$.

As in the case of geographical atlases, there are points of M belonging to more than one charts, thus they admit various systems of local coordinates. Simple algebraic computations transform one system to another. However, since our ultimate goal is to provide M with a differential structure so that, among other things, a differential calculus can be built on M, we require the change of coordinates to be differentiable in the following precise sense.

Definition 1.2.2. Two charts (U, ϕ) and (V, ψ) of M are said to be $\mathcal{C}^k$***-compatible*** $(k = 0, 1, \ldots, \infty)$ either

(i) $U \cap V = \emptyset$, or

(ii) $U \cap V \neq \emptyset$, in which case it is also assumed that $\phi(U \cap V)$ and $\psi(U \cap V)$ are open subsets of $\mathbb{R}^m$, and the ***transition functions*** or ***change of coordinates***

$$\begin{aligned} \psi \circ \phi^{-1} &\colon \phi(U \cap V) \longrightarrow \psi(U \cap V), \\ \phi \circ \psi^{-1} &\colon \psi(U \cap V) \longrightarrow \phi(U \cap V), \end{aligned} \tag{1.2.1}$$

are $\mathcal{C}^k$-differentiable.

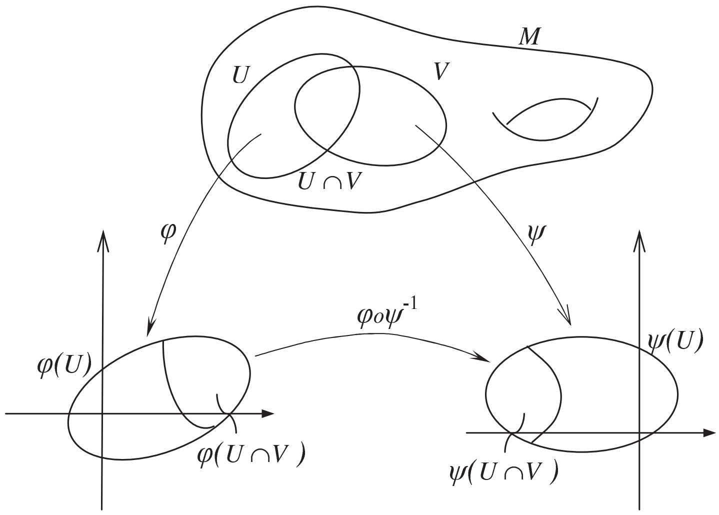

Figure 1.5 A transition function

It is worth noticing that regular surfaces in $\mathbb{R}^3$ have differentiable transition functions as a result of a proven theorem, where the fact that the surface is embedded in $\mathbb{R}^3$ plays a crucial role. It took a long time to mathematicians to understand the importance of the differentiability of the transition functions, and to realize that, in the case of absence of an ambient space, this condition should be axiomatized (see also the relevant comments in [13, pp. 2–3]).

Since the maps (1.2.1) are inverse to each other, the preceding differentiability condition implies that both mappings are $\mathcal{C}^k$***-diffeomorphisms***. We recall that a diffeomorphism is a differentiable bijection with a differentiable inverse.

For $k = 0$, $\psi \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ are only continuous (hence homeomorphisms). In this case, the charts (U, ϕ) and (V, ψ) are called ***topologically compatible***. If $k = \infty$, the charts are called ***smoothly compatible***. Clearly, smooth compatibility implies $\mathcal{C}^k$–compatibility, for every $k \in \mathbb{N}$. In turn, the latter implies the topological compatibility of charts.

If we want to be very formal, the maps (1.2.1) should be written as

$$\psi \circ \phi^{-1}|_{\phi(U \cap V)} \quad \text{and} \quad \phi \circ \psi^{-1}|_{\psi(U \cap V)}$$

respectively. However, for the sake of simplicity, we will not write down the restrictions whenever there is no danger of confusion, especially when the domain of definition is explicitly mentioned.

Remark 1.2.3. The $\mathcal{C}^k$-compatibility of two intersecting charts implies that they necessarily have the same dimension. Indeed, let (U, ϕ) and (V, ψ) be $\mathcal{C}^k$-compatible charts of dimensions m and n, respectively, with $U \cap V \neq \emptyset$. If $k \geq 1$, then, for every $x \in U \cap V$, the ordinary derivative of $\psi \circ \phi^{-1}$,

$$D(\psi \circ \phi^{-1})(\phi(x)) \colon \mathbb{R}^m \longrightarrow \mathbb{R}^n,$$

is a linear isomorphism, hence $m = n$.

However, for $k = 0$, the open sets $\phi(U \cap V)$ and $\psi(U \cap V)$ in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, are homeomorphic and the equality of the dimensions follows from Brouwer's theorem on *Invariance of Domain*. A (quite sophisticated) proof of the latter and other relevant details can be found, for instance, in [23], [36], [60] and [67].

We also note that non-intersecting charts (of the same space M) are not necessarily of the same dimension [see Exercise 1.2.14 (19)].

As a geographical atlas is a collection of charts, in our context we have the following formal notion:

Definition 1.2.4. A $\mathcal{C}^k$***-atlas*** of M is a family

$$\mathcal{A} = \{(U_i, \phi_i) \mid i \in I\}$$

of charts on M, such that:

(i) $\{U_i\}_{i \in I}$ is a *covering* of M; that is, $\bigcup_{i \in I} U_i = M$;

(ii) For every $i, j \in I$, the charts (U_i, ϕ_i) and (U_j, ϕ_j) are $\mathcal{C}^k$-compatible.

If $k = 0$ (resp. $k = \infty$), $\mathcal{A}$ will be called a ***topological*** (resp. ***differential*** or ***smooth***) ***atlas***.

If all the charts of $\mathcal{A}$ have the same dimension m, $\mathcal{A}$ is called an m***-dimensional*** atlas.

Returning to the atlases of S^2 described in Section 1.1, it is not hard to see that both satisfy the conditions of Definition 1.2.4 [details will be given in Examples 1.2.12 (D1,D2), pp. 14, 15]. Intuitively, we also see that restricting the stereographic projections to the upper or lower hemispheres, we obtain new charts which are smoothly compatible with the two charts of the original atlas, but they do not belong to it. Similarly, projecting half of the six hemispheres to the corresponding discs, we obtain new compatible charts not belonging to the initial atlas of the hemispheres.

Since we want to be able to control all the compatible charts, we consider them as members of a larger structure defined below. This is particularly useful in dealing with questions of differentiability, where we often need to shrink the domain of a chart. For this purpose we denote by $\mathfrak{A}^k_m(M)$ the set of all m-dimensional $\mathcal{C}^k$-atlases on a set $M \neq \emptyset$. If $\mathcal{A}, \mathcal{B} \in \mathfrak{A}_m^k(M)$, we say that $\mathcal{A}$ is ***smaller*** or ***coarser*** than $\mathcal{B}$ (notation: $\mathcal{A} \leq \mathcal{B}$) if $\mathcal{A}$ is (set-theoretically) contained in $\mathcal{B}$; that is,

$$\mathcal{A} \preceq \mathcal{B} \iff \mathcal{A} \subseteq \mathcal{B}. \tag{1.2.2}$$

It is clear that relation (1.2.2) induces a *partial order* in $\mathfrak{A}_m^k(M)$.

Definition 1.2.5. An atlas $\mathcal{A} \in \mathfrak{A}_m^k(M)$ is called ***maximal*** if it is a maximal element of $\mathfrak{A}_m^k(M)$, with respect to the previous partial order; that is,

$$\forall\, \mathcal{B} \in \mathfrak{A}_m^k(M) \text{ with } \mathcal{A} \preceq \mathcal{B} \implies \mathcal{A} = \mathcal{B}.$$

In other words, $\mathcal{A}$ is not contained in a larger atlas.

Proposition 1.2.6. *An atlas $\mathcal{A} \in \mathfrak{A}_m^k(M)$ is maximal if and only if it contains all the charts of M which are $\mathcal{C}^k$-compatible with every chart of $\mathcal{A}$.*

Proof. Assume that $\mathcal{A} \in \mathfrak{A}_m^k(M)$ is a maximal atlas and let (U, ϕ) be any (m-dimensional) chart on M, $\mathcal{C}^k$-compatible with every chart of $\mathcal{A}$. Then

$$\mathcal{A} \subseteq \mathcal{A} \cup \{(U, \phi)\} \in \mathfrak{A}_m^k(M).$$

Hence, the maximality condition implies that $\mathcal{A} = \mathcal{A} \cup \{(U, \phi)\}$, thus $(U, \phi) \in \mathcal{A}$.

Conversely, assume that an atlas $\mathcal{A} \in \mathfrak{A}_m^k(M)$ contains every chart $\mathcal{C}^k$-compatible with all the charts of $\mathcal{A}$. If $\mathcal{B} \in \mathfrak{A}_m^k(M)$ is any atlas with $\mathcal{A} \preceq \mathcal{B}$, then every chart of $\mathcal{B}$ is $\mathcal{C}^k$-compatible with all the charts of $\mathcal{A}$. Consequently, by the assumption, every chart of $\mathcal{B}$ belongs to $\mathcal{A}$, thus $\mathcal{B} \preceq \mathcal{A}$ and $\mathcal{B} = \mathcal{A}$, which shows that $\mathcal{A}$ is maximal. □

The next basic theorem describes the construction of a maximal atlas from a given one.

Theorem 1.2.7. *An m-dimensional atlas $\mathcal{A} \in \mathfrak{A}_m^k(M)$ is contained in a unique m-dimensional maximal atlas $\mathcal{A}'_k \in \mathfrak{A}_m^k(M)$.*

Proof. Let an arbitrary atlas $\mathcal{A} \in \mathfrak{A}_m^k(M)$. We denote by $\mathcal{A}'_k$ the set of all m-dimensional charts of M, which are $\mathcal{C}^k$-compatible with all the charts of $\mathcal{A}$. Clearly $\mathcal{A} \subseteq \mathcal{A}'_k$. We have to verify the following assertions:

$\mathcal{A}'_k$ is a $\mathcal{C}^k$-atlas: Condition (i) of Definition 1.2.4 is immediate. For condition (ii), we consider two arbitrary charts (U, ϕ) and (V, ψ) of $\mathcal{A}'_k$ with $U \cap V \neq \emptyset$. First we prove that $\phi(U \cap V)$ is an open subset of $\mathbb{R}^m$. To this end it suffices to show that, for every $a \in \phi(U \cap V)$, there exists an open subset B of $\mathbb{R}^m$ such that $a \in B \subset \phi(U \cap V)$. Set $x := \phi^{-1}(a) \in U \cap V$. Since the charts of $\mathcal{A}$ cover M, there exists $(W, \chi) \in \mathcal{A}$, with $x \in W$; hence, $x \in A := U \cap V \cap W$ (see also the next picture).

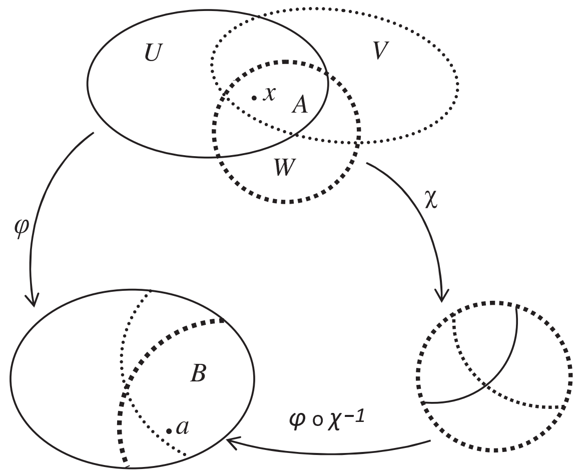

Figure 1.6 Chart compatibility of a maximal atlas

Therefore, we can write:

$$\begin{aligned}\phi(A) &= \phi(U \cap V \cap W)\\ &= (\phi \circ \chi^{-1})\big(\chi(U \cap V \cap W)\big)\\ &= (\phi \circ \chi^{-1})\big(\chi(U \cap W) \cap \chi(V \cap W)\big).\end{aligned}$$

The charts (U, ϕ) and (W, χ) are $\mathcal{C}^k$-compatible (by the construction of $\mathcal{A}'$), thus $\chi(U \cap W)$ is an open subset of $\mathbb{R}^m$. Similarly, $\chi(V \cap W)$ is an open subset of $\mathbb{R}^m$ by of the compatibility of (V, ψ) and (W, χ). Consequently, $\chi(U \cap W) \cap \chi(V \cap W)$ is an open subset of $\chi(U \cap W)$. Applying the homeomorphism $\phi \circ \chi^{-1}$, we see that

$$\begin{aligned}\phi(A) &= \big(\phi \circ \chi^{-1}\big)\big(\chi(U \cap W) \cap \chi(V \cap W)\big)\\ &\underset{\text{open}}{\subset} \big(\phi \circ \chi^{-1}\big)\big(\chi(U \cap W\big) = \phi(U \cap W) \underset{\text{open}}{\subset} \mathbb{R}^m;\end{aligned}$$

that is, $\phi(A)$ is an open subset of $\phi(U \cap W)$, thus an open subset of $\mathbb{R}^m$. Since $a \in \phi(A) \subseteq \phi(U \cap V)$, setting $B := \phi(A)$, we infer that $\phi(U \cap V)$ is an open subset of $\mathbb{R}^m$. A similar reasoning proves that $\psi(U \cap V)$ is open in $\mathbb{R}^m$.

Next we need to show the $\mathcal{C}^k$-differentiability of the transition functions (1.2.1). For the first of them we proceed as follows (analogously for the second): Since the differentiability of a function is a local property, it suffices to show that, for every $a \in \phi(U \cap V)$, there is an open subset B of $\phi(U \cap V)$, such that $a \in B$ and the restriction of $\psi \circ \phi^{-1}$ to B, i.e. $\psi \circ \phi^{-1}|_B$, is a $\mathcal{C}^k$-map. As before, we consider the chart $(W, \chi) \in \mathcal{A}$ with $x := \phi^{-1}(a) \in W$, and the set $A := U \cap V \cap W$. We have already proved that $\phi(A)$ and $\chi(A) = \chi(U \cap W) \cap \chi(V \cap W)$ are open subsets of $\mathbb{R}^m$. On the other hand, the $\mathcal{C}^k$-compatibility of (U, ϕ) with (W, χ), and (V, ψ) with

(W, χ), implies the $\mathcal{C}^k$-differentiability of the maps

$$\chi \circ \phi^{-1} \colon \phi(U \cap W) \longrightarrow \chi(U \cap W),$$
$$\psi \circ \chi^{-1} \colon \chi(V \cap W) \longrightarrow \psi(V \cap W).$$

As a result, their restrictions

$$\chi \circ \phi^{-1}|_{\phi(A)} \colon \phi(A) \longrightarrow \chi(A),$$
$$\psi \circ \chi^{-1}|_{\chi(A)} \colon \chi(A) \longrightarrow \psi(A)$$

are $\mathcal{C}^k$-differentiable, as well as their composite

$$\psi \circ \phi^{-1}|_{\phi(A)} = \big(\psi \circ \chi^{-1}|_{\chi(A)}\big) \circ \big(\chi \circ \phi^{-1}|_{\phi(A)}\big).$$

Setting again $B = \phi(A)$, we obtain the desired differentiability, thus proving the first assertion of the statement.

$\underline{\mathcal{A}'_k \text{ is maximal}}$: Indeed, if (U, ϕ) is an arbitrary chart of M, $\mathcal{C}^k$-compatible with the charts of $\mathcal{A}'_k$, then (U, ϕ) is $\mathcal{C}^k$-compatible with the charts of $\mathcal{A}$ (since $\mathcal{A} \subseteq \mathcal{A}'_k$); hence, by Proposition 1.2.6, we prove the claim.

$\underline{\mathcal{A}'_k \text{ is the unique maximal atlas containing } \mathcal{A}}$: Assume that $\mathcal{A}''_k \in \mathfrak{A}^k_m(M)$ is another maximal atlas with $\mathcal{A} \subseteq \mathcal{A}''_k$. This means that the charts of $\mathcal{A}''_k$ are $\mathcal{C}^k$-compatible with all the charts of $\mathcal{A}$, hence $\mathcal{A}''_k \subseteq \mathcal{A}'_k$. However, $\mathcal{A}''_k$ is maximal, thus $\mathcal{A}''_k = \mathcal{A}'_k$, which completes the proof. □

Another way to define a maximal atlas is by inducing an equivalence relation between atlases. More precisely, two atlases $\mathcal{A}$, $\mathcal{B} \in \mathfrak{A}^k_m(M)$ are called $\mathcal{C}^k$***-compatible*** (in symbols: $\mathcal{A} \overset{k}{\sim} \mathcal{B}$) if $\mathcal{A} \cup \mathcal{B}$ is also an m-dimensional $\mathcal{C}^k$-atlas; that is,

$$\mathcal{A} \overset{k}{\sim} \mathcal{B} \iff \mathcal{A} \cup \mathcal{B} \in \mathfrak{A}^k_m(M). \tag{1.2.3}$$

We show that this is an equivalence relation in the next proposition.

The union of two $\mathcal{C}^k$-atlases obviously covers M. On the other hand, if $(U, \phi), (V, \psi) \in \mathcal{A} \cup \mathcal{B}$, then either both charts belong to the same atlas, or one belongs to $\mathcal{A}$ and the other to $\mathcal{B}$. To ensure the compatibility of the atlases, we should check the compatibility of the preceding charts only in the second case (in the first case this is automatically satisfied). In other words,

(1.2.4) *$\mathcal{A} \overset{k}{\sim} \mathcal{B}$ if and only if every chart of $\mathcal{A}$ is $\mathcal{C}^k$-compatible with every chart of $\mathcal{B}$.*

On the other hand, the comparison of the two relations in $\mathfrak{A}^k_m(M)$ leads to

$$\mathcal{A} \preceq \mathcal{B} \implies \mathcal{A} \cup \mathcal{B} = \mathcal{B} \in \mathfrak{A}^k_m(M), \tag{1.2.5}$$

i.e. *the order relation (1.2.2) implies the equivalence relation (1.2.3).*

Proposition 1.2.8. *The following assertions hold true:*

(i) If $\mathcal{A}, \mathcal{B} \in \mathfrak{A}^k_m(M)$, then $\mathcal{A} \overset{k}{\sim} \mathcal{B} \iff \mathcal{A}'_k = \mathcal{B}'_k$.

(ii) $\stackrel{k}{\sim}$ *is an equivalence relation.*

(iii) If $[\mathcal{A}]$ *is the equivalence class of* $\mathcal{A} \in \mathfrak{A}^k_m(M)$*, with respect to* $\stackrel{k}{\sim}$*, then*

$$\mathcal{A}^* := \left(\bigcup_{\mathcal{B} \in [\mathcal{A}]} \mathcal{B} \right) \in \mathfrak{A}^k_m(M).$$

(iv) For every $\mathcal{A} \in \mathfrak{A}^k_m(M)$*, it follows that* $\mathcal{A}^* = \mathcal{A}'_k$*.*

Proof. (i) If $\mathcal{A} \stackrel{k}{\sim} \mathcal{B}$, then $\mathcal{A} \cup \mathcal{B} \in \mathfrak{A}^k_m(M)$. Therefore, in virtue of Theorem 1.1.7, we obtain the maximal atlas $(\mathcal{A} \cup \mathcal{B})'_k$. Because

$$\mathcal{A}, \mathcal{B} \subseteq \mathcal{A} \cup \mathcal{B} \subseteq (\mathcal{A} \cup \mathcal{B})'_k,$$

Definition 1.2.5 and the uniqueness of the maximal atlases corresponding to $\mathcal{A}$ and $\mathcal{B}$ imply that

$$\mathcal{A}'_k = (\mathcal{A} \cup \mathcal{B})'_k = \mathcal{B}'_k.$$

Conversely, if $\mathcal{A}'_k = \mathcal{B}'_k$, then the charts of $\mathcal{A}$ and those of $\mathcal{B}$ belong to the same atlas $\mathcal{A}'_k = \mathcal{B}'_k$; hence, they are mutually $\mathcal{C}^k$-compatible. Consequently $\mathcal{A} \stackrel{k}{\sim} \mathcal{B}$.

(ii) The reflexivity and symmetry of $\stackrel{k}{\sim}$ are obvious. Now, assume that $\mathcal{A} \stackrel{k}{\sim} \mathcal{B}$ and $\mathcal{B} \stackrel{k}{\sim} \mathcal{C}$. Then, by i) above, $\mathcal{A}'_k = \mathcal{B}'_k$ and $\mathcal{B}'_k = \mathcal{C}'_k$, whence the transitivity property.

(iii) Obviously, the charts of $\mathcal{A}^*$ cover M. It remains to check the compatibility of any charts $(U, \phi) \in \mathcal{B}$ and $(V, \psi) \in \mathcal{C}$, where $\mathcal{B}$ and $\mathcal{C}$ are arbitrary atlases in $[\mathcal{A}]$. This is an immediate consequence of $\mathcal{B} \stackrel{k}{\sim} \mathcal{C}$ and assertion (1.2.4).

(iv) Since $\mathcal{A} \cup \mathcal{A}'_k = \mathcal{A}'_k \in \mathfrak{A}^k_m(M)$, it follows that $\mathcal{A}'_k \stackrel{k}{\sim} \mathcal{A}$, thus $\mathcal{A}'_k \subseteq \mathcal{A}^*$. The maximality of $\mathcal{A}'_k$ implies that $\mathcal{A}^* = \mathcal{A}'_k$. □

Definition 1.2.9. We say that a *maximal* atlas $\mathcal{A} \in \mathfrak{A}^k_m(M)$ defines a ***$\mathcal{C}^k$-differential structure*** or the ***structure of a $\mathcal{C}^k$-manifold*** on M, and the pair $(M, \mathcal{A})$ is a ***$\mathcal{C}^k$-manifold***. In particular, if $k = 0$, $(M, \mathcal{A})$ is called a ***topological manifold***; whereas, for $k = \infty$, $\mathcal{A}$ determines a ***smooth structure*** and $(M, \mathcal{A})$ is a ***smooth manifold***. If all the charts of $\mathcal{A}$ have the same dimension m, we say that $\mathbb{R}^m$ is the ***model*** of the manifold and m its ***dimension***. We write $\dim(M) = m$.

We note that instead of *differential* structure and manifold the terms *differentiable* structure and manifold are also used. Although grammatically differential and differentiable have different connotations, in our framework they qualify the same objects.

Remarks 1.2.10. 1) The importance of Theorem 1.2.7 it should be now obvious: In order to endow a set M with the structure of a smooth manifold, it is sufficient to find a (not necessarily maximal) atlas of M. Then the respective maximal atlas defines the desired structure.

2) A maximal atlas can be the maximal atlas of many atlases, in contrast to the uniqueness (by Theorem 1.2.7) of the maximal atlas corresponding to a *given* atlas [see Exercises 1.2.14 (7), 1.2.14 (8) and 1.2.14 (9)].

3) It is worth mentioning the following result of Differential topology (see [32, pp. 51-52]):

A $\mathcal{C}^k$-structure, $k \geq 1$, contains a $\mathcal{C}^r$-structure, for every $k < r \leq \infty$.

This is not true for a $\mathcal{C}^0$-structure. As a matter of fact, M. Kervaire [39] gave an example of a topological manifold not admitting any differential structure.

On the other hand, J. Milnor [55] discovered the existence of *exotic* 7-spheres: These are manifolds that are *homemorphic* but *not diffeomorphic* to the standard manifold structure of S^7 (the precise meaning of the previous terminology will be clarified in Definition 1.4.9). For a heuristic description of them we refer to [17, p. 6]. It is also known (see [40]) that the sphere S^7 admits 28 exotic structures, while S^{31} has more than 16 million!

Also, in the 1980's (see [21], [24]), it was proved that there are manifolds homeomorphic but not diffeomorphic to the standard structure of $\mathbb{R}^4$ (*exotic* $\mathbb{R}^4$). In contrast, for any $n \neq 4$, any smooth manifold homeomorphic to $\mathbb{R}^n$ is diffeomorphic to the latter.

Conventions 1.2.11. 1) In view of the result mentioned in in the Remark 1.2.10 (3), hereafter all the manifolds considered will be *smooth*, unless otherwise stated. If $\mathcal{A}$ is any smooth atlas, then the corresponding maximal atlas will be simply denoted by $\mathcal{A}'$ (instead of $\mathcal{A}'_\infty$, as suggested by the notation of Theorem 1.2.7).

2) If $\mathcal{A}$ is the maximal atlas inducing the manifold structure on M, we usually say (and write) *"the (smooth) manifold $M \equiv (M, \mathcal{A})$"*. If there is no ambiguity about the atlas $\mathcal{A}$ in the previous pair, we simply say *" the (smooth) manifold M"*, the adjective smooth frequently omitted.

3) Also, for the sake of simplicity, we will assume that all the charts have the same dimension, as is the case of most examples, thus $\dim(M)$ is well defined. In Proposition 1.3.18 we will give a sufficient condition ensuring the existence of $\dim(M)$. But this needs the topological background of Section 1.3.

Let us stop for a while and give some standard examples.

Examples 1.2.12.

(A) $\mathbb{R}^n$ *and its open subsets*

Let an open $A \subseteq \mathbb{R}^n$ and consider the pair (A, id_A). It is obvious that (A, id_A) is an n-dimensional *global* chart of A and $\mathcal{A} := \{(A, \mathrm{id}_A)\}$ is an n-dimensional smooth atlas of A. If $\mathcal{A}'$ is the respective maximal atlas, then $A \equiv (A, \mathcal{A}')$ is an n-dimensional smooth manifold.

The previous considerations hold true, a fortiori, for $A = \mathbb{R}^n$. Hence,

$$\big(\mathbb{R}^n, \{(\mathbb{R}^n, \mathrm{id}_A)\}'\,\big)$$

is an n-dimensional smooth manifold. This is called the ***standard or ordinary smooth structure of*** $\mathbb{R}^n$. In Exercise 1.2.14 (9) we define on $\mathbb{R}$ a smooth structure that differs from the standard one.

(B) *Finite dimensional vector spaces*

Let V be an n-dimensional real vector space. A basis $(v_1, \ldots, v_n)$ of V induces a linear isomorphism

$$\psi\colon V \longrightarrow \mathbb{R}^n\colon x = \sum_{i=1}^{n} x^i v_i \mapsto (x^1, \ldots, x^n).$$

The pair (V, ψ) is clearly an n-dimensional global chart of V. Hence the set $\mathcal{A} := \{(V, \psi)\}$ is an n-dimensional smooth atlas of V and, if $\mathcal{A}'$ is the respective maximal atlas, then $(V, \mathcal{A}')$ is an n-dimensional smooth manifold.

In particular, the space $\mathcal{M}_{m\times n}(\mathbb{R})$ of $m \times n$ real matrices is an $m\cdot n$-dimensional smooth manifold. The map $\Phi\colon \mathcal{M}_{m\times n}(\mathbb{R}) \to \mathbb{R}^{m\cdot n}$ inducing this structure is precisely the one associating to a matrix the vector obtained by arranging side by side the lines of the matrix:

$$\begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix} \longmapsto (a_{11}, \ldots, a_{1n}, \ldots, a_{m1}, \ldots, a_{mn}).$$

For later use, we set $\mathcal{M}_n(\mathbb{R}) := \mathcal{M}_{n\times n}(\mathbb{R})$, thus $\mathcal{M}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$.

In the same vein, the *space of linear maps* $\mathcal{L}(\mathbb{E}, \mathbb{F})$ between two vector spaces, of respective dimensions m and n, is also a smooth $m \cdot n$-dimensional manifold: By choosing arbitrary bases of $\mathbb{E}$ and $\mathbb{F}$,

$$\mathcal{L}(\mathbb{E}, \mathbb{F}) \cong \mathcal{M}_{m\times n}(\mathbb{R}) \cong \mathbb{R}^{m\cdot n}.$$

(C) *An extreme case: 0-dimensional manifolds*

Let M be a non empty set. For each point $x \in M$ consider the map $\phi_x\colon \{x\} \to \{0\}$. Since $\mathbb{R}^0$ is nothing but the singleton $\{0\}$, obviously $(\{x\}, \phi_x)$ is a chart on M. All the charts of this form are smoothly compatible in a trivial way, thus their collection determines a 0-dimensional atlas $\mathcal{A}$. The later is necessarily maximal (why?), thus $(M, \mathcal{A})$ is a 0-dimensional smooth manifold. A set M endowed with the preceding structure is also called a ***discrete manifold***.

(D) *The surface of the unit sphere* S^2

The unit sphere centered (for simplicity) at $0 \equiv (0, 0, 0)$,

$$S^2 := \big\{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\big\}\,,$$

will be endowed with the structure of a 2-dimensional smooth manifold in the two ways outlined in Section 1.1. It turns out that these structures coincide (see Exercise 1.2.14 (2) and Theorem 1.4.16).

(D1) *S^2 as a manifold by stereographic projections*

Fixing the poles $N = (0,0,1)$ and $S = (0,0,-1)$, we define the pairs (U_N, ϕ_N) and (U_S, ϕ_S), where

$$U_N = S^2 \setminus \{N\}, \quad U_S = S^2 \setminus \{S\},$$
$$\phi_N : U_N \longrightarrow \mathbb{R}^2 : (x,y,z) \mapsto \phi_N(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z} \right),$$
$$\phi_S : U_S \longrightarrow \mathbb{R}^2 : (x,y,z) \mapsto \phi_S(x,y,z) = \left(\frac{x}{1+z}, \frac{y}{1+z} \right).$$

The previous maps are the precise expressions of the stereographic projections from the north ant south pole, respectively [see also Exercise 1.2.14 (4)]. We check that ϕ_N is injective: If (x_1, y_1, z_1), $(x_2, y_2, z_2) \in U_N$ with

$$\phi_N(x_1, y_1, z_1) = \phi_N(x_2, y_2, z_2);$$

that is,

$$\left(\frac{x_1}{1-z_1}, \frac{y_1}{1-z_1} \right) = \left(\frac{x_2}{1-z_2}, \frac{y_1}{1-z_1} \right),$$

then adding the squares of the equal coordinates and taking into account that $x_1^2 + y_1^2 + z_1^2 = x_2^2 + y_2^2 + z_2^2$, we have that $z_1 = z_2$, whence it follows that $x_1 = x_2$ and $y_1 = y_2$. The injectivity of ϕ_S is proved analogously.

We also check that the range of ϕ_N is the whole of $\mathbb{R}^2$. Indeed, for any $(a,b) \in \mathbb{R}^2$, we look for an $(x,y,z) \in S^2$ such that

$$\phi_N(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z} \right) = (a,b).$$

Then, substituting $x = a(1-z)$ and $y = b(1-z)$ in $x^2 + y^2 + z^2 = 1$, we obtain $z = \dfrac{-1 + a^2 + b^2}{1 + a^2 + b^2}$, from which follows that $x = \dfrac{2a}{1 + a^2 + b^2}$ and $y = \dfrac{2b}{1 + a^2 + b^2}$. Therefore, the inverse of ϕ_N is given by

$$\phi_N^{-1}(a,b) = \left(\frac{2a}{1+a^2+b^2}, \frac{2b}{1+a^2+b^2}, \frac{-1+a^2+b^2}{1+a^2+b^2} \right).$$

Hence $\phi_N(U_N) = \mathbb{R}^2$ is an open set. In a similar way, we verify that, for every $(a,b) \in \mathbb{R}^2$,

$$\phi_S^{-1}(a,b) = \left(\frac{2a}{1+a^2+b^2}, \frac{2b}{1+a^2+b^2}, \frac{1-a^2-b^2}{1+a^2+b^2} \right),$$

and $\phi_S(U_S) = \mathbb{R}^2$. The preceding arguments prove that (U_N, ϕ_N), (U_S, ϕ_S) are charts of S^2.

We will show that

$$\mathcal{A} := \{(U_N, \phi_N), (U_S, \phi_S)\}$$

is a smooth atlas. It is immediate that $U_N \cup U_S = S^2$. On the other hand, $U_N \cap U_S = U_N \setminus \{S\} = U_S \setminus \{N\}$; therefore,

$$\phi^N(U^N \cap U^S) = \phi^N(U^N \setminus \{S\}) = \phi_N(U_N) \setminus \{\phi_N(S)\} = \mathbb{R}^2 \setminus \{(0,0)\},$$
$$\phi_S(U_N \cap U_S) = \phi_S(U_S \setminus \{N\}) = \phi_S(U_S) \setminus \{\phi_S(N)\} = \mathbb{R}^2 \setminus \{(0,0)\},$$

which means that the set $\phi_N(U_N \cap U_N) = \phi_S(U_N \cap U_S) = \mathbb{R}^2 \setminus \{(0,0)\}$ is open in $\mathbb{R}^2$. Moreover, an easy computation shows that the maps

$$\phi_S \circ \phi_N^{-1},\ \phi_N \circ \phi_S^{-1} : \mathbb{R}^2 \setminus \{(0,0)\} \longrightarrow \mathbb{R}^2 \setminus \{(0,0)\}$$

are given by the formula

$$\phi_S \circ \phi_N^{-1}(a,b) = \phi_N \circ \phi_S^{-1}(a,b) = \left(\frac{a}{a^2+b^2}, \frac{b}{a^2+b^2}\right),$$

for every $(a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}$, thus they both are smooth.

In conclusion, (U_N, ϕ_N) and (U_S, ϕ_S) are smoothly compatible charts and $\mathcal{A}$ is a smooth atlas. If $\mathcal{A}'$ denotes the respective maximal atlas, then $(S^2, \mathcal{A}')$ is a 2-dimensional smooth manifold.

(D2) $\boldsymbol{S^2}$ *as a manifold by hemispheres*

Recalling (1.1.1), we define the sets

$$S_z^+ := \{(x,y,z) \in S^2 : z > 0\},$$
$$S_z^- := \{(x,y,z) \in S^2 : z < 0\},$$
$$D_z := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}.$$

In other words, S_z^+ (resp. S_z^-) is the positive (resp. the negative) hemisphere over (resp. under) the xy-plane without the equator. D_z is the open disc with center $0 \equiv (0,0)$ and radius 1 on the same plane. We also define the maps

$$phi_z^+ : S_z^+ \longrightarrow D_z : (x,y,z) \mapsto (x,y),$$
$$\phi_z^- : S_z^- \longrightarrow D_z : (x,y,z) \mapsto (x,y),$$

which are the projections of the preceding hemispheres to the disc separating them. Then the pairs (S_z^+, ϕ_z^+), (S_z^-, ϕ_z^-) are charts of the sphere. Indeed, we check at once that ϕ_z^+, ϕ_z^- are 1–1 onto D_z with respective inverses

$$\left(\phi_z^+\right)^{-1}(a,b) = \left(a, b, \sqrt{1-a^2-b^2}\right),$$
$$\left(\phi_z^-\right)^{-1}(a,b) = \left(a, b, -\sqrt{1-a^2-b^2}\right),$$

for every $(a,b) \in D_z$. Hence $\phi_z^+(U_z^+) = \phi_z^-(U_z^-) = D_z \subset \mathbb{R}^2$ open.

In this way, we obtain the six charts $(S_i^\alpha, \phi_i^\alpha)$, with $i = x,y,z$ and $\alpha = +,-$, and the three discs D_i, $i = x,y,z$. Then the family

$$\mathcal{B} := \{(S_i^\alpha\ \phi_i^\alpha) \,|\, i = x,y,z;\ \alpha = +,-\}$$

is an atlas of S^2: Obviously the domains of the charts cover S^2, since for every $(x, y, z) \in S^2$, we have $x^2 + y^2 + z^2 = 1$; hence, at least one coordinate, say x, does not vanish, thus

$$(x, y, z) \in U_x^+ \quad \text{or} \quad (x, y, z) \in U_x^- .$$

We prove the compatibility, e.g., of $(S_x^+\ \phi_x^+)$ and $(S_y^-\ \phi_y^-)$. In this case,

$$S_x^+ \cap S_y^- = \big\{(x, y, z) \in S^2 : x > 0,\ y < 0\big\};$$

hence,

$$\phi_x^+ : S_x^+ \cap S_y^- \ni (x, y, z) \longmapsto \phi_x^+(x, y, z) = (y, z) \in D_x,$$

with $y < 0$, i.e.

$$\phi_x^+\big(S_x^+ \cap S_y^-\big) = D_x \cap \{(y, z) \in \mathbb{R}^2 : y < 0\}.$$

Consequently, $\phi_x^+\big(S_x^+ \cap S_y^-\big)$ is the lower half of the disc D_x, which is an open subset of $\mathbb{R}^2$. Similarly,

$$\phi_y^-\big(S_x^+ \cap S_y^-\big) = D_y \cap \{(x, z) \in \mathbb{R}^2 : x > 0\} \subset \mathbb{R}^2$$

is open in $\mathbb{R}^2$.

For the differentiability of the corresponding transition functions, we notice that, for every $(a, b) \in \phi_x^+\big(S_x^+ \cap S_y^-\big)$, we have

$$\big(\phi_y^- \circ (\phi_x^+)^{-1}\big)(a, b) = \phi_y^-\big(\sqrt{1 - a^2 - b^2},\, a,\, b\big) = \big(\sqrt{1 - a^2 - b^2},\, b\big);$$

hence, $\phi_y^- \circ (\phi_x^+)^{-1}$ is a $\mathcal{C}^\infty$-map on $\phi_x^+\big(S_x^+ \cap S_y^-\big)$. Analogously, for every $(a, b) \in \phi_y^-(U_x^+ \cap U_y^-)$,

$$\big(\phi_x^+ \circ (\phi_y^-)^{-1}\big)(a, b) = \phi_x^+\big(a,\, -\sqrt{1 - a^2 - b^2},\, b\big) = \big(-\sqrt{1 - a^2 - b^2},\, b\big),$$

showing that $\phi_x^+ \circ (\phi_y^-)^{-1}$ is also smooth on its domain.

In a similar way we prove the compatibility of any other pair of charts. Consequently $\mathcal{B}$ is an atlas of S^2 and $(S^2, \mathcal{B}')$ a 2-dimensional smooth manifold.

(E) *The unit circle* S^1

The unit circle $S^1 := \{(x, y) \in \mathbb{R}^2 \ :\ x^2 + y^2 = 1\}$ becomes a smooth manifold of dimension 1 by adaptingthe analogs of stereographic projections or suitable semicircles [see Exercise 1.2.14 (8)]. Here we describe a third atlas, which proves to be very useful in various applications. More precisely, we consider the set

$$\mathcal{C} := \big\{(U_N, \theta_N), (U_S, \theta_S)\big\},$$

where

$$U_N := S^1 \setminus \{(0, 1)\},\ U_S := S^1 \setminus \{(0, -1)\},$$

the maps $\theta_i : U_i \to \mathbb{R}$ $(i = N, S)$ sending every point $P = (x, y)$ of U_i to the angle θ_i between OP and Ox (if O denotes the origin of the axes, center of the circle), subject to the restrictions

$$\theta_N(P) \in (\pi/2, 5\pi/2)\,; \quad P \in U_N,$$
$$\theta_S(P) \in (-\pi/2, 3\pi/2)\,, \quad P \in U_S.$$

By elementary trigonometry we see that θ_N and θ_S are 1–1 maps onto the respective intervals, which are open subsets of $\mathbb{R}$; hence, the pairs (U_N, θ_N) and (U_S, θ_S) are charts of S^1. For their compatibility we first see that

$$\theta_N(U_N \cap U_S) = (\pi/2, 3\pi/2) \cup (3\pi/2, 5\pi/2)\,,$$
$$\theta_S(U_N \cap U_S) = (-\pi/2, \pi/2) \cup (\pi/2, 3\pi/2)\,,$$

which are open subsets of $\mathbb{R}$. On the other hand, the transition function

$$\theta_N \circ \theta_S^{-1} : \theta_N(U_N \cap U_S) \longrightarrow \theta_S(U_N \cap U_S)$$

is given by

$$\theta_N \circ \theta_S^{-1}(t) = \begin{cases} t, & t \in (\pi/2, 3\pi/2), \\ t + 2\pi, & t \in (-\pi/2, \pi/2), \end{cases}$$

which is a $\mathcal{C}^\infty$-diffeomorphism. As a result, $\mathcal{C}$ is a differential atlas of S^1 and the pair $(S^1, \mathcal{C}')$ is an 1-dimensional smooth manifold.

(F) *The projective plane* $\mathbb{P}_2(\mathbb{R})$

Let the Euclidean space $\mathbb{R}^3$. In $\mathbb{R}^3_* := \mathbb{R}^3 \setminus \{(0,0,0)\}$ we define the relation

$$(x^1, x^2, x^3) \sim (y^1, y^2, y^3) \Longleftrightarrow \exists\, \lambda \in \mathbb{R}_* := \mathbb{R} - \{0\} : y^i = \lambda x^i,\ i = 1,2,3. \tag{1.2.6}$$

It is easily verified that this is an equivalence relation. We denote by $[x^1, x^2, x^3]$ (instead of the more complicated $[(x^1, x^2, x^3)]$) the equivalence class of (x^1, x^2, x^3). The set of all the above classes, i.e. the quotient space

$$\mathbb{P}_2(\mathbb{R}) := \big(\mathbb{R}^3 \setminus \{(0,0,0)\}\big) \,/ \sim$$

is the ***projective plane*** (or the ***2-dimensional projective space***).

We define the sets

$$U_i := \big\{[x^1, x^2, x^3] \in \mathbb{P}_2(\mathbb{R}) \,:\, x^i \neq 0\big\},\ i = 1,2,3,$$

and the maps

$$\phi_1 : U_1 \longrightarrow \mathbb{R}^2 : [x^1, x^2, x^3] \mapsto \left(\frac{x^2}{x^1}, \frac{x^3}{x^1}\right),$$
$$\phi_2 : U_2 \longrightarrow \mathbb{R}^2 : [x^1, x^2, x^3] \mapsto \left(\frac{x^1}{x^2}, \frac{x^3}{x^2}\right),$$
$$\phi_3 : U_3 \longrightarrow \mathbb{R}^2 : [x^1, x^2, x^3] \mapsto \left(\frac{x^1}{x^3}, \frac{x^2}{x^3}\right).$$

Let us check that (U_1, ϕ_1) is a chart of $\mathbb{P}_2(\mathbb{R})$: First we notice that ϕ_1 is injective, because

$$\begin{aligned} & \phi_1([x^1, x^2, x^3]) = \phi_1([y^1, y^2, y^3]) \\ \Rightarrow\ & \frac{x^i}{x^1} = \frac{y^i}{y^1}, \quad i = 2,3, \\ \Rightarrow\ & x^i = \lambda y^i, \quad \lambda := \frac{x^1}{y^1} \\ \Rightarrow\ & [x^1, x^2, x^3] = [y^1, y^2, y^3]. \end{aligned}$$

Moreover, $\phi_1(U_1) = \mathbb{R}^2$: Indeed, for every $(a,b) \in \mathbb{R}^2$ we see that $\phi_1([1,a,b]) = (a,b)$; that is, ϕ_1 is also surjective. Hence, $\phi_1(U_1)$ is an open subset of $\mathbb{R}^2$ and (U_1, ϕ_1) is a chart. Similarly, (U_2, ϕ_2) and (U_3, ϕ_3) are charts of $\mathbb{P}_2(\mathbb{R})$.

We show that the set $\mathcal{A} := \{(U_i, \phi_i) \,|\, i = 1,2,3\}$ is a smooth atlas of $\mathbb{P}_2(\mathbb{R})$: First notice that the domains of the previous charts cover $\mathbb{P}_2(\mathbb{R})$, since, for every $[x^1, x^2, x^3] \in \mathbb{P}_2(\mathbb{R})$, at least one coordinate, say x^1, does not vanish; hence, $[x^1, x^2, x^3] \in U_1$. For the compatibility, for instance, of (U_1, ϕ_1) and (U_2, ϕ_2), we see that

$$U_1 \cap U_2 = \big\{[x^1, x^2, x^3] \in \mathbb{P}_2(\mathbb{R}) \,|\, x^1 \neq 0,\ x^2 \neq 0\big\},$$

thus, for any $[x^1, x^2, x^3] \in U_1 \cap U_2$,

$$\phi_1([x^1, x^2, x^3]) = \left(\frac{x^2}{x^1}, \frac{x^3}{x^1}\right) \in \mathbb{R}_* \times \mathbb{R},$$

i.e. $\phi_1(U_1 \cap U_2) \subseteq \mathbb{R}_* \times \mathbb{R}$. Conversely, if $(a,b) \in \mathbb{R}_* \times \mathbb{R}$, then $[1,a,b] \in U_1 \cap U_2$ and $\phi_1([1,a,b]) = (a,b)$; that is, $\mathbb{R}_* \times \mathbb{R} \subseteq \phi_1(U_1 \cap U_2)$, or $\phi_1(U_1 \cap U_2) = \mathbb{R}_* \times \mathbb{R}$, which is an open subset of $\mathbb{R}^2$. Finally, for every $(a,b) \in \mathbb{R}_* \times \mathbb{R}$,

$$(\phi_2 \circ \phi_1^{-1})(a,b) = \phi_2([1,a,b]) = \left(\frac{1}{a}, \frac{b}{a}\right) = (\phi_1 \circ \phi_2^{-1})(a,b),$$

which means that the transition functions

$$\phi_2 \circ \phi_1^{-1} = \phi_1 \circ \phi_2^{-1} \colon \mathbb{R}_* \times \mathbb{R} \longrightarrow \mathbb{R}_* \times \mathbb{R}$$

are $\mathcal{C}^\infty$-maps. Analogously, we prove the compatibility of any other pair of charts. Therefore, $\mathcal{A}$ is an atlas and $(\mathbb{P}_2(\mathbb{R}), \mathcal{A}')$ is a differential manifold of dimension 2.

Geometrically speaking, the projective plane is isomorphic (by means of an appropriate 1–1 and onto map) with the set of all straight lines of $\mathbb{R}^3$ passing through the origin [see Exercise 1.2.14 (20)].

An easy generalization of $\mathbb{P}_2(\mathbb{R})$ is the *(real)* ***n*-dimensional projective space** $\mathbb{P}_n(\mathbb{R})$. This is an n-dimensional smooth manifold [see Exercise 1.2.14 (21)].

(G) *Regular surfaces in* $\mathbb{R}^3$

A ***regular surface*** is a set $S \subset \mathbb{R}^3$ such that, for every $p \in S$, there exists a pair (U, r), called ***(local) parametrization of*** S, where U is an open subset of R^2, $r(U)$ is an open subset of S containing p, and the map $r \colon U \to r(U)$ satisfies the following properties:

i) r is a homeomorphism (i.e. 1–1, onto, continuous with a continuous inverse).

ii) r is smooth.

iii) For every $q \in U$, the derivative (or differential) of r at q,

$$Dr(q) \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

is 1–1 (equivalently, the Jacobian matrix of r at q has rank 2.

From the previous definition it is clear that the pair $(r(U), r^{-1})$ is a 2-dimensional chart, thus S is covered by a family of charts. Now, the compatibility condition of the latter is proved to be a *consequence* of the properties of r, along with the fact that S is imbedded in $\mathbb{R}^3$. Hence, all the charts of S form a 2-dimensional smooth atlas whose corresponding maximal atlas determines the structure of a 2-dimensional manifold on S. Details on the classical theory of regular surfaces and related topics can be found in many excellent sources. We refer, for instance to [13], [44], [52] and [62].

A few comments may help the reader to clarify some basic differences between regular surfaces and their abstraction, namely differential manifolds. First of all, the continuity of r and its inverse has a meaning because a surface is a topological space (endowed with the subspace topology derived from the ordinary topology of $\mathbb{R}^3$). In contrast, there is no mention of a topological structure on a differential manifold so far. This will be done in the next section, where we will show that the map ϕ of every chart (U, ϕ) is indeed a homeomorphism (relative to the canonical topology induced by the differential structure). The question of the differentiability of ϕ is settled in Section 1.4, whereas the definition of an appropriate notion of differential needs a lot of work and will be treated in Chapter 2. In any case, condition iii) of the preceding definition is nonsensical in the context of manifolds.

The differential structure on the cartesian product of two manifolds is another important example of manifold. We single it out because of its utility in numerous occasions.

Proposition 1.2.13. *Let $M \equiv (M, \mathcal{A})$ and $N \equiv (N, \mathcal{B})$ be two $\mathcal{C}^k$-manifolds of respective dimensions m and n. Then the cartesian product $M \times N$ has the structure of an $(m+n)$-dimensional $\mathcal{C}^k$-manifold.*

Proof. We consider the set

$$\mathcal{C} := \{(U \times V,\, \phi \times \psi) \,|\, (U, \phi) \in \mathcal{A},\, (V, \psi) \in \mathcal{B}\}.$$

Every pair $(U \times V,\, \phi \times \psi)$ is a chart of $M \times N$, because

$$\phi \times \psi \colon U \times V \longrightarrow \phi(U) \times \psi(V) : (x, y) \mapsto (\phi(x), \psi(y))$$

is injective as the cartesian product of injective maps, and $\phi(U) \times \psi(V)$ is an open subset of $\mathbb{R}^m \times \mathbb{R}^n$ as product of sets. Also, the charts of $\mathcal{C}$ cover $M \times N$, for if $(x, y) \in M \times N$, then there are $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ with $x \in U$ and $y \in V$, hence $(x, y) \in U \times V$.

For the $\mathcal{C}^k$-compatibility of the charts of $\mathcal{C}$, we take two arbitrary pairs

$$(U_1 \times V_1, \phi_1 \times \psi_1),\ (U_2 \times V_2,\ \phi_2 \times \psi_2) \in \mathcal{C},$$

such that $(U_1 \times V_1) \cap (U_2 \times V_2) \neq \emptyset$, thus $U_1 \cap U_2$ and $V_1 \cap V_2$ are not empty. Therefore, the sets

$$(\phi_1 \times \psi_1)\big((U_1 \times V_1) \cap (U_2 \times V_2)\big) = \phi_1(U_1 \cap U_2) \times \psi_1(V_1 \cap V_2), \tag{1.2.7}$$

$$(\phi_2 \times \psi_2)\big((U_1 \times V_1) \cap (U_2 \times V_2)\big) = \phi_2(U_1 \cap U_2) \times \psi_2(V_1 \cap V_2) \tag{1.2.8}$$

are open subsets of $\mathbb{R}^m \times \mathbb{R}^n$ (in the product topology), and the transition function

$$(\phi_2 \times \psi_2) \circ (\phi_1 \times \psi_1)^{-1} = (\phi_2 \circ \phi_1^{-1}) \times (\psi_2 \circ \psi_1^{-1}),$$

sending the set (1.2.7) onto (1.2.8) is a $\mathcal{C}^k$-diffeomorphism, as the product of $\mathcal{C}^k$-diffeomorphisms in Euclidean spaces.

We conclude that $\mathcal{C}$ is a $\mathcal{C}^k$-atlas of $M \times N$ and the respective maximal $\mathcal{C}'$ atlas induces the structure of a $\mathcal{C}^k$-manifold on $M \times N$. □

Clearly, Proposition 1.2.13 extends to any cartesian product with finitely many factors (what is the corresponding dimension?). Also, referring to the previous proof, it is interesting to note that $\mathcal{C}$ is not necessarily a maximal atlas, although it derives from maximal atlases. Exercise 1.3.23 (10) in the next section provides an example explaining this remark.

Exercises 1.2.14.

1. Prove any omitted details from the Examples 1.2.12.
2. Prove that every chart (U, ϕ) is smoothly compatible with itself.
3. Explain why the n-***torus*** $T^n := S^1 \times \cdots \times S^1$ (n-times) is a differential manifold. What is its dimension?
4. Prove the formulas expressing the stereographic projections of S^2 and their inverses, given in Example 1.2.12 (D1).
5. If $U = (0,1) \times (0, \pi/2) \subseteq \mathbb{R}^2$ and $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is the map given by $f(r,\theta) = (r\cos\theta, r\sin\theta)$, prove that (U, ϕ) is a chart belonging to the standard smooth structure of $\mathbb{R}^2$.
6. Prove that the map $g\colon \mathbb{R}^2 \to \mathbb{R}^2$, with $g(x,y) := (x^2 + 2y^2, 3xy)$ determines on suitable open subsets of $\mathbb{R}^2$ (which ones?) charts on $\mathbb{R}^2$, belonging to the standard smooth structure of $\mathbb{R}^2$.
7. Prove that the charts (U_N, ϕ_N), (U_S, ϕ_S) of S^2 in Example 1.2.12 (D1) are compatible with the charts $(U_i^\alpha, \phi_i^\alpha)$ in Example 1.2.12 (D2). What can be said about the corresponding smooth structures of S^2?
8. In analogy to the sphere S^2, define on the circle S^1 two 1-dimensional smooth atlases: one consisting of appropriate stereographic projections and another with semi-circles. Prove that these atlases are smoothly compatible with each other and with the atlas $\mathcal{C}$ of Example 1.2.12 (E).
9. Consider the charts (U_1, ϕ_1) and (U_2, ϕ_2) of S^1, where
$$U_1 := \{(\sin 2\pi t, \cos 2\pi t) \mid t \in (0,1)\},$$
$$U_2 := \{(\sin 2\pi t, \cos 2\pi t) \mid t \in (-1/2, 1/2)\},$$
$$\phi_1(\sin 2\pi t, \cos 2\pi t) = \phi_2(\sin 2\pi t, \cos 2\pi t) := t.$$
Prove that they are smoothly compatible and determine also the same differential structure described in Example 1.2.12 (E) and Exercise 8 above.
10. Let K be the surface of a cylinder and B_1, B_2 its bases. Show that $K \setminus (B_1 \cup B_2)$ is a smooth manifold of dimension 2.

11. Referring to Example 1.2.12 (A), determine the charts of the maximal atlases of $\mathbb{R}$, $\mathbb{R}^n$, and those of any open set $A \subset \mathbb{R}^n$.

12. Let $\psi \colon \mathbb{R} \to \mathbb{R}$ be the map given by $\psi(x) = x^3$. Prove that $\{(\mathbb{R}, \psi)\}$ induces a differential structure on $\mathbb{R}$. Check whether the chart $(\mathbb{R}, \psi)$ is topologically or smoothly compatible with the chart $(\mathbb{R}, \mathrm{id}_{\mathbb{R}})$. What is the conclusion about the two differential structures of $\mathbb{R}$, induced by the previous charts?

13. Prove that the $\mathcal{C}^k$-compatibility of charts *is not* an equivalence relation.

14. Let (U, ϕ) be an m-dimensional chart of M. If $a \in \mathbb{R}^m$ is a fixed element, the ***right translation*** by a is the map

$$\mu_a \colon \mathbb{R}^m \longrightarrow \mathbb{R}^m : x \mapsto x + a.$$

On the other hand, for a fixed $s \in \mathbb{R}$, the ***homothetie*** by s is the map

$$\lambda_s \colon \mathbb{R}^m \longrightarrow \mathbb{R}^m : x \mapsto sx.$$

Prove that $(U, \mu_a \circ \phi)$ and $(U, \lambda_s \circ \phi)$ are chart smoothly compatible with (U, ϕ).

15. Let M be a smooth manifold. Then, for each $x \in M$:
(a) There is a chart (U, ϕ) ***centered at*** x, i.e. $\phi(x) = 0$.
(b) There is a chart (V, ψ) such that $\psi(V) = \mathbb{R}^m$.
(c) There is a chart (W, χ) such that $\chi(V) = \mathbb{R}^m$, and $\chi(x) = 0$.
Both charts belong to the given differential structure of M.

16. Prove that every chart (U, ϕ) on a set M is smoothly compatible with uncountably many charts of M. [Hint: use Exercise 14 or decompose $\phi(U)$ into appropriate open subsets]

17. Prove that a maximal atlas $\mathcal{A}$ minus a countable number of charts remains an atlas. What conclusion can drawn from this result?

18. If a topological atlas $\mathcal{A}$ is maximal, prove that it contains charts which are not $\mathcal{C}^1$-compatible, therefore $\mathcal{A}$ is not a $\mathcal{C}^k$-atlas, for $k \geq 1$. Furthermore, $\mathcal{A}$ cannot be a differential atlas.

19. Consider the set $X = X_1 \cup X_2 \subset \mathbb{R}^2$, where $X_1 := \mathbb{R} \times \{0\}$ and $X_2 := D((0,2), 1)$ (: the *disk* of center $(0,2)$ and radius 1). Prove that:
(a) (X_1, pr_1) and (X_2, id_{X_2}) are charts of X.
(b) The set $\mathcal{A} := \{(X_1, \mathrm{pr}_1), (X_2, \mathrm{id}_{X_2})\}$ is a smooth atlas of X.
What is the dimension of $\mathcal{A}$?

20. Using the fact that a straight line in $\mathbb{R}^3$ through the origin is completely determined by an equivalence class $[a, b, c]$, where $(a, b, c) \neq (0, 0, 0)$ [see Example 1.2.12 (F)], prove that the elements of $\mathbb{P}_2(\mathbb{R})$ are in bijective correspondence with the aforementioned lines.

21. Define the projective space $\mathbb{P}_n(\mathbb{R})$ and its smooth structure.

22. Let U be an open subset of $\mathbb{R}^m$ and $f \colon U \to \mathbb{R}^n$ be a smooth function. The ***graph*** of f is the set

$$\Gamma_f := \{(x, f(x) \,|\, x \in U\} \subset U \times \mathbb{R}^n.$$

Show that Γ_f is an m-dimensional manifold.

23. Generalizing the Examples 1.2.12 (D1) and (D2), define a smooth structure on the unit sphere $S^n \subset \mathbb{R}^{n+1}$ centered at 0, using the analogs of stereographic projections, and appropriate semispheres.

1.3 The canonical topology of a manifold

In this section we show that any topological or differential atlas on a set M induces, in a natural way, a topological structure on M. For this purpose, it is sufficient to consider topological atlases; consequently, all the results proved below hold a fortiori for smooth manifolds.

Definition 1.3.1. Let $\mathcal{A}$ be a (not necessarily maximal) m-dimensional atlas on M. A subset A of M will be called ***open*** if, for every $(U, \phi) \in \mathcal{A}$, the set $\phi(U \cap A)$ is open in $\mathbb{R}^m$. The collection of all open subsets of M, induced by $\mathcal{A}$, is denoted by $\mathcal{T}_{\mathcal{A}}$.

Proposition 1.3.2. *$\mathcal{T}_{\mathcal{A}}$ determines a topology on M.*

Proof. Obviously $\emptyset \in \mathcal{T}_{\mathcal{A}}$. Also, for every $(U, \phi) \in \mathcal{A}$, by the definition of a chart, $\phi(M \cap U) = \phi(U)$ is an open subset of $\mathbb{R}^m$, thus $M \in \mathcal{T}_{\mathcal{A}}$.

$\mathcal{T}_{\mathcal{A}}$ is closed under arbitrary unions: Let $(A_i)_{i \in I}$ be an arbitrary family, with $A_i \in \mathcal{T}_{\mathcal{A}}$, for every $i \in I$. Then, for any $(U, \phi) \in \mathcal{A}$,

$$A \cap U = \left(\bigcup_{i \in I} A_i \right) \bigcap U = \bigcup_{i \in I} (A_i \cap U).$$

Therefore,

$$\phi(A \cap U) = \phi \left(\bigcup_{i \in I} (A_i \cap U) \right) = \bigcup_{i \in I} \phi(A_i \cap U).$$

Since every $\phi(A_i \cap U)$ is open in $\mathbb{R}^m$, we prove the claim.

Similarly, for every $A, B \in \mathcal{T}_{\mathcal{A}}$,

$$\phi\big((A \cap B) \cap U\big) = \phi\big((A \cap U) \cap (B \cap U)\big) = \phi(A \cap U) \cap \phi(B \cap U),$$

thus $A \cap B \in \mathcal{T}_{\mathcal{A}}$. By analogous arguments we show that $\mathcal{T}_{\mathcal{A}}$ is closed under finite intersections. This completes the proof, in virtue of Definition A.1.1. □

Proposition 1.3.3. *Let $\mathcal{A}$ be an m-dimensional atlas on M and let $\mathcal{T}_{\mathcal{A}}$ be the topology induced by $\mathcal{A}$. Then every chart $(U, \phi) \in \mathcal{A}$ has the following properties:*

(i) $U \in \mathcal{T}_{\mathcal{A}}$.

(ii) The map $\phi \colon U \to \phi(U)$ *is a homeomorphism.*

Chapter 2

The tangent bundle

One is accustomed, from calculus, to thinking of the tangent plane to a smooth surface in $\mathbb{R}^3$ as a linear subspace of $\mathbb{R}^3$ (or, perhaps, as the plane obtained by translating this subspace to the point of tangency). Since a general differentiable manifold (e.g., a projective space) need not live naturally in any ambient Euclidean space, one is forced to seek another, intrinsic, characterization of tangent vectors if one wishes to define tangent spaces to such manifolds.

G. Naber [58, p. 198]

A differentiable map between Euclidean spaces $f\colon U \subseteq \mathbb{R}^m \to \mathbb{R}^n$ ($U \subseteq \mathbb{R}^m$ open) is locally approximated by a linear map $Df(x)\colon \mathbb{R}^m \to \mathbb{R}^n$, the differential (or derivative) of f at x. In order to obtain an analogous approximation on manifolds, we first need to associate a linear space (called the tangent space) to each point of the manifold. Tangent spaces will be the domains and ranges of the differentials. While in Real Analysis the differentials are defined between the constant spaces $\mathbb{R}^m$ and $\mathbb{R}^n$, the tangent spaces of manifolds vary with respect to their reference points.

The collection of all tangent spaces on a smooth manifold is a new smooth manifold of double dimension constituting the prototype of a vector bundle. The tangent bundle is the appropriate space to define the total differentials of smooth maps between smooth manifolds.

2.1 The tangent space

We start with the *kinematic* definition of a tangent space, based on a tangency notion of smooth curves. This is a geometric approach, much more simpler than the one based on the notion of derivation. Although the latter lacks the (geometric) intuition of the former, it is used very often for brevity reasons. We will deal with derivations in a later section, after having acquired enough experience in working with tangent vectors.

• In what follows, $M \equiv (M, \mathcal{A})$ will denote an arbitrary *m–dimensional smooth manifold*, without the topological restrictions of Convention 1.3.21. As a matter of fact, the Hausdorff property and second countability are not needed throughout this chapter.

Definition 2.1.1. A ***smooth curve*** on M is a smooth map $\alpha\colon J \to M$, where J is an open interval in $\mathbb{R}$ with $0 \in J$. The smoothness of α is meant in the sense of Definition 1.3.1, where J is provided with the structure of an open submanifold of $\mathbb{R}$. We say that α ***passes through*** (or α is a ***curve at***) $x \in M$ if $\alpha(0) = x$. For convenience, we often take J to be a symmetric (open) interval of the form $(-\varepsilon, \varepsilon)$.

A curve α, as above, is also called a (smooth) ***parametrized curve*** to emphasize that α is merely a map of one variable (parameter) between manifolds and not a set of points in M. This interpretation, quite different from that of Analytic Geometry, allows one to talk about the continuity and differentiability of a curve.

Definition 2.1.2. Let α, β be smooth curves on M passing through $x \in M$. Then α and β are called ***tangent*** or ***equivalent at*** x, if there is a chart (U, ϕ) of M such that $x \in U$ and

$$D(\phi \circ \alpha)(0) = D(\phi \circ \beta)(0). \tag{2.1.1}$$

If α, β are tangent at x, we write $\alpha \sim_x \beta$

Remarks 2.1.3. 1) The terminology *equivalent curves* will be justified shortly.

2) Obviously, to define the tangency of two curves, $\mathcal{C}^1$–differentiability is sufficient.

3) Since $D(\phi \circ \alpha)(0) \in \mathcal{L}(\mathbb{R}, \mathbb{R}^m)$, we have that

$$\big(D(\phi \circ \alpha)(0)\big)(\lambda) = \lambda\big(D(\phi \circ \alpha)(0)\big)(1)$$

for every $\lambda \in \mathbb{R}$. Therefore [see also Appendix B, equality(B.1.3)],

$$\begin{aligned} D(\phi \circ \alpha)(0) = D(\phi \circ \beta)(0) \quad &\Leftrightarrow \\ \big(D(\phi \circ \alpha)(0)\big)(1) = \big(D(\phi \circ \beta)(0)\big)(1) \quad &\Leftrightarrow \\ (\phi \circ \alpha)'(0) = (\phi \circ \beta)'(0) \quad &\Leftrightarrow \qquad (2.1.2) \\ (x^i \circ \alpha)'(0) = (x^i \circ \beta)'(0) & \qquad\qquad (2.1.3) \end{aligned}$$

for all $i = 1, \ldots, m$. Recall that $(x^i) \equiv \{x^i\}_{i=1,\ldots,m}$ are the local coordinates induced by the chart (U, ϕ).

We will show that Definition 2.1.1 is independent of the choice of charts containing x, thus the tangency relation is well-defined.

Proposition 2.1.4. *If α, β are tangent at $x \in M$, then*

$$D(\psi \circ \alpha)(0) = D(\psi \circ \beta)(0),$$

for every chart (V, ψ) of M with $x \in V$.

Proof. Assume that α, β are tangent at x. Then $\alpha(0) = \beta(0) = x$, and $D(\phi\circ\alpha)(0) = D(\phi\circ\beta)(0)$, for some chart (U, ϕ) with $x \in U$. For any other chart (V, ψ), with $x \in V$, we have in virtue of the ordinary chain rule:

$$\begin{aligned}
D(\psi \circ \alpha)(0) &= D\big((\psi \circ \phi^{-1}) \circ (\phi \circ \alpha)\big)(0) \\
&= D(\psi \circ \phi^{-1})\big((\phi \circ \alpha)(0)\big) \circ D(\phi \circ \alpha)(0) \\
&= D(\psi \circ \phi^{-1})\big((\phi \circ \beta)(0)\big) \circ D(\phi \circ \beta)(0) \\
&= D\big((\psi \circ \phi^{-1}) \circ (\phi \circ \beta)\big)(0) \\
&= D(\psi \circ \beta)(0).
\end{aligned}$$

□

Corollary 2.1.5. *The tangency of curves at $x \in M$ is an equivalence relation.*

Proof. Reflection and symmetry of $\alpha \sim_x \beta$ are clear. Assume that $\alpha \sim_x \beta$ and $\beta \sim_x \gamma$. The first implies that $D(\phi \circ \alpha)(0) = D(\phi \circ \beta)(0)$ for any chart (U, ϕ) with $x \in U$. Similarly, $D(\phi\circ\beta)(0) = D(\phi\circ\gamma)(0)$ because of Proposition 2.1.4. Therefore, $\alpha \sim_x \gamma$, which implies the transitivity of the relation. □

In virtue of the preceding corollary, the set of smooth curves at $x \in M$ is partitioned into equivalence classes. The equivalence class of a curve α at x is denoted by $[(\alpha, x)]$.

Definition 2.1.6. The set of equivalence classes of all smooth curves on M, passing through x, is called the ***tangent space of*** M ***at*** x and will be denoted by T_xM. The elements of T_xM are called ***tangent vectors at*** x.

Tangent vectors will be simply denoted by u, v, w etc. if there is no need to mention explicitly the corresponding equivalence class of curves. The previous notation and the term *space* suggest that T_xM is something more than a set. Before proving that, in fact, it is a vector space, we have the following.

Proposition 2.1.7. *Let M be an m-dimensional smooth manifold and $x \in M$. If (U, ϕ) is a chart at x, then the map*

$$\overline{\phi}\colon T_xM \longrightarrow \mathbb{R}^m\colon [(\alpha, x)] \mapsto \big(D(\phi \circ \alpha)(0)\big)(1) = (\phi \circ \alpha)'(0)$$

is a well defined bijection.

Proof. i) The map $\overline{\phi}$ is well defined means that it is independent of the representative of the class $[(\alpha, x)]$. Indeed, let $\beta \in [(\alpha, x)]$. Then, by definition, $(\phi \circ \alpha)'(0) = (\phi \circ \beta)'(0)$, which proves the claim.

ii) For the injectivity of $\overline{\phi}$, assume that $[(\alpha, x)], [(\beta, x)] \in T_xM$ with $\overline{\phi}([(\alpha, x)]) = \overline{\phi}([(\beta, x)])$. Then $(\phi \circ \alpha)'(0) = (\phi \circ \beta)'(0)$ and Remark 2.1.3 (3) shows that α and β are tangent at x, thus $[(\alpha, x)] = [(\beta, x)]$.

iii) Finally, to prove that $\overline{\phi}$ is a surjection, we take any $h \in \mathbb{R}^m$, and consider the affine map (in fact the straight line through $\phi(x)$ with direction h)

$$\varepsilon \colon \mathbb{R} \longrightarrow \mathbb{R}^m \colon t \mapsto \phi(x) + th.$$

Obviously, ε is smooth with $\varepsilon(0) = \phi(x)$ and $\varepsilon'(0) = h$. Since ε is continuous, and $\phi(U)$ is an open neighborhood of $\phi(x) = \varepsilon(0)$, there exists an open interval J containing 0 such that $\varepsilon(J) \subset \phi(U)$. Hence, we can define the composite $\alpha := \phi^{-1} \circ \varepsilon|_J \colon J \to U \subset M$. Then α is a smooth curve on M, such that $\alpha(0) = x$ and

$$\overline{\phi}([(\alpha, x)]) = (\phi \circ \alpha)'(0) = \varepsilon'(0) = h.$$

This completes the proof. □

For later use we single out the following useful conclusion derived from the proof of surjectivity of $\overline{\phi}$:

Corollary 2.1.8. *For every $h \in \mathbb{R}^m$, the inverse of $\overline{\phi}$ is given by $\overline{\phi}^{-1}(h) = [(\alpha, x)]$, where $\alpha(t) = \phi^{-1}(\phi(x) + th)$, with t varying in a suitable interval $J \subseteq \mathbb{R}$ such that $\alpha(J) \subseteq \phi(U)$.*

The map $\overline{\phi}$ identifies the sets T_xM and $\mathbb{R}^m$. Therefore, we can transfer the algebraic structure of $\mathbb{R}^m$ to T_xM by setting

$$\begin{aligned} u + v &:= \overline{\phi}^{-1}\left(\overline{\phi}(u) + \overline{\phi}(v)\right) \\ \lambda u &:= \overline{\phi}^{-1}\left(\lambda \overline{\phi}(u)\right) \end{aligned} \tag{2.1.4a}$$

for every $u, v \in T_xM$ and $\lambda \in \mathbb{R}$.

We note that the previous structure of T_xM can be defined also by the relation

$$\lambda u + \mu v := \overline{\phi}^{-1}(\lambda \overline{\phi}(u) + \mu \overline{\phi}(v)),$$

whence we obtain separately the addition and the scalar multiplication by setting, respectively, $u + v := 1u + 1v$ and $\lambda u := \lambda u + 0u$.

The following result is routinely checked.

Theorem 2.1.9. *If M is an m-dimensional smooth manifold, $x \in M$ and (U, ϕ) a chart of M with $x \in U$, then the operations ((2.1.4a)) induce a vector space structure on T_xM such that $\overline{\phi}$ is a linear isomorphism.*

If $(U, \phi), (V, \psi)$ are charts with $x \in U \cap V$, then we obtain the two isomorphisms

$$\overline{\phi}, \overline{\psi} \colon T_xM \to \mathbb{R}^m.$$

In general, $\overline{\phi} \neq \overline{\psi}$, thus apparently they induce two different vector space structures on T_xM. However, this is not true. To verify this, let us denote by

$$\lambda \odot u \oplus \mu \odot v := \overline{\psi}^{-1}\big(\lambda \overline{\psi}(u) + \mu \overline{\psi}(v)\big), \tag{2.1.4b}$$

the analogous operations induced by $\overline{\psi}$. Then, to show that the two linear structures on T_xM coincide, we should show that

$$\lambda \odot u \oplus \mu \odot v = \lambda u + \mu v,$$

or, equivalently,

$$\overline{\psi}^{\,-1}\big(\lambda\overline{\psi}(u) + \mu\overline{\psi}(v)\big) = \overline{\phi}^{\,-1}\big(\lambda\overline{\phi}(u) + \mu\overline{\phi}(v)\big). \tag{2.1.5}$$

For our purpose we first need the following:

Lemma 2.1.10. *If (U, ϕ) and (V, ψ) are charts containg x, then*

$$\overline{\psi} \circ \overline{\phi}^{\,-1} = D(\psi \circ \phi^{-1})(\phi(x));$$

in other words, the following diagram is commutative.

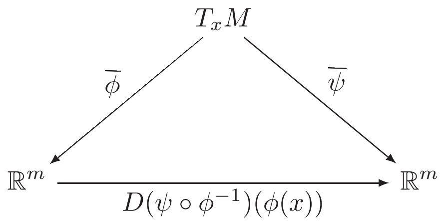

Diagram 2.1

Proof. Let $h \in \mathbb{R}^m$. Since $\overline{\phi}$ is an isomorphism, there exists a unique $[(\alpha, x)] \in T_xM$ with $\overline{\phi}([(\alpha, x)]) = h$. Consequently [see also (B.1.3)],

$$\begin{aligned}
\big(\overline{\psi} \circ \overline{\phi}^{\,-1}\big)(h) = \overline{\psi}\big([(\alpha, x)]\big) &= \big(D(\psi \circ \alpha)\big)(1) \\
&= \big(D(\psi \circ \phi^{-1} \circ \phi \circ \alpha)(0)\big)(1) \\
&= \big(D(\psi \circ \phi^{-1})(\phi \circ \alpha(0)) \circ D(\phi \circ \alpha)(0)\big)(1) \\
&= \big(D(\psi \circ \phi^{-1})(\phi(x))\big)\big((\phi \circ \alpha)'(0)\big) \\
&= \big(D(\psi \circ \phi^{-1})(\phi(x))\big)(h),
\end{aligned}$$

as desired. □

Remark 2.1.11. In the preceding proof, a representative of the class $[(\alpha, x)]$ is also the curve $\phi^{-1} \circ \varepsilon|_J$, where $\varepsilon(t) = \phi(x) + th$ (see Corollary 2.1.8. Using this particular curve, the above series of equalities can be modified as follows:

$$\begin{aligned}
\big(\overline{\psi} \circ \overline{\phi}^{\,-1}\big)(h) &= \overline{\psi}\big(\big[(\phi^{-1} \circ \varepsilon|_J, x)\big]\big) \\
&= \big(\psi \circ \phi^{-1} \circ \varepsilon\big)'(0) \\
&= \big(D(\psi \circ \phi^{-1} \circ \varepsilon)(0)\big)(1) \\
&= \big(D(\psi \circ \phi^{-1})(\varepsilon(0)) \circ D(\varepsilon)(0)\big)(1) \\
&= \big(D(\psi \circ \phi^{-1})(\phi(x))\big)(\varepsilon'(0)) \\
&= \big(D(\psi \circ \phi^{-1})(\phi(x))\big)(h).
\end{aligned}$$

Theorem 2.1.12. *The vector space structure of T_xM does not depend on the choice of the chart (U, ϕ), as defined by* (2.1.4a) *and Theorem 2.1.9.*

Proof. If (U, ϕ), (V, ψ) are charts at x, then (2.1.4b) is equivalent to

$$\lambda\overline{\psi}(u) + \mu\overline{\psi}(v) = \big(\overline{\psi} \circ \overline{\phi}^{-1}\big)\big(\lambda\overline{\phi}(u) + \mu\overline{\phi}(v)\big),$$

which holds, since $\overline{\psi} \circ \overline{\phi}^{1}$ is a linear map in virtue of Lemma 2.1.10. □

Remark 2.1.13 (The tangent space of a surface).
Let S be a *regular surface* [see Example 1.2.12 (G)]. One way to define the tangent space T_pS at a point $p \in S$ is the following: If (U, r) is a parametrization of S at p, and $q \in U$ with $r(q) = p$, then the tangent space of S at p is defined to be the space $T_pS := Dr(q)(\mathbb{R}^2)$. Since $Dr(q)$ is injective, the vectors

$$r_u(q) := \frac{\partial r}{\partial u}\bigg|_q = Dr(q)(e_1), \quad \text{and} \quad r_v(q) := \frac{\partial r}{\partial v}\bigg|_q = Dr(q)(e_2)$$

form a basis of $Dr(q)(\mathbb{R}^2)$, thus T_pS is the vector space spanned by $\{r_u(q), r_v(q)\}$, and $\mathbb{R}^2 \xrightarrow{Dr(q)} DT_pS$ is a linear isomorphism (see, for instance, [13], [52], [62]).

On the other hand, thinking of S as a smooth manifold, we obtain the tangent space of S at p as in the general case of Definition 2.1.6. To avoid confusion, let us denote the latter space by $\widetilde{T}pS$. In particular, using the chart $(\widetilde{U}, \phi)$ of S at p, induced by (U, r); that is, $\widetilde{U} = r(U)$ and $\phi = r^{-1}$, we obtain the linear isomorphism $\widetilde{T}pS \xrightarrow{\bar{\phi}} \mathbb{R}^2$. Note that since $\phi = r^{-1}$ is taking values in S, $D\phi(p)$ does not make sense, thus $\widetilde{T}pS$ cannot be interpreted as the image of it.

It is now clear that $\widetilde{T}pS \equiv T_pS$ via the isomorphism $Dr(q) \circ \bar{\phi}$.

An equivalent description of T_pS, frequently used in the theory of surfaces, is the following. First denote by $C(S, p)$ the set of all smooth curves $\beta\colon J_\beta \to \mathbb{R}^3$ such that J_β is an open interval containing 0, $\beta(J_\beta) \subset S$, $\beta(0) = p$. Then we define the set $S_p := \{\beta'(0) \mid \beta \in C(S, p)\}$. Now, consider an arbitrary tangent vector $[(\alpha, p)] \in \widetilde{T}_pS$, where $\alpha\colon J_\alpha \to S$ is a smooth curve with J_α open interval containing 0 and $\alpha(0) = p$. Then

$$(Dr(q) \circ \bar{\phi})([(\alpha, p)]) = Dr(q)\big(\phi \circ \alpha)'(0)\big) = \alpha'(0) \in S_p.$$

Note that α being an S-valued smooth curve it is also smooth as an $\mathbb{R}^3$-valued curve, because in the latter case it can be thought of as the composite of α and the canonical inclusion $i \hookrightarrow S$, thus $\alpha'(0) \in S_p$. The association $[(\alpha, p)] \mapsto \alpha'(0)$ is obviously injective. It is also surjective. Indeed, for an arbitrary $\beta'(0) \in S_p$, we check that $\alpha(t) := \phi^{-1}\big(q + t\beta'(0)\big)$ determines a smooth curve in S through p, for sufficiently small t. Therefore, $T_pS = \big(Dr(q) \circ \bar{\phi}\big)(\widetilde{T}pS) = S_p$.

The tangent space T_pS being a 3-dimensional vector subspace of $\mathbb{R}^3$ represents a plane in $\mathbb{R}^3$ passing through 0. Thus there is a unit vector N vertical to T_pS and passing through p. The parallel translation of T_pS to p is a plane passing through

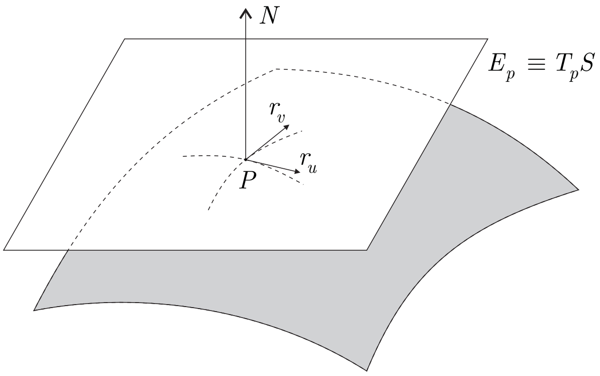

Figure 2.1 Tangent plane and norma vector

p, called the *tangent plane of* S *at* p and denoted by E_p. Figure 2.1 above is a familiar picture in which the tangent space of the surface at p is identified with the tangent plane E_p, with the vectors N, r_u and r_v also translated to p. The role of N is very important to the geometry of a surface because its "differentiation" leads to the fundamental notion of *curvature*. More details can be found in the books cited in the beginning of our comments.

Definition 2.1.14. Consider a chart (U, ϕ) at x, with coordinate functions $\{x^i\}_{i=1,\dots,m}$. If $\{e_i\}_{i=1,\dots,m}$ is the the canonical basis of $\mathbb{R}^m$, then the set $\{\overline{\phi}^{-1}(e_i)\}_{i=1,\dots,m}$ is a basis of T_xM, called the ***canonical basis of*** T_xM ***with respect to*** (U, ϕ).

We customarily set

$$\left.\frac{\partial}{\partial x^i}\right|_x := \overline{\phi}^{-1}(e_i), \qquad i = 1, \dots, m. \tag{2.1.6}$$

The reader probably suspects that the left-hand side of (2.1.6) might have the meaning of partial differentiation. Indeed this is true in a sense that will be explained in § 2.3 below.

A consequence of the preceding considerations is that every vector $u \in T_xM$ is written in the form

$$u = \sum_{i=1}^{m} \lambda^i \left.\frac{\partial}{\partial x^i}\right|_x, \quad \lambda^i \in \mathbb{R}. \tag{2.1.7}$$

In the particular case of the (trivial) 1–dimensional manifold $\mathbb{R}$, the canonical basis of $T_s\mathbb{R}$ ($s \in \mathbb{R}$), with respect to the standard chart $(\mathbb{R}, \mathrm{id}_{\mathbb{R}})$, consists of the vector

$$\left.\frac{d}{dt}\right|_s := \left(\overline{\mathrm{id}}_{\mathbb{R}}\right)^{-1}(1). \tag{2.1.8}$$

The tangent space at a point of the cartesian product of two manifolds or at a point of an open submanifold will be computed in the next section.

Exercises 2.1.15. **1.** Explain why the tangent space at any point f a vector space identifies with the vector space itself.

2. Let $u \in T_xM$. Can you specify a curve representing u in terms of the vector u itself?

3. Let $u, v \in T_xM$ and $\lambda \in \mathbb{R}$. Find curves γ, δ on M, such that $u + v = [(\gamma, x)]$ and $\lambda u = [(\delta, x)]$.

3. Find a curve representing the zero element $0_x \in T_xM$.

3. Find a curve representing the basic vector $\left.\dfrac{\partial}{\partial x^i}\right|_x$ corresponding to the chart (U, ϕ)..

3. Let the unit sphere and its point $p = (0, 1, 0)$. If $v \in T_pS^2$ is the tangent vector with coordinates $(2, -1)$, with respect to the chart (U_y^+, ϕ_y^+) [see Example 1.2.12 (D2)], find a smooth curve representing v.

3. Show that the curves $\alpha, \beta \colon \mathbb{R} \to \mathbb{R}^3$, with $\alpha(t) = (t, 1 + t^2, t)$ and $\beta(t) = (\sin t, \cos t, t)$ define the same tangent vector $u \in T_{(0,1,0)}\mathbb{R}^3$. Compute the coordinates of u.

8. Does the curve $\alpha \colon \mathbb{R} \to S^2$ defines a tangent vector of S^2? Find its coordinates with respect to the appropriate charts derived from the stereographic projections and hemispheres.

9. If $\alpha \colon \mathbb{R} \to \mathcal{M}_{2\times 2}(\mathbb{R})$ is the smooth curve given by

$$\alpha(t) := \begin{pmatrix} t+1 & 2t^2 \\ 3t & 2t+1 \end{pmatrix},$$

compute the coordinates of the tangent vector $u = [(\alpha, I)] \in T_I\mathcal{M}_{2\times 2}(\mathbb{R})$, with respect to the standard chart of $\mathcal{M}_{2\times 2}(\mathbb{R})$. Here I denotes the unit matrix.

10. Let $\phi \colon \mathbb{R}^2 \to \mathbb{R}^2$ the map given by check map

$$\phi(t, s) := \left(\frac{2t+s}{3}, \frac{2t-s}{3}\right).$$

Prove that $(\mathbb{R}^2, \phi)$ is a chart of the standard smooth structure of $\mathbb{R}^2$, and find a curve representing the tangent vector

$$u = 2\left.\frac{\partial}{\partial x^1}\right|_p - 3\left.\frac{\partial}{\partial x^2}\right|_p,$$

where $p = \left(\frac{1}{2}, 0\right)$, and x^1, x^2 are the coordinates of the previous chart.

11. Let $\alpha \colon \mathbb{R} \to \mathbb{P}_2(\mathbb{R})$ be given by $f(t) = [(2t+1, 1, t^2)]$. Show that f is smooth and determines a tangent vector u of $\mathbb{P}_2(\mathbb{R})$. What are the coordinates of the latter?

2.2 Differentials or tangent maps

Tangent spaces provide the appropriate vector spaces needed to define the differential of smooth maps, generalizing thus the ordinary derivative in Euclidean spaces. More precisely:

Chapter 4

Lie groups

Lie groups are of the greatest importance in modern theoretical physics and the subject can be approached from a variety of perspectives.

C.J. Isham [37, p. 61]

Lie groups are groups provided with a smooth structure such that the operations of multiplication and inversion are smooth maps. A particular feature of a Lie group is the structure of the Lie algebra defined on the tangent space at the identity. It is identified with the Lie algebra of all left (or right)-invariant vector fields of the group. By means of its Lie algebra, a Lie group admits an exponential map, generalizing the ordinary matrix exponential. Moreover, a Lie group has a natural representation into its Lie algebra (the adjoint representation). Another interesting property of Lie groups is that the left-invariant vector fields are complete and their integral curves reflect the group structure. The aim of the present chapter is to discuss these fundamental concepts.

4.1 Lie groups and Lie group morphisms

Throughout this chapter, G will mainly denote a multiplicative group with identity (neutral) element e. The ***multiplication*** or ***product map*** will be denoted by

$$\boldsymbol{\gamma}\colon G \times G \longrightarrow G\colon (x,y) \mapsto \boldsymbol{\gamma}(x,y) := xy,$$

while the ***inversion map*** is

$$\boldsymbol{\alpha}\colon G \longrightarrow G\colon x \mapsto \alpha(x) := x^{-1}$$

Scholium on the notation: $\boldsymbol{\gamma}$ comes from the Greek word γινόμενο, meaning product; $\boldsymbol{\alpha}$ is the first letter of αντιστροφή, the Greek word for inversion. We use for both symbols bold typefaces to distinguish them from ordinary α and γ, usually denoting curves. Occasionally, we will write $x \cdot y$, if the factors have a complicated form.

A Lie group combines its algebraic structure with a differential structure in the following sense of compatibility.

Definition 4.1.1. A ***Lie group*** is a group (in the algebraic sense) G together with a smooth structure such that the operations of multiplication $\boldsymbol{\gamma}$ and inversion $\boldsymbol{\alpha}$ are smooth maps.

We will soon ascertain that it is redundant to incorporate the smoothness of $\boldsymbol{\alpha}$ in the very definition of a Lie group, because this is a consequence of the smoothness of $\boldsymbol{\gamma}$. However, for convenience, Definition 4.1.1 is in use in the majority of books.

In preparation to the proof of the above assertion, we introduce the following notions of translation. More precisely, the ***left translation*** of G by an element $g \in G$ is the map

$$L_g \colon G \longrightarrow G \colon x \longmapsto L_g(x) := gx,$$

whereas the ***right translation*** by $g \in G$ is defined by

$$R_g \colon G \longrightarrow G \colon x \longmapsto R_g(x) := xg.$$

Proposition 4.1.2. *For every $g \in G$, the translations L_g and R_g are diffeomorphisms.*

Proof. Since $L_g = \boldsymbol{\gamma}(g, \cdot) = \boldsymbol{\gamma}_g$, Exercise 1.4.17 (9) implies that L_g is smooth. On the other hand, $L_g \circ L_{g^{-1}} = \mathrm{id}_G$ and $L_{g^{-1}} \circ L_g = \mathrm{id}_G$, thus $(L_g)^{-1} = L_{g^{-1}}$ is also smooth, and L_g is a diffeomorphism. Similar arguments hold for R_g. ☐

If U and V are subsets of any group G, then we set

$$U \cdot V := \boldsymbol{\gamma}(U \times V) = \{xy \mid x \in U, y \in V\}.$$

Lemma 4.1.3. *Let G be a Lie group. Then*

(i) For every open $U \subset G$ and every $g \in G$, the set $gU := L_g(U)$ is an open subset of G. Similarly for $Ug := R_g(U)$.
(ii) For every open subsets U and V of G, $U \cdot V$ is also open in G.
(iii) For any neighborhood U of e, there is a neighborhood $V \subset U$ of e such that $V \cdot V \subset U$.

Proof. The sets gU and Ug are homeomorphic images of U (by L_g and R_g, respectively), whence the first assertion. The second is a consequence of equality

$$U \cdot V = \bigcup_{g \in V} Ug.$$

Finally, given a neighborhood U of e, the continuity of $\boldsymbol{\gamma}$ and the product topology imply the existence of two neighborhoods V_1 and V_2 of e such that $\boldsymbol{\gamma}(V_1 \times V_2) \subseteq U$. Setting $V = V_1 \cap V_2$ we prove the last assertion. ☐

We notice that the previous lemma is valid for every ***topological group***, i.e. a group (in the algebraic sense) equipped with a topology such that $\boldsymbol{\gamma}$ and $\boldsymbol{\alpha}$ are continuous. Topological groups form an important category of groups. Details can be found, e.g., in [31] and [11].

Theorem 4.1.4. *Let G be a group equipped with a smooth structure such that the multiplication $\boldsymbol{\gamma}$ is smooth. Then the inversion map α is a diffeomorphism and G becomes a Lie group.*

Proof. ▼ We first prove that the inversion $\boldsymbol{\alpha}$ is smooth at e: Let (U, ϕ) be a chart of G with $\phi(e) = 0$ [see Exercise 1.2.15(15)] and $V \subset U$ an neighborhood of e such that $V \cdot V \subset U$ (by Lemma 1.4.3). We define the chart (V, ψ), where $\psi = \phi|_V$. Considering now the charts $(V \times V, \psi \times \psi)$ and (U, ϕ), of $G \times G$ and G, respectively, with $(e, e) \in G \times G$ and $e \in G$, we see that $\boldsymbol{\gamma}(V \times V) = V \cdot V \subset U$, thus we obtain the local representation of $\boldsymbol{\gamma}$

$$\Gamma = \phi \circ \boldsymbol{\gamma} \circ (\psi \times \psi)^{-1} \colon \psi(V) \times \psi(V) \longrightarrow \phi(U).$$

shown also in the next commutative diagram.

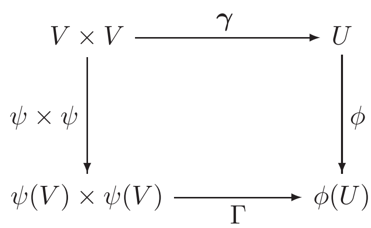

Diagram 4.1

Then, for every $(0, u) \in \psi(V) \times \psi(V)$,

$$\begin{aligned} \Gamma(0,u) &= \left(\phi \circ \boldsymbol{\gamma} \circ (\psi \times \psi)^{-1}\right)(0,u) = \\ &= (\phi \circ \boldsymbol{\gamma})\left(e, \psi^{-1}(0)\right) = \phi\left(\psi^{-1}(u)\right) = u. \end{aligned}$$

Since we want to apply the implicit function theorem in Euclidean spaces (see Theorem B.4.3), we compute the Jacobian matrix

$$J = \left(\left. \frac{\partial \Gamma^i}{\partial u^j} \right|_{(0,0)} \right)$$

where $i, j = 1, \ldots, n = \dim(G)$. It should be noted that, for the correct application of the implicit function theorem, the partial derivatives are taken with respect to the n variables of the *second* factor in $\mathbb{R}^n \times \mathbb{R}^n$. With this remark in mind, we check that

$$\left. \frac{\partial \Gamma^i}{\partial u^j} \right|_{(0,0)} = \left. \frac{\partial}{\partial u^j} \right|_0 \left(\Gamma^i(0, \cdot) \right) = \delta_{ij},$$

thus $\det J \neq 0$. As a result, by the aforementioned theorem, there are neighborhoods A and B of 0, and a smooth function $f\colon A \to B$, such that $A \times B \subseteq \psi(V) \times \psi(V)$, $f(0) = 0$ and $\Gamma(a, f(a)) = 0$, for every $a \in A$. Clearly, the sets $W_1 = \psi^{-1}(A) \subset V$ and $W_2 = \psi^{-1}(B) \subset V$ are neighborhoods of e and

$$F = \psi^{-1} \circ f \circ \psi|_{W_1} \colon W_1 \longrightarrow W_2$$

is a smooth map. Moreover, by Diagram 4.1 on the preceding page, and for every $x \in W_1$,

$$\begin{aligned} \boldsymbol{\gamma}\big(x, F(x)\big) &= \big(\phi^{-1} \circ \Gamma \circ (\psi \times \psi)\big)\big(x, F(x)\big) \\ &= \big(\phi^{-1} \circ \Gamma\big)\big(\psi(x), \psi(F(x)\big) \\ &= \big(\phi^{-1} \circ \Gamma\big)\big(\psi(x), f(\psi(x)\big) \\ &= \phi^{-1}(0) = e; \end{aligned}$$

in other words, $x \cdot F(x) = e$, or $F(x) = x^{-1}$, which implies that $F = \boldsymbol{\alpha}|_{W_1}$; hence, $\boldsymbol{\alpha}$ is smooth at e.

We now show that α is smooth at an arbitrary $g \in G$: With F as before, we consider the smooth map (see Proposition 4.1.2)

$$R_{g^{-1}} \circ F \circ L_{g^{-1}} \colon g\,W_1 \longrightarrow W_2 \cdot g,$$

shown also in the following diagram.

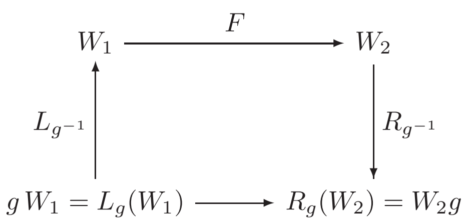

Diagram 4.2

Then, for every $x \in g\,W_1$ (thus $g^{-1}x \in W_1$),

$$\begin{gathered} \big(R_{g^{-1}} \circ F \circ L_{g^{-1}}\big)\,(x) = \big(R_{g^{-1}} \circ F\big)\big(g^{-1}x\big) = \\ = F\big(g^{-1}x\big) \cdot g^{-1} = \boldsymbol{\alpha}\,\big(g^{-1}x\big) \cdot g^{-1} = x^{-1}gg^{-1} = x^{-1}. \end{gathered}$$

Therefore, $\boldsymbol{\alpha}|_{gW_1} = R_{g^{-1}} \circ F \circ L_{g^{-1}}|_{gW_1}$, which shows that $\boldsymbol{\alpha}$ is smooth at g, since $g\,W_1$ is a neighborhood of g by Lemma 4.1.3.

Finally, it is trivially checked that $\boldsymbol{\alpha}$ is a bijection whose inverse coincides with $\boldsymbol{\alpha}$ itself, therefore it is a diffeomorphism. ▲ □

Corollary 4.1.5. *Let G be a Lie group. If $U \subseteq G$ and*

$$U^{-1} := \{x^{-1} \,|\, x \in U\} = \boldsymbol{\alpha}(U),$$

then, the following assertions are true:

(i) If U is open, then so is U^{-1}.

(ii) For any neighborhood U of e, there exists a neighborhood $V \subset U$ of e such that $V \cdot V^{-1} \subset U$.

Proof. Since $\boldsymbol{\alpha}$ is a diffeomorphism, (i) clear. To prove (ii) we consider a neighborhood W of e such that $W \cdot W \subset U$ (see Lemma 4.1.3). Setting $V = W \cap W^{-1}$, we check that $V = V^{-1}$; hence, $V \cdot V^{-1} \subset W \cdot W \subset U$. □

By Convention 1.3.21, a smooth manifold is always assumed to be a Hausdorff space. In the following result, we prove that this topological property is automatically satisfied by a Lie group, as already noted in the comments following the above convention.

Theorem 4.1.6. *Let G be a group together with a smooth structure not satisfying à priori Convention 1.3.21. Then G is a Hausdorff space.*

Proof. Let g and h be two arbitrary distinct elements of G. Then $g^{-1}h \neq e$. Since G is a T_1 space (see Proposition 1.3.13), there is a neighborhood U of e such that

$$g^{-1}h \notin U. \tag{4.1.1}$$

On the other hand, if W is a neighborhood of e such that $W \cdot W^{-1} \subset U$ (as assured by Corollary 4.1.5), then $(g^{-1}h) \cdot W$ is a neighborhood of $g^{-1}h$ such that

$$W \cap (g^{-1}h) \cdot W = \emptyset. \tag{4.1.2}$$

Indeed, if there were an element y in the previous intersection, then $y = (g^{-1}h)x$, for some $x \in W$; hence, $g^{-1}h = yx^{-1} \in W \cdot W^{-1} \subset U$, which contradicts (4.1.1).

We now define the neighborhoods of g and h, respectively, $U_1 = g\,W$ and $U_2 = h\,W$. We claim that $U_1 \cap U_2 = \emptyset$, for if there is a $z \in U_1 \cap U_2$, then $z = gx = hy$, for some $x, y \in W$. As a result, $x = (g^{-1}h)y \in (g^{-1}h) \cdot W$, i.e. $x \in W \cap (g^{-1}h)W$, a contradiction to (4.1.2). Therefore, U_1 and U_2 satisfy the Hausdorff property for the elements g, h, and prove the theorem. □

It is clear that the preceding proof relies, from its very beginning, on the basic property of a smooth structure, namely the separation property T_1. This is not a property shared by every topological group, although most of the well known examples are T_1 spaces.

Examples 4.1.7.

(A) $(\mathbb{R}^n, +)$ *is a Lie group*

Taking into account the standard differential structure of $\mathbb{R}^n$ [see Example 1.2.13 (A)], the claim is obvious.

Note that this is a typical example of an ***abelian*** Lie group. We recall that a group is abelian if the multiplication is commutative, whence the alternative term ***commutative***. In this case, we usually write $x + y$ instead of xy.

(B) *The general linear group* $\mathrm{GL}(n, \mathbb{R})$

We recall that $\mathrm{GL}(n,\mathbb{R})$ consists of all invertible $n \times n$ real matrices provided with the operation of matrix multiplication. Then $\mathrm{GL}(n,\mathbb{R})$, equipped with the open submanifold structure induced from the vector space $\mathcal{M}_{n\times n}(\mathbb{R})$ [see Example 1.3.17 (C)], is a Lie group, as we routinely check in virtue of Example 1.4.8 (C). For the sake of completeness, let us verify the smoothness of the inversion map $A \mapsto A^{-1}$. If $A = (a_{ij})$ and $A^{-1} = (b_{ij})$, then

$$b_{ij} = \frac{1}{\det A}(-1)^{i+j} \det A_{ij},$$

where A_{ij} is the $(n-1)\times(n-1)$ submatrix of A obtained by deleting the ith row and the jth column of A. Since det is a smooth function of the a_{ij}'s [see Exercise 1.4.17 (3)], so is every entry b_{ij}. Applying the global chart $(\mathrm{GL}(n,\mathbb{R}),\Phi)$ we immediately see that the local representation of the inversion is smooth, and so is the inversion itself.

(C) *The multiplicative group* $\mathbb{R}_* = \mathbb{R}\setminus 0 = \mathbb{R}$-$\{0\}$

This is a special case of (B), for $n = 1$.

(D) *The unit circle* $S^1 \subset \mathbb{R}^2$

We consider the following multiplication in $\mathbb{R}^2$:

$$(*) \qquad (a,b)\cdot(c,d) = (ac - bd, ad + bc).$$

An immediate computation shows that $(a,b)\cdot(c,d) \in S^1$, for every (a,b) and (c,d) in S^1. Therefore, restricting the previous multiplication to S^1, we define the map

$$\boldsymbol{\gamma}\colon S^1 \times S^1 \longrightarrow S^1 : \boldsymbol{\gamma}((a,b),(c,d)) = (a,b)\cdot(c,d).$$

Using one of the differential structures of S^1, defined in Example 1.2.12 (E), or in Exercises 1.2.14 (8) and 1.2.14 (9), we see that $\boldsymbol{\gamma}$ is smooth, thus S^1 becomes a Lie group.

Although Theorem 4.1.4 ensures the smoothness of the inversion map, we can verify it directly from the definition of the above product rule. As matter of fact, an easy computation yields

$$(a,b)^{-1} = \frac{1}{a^2+b^2}(a,-b).$$

The reader undoubtedly recognizes that $(*)$ is the multiplication of complex numbers restricted to S^1, since $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, thus he/she may may use this approach in the study of S^1.

In Example 4.1.10 (A) we will show that the tangent bundle of a Lie group is also a Lie group. The list of examples will be completed (not exhaustively) much later, in § 5.4, by a few subgroups of $\mathrm{GL}(n\mathbb{R})$.

Definition 4.1.8. Let G, H be Lie groups. A map $f\colon G \to H$ is a ***morphism of Lie groups***, if it is a smooth group homomorphism.

In most books a morphism of Lie group is called ***Lie group homomorphism***. We prefer the terminology of the above definition because of its categorical flavor.

Examples 4.1.9 (of morphisms). The *determinant map*

$$\det\colon \mathrm{GL}(n,\mathbb{R}) \longrightarrow \mathbb{R}_*\colon A \mapsto \det(A)$$

and the ordinary *exponential map*

$$\exp\colon \mathbb{R} \longrightarrow \mathbb{R}_*\colon t \mapsto e^t$$

are group homomorphisms, since $\det(A \cdot B) = \det(A)\det(B)$ and $\exp(t+s) = \exp(t)\exp(s)$. Their differentiability is immediately checked by routine application of the standard differential structures of the spaces involved.

To complete the preceding examples of Lie groups and morphisms, we need to compute the differentials of $\boldsymbol{\gamma}$ and $\boldsymbol{\alpha}$. First, Combining the Leibniz formula (2.2.14) with the definition of the left and right translations, we obtain

$$\begin{aligned} T_{(x,y)}\boldsymbol{\gamma}(u,v) &= T_x\boldsymbol{\gamma}_y(u) + T_y\boldsymbol{\gamma}_x(u) \\ &= T_xR_y(u) + T_yL_x(v), \end{aligned} \tag{4.1.3}$$

for every $(x,y) \in G \times G$ and $(u,v) \in T_xG \times T_yG$.

Next, to find $T_x\boldsymbol{\alpha}$, we consider the commutative diagram

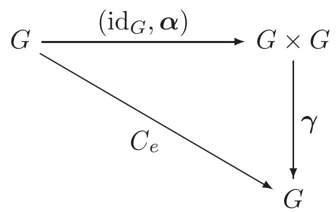

Diagram 4.3

where $C_e\colon G \to G$ is the constant map $C_e(x) = e$ As a result, for every $x \in G$ and every $u \in T_eG$,

$$\begin{aligned} 0 = T_xC(u) &= T_x\big(\boldsymbol{\gamma} \circ (\mathrm{id}_G, \boldsymbol{\alpha})\big)(u) \\ &= T_{(x,x^{-1})}\boldsymbol{\gamma}\big(T_x\,\mathrm{id}_G(u), T_x\boldsymbol{\alpha}(u)\big) \\ &= T_xR_{x^{-1}}(u) + T_{x^{-1}}L_x(T_x\boldsymbol{\alpha}(u)), \end{aligned}$$

which implies that

$$T_{x^{-1}}L_x(T_x\boldsymbol{\alpha}(u)) = -T_xR_{x^{-1}}(u).$$

Taking into account that L_x is a diffeomorphism with $(L_x)^{-1} = L_{x^{-1}}$, it follows that [see Example 2.2.9 (C)]

$$\big(T_{x^{-1}}L_x\big)^{-1} = T_{L_x(x^{-1})}L_{x^{-1}} = T_eL_{x^{-1}};$$

hence,

$$T_x\boldsymbol{\alpha}(u) = -T_eL_{x^{-1}} \circ T_xR_{x^{-1}}(u) = -T_x(L_{x^{-1}} \circ R_{x^{-1}})(u), \tag{4.1.4}$$

for every $x \in G$ and every $u \in T_xG$.

We are now in a position to work out two more examples, as promised.

Examples 4.1.10.

(A) *The Lie group TG*
We will prove that the tangent bundle TG of a Lie group G is a Lie group with multiplication the differential of $\boldsymbol{\gamma}$

$$T\boldsymbol{\gamma}\colon TG \times TG \equiv T(G \times G) \longrightarrow TG.$$

For convenience, the product $T\boldsymbol{\gamma}(u,v)$ of any two tangent vectors $u, v \in TG$ will be denoted by $u \cdot v$.

Since $T\boldsymbol{\gamma}$ and $T\boldsymbol{\alpha}$ are smooth maps (see Proposition 2.4.10), we only need to show that TG is an (algebraic) group with multiplication map $T\boldsymbol{\gamma}$.
i) $T\boldsymbol{\gamma}$ is associative: The associativity law $(xy)z = x(yz)$ of G means that

$$\boldsymbol{\gamma} \circ (\boldsymbol{\gamma} \times \mathrm{id}_G) = \boldsymbol{\gamma} \circ (\mathrm{id}_G \times \boldsymbol{\gamma}). \tag{4.1.5}$$

In virtue of Definition 2.4.8, combined with Theorem 2.2.6 and Exercise 2.2.15 (4), differentiation of (4.1.5) leads to

$$T\boldsymbol{\gamma} \circ (T\boldsymbol{\gamma} \times \mathrm{id}_{TG}) = T\boldsymbol{\gamma} \circ (\mathrm{id}_{TG} \times T\boldsymbol{\gamma}).$$

Then, for every $u, v, w \in TG$, we have that

$$\begin{aligned} (u \cdot v) \cdot w &= T\boldsymbol{\gamma}\big(T\boldsymbol{\gamma}(u,v), w\big) \\ &= T\boldsymbol{\gamma} \circ \big(T\boldsymbol{\gamma} \times \mathrm{id}_{TG}\big)\big((u,v), w\big) \\ &= T\boldsymbol{\gamma} \circ \big(\mathrm{id}_{TG} \times T\boldsymbol{\gamma}\big)\big(u, (v,w)\big) \\ &= u \cdot (v \cdot w). \end{aligned}$$

ii) $0_e \in T_eG$ is the unit of TG: Indeed, for every $u \in TG$ with $\pi(u) = x$, (4.1.3) implies that

$$\begin{aligned} u \cdot 0_e &= T\boldsymbol{\gamma}(u, 0_e) = T_{(x,e)}\boldsymbol{\gamma}(u, 0_e) \\ &= T_x R_e(u) + T_e L_x(0_e) \\ &= T_x \,\mathrm{id}_G(u) + 0_x = u, \end{aligned}$$

and, similarly,

$$\begin{aligned} u \cdot 0_e &= T\boldsymbol{\gamma}(0_e, u) = T_{(e,x)}\boldsymbol{\gamma}(0_e, u) \\ &= T_e R_x(0_e) + T_x L_e(u) \\ &= 0_x + T_x \,\mathrm{id}_G(u) = u. \end{aligned}$$

iii) The inverse of u is $u^{-1} = T\boldsymbol{\alpha}(u)$, for any $u \in TG$. As a matter of fact, assuming that $\pi(u) = x$, (4.1.3) and (4.1.4) yield

$$\begin{aligned} u \cdot u^{-1} &= T\boldsymbol{\gamma}\big(u, u^{-1}\big) = T_x R_{x^{-1}}(u) + T_{x^{-1}} L_x(T_x\boldsymbol{\alpha}(u)) \\ &= T_x R_{x^{-1}}(u) - T_{x^{-1}} L_x\big(T_x(L_{x^{-1}} \circ R_{x^{-1}})(u)\big) \\ &= T_x R_{x^{-1}}(u) - T_x R_{x^{-1}}(u) = 0_{R_{x^{-1}}(x)} = 0_e, \end{aligned}$$

and, analogously,

$$\begin{aligned} u^{-1} \cdot u &= T\boldsymbol{\gamma}\big(u^{-1}, u\big) = T_{x^{-1}} R_x(T_x\boldsymbol{\alpha}(u)) + T_x L_{x^{-1}}(u) \\ &= -T_{x^{-1}} R_x\big(T_x(L_{x^{-1}} \circ R_{x^{-1}})(u)\big) + T_x L_{x^{-1}}(u) \\ &= -T_x L_{x^{-1}}(u) + T_x L_{x^{-1}}(u) = 0_{L_{x^{-1}}(x)} = 0_e. \end{aligned}$$

Therefore, $u^{-1} = T\boldsymbol{\alpha}(u)$ is indeed the inverse of u in TG.

(B) *The zero section of TG is a Lie group morphism*

We recall that [see Example 3.1.3 (A) and Remark 3.1.5 (1)] the zero section of TG is the vector field $\Omega\colon G \to TG$, given by $\Omega(x) = \Omega_x = 0_x$, for every $x \in G$. Since it is a smooth map, it suffices to show that Ω is a group homomorphism. Indeed, for any $x, y \in G$, we have

$$\begin{aligned} \Omega(x) \cdot \Omega(y) &= T\boldsymbol{\gamma}(\Omega_x, \Omega_y) = T_{(x,y)}\boldsymbol{\gamma}(0_x, 0_y) = \\ &= T_x R_y(0_x) + T_y L_x(0_y) = 0_{xy} = \Omega(\boldsymbol{\gamma}(x, y)), \end{aligned}$$

as desired.

Exercises 4.1.11.

1. Complete the details of Example 4.1.9.
2. Prove that
$$L_x \circ L_y = L_{xy}, \quad R_x \circ R_y = R_{yx}, \quad L_x \circ R_y = R_y \circ L_x,$$
for every $x, y \in G$.
3. Prove the equalities
$$(T_x L_y)^{-1} = T_{yx} L_{y^{-1}} \quad \text{and} \quad (T_x R_y)^{-1} = T_{yx} R_{y^{-1}}.$$
4. Verify equality (4.1.5) in Example 4.1.10 (A).
5. Show that (4.1.4) takes also the form
$$T_x\boldsymbol{\alpha}(u) = -T_x(R_{x^{-1}} \circ L_{x^{-1}})(u),$$
for every $x \in G$ and every $u \in T_x G$. Hence, $T_e\boldsymbol{\alpha}(v) = -v$, for every $v \in T_e G$.
6. Find the expressions of the left and right translations of TG by any $u \in TG$, denoted by $\overline{L}_u$ and $\overline{R}_u$, respectively.
7. If G is an abelian Lie group, prove that TG is also an abelian Lie group.
8. Prove that the inversion map of an abelian Lie group is an (iso)morphism of Lie groups.
9. If G, H are Lie groups, prove that $G \times H$ is a Lie group.
10. Explain why the n-torus T^n is an abelian Lie group.
11. Let G be a smooth manifold that is also a group. Prove that G is a Lie group if and only if the map
$$f\colon G \times G \longrightarrow G\colon (x, y) \mapsto xy^{-1}$$
is smooth.

4.2 The Lie algebra of a Lie group

The left invariant vector fields of a Lie group G, defined right below, form a Lie algebra $\mathcal{L}(G)$, subalgebra of the the Lie algebra $\mathcal{X}(G)$ of all smooth vector fields of G (see also the comments preceding Remark 3.3.3, p. 108). The algebra $\mathcal{L}(G)$ induces an isomorphic Lie algebra structure on the tangent space T_eG, allowing to view the Lie algebra of a Lie group in a different (but equivalent) way. The latter interpretation turns out to be quite convenient in many cases.

The vector fields considered throughout this section are *smooth* (recall the convention preceding Proposition 3.1.6 on page 97).

Definition 4.2.1. Let G be a Lie group. A vector field $\xi \in \mathcal{X}(G)$ is called ***left-invariant*** if

$$TL_g \circ \xi = \xi \circ L_g, \tag{4.2.1}$$

for every $g \in G$. Equivalently, by Definition 3.5.1, ξ *is* L_g*-related to itself.*

Right-invariant vector fields are defined analogously by right translations. The set of all left-invariant (resp. right-invariant) vector fields on G is denoted by $\mathcal{L}(G)$ [resp. $\mathcal{R}(G)$]. It is also customary to denote the sets of left-invariant vector fields $\mathcal{L}(G)$, $\mathcal{L}(H)$ etc. by $\mathfrak{g}$, $\mathfrak{h}$ (: the Gothic characters for g, h, respectively). We will follow this trend, although, occasionally, we will use the notation $\mathcal{L}(\cdot)$.

Applying (4.2.1) at any $x \in G$, we see that ξ is left-invariant if and only if

$$T_xL_g(\xi_x) = \xi(L_g(x)) = \xi_{gx}; \qquad x, g \in G. \tag{4.2.2}$$

We will show that a left-invariant vector field of G is necessarily smooth. As we have seen in the beginning of § 3.1, vector fields are sections of the tangent bundle. Hence, when we consider a section of the tangent bundle without further qualifications, we mean a vector field not necessarily smooth from the outset. With this terminology in mind we have:

Proposition 4.2.2. *A section* $\xi\colon G \to TG$ *of the tangent bundle of* G *is a left-invariant smooth vector field if and only if there exists a tangent vector* $v \in T_eG$*, such that*

$$\xi_x = T_eL_x(v), \tag{4.2.3}$$

holds for every $x \in G$.

Proof. Let $\xi \in \mathfrak{g}$. Then, for every $x \in G$, Definition 4.2.1 implies that

$$\xi_x = \xi(xe) = (\xi \circ L_x)(e) = (TL_x \circ \xi)(e) = T_eL_x(\xi_e), \tag{4.2.4}$$

thus we obtain (4.2.3) for $v = \xi_e$.

Conversely, let $\xi\colon G \to TG$ be a section satisfying (4.2.3). Obviously,

$$\xi_e = T_eL_e(v) = T_e\,\mathrm{id}_G(v) = \mathrm{id}_{T_eG}(v) = v.$$

On the other hand,

$$\begin{aligned}\xi(x) = \xi_x = T_eL_x(\xi_e) &= T_eL_x(\xi_e) + T_xR_e(0_x)\\ &= T_{(e,x)}\boldsymbol{\gamma}(\xi_e, \Omega_x) = (T\boldsymbol{\gamma} \circ (C_e, \Omega))(x),\end{aligned}$$

where $C_e \colon G \to TG$ is the constant map with $C_e(x) = \xi_e$, and Ω is, as usual, the (smooth) zero vector field on G. Thus ξ is smooth as the composite of smooth maps. Finally, ξ is left-invariant. Indeed, for every $g, x \in G$, (4.2.3) and Exercise 4.1.11 (4) yield

$$\begin{aligned}\xi_{gx} &= T_eL_{gx}(\xi_e) = T_e(L_g \circ L_x)(\xi_e)\\ &= T_{L_x(e)}L_g \circ T_eL_x(\xi_e) = T_xL_g(\xi_x).\end{aligned}$$ □

Corollary 4.2.3. *Every left-invariant vector field ξ is completely determined by its value ξ_e at the unit e of G.*

As a consequence of the above corollary, we will see that there are as many vector fields in $\mathfrak{g}$ as the vectors of T_eG. The resulting correspondence preserves the algebraic structure of the spaces involved. Before this we prove the following.

Proposition 4.2.4. *The set $\mathfrak{g}$ is a Lie subalgebra of $\mathcal{X}(G)$.*

Proof. Let $\xi, \eta \in \mathfrak{g}$ and $\lambda, \mu \in \mathbb{R}$. Then

$$\begin{aligned}(\lambda\xi + \mu\eta)(x) &= \lambda\xi_x + \mu\eta_x = \lambda T_eL_x(\xi_e) + \mu T_eL_x(\eta_e)\\ &= T_eL_x(\lambda\xi_e + \mu\eta_e) = T_eL_x\left((\lambda\xi + \mu\eta)_e\right);\end{aligned}$$

that is, $\lambda\xi + \mu\eta \in \mathfrak{g}$, thus $\mathfrak{g}$ is a vector subspace of $\mathcal{X}(G)$.

On the other hand, let two arbitrary $\xi, \eta \in \mathfrak{g}$. For every $g \in G$, ξ is L_g-related to ξ, and η is L_g-related to η. In virtue of Corollary 3.5.3, $[\xi, \eta]$ is L_g-related to $[\xi, \eta]$. In other words, $[\xi, \eta]$ is left-invariant, i.e. $[\xi, \eta] \in \mathfrak{g}$. Therefore, $\mathfrak{g}$ is a Lie subalgebra of $\mathcal{X}(G)$. □

The space $\mathfrak{g}$, with the structure induced by Proposition 4.2.4, is called the ***Lie algebra of*** G.

Theorem 4.2.5. *Let G be a Lie group. Then the map*

$$\boldsymbol{h} \colon \mathfrak{g} \longrightarrow T_eG \colon \xi \mapsto \boldsymbol{h}(\xi) := \xi_e \tag{4.2.5}$$

is a linear isomorphism.

Proof. Clearly, $\boldsymbol{h}$ is linear, for if $\xi, \eta \in \mathfrak{g}$ and $\lambda, \mu \in \mathbb{R}$, then

$$\boldsymbol{h}(\lambda\xi + \mu\eta) = (\lambda\xi + \mu\eta)_e = \lambda\xi_e + \mu\eta_e = \lambda\boldsymbol{h}(\xi) + \mu\boldsymbol{h}(\eta).$$

Moreover, $\boldsymbol{h}$ is an injection because $\boldsymbol{h}(\xi) = \xi_e = 0_e$ implies that

$$\xi_x = T_eL_x(0_e) = 0_x = \Omega_x; \qquad x \in G,$$

i.e., ξ coincides with the zero element of $\mathfrak{g} \leq \mathcal{X}(G)$. Finally, $\boldsymbol{h}$ is a surjection: Given a vector $v \in T_eG$, we define the left-invariant vector field $\xi^v \in \mathfrak{g}$ by setting $\xi^v_x = T_eL_x(v)$ [see also (4.2.3)]. Then

$$\boldsymbol{h}(\xi^v) = \xi^v_e = v,$$

which proves the last assertion and completes the proof. □

From the preceding proof we single out the following useful formula:

$$\boldsymbol{h}^{-1}(v) = \xi^v, \quad \text{with} \quad \xi^v_x = \xi^v(x) = T_eL_x(v), \tag{4.2.6}$$

for every $v \in T_eG$ and $x \in G$.

Corollary 4.2.6. *The tangent space T_eG is endowed with the structure of a Lie algebra such that the map* (4.2.5) *is a Lie algebra isomorphism. Thus, with this structure, T_eG is identified with $\mathfrak{g}$ and may be interpreted as the* **Lie algebra of** G.

Proof. For every $u, v \in T_eG$, we define their bracket by setting

$$[u, v] := [\xi^u, \xi^v]_e = \boldsymbol{h}\big([\boldsymbol{h}^{-1}(u), \boldsymbol{h}^{-1}(v)]\big), \tag{4.2.7}$$

Therefore, for arbitrary $\xi, \eta \in \mathfrak{g}$, we have $\boldsymbol{h}(\xi) = \xi_e$, $\boldsymbol{h}(\eta) = \eta_e$, and

$$[\boldsymbol{h}(\xi), \boldsymbol{h}(\eta)] = [\,\xi_e, \eta_e\,] = \boldsymbol{h}\big([\boldsymbol{h}^{-1}(\xi_e), \boldsymbol{h}^{-1}(\eta_e)]\big) = \boldsymbol{h}[\,\xi, \eta\,] \tag{4.2.8}$$

which proves that $\boldsymbol{h}$ is a Lie algebra (iso)morphism. □

As we have commented in Example 3.1.8 (B), $\mathcal{X}(M)$ is an infinite-dimensional vector space over $\mathbb{R}$. The same is true for $\mathcal{X}(G)$ if G is a Lie group. However, the situation radically changes if we are restricted to the subspace $\mathfrak{g} = \mathcal{L}(G)$ of $\mathcal{X}(G)$. In fact, we obtain the following consequence of Theorem 4.2.5.

Corollary 4.2.7. *The Lie algebra $\mathfrak{g}$ of an n-dimensional Lie group G is an n-dimensional vector space. Therefore, every Lie group is a parallelizable manifold.*

Proof. Let (U, ϕ) be a chart on G at e and $\left\{ \dfrac{\partial}{\partial x^i}\Big|_e \right\}_{1 \leq i \leq n}$ the induced canonical basis of T_eG. Then the left-invariant vector fields

$$X_i := \boldsymbol{h}^{-1}\Big(\frac{\partial}{\partial x^i}\Big|_e\Big); \qquad i = 1, \ldots, n,$$

form a basis for $\mathfrak{g}$, thus $\mathfrak{g}$ is an n-dimensional vector space. Moreover, for each $x \in G$, (4.2.6) implies that the vectors

$$X_i(x) = T_eL_x \left(\frac{\partial}{\partial x^i}\Big|_e \right)$$

form a basis for T_xG. Hence, by Example 3.1.8 (C), G is parallelizable. □

Corollary 4.2.8. *There is a diffeomorphism* $\widetilde{F}\colon TG \to G \times \mathfrak{g}$ *such that* $\widetilde{F}\big|_{T_gG}\colon T_gG \to \{g\} \times G$ *is a linear isomorphism and the diagram*

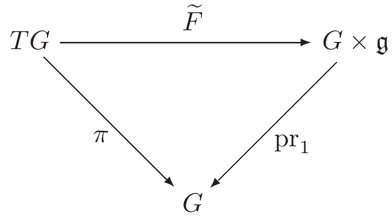

Diagram 4.4

is commutative.

Proof. Assume that $\dim G = n$. Since G is parallelizable, there exists a diffeomorphism $F\colon G\colon TG \to G \times \mathbb{R}^n$ satisfying the properties listed in Example 3.1.8 (C). The desired diffeomorphism is the map $\widetilde{F}$ closing the next diagram.

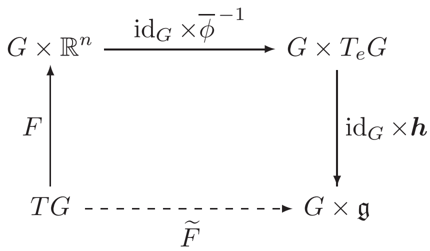

Diagram 4.5

In other words, $\widetilde{F} = (\mathrm{id}_G \times \boldsymbol{h}) \circ (\mathrm{id}_G \times \overline{\phi}^{-1}) \circ F$, where $\overline{\phi}$ is the linear map induced by a chart (U, ϕ) of G at e. □

Example 4.2.9. *The Lie algebra* $\mathcal{L}(\mathbb{R}_*)$

We give an explicit description of the left-invariant vector fields of the multiplicative group $\mathbb{R}_*$ [see also Example 4.1.7 (C)].

The left translation of $\mathbb{R}_*$ by $s \in \mathbb{R}_*$ is the maps $L_s\colon \mathbb{R}_* \to \mathbb{R}_*$ with $L_s(t) = st$, for every $t \in \mathbb{R}_*$. We already know that $\mathbb{R}$ admits the basic basic vector field d/dt [see (3.1.2b)]. Since $\mathbb{R}_*$ is an open submanifold of $\mathbb{R}$, the restriction of d/dt to $\mathbb{R}_*$ induces a basic vector field on $\mathbb{R}_*$, denoted (for simplicity) by the same symbol.

If $\xi \in \mathcal{L}(\mathbb{R}_*)$, then

$$\xi = f\frac{d}{dt}, \tag{4.2.9}$$

where $f \in \mathcal{C}^\infty(\mathbb{R}_*, \mathbb{R})$ is the (global) coordinate of ξ. On the other hand, the left invariance of ξ implies that [see (4.2.4)]

$$\xi(s) = T_1L_s(\xi(1)), \qquad s \in \mathbb{R}_*. \tag{4.2.10}$$

Taking into account (2.2.12a), it follows from (4.2.10) that

$$\xi(s) = T_1 L_s(\xi(1)) = T_1 L_s \left(f(1) \frac{d}{dt}\bigg|_1 \right) = f(1) s \left.\frac{d}{dt}\right|_s \tag{4.2.11}$$

Also, (4.2.9) implies

$$\xi(s) = f(s) \left.\frac{d}{dt}\right|_s, \qquad s \in \mathbb{R}_*. \tag{4.2.12}$$

Comparing (4.2.10) and (4.2.12), we see that

$$f(s) = f(1)s = f(s)\,\mathrm{id}_{\mathbb{R}_*}(s); \qquad s \in \mathbb{R}_*,$$

or $f = f(1)\,\mathrm{id}_{\mathbb{R}_*}$. Setting $\lambda = f(1)$, (4.2.9) takes the form

$$\xi = \lambda\, \mathrm{id}_{\mathbb{R}_*} \cdot \frac{d}{dt}. \tag{4.2.13}$$

Note that equality $f(s) = f(1)s$ found earlier implies that f is a linear function.

Conversely, let ξ be a vector field of $\mathbb{R}_*$ given by (4.2.13), for any $\lambda \in \mathbb{R}$. This is a left-invariant vector field on $\mathbb{R}_*$ because, for every $t, s \in \mathbb{R}_*$,

$$\begin{aligned} T_t L_s(\xi_t) &= T_t L_s \left(\lambda t \left.\frac{d}{dt}\right|_t \right) = \lambda t\, T_t L_s \left(\left.\frac{d}{dt}\right|_t \right) \\ &= \lambda t s \left.\frac{d}{dt}\right|_{st} = \lambda \left(ts \left.\frac{d}{dt}\right|_{st} \right) = \xi(st). \end{aligned}$$

Therefore, we conclude that

$$\mathcal{L}(\mathbb{R}_*) = \left\{ \xi = \lambda\, \mathrm{id}_{\mathbb{R}_*} \cdot \frac{d}{dt} \;\middle|\; \lambda \in \mathbb{R} \right\}.$$

The general linear group $\mathrm{GL}(n, \mathbb{R})$, mentioned in Example 4.1.7 (B), is a very important Lie group. Along with some of its subgroups, it is involved in many geometrical structures and applications. The same applies to its Lie algebra, denoted by $\mathfrak{gl}(n, \mathbb{R})$. The structure of the later merits a more detailed treatment.

Theorem 4.2.10. *The Lie algebra* $\mathfrak{gl}(n, \mathbb{R})$ *of* $\mathrm{GL}(n, \mathbb{R})$ *can be identified with the Lie algebra of* $n \times n$ *matrices* $\mathcal{M}_n(\mathbb{R})$. *Under this identification, the bracket of* $\mathfrak{gl}(n, \mathbb{R})$ *coincides with the commutator of matrices [see Exercise 3.3.5 (6)].*

Before the proof, we recall a few facts from earlier sections and adapt them to the present framework.

As we know, $\mathcal{M}_n(\mathbb{R})$ is an n^2-dimensional manifold whose structure is defined by the global chart $(\mathcal{M}_n(\mathbb{R}), \Phi)$, where

$$\Phi(A) = (a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{nn}),$$

for every matrix $A = (a_{ij})$. The corresponding coordinate functions are now denoted by x^{ij}, for all $i, j = 1, \ldots, n$, thus

$$x^{ij}(A) = a_{ij}.$$

On the other hand, the tangent space $T_I\mathcal{M}_n(\mathbb{R})$ of $\mathcal{M}_n(\mathbb{R})$ at the identity matrix $I = (\delta_{ij})$ can be identified with R^{n^2} by means of the linear isomorphism

$$\overline{\Phi}\colon T_I\mathcal{M}_n(\mathbb{R}) \longrightarrow R^{n^2}$$

induced by Φ. If $\{e_{ij}\}_{i,j=1,\dots,n}$ is the natural basis of $\mathbb{R}^{n^2}$, then the vectors

$$\left.\frac{\partial}{\partial x^{ij}}\right|_I = \overline{\Phi}^{-1}(e_{ij}); \qquad i,j = 1,\dots,n,$$

determine the canonical basis of $T_I\mathcal{M}_n(\mathbb{R})$, with respect to the chosen chart. Thus, by Example 2.3.8 (C), any $V \in T_I\mathcal{M}_n(\mathbb{R})$ is written as

$$V = \sum_{i,j=1}^{n} u^{ij} \left.\frac{\partial}{\partial x^{ij}}\right|_I = \sum_{i,j=1}^{n} u(x^{ij}) \left.\frac{\partial}{\partial x^{ij}}\right|_I . \tag{4.2.14}$$

We now define the linear isomorphism

$$f = \Phi^{-1} \circ \overline{\Phi}\colon T_I(\mathcal{M}_{n\times n}(\mathbb{R})) \longrightarrow \mathcal{M}_n(\mathbb{R}). \tag{4.2.15}$$

Taking into account (4.2.14), we have that

$$f(V) = \Phi^{-1}(u^{11},\dots,u^{nn}) = \begin{pmatrix} u^{11} & \cdots & u^{1n} \\ \cdots & \cdots & \cdots \\ u^{n1} & \cdots & u^{nn} \end{pmatrix} =: \widehat{V};$$

therefore, the ij-th entry of $f(V)$ is precisely

$$f(V)_{ij} = u(x^{ij}) = u^{ij}, \tag{4.2.16a}$$

which implies that

$$T_I\mathcal{M}_n(\mathbb{R}) \ni V \overset{f}{\equiv} \widehat{V} \in \mathcal{M}_n(\mathbb{R}) \tag{4.2.16b}$$

We also remind that, since $\mathrm{GL}(n,\mathbb{R})$ is open in $\mathcal{M}_n(\mathbb{R})$,

$$T_I GL(n,\mathbb{R}) \equiv T_I\mathcal{M}_n(\mathbb{R}) \tag{4.2.17}$$

(see Proposition 2.2.10). As a result, combining the linear isomorphisms (4.2.5) and (4.2.15), along with (4.2.17), we obtain the linear isomorphism

$$F := f \circ \boldsymbol{h}\colon \mathfrak{gl}(n,\mathbb{R}) \longrightarrow \mathcal{M}_n(\mathbb{R}).$$

With the previous preparation we are in a position to give the

Proof of Theorem 4.2.10. First we find the entries of the matrix $F(\xi)$ for any $\xi \in \mathfrak{gl}(n,\mathbb{R})$: Since $F(\xi) = f(\boldsymbol{h}(\xi)) = f(\xi_I)$, (4.2.16) implies

$$F(\xi)_{ij} = f(\xi_I)_{ij} = \xi_I\left(x^{ij}\right). \tag{4.2.18}$$

Also, for an $\xi \in \mathfrak{gl}(n,\mathbb{R})$, its value $\xi(A)$ at any $A \in GL(n,\mathbb{R})$ is given by

$$\xi_A = \xi(A) = T_I L_A(\xi_I);$$

hence, evaluation of the latter at the coordinate functions, together with (2.3.25), yields

$$\xi_A\left(x^{ij}\right) = \big(T_I L_A(\xi_I)\big)\left(x^{ij}\right) = \xi_I\left(x^{ij}\circ L_A\right). \tag{4.2.19}$$

But, for every $B\in \mathrm{GL}(n,\mathbb{R})$,

$$\begin{aligned}\big(x^{ij}\circ L_A\big)(B) &= x^{ij}\big(L_A(B)\big) = x^{ij}(AB)\\ &= x^{ij}\big((a_{ij})\cdot(b_{ij})\big) = x^{ij}\left(\Big(\sum_{k=1}^{n} a_{ik}b_{kj}\Big)\right)\\ &= \sum_{k=1}^{n} a_{ik}b_{kj} = \sum_{k=1}^{n} x^{ik}(A)\,x^{kj}(B);\end{aligned}$$

that is,

$$x^{ij}\circ L_A = \sum_{k=1}^{n} x^{ik}(A)\cdot x^{kj}, \tag{4.2.20}$$

where, clearly, $x^{ik}(A)\cdot x^{kj}$ denotes the product of a real number with a real-valued functions. In virtue of (4.2.20) and (4.2.18), we transform (4.2.19) into

$$\begin{aligned}\xi_A\left(x^{ij}\right) &= \xi_I\left(x^{ij}\circ L_A\right) = \xi_I\left(\sum_{k=1}^{n} x^{ik}(A)\cdot x^{kj}\right)\\ &= \sum_{k=1}^{n} x^{ik}(A)\cdot \xi_I\left(x^{kj}\right) = \sum_{k=1}^{n} x^{ik}(A)\cdot F(\xi)_{kj},\end{aligned}$$

for every $A\in \mathrm{Gl}(n,\mathbb{R})$, whence the equality

$$\xi\left(x^{ij}\right) = \sum_{k=1}^{n} x^{ik}\cdot F(\xi)_{kj}. \tag{4.2.21}$$

Applying now (4.2.18), Example 3.3.4 (A), and (4.2.21), we find for every $\xi,\eta\in \mathfrak{gl}(n,\mathbb{R})$:

$$\begin{aligned}F\big([\,\xi,\eta\,]\big)_{ij} &= [\,\xi,\eta\,]_I\big(x^{ij}\big) = \xi_I\big(\eta(x^{ij})\big) - \eta_I\big(\xi(x^{ij})\big)\\ &= \xi_I\left(\sum_{k=1}^{n} x^{ik}\cdot F(\eta)_{kj}\right) - \eta_I\left(\sum_{k=1}^{n} x^{ik}\cdot F(\xi)_{kj}\right)\\ &= \sum_{k=1}^{n}\Big(\xi_I\big(x^{ik}\cdot F(\eta)_{kj}\big) - \eta_I\big(x^{ik}\cdot F(\xi)_{kj}\big)\Big)\\ &= \sum_{k=1}^{n}\Big(\xi_I\big(x^{ik}\big)F(\eta)_{kj} - \eta_I\big(x^{ik}\big)F(\xi)_{kj}\Big)\\ &= \sum_{k=1}^{n}\big(F(\xi)_{ik}F(\eta)_{kj} - F(\eta)_{ik}F(\xi)_{kj}\big)\\ &= \big(F(\xi)\cdot F(\eta)\big)_{ij} - \big(F(\eta)\cdot F(\xi)\big)_{ij},\end{aligned}$$

from which follows that

$$F\big([\,\xi,\eta\,]\big) = F(\xi)\cdot F(\eta) - F(\eta)\cdot F(\xi). \tag{4.2.22}$$

The preceding equality shows that the linear isomorphism F is a morphism of Lie algebras, thus an isomorphism of Lie algebras identifying $\mathfrak{gl}(n,\mathbb{R})$ with $\mathcal{M}_n(\mathbb{R}))$, as desired. Omitting F, we may write (4.2.22) as

$$[\,\xi,\eta\,] = \xi\cdot\eta - \eta\cdot\xi; \qquad \xi,\eta\in\mathfrak{gl}(n,\mathbb{R}),$$

but the isomorphism F should be kept in mind. □

To complete our definitions, we say that a Lie algebra $\mathbb{A}$ is called ***abelian*** if $[a,b]=0$, for every $a,b\in\mathbb{A}$. This is an obvious consequence of the antisymmetry of the bracket.

Example 4.2.11. *The differential* $T_I\det\colon T_I\mathrm{GL}(n,\mathbb{R})\to T_1\mathbb{R}$.

We want to compute $T_I\det$ after the identifications $T_I\mathrm{GL}(n,\mathbb{R})\equiv T_I\mathcal{M}_n(\mathbb{R})\equiv \mathcal{M}_n(\mathbb{R})$ and $T_1\mathbb{R}\equiv\mathbb{R}$. The trick working in this case is to use the well-known equality $\det\big(e^X\big) = e^{\mathrm{tr}(X)}$, for every $X\in\mathcal{M}_n(\mathbb{R})$, where e^X and $\mathrm{tr}(X)$ denote, respectively, the exponential and the trace of X.

Let an arbitrary $X\in\mathrm{GL}(n,\mathbb{R})\equiv\mathcal{M}_n(\mathbb{R})$. As a tangent vector, X can be realized as the velocity vector of a smooth curve passing through I. A convenient curve for our purpose is given by $\alpha(t):=e^{tX}$, $t\in\mathbb{R}$, which yields

$$\alpha(0)=I, \ \text{ and } \dot\alpha(0)\equiv\alpha'(0) = \frac{d}{dt}\left(e^{tX}\right)\Big|_{t=0}.$$

Therefore,

$$\begin{aligned}(T_I\det)(X) &= (T_I\det)\big(\dot\alpha(0)\big) = T_0(\det\circ\alpha)\left(\frac{d}{dt}\Big|_{t=0}\right)\\ &\equiv \frac{d}{dt}(\det\circ\alpha)\Big|_{t=0} = \frac{d}{dt}\left(e^{tX}\right)\Big|_{t=0} = \mathrm{tr}(X).\end{aligned}$$

Exercises 4.2.12.

1. Explain why condition $[\ ,\ ]=0$ is necessary in the definition of an abelian Lie algebra.
2. Find the Lie algebra of the Lie group $(\mathbb{R},+)$.
3. Show that the left-invariant vector fields of $\mathbb{R}^n\equiv(\mathbb{R}^n,+)$ are in bijective correspondence with the vectors $v\in\mathbb{R}^n$, such that
$$\xi_a = \sum_{i=1}^n v^i\,\frac{\partial}{\partial x^i}\Big|_a, \qquad \forall\, a\in\mathbb{R}^n.$$
4. Prove that every morphism of Lie groups $f\colon G\to H$ induces a morphism of Lie algebras $\widetilde{f}\colon\mathfrak{g}\to\mathfrak{h}$.

5. Continuing the preceding exercise, prove the following properties:
(i) If id_G is the identity isomorphism of G, then $\widetilde{\mathrm{id}_G} = 1_{\mathfrak{g}}$.
(ii) If $f\colon G \to H$ and $g\colon H \to K$ are morphisms of Lie groups, then $\widetilde{g \circ f} = \widetilde{g} \circ \widetilde{f}$.
(iii) If $f\colon G \to H$ is an isomorphism of Lie groups, then $\widetilde{f}\colon \mathfrak{g} \to \mathfrak{h}$ is a morphism of Lie algebras.

6. Prove that that the Lie algebra of an abelian Lie group is also abelian.

7. Let a matrix $A \in \mathrm{GL}(n, \mathbb{R})$. Using the analogs of (4.2.16a) and (4.2.16b) prove that the differential $T_I L_A$ identifies with the translation L_A of matrices. As a result, there is a bijective correspondence between vector fields $\xi \in \mathfrak{gl}(n, \mathbb{R})$ and matrices $B \in \mathcal{M}_n(\mathbb{R})$ such that, when $\xi_A \in T_A\mathrm{GL}(n, \mathbb{R})$ is viewed as a matrix in $\mathcal{M}_n(\mathbb{R})$, then $\xi_A = A \cdot B$, for every $A \in \mathrm{GL}(n, \mathbb{R})$.

8. Let G be an n-dimensional Lie group and $\mathcal{B} = \{X_i \,|\, i = 1, \ldots, n\}$ any basis of $\mathfrak{g}$ (parallelization of G). Show that there are n^3 constants C_{ij}^k given by

$$[X_i, X_j] = \sum_{k=1}^{n} C_{ij}^k X_k; \qquad i, j = 1, \ldots, n. \tag{4.2.23}$$

They are called the ***structural constants*** of G, relative to $\mathcal{B}$. Then verify the following equalities:

$$C_{ij}^k = -C_{ji}^k, \tag{4.2.24}$$

$$\sum_{l=1}^{n} \left(C_{il}^m C_{jk}^l + C_{jl}^m C_{ki}^l + C_{kl}^m C_{ij}^l \right) = 0, \tag{4.2.25}$$

for all indices $i, j, k, m = 1, \ldots, n$.

9. Following the pattern of Example 4.2.11, prove that $(T_A \det)(AX) = (\det A) \cdot \mathrm{tr}(X)$, for every $A \in \mathrm{GL}(n, \mathbb{R})$ and $X \in T_A\mathrm{GL}(n, \mathbb{R}) \equiv \mathcal{M}_n(\mathbb{R})$. Based on the preceding result, compute $(T_A \det)(Y)$ for any $Y \in T_A\mathrm{GL}(n, \mathbb{R})$.

4.3 One-parameter subgroups

An interesting property of the integral curves of the left-invariant vector fields of a Lie group G is that they are complete, and their set is in a bijective correspondence with the Lie algebra of G.

Definition 4.3.1. An ***one-parameter subgroup*** of a Lie group G is a morphism of Lie groups $\alpha\colon \mathbb{R} \to G$.

Clearly, $\mathbb{R}$ is considered with the structure of the additive Lie group $(\mathbb{R}, +)$. Stated differently, an one-parameter subgroup is a smooth curve defined on the whole of $\mathbb{R}$ and satisfies the condition

$$\alpha(t + s) = \alpha(t)\alpha(s), \qquad t, s \in \mathbb{R}.$$

By the very definition $\alpha(0) = e$; hence, all one-parameter subgroups of G pass through e.

The set of all one-parameter subgroups of G will be denoted by $\mathcal{H}om(\mathbb{R}, G)$. The terminology is due to the fact that $\alpha(\mathbb{R})$ is an (abelian) subgroup of G parametrized by $\mathbb{R}$.

Example 4.3.2. The ordinary exponential map $\exp\colon \mathbb{R} \ni t \mapsto e^t \in \mathbb{R}_*$ is a one-parameter subgroup of $\mathbb{R}_*$.

Our aim is to prove that there is a 1–1 correspondence between the Lie algebra $\mathfrak{g}$ of G and the set $\mathcal{H}om(\mathbb{R}, G)$. First, with the notation of Theorem 4.2.5 [see also equality (4.2.6)], we prove the following.

Proposition 4.3.3. *Let $\alpha \in \mathcal{H}om(\mathbb{R}, G)$ and $v := \dot{\alpha}(0)$. If $\xi^v \in \mathfrak{g}$ is the left-invariant vector field with $\xi^v(e) = v$, then α is the integral curve of ξ^v with initial condition $\alpha(0) = e$.*

Proof. We have to show that $\dot{\alpha}(t) = \xi^v(\alpha(t))$, for every $t \in \mathbb{R}$. In virtue of (2.2.13) and (4.2.5), we have:

$$\begin{aligned} \xi^v(\alpha(t)) &= T_e L_{\alpha(t)}(v) = T_e L_{\alpha(t)}\big(\dot{\alpha}(0)\big) = \\ &= T_e L_{\alpha(t)} \left(T_0 \alpha \left(\frac{d}{dt}\bigg|_0 \right) \right) = T_0\big(L_{\alpha(t)} \circ \alpha\big) \left(\frac{d}{dt}\bigg|_0 \right). \end{aligned} \tag{4.3.1}$$

The composite map in the last differential, evaluated at any $s \in \mathbb{R}$, transforms as follows:

$$\big(L_{\alpha(t)} \circ \alpha\big)(s) = \alpha(t)\alpha(s) = \alpha(t+s) = (\alpha \circ \lambda_t)(s),$$

where λ_t, with $\lambda_t(s) = t + s$, is the left translation of $(\mathbb{R}, +)$ by t. Hence, (4.3.1) yields

$$\begin{aligned} \xi^v(\alpha(t)) = T_0\,(\alpha \circ \lambda_t) \left(\frac{d}{dt}\bigg|_0 \right) &= T_t\alpha \left(T_0\lambda_t \left(\frac{d}{dt}\bigg|_0 \right) \right) \\ &= T_t\alpha \left(\lambda_t'(0)\, \frac{d}{dt}\bigg|_{\lambda_t(0)} \right) = T_t\alpha\Big(\frac{d}{dt}\bigg|_t \Big) = \dot{\alpha}(t), \end{aligned}$$

which proves the statement. □

The preceding proposition implies that every $\alpha \in \mathcal{H}om(\mathbb{R}, G)$ determines a left-invariant vector field $\xi^v \in \mathfrak{g}$ with $\xi^v(e) = v = \dot{\alpha}(0) \in T_eG$, whose integral curve with initial condition e is precisely α. Conversely, we will prove that the integral curve of any $\xi \in \mathfrak{g}$, with initial condition e, determines an one-parameter subgroup. Before this we need the following auxiliary result:

Lemma 4.3.4. *Let $\xi \in \mathfrak{g}$. If $\alpha\colon J = (-\varepsilon, \varepsilon) \to G$ is the integral curve of ξ satisfying $\alpha(0) = e$, then equality $\alpha(s+t) = \alpha(s)\alpha(t)$ holds for every $t, s \in J$ such that $t+s \in J$.*

Proof. We fix an $s \in J$ and consider the curves

$$\beta(t) := \alpha(s+t) = (\alpha \circ \lambda_s)(t), \qquad t \in -s + J$$
$$\gamma(t) := \alpha(s)\alpha(t) = (L_{\alpha(s)} \circ \alpha)(t), \qquad t \in J$$

where $-s + J = \lambda_{-s}(J)$ and, as before, λ_s is the left translation $(\mathbb{R}, +)$ by s. Obviously, β and γ are smooth curves such that

$$\beta(0) = \alpha(s) = \alpha(s) \cdot e = \gamma(0).$$

We show that both β and γ are integral curves of ξ. For β we have:

$$\begin{aligned}\dot{\beta}(t) &= T_t\beta\left(\left.\frac{d}{dt}\right|_t\right) = T_t\left(\alpha \circ \lambda_s\right)\left(\left.\frac{d}{dt}\right|_t\right) = T_{s+t}\alpha\left(T_t\lambda_s\left(\left.\frac{d}{dt}\right|_t\right)\right) \\ &= T_{s+t}\alpha\left(\lambda_s'(t)\left.\frac{d}{dt}\right|_{\lambda_s(t)}\right) = T_{s+t}\alpha\left(\left.\frac{d}{dt}\right|_{s+t}\right) = \dot{\alpha}(s+t),\end{aligned}$$

or, since α is an integral curve of ξ,

$$\dot{\beta}(t) = \dot{\alpha}(s+t) = \xi(\alpha(s+t)) = \xi(\beta(t));$$

that is, β is an integral curve of ξ. Similarly,

$$\begin{aligned}\dot{\gamma}(t) &= T_t\gamma\left(\left.\frac{d}{dt}\right|_t\right) = T_t\left(L_{\alpha(s)} \circ \alpha\right)\left(\left.\frac{d}{dt}\right|_t\right) \\ &= T_{\alpha(t)}L_{\alpha(s)}\left(T_t\alpha\left(\left.\frac{d}{dt}\right|_t\right)\right) \\ &= T_{\alpha(t)}L_{\alpha(s)}\big(\dot{\alpha}(t)\big) = T_{\alpha(t)}L_{\alpha(s)}\big(\xi(\alpha(t))\big).\end{aligned}$$

Since ξ is left-invariant, (4.2.2) implies that

$$\dot{\gamma}(t) = T_{\alpha(t)}L_{\alpha(s)}\big(\xi(\alpha(t))\big) = \xi\big(\alpha(s)\alpha(t)\big) = \xi(\gamma(t)),$$

showing that γ is also an integral curve of ξ, with the same initial condition as β; therefore, $\beta = \gamma$ on their common domain. As a result, $\alpha(s+t) = \alpha(s)\alpha(t)$, for the chosen s and every suitable t. The same holds for every $s \in J$. This proves the lemma. □

The integral curve in the preceding result seems not to be necessarily an one-parameter subgroup, because we do not know yet that it is defined on the whole of $\mathbb{R}$. However, we do have the following important result.

Theorem 4.3.5. *Every left-invariant vector field is complete.*

Proof. Let $\xi \in \mathfrak{g}$ be an arbitrary left-invariant vector field of G and $\alpha_o \colon (-\varepsilon, \varepsilon) \to G$ its integral curve with initial condition $\alpha_o(0) = e$. We will prove that α_o can be extended to an integral curve α defined on the whole of $\mathbb{R}$, with the same initial condition. To this end we consider a point $t_o \in (-\varepsilon, \varepsilon)$, an integer $n \in \mathbb{Z}$, and the curve

$$\alpha_n \colon (nt_o - \varepsilon,\, nt_o + \varepsilon) \longrightarrow G \colon t \mapsto \alpha_n(t) := \alpha_o(t_o)^n \cdot \alpha_o(t - nt_o).$$

If $\mu(t) = t - nt_o$ (the right translation of $\mathbb{R}$ by $-nt_o$), $\alpha_n(t)$ takes the form

$$\alpha_n(t) = L_{\alpha_o(t_o)^n}\big(\alpha_o(t - nt_o)\big) = \big(L_{\alpha_o(t_o)^n} \circ \alpha_o \circ \mu\big)\,(t)$$

as shown also in the next commutative diagram.

$$\begin{array}{ccc}
(nt_o - \varepsilon,\, nt_o + \varepsilon) \ni t & \xrightarrow{\quad\mu\quad} & t - nt_o \in (-\varepsilon, \varepsilon) \\
\Big\downarrow \alpha_n & & \Big\downarrow \alpha_o \\
G \ni L_{\alpha_o(t_o)^n(\alpha_o(t - nt_o))} & \xleftarrow[\;L_{\alpha_o(t_o)^n}\;]{} & \alpha_o(t - nt_o) \in G
\end{array}$$

Diagram 4.6

Then every curve α_n is smooth, and its velocity vectors are computed as follows:

$$\begin{aligned}
\dot\alpha_n(t) &= T_t\alpha_n\left(\left.\frac{d}{dt}\right|_t\right) = T_t\big(L_{\alpha_o(t_o)^n} \circ \alpha_o \circ \mu\big)\left(\left.\frac{d}{dt}\right|_t\right) \\
&= \big(T_{\alpha(t-nt_o)}L_{\alpha_o(t_o)^n} \circ T_{t-nt_o}\,\alpha_o\big)\left(T_t\mu\left(\left.\frac{d}{dt}\right|_t\right)\right) \\
&= \big(T_{\alpha_o(t-nt_o)}L_{\alpha_o(t_o)^n} \circ T_{t-nt_o}\alpha_o\big)\left(\left.\frac{d}{dt}\right|_{t-nt_o}\right) \\
&= T_{\alpha_o(t-nt_o)}L_{\alpha_o(t_o)^n}\left(T_{t-nt_o}\alpha_o\left(\left.\frac{d}{dt}\right|_{t-nt_o}\right)\right) \\
&= T_{\alpha_o(t-nt_o)}L_{\alpha_o(t_o)^n}\big(\dot\alpha_o(t - nt_o)\big)
\end{aligned}$$

Since $t \in (nt_o - \varepsilon, nt_o + \varepsilon)$, it follows that $t - nt_o \in (-\varepsilon, \varepsilon)$. Taking into account that ξ is left-invariant, and α_o is an integral curve of ξ, we obtain:

$$\begin{aligned}
\dot\alpha_n(t) &= T_{\alpha_o(t-nt_o)}L_{\alpha_o(t_o)^n}\big(\dot\alpha_o(t - nt_o)\big) \\
&= T_{\alpha_o(t-nt_o)}L_{\alpha_o(t_o)^n}\big(\xi(\alpha_o(t - nt_o))\big) \\
&= \xi\big(\alpha_o(t_o)^n \cdot \alpha_o(t - nt_o)\big) = \xi(\alpha_n(t));
\end{aligned}$$

that is, α_n is an integral curve of ξ. On the other hand,

$$nt_o \in \big((n-1)t_o - \varepsilon,\, (n-1)t_o + \varepsilon\big) = \mathrm{Dom}(\alpha_{n-1}),$$

as well as,

$$\begin{aligned}
\alpha_{n-1}(nt_o) &= \alpha_o(t_o)^{n-1} \cdot \alpha_o\big(nt_o - (n-1)t_o\big) = \\
&= \alpha_o(t_o)^{n-1} \cdot \alpha_o(t_o) = \alpha_o(t_o)^n = \alpha_n(nt_o),
\end{aligned}$$

whence we deduce that the curves α_{n-1} and α_n are integral curves of ξ coinciding at nt_o. According to Exercise 3.6.7 (6), the curves coincide on the intersection of their domains and they extend to a unique integral curve of ξ on the union of their domains. The union α of all such curves is defined on $\mathbb{R}$ and it is an integral curve of ξ with $\alpha(0) = \alpha_o(0) = e$.

We prove now that every integral curve of ξ is defined on the whole of $\mathbb{R}$. Indeed, for an $x \in G$, we set

$$\beta(t) := x\alpha(t) \equiv x \cdot \alpha(t) = (L_x \circ \alpha)(t), \qquad t \in \mathbb{R}.$$

Then $\beta(0) = x$, and

$$\begin{aligned}\dot{\beta}(t) &= T_{\alpha(t)}L_x\left(T_t\alpha\left(\left.\frac{d}{dt}\right|_t\right)\right) = T_{\alpha(t)}L_x\big(\dot{\alpha}(t)\big)\\ &= T_{\alpha(t)}L_x\big(\xi(\alpha(t)\big) = \xi\big(x\alpha(t)\big) = \xi(\beta(t));\end{aligned}$$

hence, β is an integral curve of ξ. By the uniqueness of integral curves, β is the unique integral curve of ξ with initial condition x and domain the whole of $\mathbb{R}$. □

An immediate consequence of Lemma 4.3.4 and Theorem4.3.5 is:

Corollary 4.3.6. *Let $\xi \in \mathfrak{g}$ and let $\alpha\colon \mathbb{R} \to G$ be the integral curve of ξ, with $\alpha(0) = e$. Then $\alpha \in \mathcal{H}om(\mathbb{R}, G)$.*

With the same assumptions as before, the last part of the proof of Theorem 4.3.5 yields:

Corollary 4.3.7. *The integral curve of ξ, with initial condition $x \in G$, is the curve $\beta\colon \mathbb{R} \to G$ given by $\beta(t) = x\alpha(t)$.*

The preceding results establish now the relation between $\mathfrak{g}$ and $\mathcal{H}om(\mathbb{R}, G)$ announced in the beginning of this section. Namely, we prove:

Theorem 4.3.8. *There is a bijective correspondence $\mathcal{F}$ between the sets $\mathfrak{g}$ and $\mathcal{H}om(\mathbb{R}, G)$.*

Proof. We define the map $\mathcal{F}\colon \mathfrak{g} \to \mathcal{H}om(\mathbb{R}, G)$ assigning to each $\xi \in \mathfrak{g}$ the integral curve of ξ with initial condition e. This is a uniquely determined curve and, according to Corollary 4.3.6, it is an element of $\mathcal{H}om(\mathbb{R}, G)$. This shows that $\mathcal{F}$ is a well defined map. Furthermore:
$\mathcal{F}$ is injective: Indeed, if $\xi, \eta \in \mathfrak{g}$ and $\mathcal{F}(\xi) = \mathcal{F}(\eta) = \alpha$, then

$$\xi(x) = T_eL_x(\xi(e)) = T_eL_x(\dot{\alpha}(0)) = T_eL_x(\eta(e)) = \eta(x).$$

for every $x \in G$; hence, $\xi = \eta$.
$\mathcal{F}$ is surjective: Let $\alpha \in \mathcal{H}om(\mathbb{R}, G)$. Since $\dot{\alpha}(o) \in T_eG$, Theorem 4.2.5 implies the existence of a $\xi \in \mathfrak{g}$, with $\xi(e) = \dot{\alpha}(0)$. Then $\mathcal{F}(\xi) = \alpha$, as an immediate consequence of Proposition 4.3.3. □

Let $\xi \in \mathfrak{g}$. Since ξ corresponds bijectively to a one-parameter subgroup $\alpha \in \mathcal{H}om(\mathbb{R}, G)$, which is the unique integral curve of ξ with $\alpha(0) = e$, we also write α_ξ instead of α, to remind us the correspondence of Theorem 4.3.8. Therefore,

$$\alpha_\xi = \mathcal{F}(\xi), \;\; \alpha_\xi(0) = e, \;\; \dot{\alpha}_\xi(0) = \xi_e. \tag{4.3.2}$$

It is convenient to call α_ξ the one-parameter subgroup ***generated*** by ξ.

Let now any $s \in \mathbb{R}$. Then $s\xi$ is a left-invariant vector field and determines the one-parameter subgroup $\alpha_{s\xi}$. The next result describes the relation between the curves $\alpha_{s\xi}$ and α_ξ

Proposition 4.3.9. *For every $s, t \in \mathbb{R}$, the following equality holds true:*

$$\alpha_{s\xi}(t) = \alpha_\xi(st). \tag{4.3.3}$$

Proof. We fix an $s \in \mathbb{R}$, and consider the curve

$$\beta\colon \mathbb{R} \longrightarrow G\colon t \mapsto \beta(t) := \alpha_\xi(st).$$

Then $\beta = \alpha_\xi \circ l_s$, where $l_s(t) := st$ is the left translation of $\mathbb{R}$ by s. Moreover, $\beta \in \mathcal{H}om(\mathbb{R}, G)$ as the composite of the Lie group morphisms $l_s\colon \mathbb{R} \to \mathbb{R}$ and $\alpha_\xi\colon \mathbb{R} \to G$. Consequently, β corresponds bijectively to a unique $\eta \in \mathfrak{g}$ such that $\eta_e = \dot{\beta}(0)$. Since

$$\begin{aligned}\dot{\beta}(0) &= T_0\big(\alpha_\xi \circ \lambda_s\big)\left(\left.\frac{d}{dt}\right|_0\right) = T_0\alpha_\xi\left(T_0\lambda_s\left(\left.\frac{d}{dt}\right|_0\right)\right)\\ &= T_0\alpha_\xi\left(s\left.\frac{d}{dt}\right|_0\right) = s\dot{\alpha}_\xi(0) = s\xi_e,\end{aligned}$$

it follows that $\eta_e = s\xi_e$, thus, for every $x \in G$,

$$\eta(x) = \eta(xe) = T_eL_x(\eta_e) = T_eL_x(s\xi_e) = sT_eL_x(\xi_e) = s\xi(x).$$

As a result, $\eta = \xi$, whence $\beta = \alpha_{s\xi}$. Varying s in $\mathbb{R}$, we obtain (4.3.3). $\square$

Example 4.3.10. Let $f\colon G \to H$ be a morphism of Lie groups. If $\alpha \in \mathcal{H}om(\mathbb{R}, G)$, there exists a unique $\xi \in \mathfrak{g}$, with $\dot{\alpha}(0) = \xi_e$. Then $\gamma := f \circ \alpha\colon \mathbb{R} \to H$ is a Lie group morphism, as the composite of such morphisms; thus, it is a one-parameter subgroup of H. We want to find the left-invariant vector field η of H, corresponding to γ. The former is completely determined by its value at the unit e of H (for convenience, we denote by e the unit of all Lie groups). Therefore,

$$\begin{aligned}\eta_e = \dot{\gamma}(0) &= T_0(f \circ \alpha)\left(\left.\frac{d}{dt}\right|_0\right)\\ &= T_ef\big(\dot{\alpha}(0)\big) = T_ef(\xi_e).\end{aligned}$$

As a result, if L'_y is the left translation of H by y, we find that

$$\eta_y = T_eL'_y\big(T_ef(\xi_e)\big) = T_e\big(L'_y \circ f\big)(\xi_e),$$

for every $y \in H$.

Remark 4.3.11. Instead of the notation α_ξ, denoting the one-parameter subgroup generated by ξ, one could equally write α_v, where $v = \xi_e \in T_eG$, a fact justified by the identification $\mathfrak{g} \equiv \mathcal{H}om(\mathbb{R}, G)$ induced by Theorem 4.3.8.

Exercises 4.3.12.

1. Let any $\xi \in \mathfrak{g}$. If α_ξ is the one-parameter subgroup of G corresponding to ξ, explain why $\xi_e = [(\alpha_\xi, e)]$.
2. Let G be an abelian Lie group. Given $\xi, \eta \in \mathfrak{g}$, we denote by $\alpha, \beta \in \mathcal{H}om(\mathbb{R}, G)$ their respective one-parameter subgroups. Prove that the map
$$\alpha \cdot \beta \colon \mathbb{R} \longrightarrow G \colon t \mapsto \alpha(t) \cdot \beta(t)$$
is a one-parameter subgroup of G, and find the corresponding left-invariant vector field of it. Moreover, if $\mathcal{H}om(\mathbb{R}, G)$ is considered as an (abelian) group with multiplication defined by $\alpha \cdot \beta$, for every $\alpha, \beta \in \mathcal{H}om(\mathbb{R}, G)$, conclude that the map $\mathcal{F}$ of Theorem 4.3.8 is a morphism of abelian groups.
3. Let G be an abelian Lie group. Show that $\alpha_{\xi+\eta} = \alpha_\xi \cdot \alpha_\eta$, for every $\xi, \eta \in \mathfrak{g}$. Equivalently, $\alpha_{u+v} = \alpha_u \cdot \alpha_\xi$, for every $u, v \in T_eG$.
4. Prove that a morphism of Lie groups $f \colon G \to H$ induces a group morphism $\widetilde{\mathcal{F}} \colon \mathcal{H}om(\mathbb{R}, G) \to \mathcal{H}om(\mathbb{R}, H)$ (see Exercise 1 above). Relate $\widetilde{\mathcal{F}}$ with the morphism of Lie algebras $\widetilde{f} \colon \mathfrak{g} \to \mathfrak{h}$ of Exercise 4.2.12 (4).

4.4 The exponential map of a Lie group

We generalize here the ordinary exponential map $\exp \colon \mathbb{R} \to \mathbb{R}_* \colon t \mapsto e^t$, and the matrix exponential $\exp \colon \mathcal{M}_n(\mathbb{R}) \to \mathrm{GL}(n, \mathbb{R}) \colon A \mapsto e^A$. The exponential map of a Lie group G induces a special chart, which sends the one-parameter subgroups of G to straight lines in the model space of the group.

We fix a Lie group G of dimension n, with respective algebra Lie $\mathfrak{g}$.

Definition 4.4.1. With the notation of (4.3.2), the ***exponential map*** of G is defined to be the map
$$\exp \colon \mathfrak{g} \longrightarrow G \colon \xi \mapsto \exp(\xi) := \alpha_\xi(1).$$

The linear isomorphism $\boldsymbol{h} \colon \mathfrak{g} \to T_eG$ [see Theorem 4.2.5 and equality (4.2.5)] determines the map

$$\widetilde{\exp} := \exp \circ \boldsymbol{h}^{-1} \colon T_eG \longrightarrow G; \tag{4.4.1a}$$

therefore,

$$\exp(\xi) = \widetilde{\exp}(\xi_e). \tag{4.4.1b}$$

We notice that some authors consider $\widetilde{\exp}$ as the exponential map of G. Although we mainly use Definition 4.4.1, it will be convenient to identify the two exponentials and write, without further warning,

$$\exp(\xi) = \exp(\xi_e) \tag{4.4.1c}$$

as a consequence of the isomorphism $\boldsymbol{h}$.

Our immediate goal is to prove the smoothness of exp. Before this we recall that T_eG, as an n-dimensional vector space, is a smooth manifold [see Example 1.2.12 (B)]. More explicitly: Fixing a chart (U, ϕ) of G at e, we consider the induced linear isomorphism $\overline{\phi}\colon T_eG \to \mathbb{R}^n$(recall Theorem 2.1.9). Then $\big(T_eG, \overline{\phi}\,\big)$ is a global chart on T_eG and the corresponding maximal atlas $\big\{\big(T_eG, \overline{\phi}\,\big)\big\}'$ determines a smooth structure on T_eG. Analogously, $\boldsymbol{h}$ determines on $\mathfrak{g}$ the global chart $(\mathfrak{g}, \Psi)$, where $\Psi = \overline{\phi} \circ \boldsymbol{h}$, thus $\{(\mathfrak{g}, \Psi)\}'$ induces a smooth structure on $\mathfrak{g}$ with respect to which $\boldsymbol{h}$ is a diffeomorphism.

Theorem 4.4.2. *The exponential map* $\exp\colon \mathfrak{g} \to G$ *is smooth.*

Proof. The following proof, though a bit tricky, can be easily understood (see, for instance, [49], [70]), and avoids the use of geodesics (as in [12], [46], [68]) or analytic methods (e.g. [54]).

We define the map [see also Exercise 2.4.12 (5)]

$$\Xi\colon G \times \mathfrak{g} \to T(G \times \mathfrak{g}) \equiv TG \times T\mathfrak{g}$$

by setting

$$\Xi(g, \xi) := (\xi_g, 0_\xi) \in T_gG \times T_\xi\mathfrak{g},$$

where, as usual, 0_ξ is the zero vector of $T_\xi\mathfrak{g}$. We intend to show that Xi is a vector field on $G \times \mathfrak{g}$. It is immediate that Ξ is a section of $T(G \times \mathfrak{g})$. We check that it is a smooth map by showing the smoothness of each component. Indeed, following the proof of part ii) in Example 4.1.10 (A), we see that

$$\xi_g = T_eL_g(\xi_e) = T_{(g,e)}\boldsymbol{\gamma}(0_g, \xi_e),$$

or, if $\Omega\colon G \to TG$ is the zero section (zero vector field) of G, and $p_G\colon G \times \mathfrak{g} \to G$, $p_\mathfrak{g}\colon G \times \mathfrak{g} \to \mathfrak{g}$ are the projections to the respective factors,

$$\xi_g = T\boldsymbol{\gamma}\left(\Omega(g), \boldsymbol{h}^{-1}(\xi)\right) = \left(T\gamma \circ \left(\Omega \circ p_G, h^{-1} \circ p_\mathfrak{g}\right)\right)(g, \xi),$$

which proves that the first component of Ξ is a smooth map of (g, ξ) as a composite of smooth maps. The second component is also smooth since it takes the form

$$0_\xi = \big(\widetilde{\Omega} \circ p_\mathfrak{g}\big)(g, \xi),$$

where now $\widetilde{\Omega}\colon \mathfrak{g} \to T\mathfrak{g}$ is the zero section of $\mathfrak{g}$. Therefore, Ξ is a vector field on $G \times \mathfrak{g}$, as desired.

Next we find the integral curve $\beta := \beta_{(g,\xi)}$ of Ξ, with initial condition $\beta(0) = (g, \xi)$. Such a curve has the form $\beta(t) = \big(\beta_1(t), \beta_2(t)\big)$, with

$$\beta_1(0) = g, \qquad \beta_2(0) = \xi. \tag{4.4.2}$$

On the other hand, condition $\dot{\beta}(t) = \Xi(\beta(t))$ implies that

$$\big(\dot{\beta}_1(t), \dot{\beta}_2(t)\big) = \Xi\big(\beta_1(t), \beta_2(t)\big) = \big(\beta_2(t)\big(\beta_1(t)\big), 0\big);$$

that is,

$$\dot{\beta}_1(t) = \beta_2(t)\big(\beta_1(t)\big), \tag{4.4.3}$$

$$\dot{\beta}_2(t) = 0. \tag{4.4.4}$$

Equality (4.4.4), along with the second of (4.4.2), yields

$$\beta_2(t) = \xi, \qquad \forall\, t \in \mathbb{R}. \tag{4.4.5}$$

Substituting (4.4.5) in (4.4.3), we find that $\dot{\beta}_1(t) = \xi\big(\beta_1(t)\big)$, which means that β_1 is the integral curve of ξ, with initial condition $\xi(0) = g$. Note that, since ξ is complete, t varies also in the whole of $\mathbb{R}$, thus the domain of β is $\mathbb{R}$.

Let us denote by

$$\theta \colon \mathbb{R} \times G \to G \quad \text{and} \quad \Theta \colon \mathbb{R} \times G \times \mathfrak{g} \to G \times \mathfrak{g}$$

the flows of ξ and Ξ, respectively. Then, by Definition 3.7.1

$$\Theta_t(g, \xi) = \beta(t) = (\beta_1(t), \xi) = \big(\theta(t, g), \xi\big),$$

thus Θ_t is a smooth transformation of $G \times \mathfrak{g}$, for every $t \in \mathbb{R}$; hence,

$$\Theta_1(g, \xi) = \big(\theta(1, g), \xi\big), \tag{4.4.6}$$

showing that Θ_1 smooth. But $\theta(1, g) = \theta_g(1)$ is the value at 1 of the integral curve of ξ, with initial condition $\theta_g(0) = g$; therefore, by Corollary 4.3.7, $\theta_g = g\,\theta_e$, where θ_e is the integral curve of ξ with $\theta_e(0) = e$. As a result,

$$\theta(1, g) = \theta_g(1) = g\,\theta_e(1) = g\alpha_\xi(1) = g \exp \xi.$$

Consequently, for $g = e$, (4.4.6) yields $\Theta_1(e, \xi) = (\exp \xi, \xi)$, whence

$$\exp \xi = (p_{\mathfrak{g}} \circ \Theta_1)(e, \xi).$$

This proves the smoothness of exp and completes the proof. □

Proposition 4.4.3. *The exponential map has the following properties:*

$$\exp(t\xi) = \alpha_\xi(t), \tag{4.4.7}$$

$$\exp(0) = e, \tag{4.4.8}$$

$$\exp\big((t + s)\xi\big) = \exp(t\xi) \cdot \exp(s\xi) = \exp(s\xi) \cdot \exp(t\xi), \tag{4.4.9}$$

$$\exp(n\xi) = \exp(\xi)^n, \tag{4.4.10}$$

for all $s, t \in \mathbb{R}$, $n \in \mathbb{N}$ and $\xi \in \mathfrak{g}$.

Proof. Combining the definition of exp with (4.3.3), we obtain (4.4.7); namely,

$$\exp(t\xi) = \alpha_{t\xi}(1) = \alpha_\xi(t).$$

From the preceding equality it follows that

$$\exp(0) = \exp(0\xi) = \alpha_\xi(0) = e,$$

proving (4.4.8). On the other hand,

$$\exp\big((t + s)\xi\big) = \alpha_\xi(t + s) = \alpha_\xi(t) \cdot \alpha_\xi(s) = \exp(t\xi) \cdot \exp(s\xi),$$
$$\exp\big((s + t)\xi\big) = \alpha_\xi(s + t) = \alpha_\xi(s) \cdot \alpha_\xi(t) = \exp(s\xi) \cdot \exp(t\xi).$$

Since the left-hand sides of the above equalities coincide, the proof of (4.4.9) is now clear, as well as that of (4.4.10). □

Thinking again of $\mathfrak{g}$ as a smooth manifold in the way described before Theorem 4.4.2, from the chart $(\mathfrak{g}, \Psi)$, with $\Psi\colon \mathfrak{g} \to \mathbb{R}^n$, we obtain the induced linear isomorphism $\overline{\Psi}\colon T_0\mathfrak{g} \to \mathbb{R}^n$. Thus $T_0\mathfrak{g}$ is also identified with $\mathfrak{g}$ by means of the linear isomorphism $\Psi^{-1} \circ \overline{\Psi}\colon T_0\mathfrak{g} \to \mathfrak{g}$, as in the following diagram.

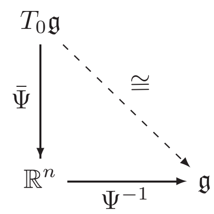

Diagram 4.7

Theorem 4.4.4. *The differential* $T_0 \exp\colon T_0\mathfrak{g} \to T_eG$ *of* $\exp$ *at* $0 \in \mathfrak{g}$ *is a linear isomorphism. In particular, after the identification* $T_0\mathfrak{g} \equiv \mathfrak{g}$*, one has that* $T_0 \exp = id_{\mathfrak{g}}$*. Hence,* $\exp$ *is a local diffeomorphism at 0.*

Proof. First observe that the range of $T_0 \exp$ is T_eG since $\exp(0) = e$. For any fixed $\xi \in \mathfrak{g}$, we consider the map

$$\beta\colon \mathbb{R} \longrightarrow \mathfrak{g} : t \mapsto \beta(t) := t\xi. \tag{4.4.11}$$

It is a smooth curve in $\mathfrak{g}$ with $\beta(0) = 0$, thus the one-parameter subgroup α_ξ generated by ξ, takes the form

$$\alpha_\xi(t) = \exp(t\xi) = (\exp \circ \beta)(t), \qquad t \in \mathbb{R}. \tag{4.4.12}$$

Therefore,

$$\begin{aligned} \xi_e = \dot{\alpha}_\xi(0) &= \Big(\dot{\overbrace{\exp \circ \beta}}\Big)(0) = T_0\big(\exp \circ \beta\big)\left(\frac{d}{dt}\bigg|_0\right) \\ &= T_0 \exp\left(T_0\beta\left(\frac{d}{dt}\bigg|_0\right)\right) = T_0 \exp\big(\dot{\beta}(0)\big) \end{aligned} \tag{4.4.13}$$

Since, by Exercise 2.2.15 (5), $\dot{\beta}(0) = [(\beta, \beta(0))] = [(\beta, 0)]$, we obtain

$$\xi_e = T_0 \exp([(\beta, 0)]). \tag{4.4.14}$$

Applying $\overline{\Psi}$ to $[(\beta, 0)]$, we have that (see Proposition 2.1.7)

$$\overline{\Psi}\big([(\beta, 0)]\big) = \big(\Psi \circ \beta\big)'(0),$$

while, by the linearity of Ψ,

$$(\Psi \circ \beta)(t) = \Psi(t\xi) = t\Psi(\xi) \in \mathbb{R}^n.$$

As a consequence,

$$\overline{\Psi}\big([(\beta, 0)]\big) = \big(t\Psi(\xi)\big)'_{t=0} = \Psi(\xi),$$

or, equivalently,

$$[(\beta, 0)] = \big(\overline{\Psi}^{-1} \circ \Psi\big)(\xi).$$

Combining the preceding equality with (4.4.14) and (4.2.5), we obtain

$$\xi_e = \boldsymbol{h}(\xi) = \Big(T_0 \exp) \circ \big(\overline{\Psi}^{-1} \circ \Psi\big)\Big)(\xi). \tag{4.4.15}$$

Varying ξ in the definition of β, we see that (4.4.15) holds for every $\xi \in \mathfrak{g}$, whence (see also the notations before Theorem 4.4.2)

$$T_0 \exp = \boldsymbol{h} \circ \Psi^{-1} \circ \overline{\Psi} = \overline{\phi}^{-1} \circ \overline{\Psi}. \tag{4.4.16}$$

Since $\overline{\phi} \circ \overline{\Psi}$ is a linear isomorphism, we have the first assertion of the statement.

Considering now the identification $T_0\mathfrak{g} \equiv \mathfrak{g}$ and omitting the term $\overline{\phi} \circ \overline{\Psi}$, we can write $T_0 \exp \equiv \mathrm{id}_{\mathfrak{g}}$, i.e. $T_0 \exp$ is a linear isomorphism; hence, in virtue of the Inverse Function Theorem 2.2.7, exp is a local diffeomorphism at 0. □

By the second assertion of Theorem 4.4.4, there exists a neighborhood V_o of 0 in $\mathfrak{g}$, and a neighborhood N_e of e in G such that $\exp|_{V_o} \colon V_o \to N_e$ is a diffeomorphism, whose inverse is the map $\exp^{-1} \colon N_e \to V_o$ [for simplicity, we write $\exp^{-1}$ instead of the more accurate $(\exp|_{V_o})^{-1}$]. Composing $\exp^{-1}$ with Ψ, we obtain the map

$$\nu := \Psi \circ \exp^{-1} \colon N_e \longrightarrow \Psi(V_o),$$

also shown in the next diagram.

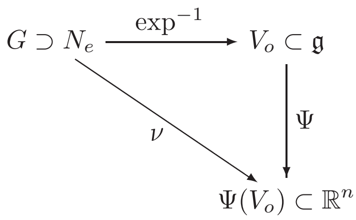

Diagram 4.8

The set $\Psi(V_o)$ is open in $\mathbb{R}^n$ and ν is a diffeomorphism; hence, by Proposition 1.4.12, the pair (N_e, ν) is a chart of G with $e \in N_e$, called the ***normal chart*** at e.

Proposition 4.4.5. *Let any one-parameter subgroup $\alpha \in \mathcal{H}om(\mathbb{R}, G)$. Then the map ν of the normal chart sends the part of $\alpha(\mathbb{R})$ contained in N_e to a line segment in $\mathbb{R}^n$ passing through 0.*

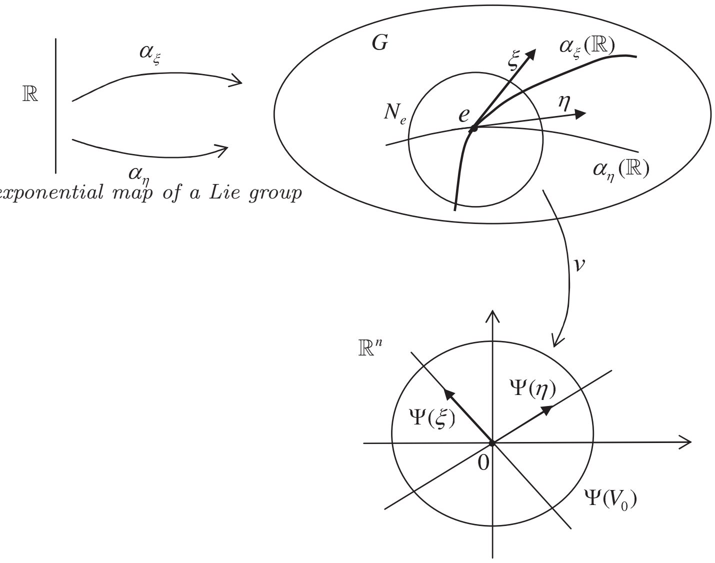

Figure 4.1 The normal chart at $e \in G$

Proof. In virtue of (4.3.2) and Theorem 4.3.8, we may write $\alpha = \alpha_\xi$ for some $\xi \in \mathfrak{g}$. Then, for every $t \in \mathbb{R}$ such that $\alpha_\xi(t) \in N_e$, we see that

$$\begin{aligned} \nu\big(\alpha_\xi(t)\big) &= \big(\Psi \circ \exp^{-1}\big)\big(\alpha_\xi(t)\big) \\ &= \Psi\big(\exp^{-1}(\exp(t\xi))\big) \\ &= \Psi(t\xi) = t\Psi(\xi), \end{aligned}$$

which proves the statement. □

Examples 4.4.6.

(A) *Comparison of* $\overline{\phi}$ *and* $\overline{\nu}$

We will show that the linear isomorphisms $\overline{\phi}\colon T_eG \to \mathbb{R}^n$ and $\overline{\nu}\colon T_eG \to \mathbb{R}^n$, induced respectively by the charts (U, ϕ) and (N_e, ν) coincide. Recall that (U, ϕ) is the chart of G at e, mentioned in the discussion before Theorem 4.4.2 and used throughout the ensuing results.

Indeed, by definition, $\nu \circ \exp = \Psi$. Differentiating the previous equality at 0, we obtain in virtue of (4.4.16):

$$T_0\Psi = T_0(\nu \circ \exp) = T_e\nu \circ T_0 \exp = T_e\nu \circ \overline{\phi}^{\,-1} \circ \overline{\Psi}.$$

Applying (2.2.4a), the preceding equality leads to

$$\overline{\Psi} = \overline{\mathrm{id}}_{\mathbb{R}^n} \circ T_0\Psi = \overline{\mathrm{id}}_{\mathbb{R}^n} \circ T_e\nu \circ \overline{\phi}^{\,-1} \circ \overline{\Psi} = \overline{\nu} \circ \overline{\phi}^{\,-1} \circ \overline{\Psi}$$

which implies that $\overline{\phi} = \overline{\nu}$.

Note. Let (x^i) and (y^i), $i = 1, \ldots, n$, be the coordinates of the charts (U, ϕ) and (N_e, ν), respectively. Then, the corresponding bases of T_eG coincide, because

$$\left.\frac{\partial}{\partial x^i}\right|_e = \overline{\phi}^{\,-1}(e_i) = \overline{\nu}^{\,-1}(e_i) = \left.\frac{\partial}{\partial y^i}\right|_e$$

holds for every $i = 1, \ldots, n$.

(B) *Lie group morphisms and exponential maps*

Let G, H be Lie groups with corresponding Lie algebras $\mathfrak{g}$, $\mathfrak{h}$ and exponential maps $\exp_G$, $\exp_H$. Let $f\colon G \to H$ be a Lie group morphism. Then, in virtue of (**??**), we will show that the following diagram is commutative:

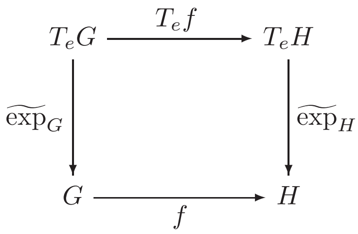

Diagram 4.9

Recall that e denotes the unit of both G and H.

Indeed, for any $u \in T_eG$, we set $v := T_ef(u) \in T_eH$. By Theorem 4.2.5 there are $\xi \in \mathfrak{g}$ and $\eta \in \mathfrak{h}$, such that $u = \xi_e$ and $v = \eta_e$, respectively. Moreover [see (4.3.2) and Theorem 4.3.8], we find one-parameter subgroups $\alpha_\xi \in \mathcal{H}om(\mathbb{R}, G)$ and $\beta_\eta \in \mathcal{H}om(\mathbb{R}, H)$ such that $\alpha_\xi(0) = e$, $\dot{\alpha}_\xi(0) = \xi_e = u$, and $\beta_\eta(0) = e$, $\dot{\beta}_\eta(0) = \beta_e = v$. Then (see also Example 4.3.10)

$$(4.4.17)\qquad \big(\widetilde{\exp}_H \circ T_ef\big)(u) = \widetilde{\exp}_H\big(T_ef(\xi_e)\big) = \widetilde{\exp}_H(\eta_e) = \beta_\eta(1).$$

On the other hand,

$$\Big(\dot{\widetilde{f \circ \alpha_\xi}}\Big)(0) = T_0(f \circ \alpha_\xi)\left(\frac{d}{dt}\Big|_0\right) = T_ef\big(\dot{\alpha}_\xi\big)(0) = T_ef(u) = \eta_e;$$

therefore, $f \circ \alpha_\xi$ is the unique one-parameter subgroup of H with $\Big(\dot{\widetilde{f \circ \alpha_\xi}}\Big)(0) = \eta_e$, thus $f \circ \alpha_\xi = \beta_\eta$. Consequently, (4.4.17) transforms into

$$(4.4.18a)\qquad \begin{aligned} \big(\widetilde{\exp}_H \circ T_ef\big)(u) &= \beta_\eta(1) = (f \circ \alpha_\xi)(1) = \\ &= f\big(\alpha_\xi(1)\big) = f\big(\exp_G(\xi)\big) = \big(f \circ \widetilde{\exp}_G\big)(u), \end{aligned}$$

for every $u \in T_eG$, whence the commutativity of Diagram 4.9.

If we consider the Lie algebra morphism $\widetilde{f}\colon \mathfrak{g} \to \mathfrak{h}$ induced by f [see Exercise 4.2.12 (4)], then we obtain equality

$$(4.4.18b)\qquad \exp_H \circ \widetilde{f} = f \circ \exp_G,$$

which amounts to the commutativity of the next diagram.

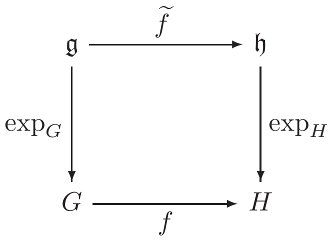

Diagram 4.10

We leave the proof of (4.4.18b) to the reader.

(C) *Exponentials and derivations induced by vector fields*

Viewing a vector field $\xi \in \mathfrak{g}$ as a derivation (see Theorem 3.2.3), we will prove that

$$\xi_x(f) = \Big(f\big(x\exp(t\xi)\big)\Big)'(0) = \lim_{t\to 0}\frac{1}{t}[f(x\exp(t\xi)) - f(x)]. \tag{4.4.19}$$

for every $x \in G$ and $f \in \mathcal{C}^\infty(G,\mathbb{R})$.

To this end observe that the left invariance of ξ and (2.3.25) yield

$$\xi_x(f) = \big(T_eL_x(\xi_e)\big)(f) = \xi_e(f\circ L_x),$$

or, because $\xi_e = \dot{\alpha_\xi}(0)$,

$$\begin{aligned}\xi_x(f) = \big(\dot{\alpha}_\xi(0)\big)(f\circ L_x) &= \left(T_0\alpha_\xi\left(\frac{d}{dt}\bigg|_0\right)\right)(f\circ L_x)\\ &= \frac{d}{dt}\bigg|_0\big(f\circ L_x\circ\alpha_\xi\big) = \big(f\circ L_x\circ\alpha_\xi\big)'(0).\end{aligned}$$

Furthermore, since $\alpha_\xi(t) = \exp(t\xi)$, we see that

$$\big(f\circ L_x\circ\alpha_\xi\big)(t) = f\big(L_x(\alpha_\xi(t))\big) = f\big(x\exp(t\xi)\big).$$

Therefore,

$$\xi_x(f) = \big(f\circ L_x\circ\alpha_\xi\big)'(0) = \big(f(x\exp(t\xi))\big)'(0).$$

(D) *The flow* $\theta\colon \mathbb{R}\times G\to G$ *of a* $\xi\in\mathfrak{g}$ *in terms of* exp

Since $\theta_x(t)$ is the integral curve of ξ with initial condition $x \in G$, equality (4.4.7) implies that $\theta_x(t) = x\alpha_\xi(t) = x\exp(t\xi)$. Therefore, after (4.4.1c),

$$\theta(t,x) = \theta_x(t) = \theta_t(x) = x\exp(t\xi) = x\exp(t\xi_e). \tag{4.4.20}$$

In particular,

$$\theta_t(x) = x\exp(t\xi) = R_{\exp(t\xi)}(x); \qquad x\in G,$$

implying that

$$\theta_t = R_{\exp(t\xi)} = R_{\exp(t\xi_e)}. \tag{4.4.21}$$

(E) *The exponential of* $\mathrm{GL}(n,\mathbb{R})$

Let $\xi \in \mathfrak{gl}(n,\mathbb{R})$ be a left-invariant vector field on $\mathrm{GL}(n,\mathbb{R})$, and let us denote by $A\colon \mathbb{R}\to\mathrm{GL}(n,\mathbb{R})$ the one-parameter subgroup generated by ξ. Since A is the integral curve of ξ with $A(0) = I$,

$$\xi_{A(t)} = \xi(A(t)) = \dot{A}(t). \tag{4.4.22}$$

By Exercise 4.2.12 (7) and the identification $f_A\colon T_{A(t)}\mathrm{GL}(n,\mathbb{R}) \xrightarrow{\cong} \mathcal{M}_n(\mathbb{R})$, $\xi_{A(t)}$ identifies with the product of matrices $A(t)\cdot B$, where B is the matrix corresponding to ξ_I [via the identification $f_I\colon T_I\mathrm{GL}(n,\mathbb{R}) \xrightarrow{\cong} \mathcal{M}_n(\mathbb{R})$].

On the other hand, to find the matrix corresponding to$\dot{A}(t)$, we check that

$$\begin{aligned}\dot{A}(t) &= T_tA\left(\left.\frac{d}{dt}\right|_t\right)\\ &= \sum_{i,j=1}^n \left[T_tA\left(\left.\frac{d}{dt}\right|_t\right)\right](x^{ij})\left.\frac{\partial}{\partial x^{ij}}\right|_{A(t)}\\ &= \sum_{i,j=1}^n \left.\frac{d}{dt}\right|_t (x^{ij}\circ A)\left.\frac{\partial}{\partial x^{ij}}\right|_{A(t)}\\ &= \sum_{i,j=1}^n A'_{ij}(t)\left.\frac{\partial}{\partial x^{ij}}\right|_{A(t)},\end{aligned}$$

where $A_{ij}(t) = (x^{ij}\circ A)(t) = x^{ij}(A(t))$. Hence, the matrix corresponding to $\dot{A}(t)$ is precisely $\big(A'_{ij}(t)\big)$, thus (4.4.22) identifies with

$$A'(t) = A(t)\cdot B,$$

which implies $A(t) = e^{tB}$, the analog of $\alpha_\xi(t) = \exp(t\xi_e)$. In particular, $e^B = A(1)$, corresponds to $\alpha_\xi(1) = \exp(\xi_e)$.

In conclusion, the exponential of $\mathrm{GL}(n,\mathbb{R})$ is identified with the matrix exponential. For the properties of the latter we refer to [4], [43], and [70].

Exercises 4.4.7.

1. Prove that the smooth structure of T_eG does not depend on the choice of the chart (U,ϕ) containing e.
2. Prove that $\exp(-t\xi) = \big(\exp(t\xi)\big)^{-1}$, for every $t\in\mathbb{R}$ and $\xi\in\mathfrak{g}$.
3. If G is an abelian Lie group, prove that
$$\exp(\xi+\eta) = \exp(\xi)\cdot\exp(\eta)$$
for every $\xi,\eta\in\mathfrak{g}$.
4. Prove equality (4.4.18b).
5. Let (N_e,ν) be the normal chart on G. If $x\in G$, prove that the pair (N_x,ν_x), where $N_x = L_x(N_e)$ and $\nu_x = \nu\circ L_{x^{-1}}$, is a chart on G containing x. Moreover, if β is the integral curve of some $\xi\in\mathfrak{g}$, with $\beta(0)=x$, prove that $\nu_x\big(\beta(\mathbb{R})\cap N_x\big)$ is contained in a straight line in $\mathbb{R}^n$ passing through 0.

4.5 The adjoint representation

The representation of the title turns out to be of particular importance in the theory of connections on principal bundles and in gauge theories, subjects tackled in advanced treatises.

Remark 6.2.8. If we start with a connected smooth manifold M without assuming beforehand any other topological property, it is clear that the existence of a Riemannian structure on M imposes on it also the Hausdorff property. Thus, in the case of such a manifold, Corollary 6.2.7 has an *if and only if* expression.

Exercises 6.2.9.

1. Verify equalities (6.2.4) and (6.2.5).

2. We define the following relation in a manifold M:

$$x \sim y \quad \Leftrightarrow \quad x \text{ and } y \text{ are joined by a piecewise smooth curve.}$$

Show that $\sim$ is an equivalence relation in M and all the equivalence classes $[x]$, $x \in M$, are open and closed subsets of M. How can this result be used to prove Lemma 6.2.2?

3. Let $\alpha\colon [a,b] \to M$ be a (piecewise) smooth curve. A ***reparametrization*** of α is a (piecewise) smooth curve $\beta\colon [a',b'] \to M$ such that $\beta = \alpha \circ h$, where $h\colon [a',b'] \ni s \mapsto h(s) = t \in [a,b]$ is a diffeomorphism (in the ordinary sense of calculus). Then show that $L_g(\alpha) = L_g(\beta)$, which means that

the length of a (piecewise) smooth curve is independent of the parametrization of the curve.

4. Show that the distance function $d_g\colon M \times M \to \mathbb{R}$, induced by a Riemannian structure g, is continuous.

5. Let (M,g) and (N,g') be Riemannian manifolds. An ***isometry*** of M onto N is a diffeomorphism $f\colon M \to N$ such that

$$g_x(u,v) = g'_{f(x)}\big(T_xf(u), T_xf(v)\big),$$

for every $x \in M$, and every $u, v \in T_xM$. Then prove that equality

$$L_{g'}(f \circ \gamma) = L_g(\gamma)$$

holds for every (piecewise) smooth curve in M; in other words,

isometries preserve the length.

6. Let $f\colon M \to N$ be an isometry as in the preceding exercise. Then show that f^{-1} is also an isometry

7. With the notation of the Exercise 5, prove that $d_{g'}\big(f(x), f(y)\big) = d_g(x,y)$, for every $x, y \in M$. Therefore,

isometries preserve the distance.

8. Prove that the isometries of a Riemannian manifold (M,g) (onto itself) form a group.

6.3 Connections

The notion of a connection can be traced back to the early attempts of E. B. Christoffel (ca. 1869), M.M.C. Ricci and T. Levi-Cività (between 1901–1917) to differentiate vector and tensor fields in a "covariant" way to preserve, so to speak, the character of the fields. The geometric interpretation of such a covariant derivation was given by Levi-Cività in terms the notion of parallel displacement on a surface. Since then, connections were defined on manifolds and fiber bundles (a short note on this development can be found e.g. in [41]), and they have had a profound effect on the development of differential geometry itself and related disciplines, as well as to modern theories of physics.

In this section we treat connections in the sense of J.L. Koszul (ca. 1950). This approach is quite elegant and easily manipulated. With this background the reader may grasp many related notions (such as the parallel displacement and the holonomy groups), other generalizations to fiber bundles, as well as applications to physics, mechanics and elsewhere, which are beyond the scope of this book.

Nowadays there exist so many definitions of a connection (within various frameworks, of course), that M. Spivak writes, in his particular style (see [68, Vol. V, p. 602]:

> *I personally feel that the next person to propose a new definition of a connection should be summarily executed.*

Definition 6.3.1. Let M be a smooth manifold. A ***connection*** or ***covariant derivation*** on M is an $\mathbb{R}$-bilinear map

$$\nabla\colon \mathcal{X}(M)\times\mathcal{X}(M)\longrightarrow\mathcal{X}(M)\colon (\xi,\eta)\mapsto\nabla_\xi\eta := \nabla(\xi,\eta),$$

satisfying the following conditions:

$$\text{(C. 1)}\qquad \nabla_{f\xi}\eta = f\,\nabla_\xi\eta,$$

$$\text{(C. 2)}\qquad \nabla_\xi(f\eta) = f\,\nabla_\xi\eta + \xi(f)\cdot\eta,$$

for every $\xi,\eta\in\mathcal{X}(M)$ and $f\in\mathcal{C}^\infty(M,\mathbb{R})$. $\nabla_\xi\eta$ is called the ***covariant derivative of*** η ***in the direction of*** ξ.

Condition (C. 1) also implies that ∇ is $\mathcal{C}^\infty(M,\mathbb{R})$-linear with respect to the first variable, that is $\nabla_{f\xi+g\eta}\zeta = f\,\nabla_\xi\zeta + g\nabla_\eta\zeta$, for every $f,g\in\mathcal{C}^\infty(M,\mathbb{R})$, and $\xi,\eta,\zeta\in\mathcal{X}(M)$. This is not true for the second variable due to the *Leibniz* (or *product*) rule expressed by condition (C. 2).

A connection ∇ (read *nabla* or *del*) is also called a *linear* or *affine* connection because, in the case of a finite-dimensional manifold M, a covariant derivative is equivalent to the former, determined by an appropriate *horizontal distribution* on $T(TM)$ or by a *connection map.* For details we refer, e.g. to [12], or the more advanced [27], [41], [71].

Examples 6.3.2.

(A) *Connections over charts*

Consider a chart (U,ϕ) of an m-dimensional manifold M. To simplify our computations, the corresponding basic vector fields (3.1.3) are abbreviated as follows:

$$\text{(6.3.1)}\qquad \partial_i := \frac{\partial}{\partial x^i},\qquad 1\le i\le m.$$

Then, for two vector fields ξ and η in $\in \mathcal{X}(U)$, we write

$$\xi = \sum_{i=1}^{m} \xi^i \,\partial_i \,, \;\; \eta = \sum_{i=1}^{m} \eta^i \,\partial_i,$$

where the coefficients are the (smooth) coordinates of the fields. Setting

$$\nabla_\xi \eta := \sum_{i=1}^{m} \xi(\eta^i) \cdot \partial_i, \tag{6.3.2}$$

we obtain a connection on the open submanifold U of M. Indeed, $\mathbb{R}$-bilinearity and (C. 1) are immediately checked. On the other hand, for every $\xi, \eta \in \mathcal{X}(U)$, as above, and every $f \in \mathcal{C}^\infty(U, \mathbb{R})$, we have that

$$\nabla_{f\xi}\eta = \sum_{i=1}^{m} f\xi(\eta^i) \cdot \partial_i = f \sum_{i=1}^{m} \xi(\eta^i) \cdot \partial_i = f\, \nabla_\xi \eta,$$

whereas [see also (3.2.1b)]

$$\begin{aligned} \nabla_\xi(f\eta) &= \sum_{i=1}^{m} \xi(f\eta^i) \cdot \partial_i = \sum_{i=1}^{m} \big(\xi(f)\eta^i + f\xi(\eta^i)\big) \cdot \partial_i \\ &= \sum_{i=1}^{m} \xi(f)\eta^i \cdot \partial_i + \sum_{i=1}^{m} f\xi(\eta^i) \cdot \partial_i \\ &= \xi(f) \sum_{i=1}^{m} \eta^i \cdot \partial_i + f \sum_{i=1}^{m} \xi(\eta^i) \cdot \partial_i \\ &= \xi(f)\eta + f\, \nabla_\xi \eta. \end{aligned}$$

Connection (6.3.2) is a particular case of the following example.

(B) *Connections on parallelizable manifolds*
Let M be an m-dimensional parallelizable manifold with a parallelization $\mathcal{B} = \{X_1, \ldots, X_m\}$ [recall Example 3.1.8 (C)]. Given two vector fields of M

$$\xi = \sum_{i=1}^{m} \xi^i X_i, \;\; \eta = \sum_{i=1}^{m} \eta^i X_i,$$

we set

$$\nabla_\xi \eta := \sum_{i=1}^{m} \xi(\eta^i) \cdot X_i.$$

Working exactly as in the preceding example, we check that ∇ is indeed a connection on M. This is the so-called ***direct connection***, relative to the parallelization $\mathcal{B}$.

Proposition 6.3.3. *Let ∇ be a connection on a smooth manifold M and $\xi, \eta \in \mathcal{X}(M)$. If A is an open subset of M such that $\xi|_A = 0$ or $\eta|_A = 0$, then $\nabla_\xi \eta\big|_A = 0$. This property characterizes ∇ as a* ***local operator****.*

We clarify that $\nabla_\xi \eta\big|_A \equiv \big(\nabla_\xi \eta\big)\big|_A$.

Proof. Let any $x \in A$. By Proposition 1.5.2, there are open sets V, W in M, with $x \in W \subset \overline{W} \subset V \subset \overline{V} \subset A$, and a smooth bump function $f\colon M \to \mathbb{R}$ such that $f|_W = 1$ and $f|_{\overline{V}^c} = 0$. We consider the following cases:

i) Assume that $\xi|_A = 0$. Then, taking into account the assumption and the form of f, the evaluation of $f\xi$ on A and A^c implies that $(f\xi)(x) = 0$, for every $x \in M$, i.e. $f\xi = 0$. Thus (C. 1) and the $\mathbb{R}$-bilinearity of ∇ yield

$$\nabla_{f\xi}\eta = f\,\nabla_\xi\eta = 0;$$

therefore, for every $x \in A$,

$$\big(\nabla_{f\xi}\eta\big)(x) = \big(f\,\nabla_\xi\eta\big)(x) = f(x)\cdot(\nabla_\xi\eta)(x) = 1\cdot(\nabla_\xi\eta)(x) = 0,$$

whence $(\nabla_\xi\eta)(x)|_A = 0$.

ii) If $\eta|_A = 0$, again $f\eta = 0$, therefore

$$\nabla_\xi(f\eta) = f\,\nabla_\xi\eta + \xi(f)\cdot\eta = 0,$$

or, by evaluating the latter at any $x \in A$,

$$\big(\nabla_\xi(f\eta)\big)(x) = f(\nabla_\xi\eta)(x) + \xi_x(f)\cdot\eta(x) = 0.$$

Since f is constantly 1 in the neighborhood V of x, equality (2.3.10) implies that $\xi_x(f) = 0$. As a result,

$$(\nabla_\xi\eta)(x) = (\nabla_\xi f\eta)(x) = 0; \qquad x \in A,$$

showing also that $(\nabla_\xi\eta)|_A = 0$. □

An immediate consequence of Proposition 6.3.3 is the following:

Corollary 6.3.4. *Let ∇ be a connection on M and $A \subset M$ open. If $\xi_1, \xi_2 \in \mathcal{X}(M)$, with $\xi_1|_A = \xi_2|_A$, then*

$$(\nabla_{\xi_1}\eta)|_A = (\nabla_{\xi_2}\eta)|_A\,, \quad \textit{for every } \eta \in \mathcal{X}(M).$$

Similarly, if $\eta_1, \eta_2 \in \mathcal{X}(M)$, with $\eta_1|_A = \eta_2|_A$, then

$$(\nabla_\xi\eta_1)|_A = (\nabla_\xi\eta_2)|_A\,, \quad \textit{for every } \xi \in \mathcal{X}(M).$$

Proposition 6.3.5. *Let ∇ be a connection on M and $x \in M$. If $\xi_1, \xi_2 \in \mathcal{X}(M)$ are vector fields such that $\xi_1(x) = \xi_2(x)$, then*

$$(\nabla_{\xi_1}\eta)(x) = (\nabla_{\xi_2}\eta)(x),$$

for every $\eta \in \mathcal{X}(M)$.

Proof. In virtue of the $\mathbb{R}$-linearity of ∇ with respect to the first variable, it suffices to prove that, if $\xi \in \mathcal{X}(M)$ with $\xi(x) = 0$, then $(\nabla_\xi\eta)(x) = 0$, for every $\eta \in \mathcal{X}(M)$. To this end we consider a coordinate chart $(U, \phi) = (U; x^1, \ldots, x^m)$ of M at x. Then, by (6.3.1),

$$\xi|_U = \sum_{i=1}^{m} \xi^i\,\partial_i.$$

As in the proof of Proposition 6.3.3, there are (open) neighborhoods W, V of x, such that $x \in W \subset \overline{W} \subset V \subset \overline{V} \subset U$ and a smooth bump function f with $f|_V = 1$ and $f|_{\overline{V}^c} = 0$. We define the following extensions of ξ^i and ∂_i, respectively:

$$\widetilde{\xi^i}(x) = \begin{cases} f(x)\xi^i(x), & x \in U, \\ 0, & x \notin U, \end{cases}$$

$$\widetilde{\partial_i}(x) = \begin{cases} f(x)\partial_i, & x \in U, \\ 0, & x \notin U. \end{cases}$$

Obviously $\widetilde{\xi_i} \in \mathcal{C}^\infty(M, \mathbb{R})$ and $\widetilde{\partial_i} \in \mathcal{X}(M)$. Setting

$$\widetilde{\xi} := \sum_{i=1}^m \widetilde{\xi^i}\, \widetilde{\partial_i},$$

we see that $\widetilde{\xi} \in \mathcal{X}(M)$, with $\widetilde{\xi} = \xi|_W$. Therefore, for every $\eta \in \mathcal{X}(M)$ and every $z \in W$, Proposition 6.3.3 implies that

$$\big(\nabla_\xi \eta\big)(z) = \big(\nabla_{\widetilde{\xi}}\, \eta\big)(z) = \Big(\nabla_{\left(\sum_{i=1}^m \widetilde{\xi^i}\widetilde{\partial_i}\right)}\, \eta\Big)(z) = \sum_{i=1}^m \widetilde{\xi^i}(z) \cdot \big(\nabla_{\widetilde{\partial_i}}\, \eta\big)(z).$$

Consequently, for $z = x$, the assumption $\xi(x) = 0$ yields $\widetilde{\xi^i}(x) = \xi^i(x) = 0$, thus the above series of equalities lead to $(\nabla_\xi \eta)(x) = 0$, completing the proof. $\square$

The next result clarifies the exact way $\big(\nabla_\xi \eta\big)(x)$ depends on ξ and η.

Corollary 6.3.6. *The value $\big(\nabla_\xi \eta\big)(x)$ of $\nabla_\xi \eta$ at $x \in M$ depends only on the value of ξ at x and the values of η on an (arbitrarily small) neighborhood of x. In other words, if $\widetilde{\xi}$, $\widetilde{\eta}$ are vector fields on M such that $\widetilde{\xi}(x) = \xi(x)$ and $\widetilde{\eta}|_V = \eta|_V$, for any $V \in \mathcal{N}_x$, then*

$$\big(\nabla_{\widetilde{\xi}}\, \widetilde{\eta}\big)(x) = \big(\nabla_\xi \eta\big)(x).$$

Proof. By Corollary 6.3.4 and Proposition 6.3.5,

$$\big(\nabla_{\widetilde{\xi}}\, \widetilde{\eta}\big)(x) = \big(\nabla_{\widetilde{\xi}}\, \eta\big)(x) = \big(\nabla_\xi \eta\big)(x). \qquad \square$$

A consequence of the preceding result is that it makes sense to set

$$\nabla_{\xi_x} \eta := \big(\nabla_\xi \eta\big)(x).$$

Applying now conditions (C. 1) and (C. 2) of Definition 6.3.1, we see that $\nabla_{\xi_x} \eta$ satisfies the analogs of the properties of the ordinary directional derivative (see Appendix B, Definition B.1.9); hence,

$\nabla_{\xi_x} \eta$ can be interpreted as a directional derivative of η in the direction of ξ_x.

Concerning the existence of connections, we prove the following.

Theorem 6.3.7. *Every (second countable and Hausdorff) smooth manifold M admits a connection.*

Proof. Let $\{(U_\alpha, \phi_\alpha),\ \alpha \in I$, be a locally finite family of charts in $\mathcal{A}$ and $\{\psi_\alpha\}_{i\in I}$ a subordinate smooth partition of unity, as ensured by Theorem 1.5.6. Note that the indices of I are denoted by lower Greek letters to avoid confusion with the indices $i, j, k, \ldots$ running in the set $\{1, \ldots, m = \dim M\}$.

For every $\alpha \in I$ and $\xi, \eta \in \mathcal{X}(M)$, we consider the local connection, over U_α, [see also (6.3.2)]

$$\nabla^\alpha_\xi \eta := \sum_{i=1}^m \xi(\eta^i) \cdot \partial_i,$$

where ξ and η in the left-hand side are restricted to U_α (to simplify the subsequent computations, we omit restrictions). Then we set

$$\big(\nabla_\xi \eta\big)(x) := \sum_{\alpha\in I} \psi_\alpha(x) \cdot \big(\nabla^\alpha_\xi \eta\big)(x), \qquad x \in M.$$

It is immediately verified that $\nabla_\xi \eta \in \mathcal{X}(M)$, for every $\xi, \eta \in \mathcal{X}(M)$, and ∇ is an $\mathbb{R}$-bilinear map. Moreover, for any $\xi, \eta \in \mathcal{X}(M)$ and $f \in \mathcal{C}^\infty(M, \mathbb{R})$,

$$\begin{aligned}\nabla_{f\xi}\eta &= \sum_{\alpha\in I} \psi_\alpha \cdot \nabla^\alpha_{f\xi}\eta = \sum_{\alpha\in I} \psi_\alpha \cdot \big(f\, \nabla^\alpha_\xi \eta\big) \\ &= f \sum_{\alpha\in I} \psi_\alpha \cdot \nabla^\alpha_\xi \eta = f\, \nabla_\xi \eta,\end{aligned}$$

whereas

$$\begin{aligned}\nabla_\xi(f\eta) &= \sum_{\alpha\in I} \psi_\alpha \cdot \nabla^\alpha_\xi (f\eta) \\ &= \sum_{\alpha\in I} \psi_\alpha \cdot \big(f\, \nabla^\alpha_\xi \eta + \xi(f)\eta\big) \\ &= f \sum_{\alpha\in I} \psi_i \cdot \nabla^\alpha_\xi \eta + \left(\sum_{\alpha\in I} \psi_\alpha\right) \cdot \xi(f)\eta \\ &= f\, \nabla_\xi \eta + \xi(f)\eta.\end{aligned}$$

□

To describe a connection locally, we need the next result:

Lemma 6.3.8. *Let ∇ be a connection an a manifold M. If A is any open subset of M, then ∇ induces on A a (local) connection $\nabla^A \colon \mathcal{X}(A) \times \mathcal{X}(A) \to \mathcal{X}(A)$ such that $\nabla^A\big(\xi|_A, \eta|_A\big) = (\nabla_\xi \eta)|_A$, for every $\xi, \eta \in \mathcal{X}(M)$.*

∇^A, also denoted by $\nabla|_A$, is called the ***restriction of*** ∇ ***to*** A.

Proof. We define ∇^A in the following way: Let any $X, Y \in \mathcal{X}(A)$ and $x \in A$. By means of a smooth bump function (as in the proofs of Propositions 6.3.3 and 6.3.5), we extend X and Y to the vector fields $\widetilde{X}, \widetilde{Y} \in \mathcal{X}(M)$, such that $\widetilde{X} = X|_V$ and $\widetilde{Y} = Y|_V$, where V is a small enough open set, with $x \in V \subset A$. Then we set

$$\big(\nabla^A_X Y\big)(x) \equiv \nabla^A(X, Y)(x) := \big(\nabla_{\widetilde{X}} \widetilde{Y}\big)(x), \qquad x \in A.$$

In virtue of Corollary 6.3.4, ∇^A is a well-defined (i.e. independent of the extensions of X and Y) connection on A. Now, arbitrary vector fields $\xi, \eta \in \mathcal{X}(M)$ are extensions of $\xi|_A$ and $\eta|_A$, respectively; thus ∇^A satisfies the equality of the statement. □

Consider now a connection ∇ defined on the domain of a coordinate chart $(U; x^1, \dots, x^m)$ of an m-dimensional manifold M. As in Example 6.3.2 (A), $\{\partial_i\}_{1\leq i\leq m}$ is the canonical basis of $\mathcal{X}(U)$. Since $\nabla_{\partial_i}\partial_j \in \mathcal{X}(U)$, there exist m^3 *smooth* functions

$$\Gamma_{ij}^k : U \longrightarrow \mathbb{R}; \qquad k = 1, \dots, m, \tag{6.3.3}$$

such that

$$\nabla_{\partial_i}\partial_j = \sum_{k=1}^m \Gamma_{ij}^k \, \partial_k; \qquad i, j, k = 1, \dots, m. \tag{6.3.4}$$

The functions Γ_{ij}^k are called the ***Christoffel symbols*** of ∇ with respect to U. Other terms, also in use, are: ***Christoffel functions*** or ***coefficients*** of ∇.

Let any $\xi, \eta \in \mathcal{X}(U)$. Taking into account equalities

$$\xi = \sum_{i=1}^m \xi^i \partial_i \quad \text{and} \quad \eta = \sum_{j=1}^m \eta^j \partial_j,$$

along with conditions (C. 1) and (C. 2) of Definition 6.3.1, we obtain:

$$\begin{aligned}
\nabla_\xi \eta = \nabla_\xi \left(\sum_{j=1}^m \eta^j \partial_j \right) &= \sum_{j=1}^m \nabla_\xi \, \eta^j \partial_j \\
&= \sum_{j=1}^m \big(\xi(\eta^j) \cdot \partial_j + \eta^j \nabla_\xi \partial_j \big) \\
&= \sum_{j=1}^m \xi(\eta^j) \cdot \partial_j + \sum_{j=1}^m \eta^j \nabla_{\left(\sum_{i=1}^m \xi^i \partial_i\right)} \partial_j \\
&= \sum_{j=1}^m \xi(\eta^j) \cdot \partial_j + \sum_{i,j=1}^m \xi^i \eta^j \nabla_{\partial_i} \partial_j \\
&= \sum_{k=1}^m \xi(\eta^k) \cdot \partial_k + \sum_{i,j=1}^m \xi^i \eta^j \sum_k \Gamma_{ij}^k \partial_k.
\end{aligned}$$

As a result, a connection ∇ on (U, ϕ) is expressed by means of the Christoffel symbols by

$$\nabla_\xi \eta = \sum_{k=1}^m \left(\xi(\eta^k) + \sum_{i,j=1}^m \xi^i \eta^j \Gamma_{ij}^k \right) \partial_k. \tag{6.3.5}$$

The same formula holds for the restriction $\nabla^U \equiv \nabla|_U$ of any connection ∇ of M to the domain of a chart (U, ϕ).

Conversely, any family of m^3 smooth functions $\Gamma_{ij}^k\colon U \longrightarrow \mathbb{R}$ $(i,j,k = 1,\dots,m)$ determines a connection on U by (6.3.5). Indeed, if $\xi, \eta, \zeta \in \mathcal{X}(U)$ and $f, g \in \mathcal{C}^\infty(U,\mathbb{R})$, then

$$\begin{aligned}
\nabla_{f\xi+g\eta}\zeta &= \sum_{k=1}^m \Big((f\xi + g\eta)(\zeta^k) + \sum_{i,j=1}^m (f\xi^i + g\eta^i)\zeta^k \Gamma_{ij}^k \Big)\partial_k \\
&= f\sum_{k=1}^m \Big(\xi(\zeta^k) + \sum_{i,j=1}^m \xi^i\zeta^j\Gamma_{ij}^k\Big)\partial_k \\
&\qquad\qquad + g\sum_{k=1}^m \Big(\eta(\zeta^k) + \sum_{i,j=1}^m \eta^i\zeta^j\Gamma_{ij}^k\Big)\partial_k \\
&= f\,\nabla_\xi\zeta + g\nabla_\eta\zeta,
\end{aligned}$$

thus ∇ is $\mathcal{C}^\infty(U,\mathbb{R})$-linear with respect to the first variable. On the other hand, for ξ, η, ζ as before and $\lambda, \mu \in \mathbb{R}$,

$$\begin{aligned}
\nabla_\xi(\lambda\eta + \mu\zeta) &= \sum_{k=1}^m \Big(\xi\big(\lambda\eta^k + \mu\zeta^k\big) + \sum_{i,j=1}^m \xi^i\big(\lambda\eta^j + \mu\zeta^j\big)\Gamma_{ij}^k \Big)\partial_k \\
&= \lambda\sum_{k=1}\Big(\xi(\eta^k) + \sum_{i,j=1}^m \xi^i\eta^j\Gamma_{ij}^k\Big)\partial_k \\
&\qquad\qquad + \mu\sum_{k=1}^m \Big(\xi(\zeta^k) + \sum_{i,j=1}^m \xi^i\zeta^j\Gamma_{ij}^k\Big)\partial_k \\
&= \lambda\nabla_\xi\eta + \mu\nabla_\xi\zeta;
\end{aligned}$$

hence, ∇ is $\mathbb{R}$-linear with respect to the second variable. Finally,

$$\begin{aligned}
\nabla_\xi(f\eta) &= \sum_{k=1}^m \Big(\xi(f\eta_k) + \sum_{i,j=1}^m \xi_i\cdot f\eta_j\Gamma_{i,j}^k\Big)\partial_k \\
&= \sum_{k=1}^m \Big(\xi(f)\cdot\eta_k + f\xi(\eta_k) + \sum_{i,j=1}^m f\xi_i\eta_j\Gamma_{i,j}^k\Big)\partial_k \\
&= \xi(f)\sum_{k=1}^m \eta_k\partial_k + f\sum_{k=1}^m \Big(\xi(\eta_k) + \sum_{i,j=1}^m \xi_i\eta_j\Gamma_{ij}^k\Big)\partial_k \\
&= \xi(f)\cdot\eta + f\,\nabla_\xi\eta,
\end{aligned}$$

which shows the Leibniz condition of a connection. The previous arguments along with the obvious $\mathbb{R}$-bilinearity prove the claim.

Summarizing, we state:

Proposition 6.3.9. *A connection ∇ over a chart (U,ϕ) is completely determined by its Christoffel symbols. As a matter of fact, there is a bijective correspondence between connections over (U,ϕ) and families of smooth functions*

$$\big\{\Gamma_{ij}^k\colon U \to \mathbb{R}\,|\, i,j,k = 1,\dots,m\big\}.$$

Proof. By the discussion preceding the statement, it remains to prove that the association $\nabla \to \{\Gamma_{ij}^k\}$ is an injection. But this is an immediate consequence of equality (6.3.5). □

Proposition 6.3.9 provides yet another example of (a construction of) a local connection.

Remark 6.3.10. Let $\mathcal{A} = \{(U_\alpha, \phi_\alpha) \mid \alpha \in I\}$ be the maximal atlas defining the smooth structure of an m-dimensional manifold M (recall the comments about the index space I in the proof of Theorem 6.3.7). If ∇ is any connection on M, then, in virtue of Lemma 6.3.8, we obtain the restriction $\nabla^\alpha := \nabla|_{U_\alpha}$ of ∇, over each U_α. Let us denote by

$$^{\alpha}\Gamma_{ij}^k \colon U_\alpha \longrightarrow \mathbb{R}; \qquad \alpha \in I,$$

the Christoffel symbols corresponding to ∇^α

Naturally, if $U_\alpha \cap U_\beta \neq \emptyset$, the Christoffel symbols $^{\alpha}\Gamma_{ij}^k$ and $^{\beta}\Gamma_{ij}^k$ should be related to each other on the overlapping. As a matter of fact, if $(x^1, \ldots, x^m)$ are the local coordinates of (U_α, ϕ_α) and $(y^1, \ldots, y^m)$ those of (U_β, ϕ_β), then

$$\boxed{^{\beta}\Gamma_{pq}^r = \sum_{i,j,k=1}^{m} {}^{\alpha}\Gamma_{ij}^k \frac{\partial x^i}{\partial y^p}\frac{\partial x^j}{\partial y^q}\frac{\partial y^r}{\partial x^k} + \sum_{i=1}^{m} \frac{\partial^2 x^k}{\partial y^p \partial y^q}\frac{\partial y^r}{\partial x^k}} \tag{6.3.6}$$

where $p, q, r = 1, \ldots, m$. The ***compatibility condition*** or ***transformation of Christoffel symbols*** (6.3.6) arises from a careful application of typical differentiations that the reader might like to try (see Exercise 6.3.12 (5). Therefore,

> ∇ *is completely determined by a collection of families of smooth functions*
>
> $$\left\{ {}^{\alpha}\Gamma_{ij}^k \colon U_\alpha \longrightarrow \mathbb{R} \mid i, j, k = 1, \ldots, m \right\},$$
>
> *for all $\alpha \in I$, satisfying the compatibility condition* (6.3.6).

Associated to a connection are the notion of torsion and curvature defined below. In spite of their importance (especially of the second in the framework of Riemannian geometry), we do not go into details since only the definition of the first will be needed in the next section. More precisely:

Definitions 6.3.11. The ***torsion*** of a connection ∇ on M is the map

$$T \colon \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{X}(M),$$

given by

$$T(\xi, \eta) := \nabla_\xi \eta - \nabla_\eta \xi - [\xi, \eta]; \qquad \xi, \eta \in \mathcal{X}(M).$$

On the other hand, if $L(\mathcal{X}(M))$ denotes the $\mathcal{C}^\infty(M, \mathbb{R})$-module of linear transformations of $\mathcal{X}(M)$, then the ***curvature*** of ∇ is the map

$$R \colon \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow L(\mathcal{X}(M))$$

defined by

$$R(\xi, \eta) = \nabla_\xi \nabla_\eta - \nabla_\eta \nabla_\xi - \nabla_{[\xi, \eta]}; \qquad \xi, \eta \in \mathcal{X}(M).$$

More explicitly, the above formula of R means that, for every $\zeta \in \mathcal{X}(M)$,

$$R(\xi,\eta)\zeta \equiv \big(R(\xi,\eta)\big)(\zeta) = \nabla_\xi(\nabla_\eta\zeta) - \nabla_\eta(\nabla_\xi\zeta) - \nabla_{[\,\xi,\eta\,]}\zeta.$$

Other properties of connections and of their torsion and curvature are listed in the next exercises.

Exercises 6.3.12.

1. Define a direct connection on a Lie group G.
2. The direct connection ∇, determined by a parallelization $\mathcal{B} = \{X_1,\dots,X_m\}$ on an m-dimensional manifold M, is characterized by the property $\nabla_{X_i}X_j = 0$, for every $i,j = 1,\dots,m$. Is there any analog for the the connection of Example 6.3.2 (A)?
3. Find an explicit expression of the symbols Γ^k_{ij}, as opposed to their indirect definition of (6.3.4).
4. Compute the Christoffel symbols of the connections of Examples 6.3.2, over a coordinate chart $(U; x^1,\dots,x^m)$.
5. Complete the concluding arguments in the proof of Proposition 6.3.9.
6. Verify the compatibility condition (transformation) of the Christoffel symbols (6.3.6).
7. Prove the following properties of the torsion T:
$$T(\xi,\eta) = -T(\eta,\xi),$$
$$T(f\xi,\eta) = T(\xi,f\eta) = fT(\xi,\eta),$$
for every $\xi,\eta \in \mathcal{X}(M)$ and $f \in \mathcal{C}^\infty(M,\mathbb{R})$. Furthermore, prove that T is bilinear with respect to $\mathcal{C}^\infty(M,\mathbb{R})$.
8. Prove the following properties of the curvature R:
$$R(\xi,\eta) = -R(\eta,\xi),$$
$$R(f\xi,\eta) = R(\xi,f\eta) = fR(\xi,\eta),$$
$$R(\xi,\eta)(f\zeta) = fR(\xi,\eta)(\zeta),$$
for every $\xi,\eta,\zeta \in \mathcal{X}(M)$ and $f \in \mathcal{C}^\infty(M,\mathbb{R})$. Also, prove that R is bilinear with respect to $\mathcal{C}^\infty(M,\mathbb{R})$.
9. A connection is said to be ***flat*** if its curvature vanishes identically. Prove that the connections defined in Examples 6.3.2 are flat.
10. A connection is said to be of ***symmetric*** if its torsion vanishes identically. Are the connections of Examples 6.3.2 symmetric? If not, when does this happen?
11. Let $(U; x^1,\dots,x^m)$ be a coordinate chart of an m-dimensional manifold, and let $\left\{\partial_i = \dfrac{\partial}{\partial^i_x} \mid i = 1,\dots,m\right\}$ be the corresponding basis of local vector fields. If ∇ is a connection on M, with torsion T and curvature R, we define the

components $T_{ij}^k\colon U \to \mathbb{R}$ and $R_{ijk}^l\colon \to \mathbb{R}$ of T and R, respectively, by

$$T(\partial_i, \partial_j) = \sum_{i=1}^{m} T_{ij}^k \partial_k, \quad R(\partial_i, \partial_j)(\partial_k) = \sum_{i=1}^{l} R_{ijk}^l \partial_l.$$

Express these components in terms of the Christoffel symbols of ∇.
[Note. In many books one can find the following formula defining the components of the curvature: $\sum_{i=1}^m R_{jkl}^i = R(\partial_k, \partial_l)(\partial_j)$. This goes back to older days, long before the definition of ∇, when the curvature was treated (as a tensor with specific properties) by means of the components. Later, with the introduction of ∇, it turned out that the components R_{jkl}^i were related with R by the aforementioned formula. For the sake of convenience, here we use the formula of the exercise, without loosing the essential features of the components.]

12. Prove that $T = 0$ if and only if, over each chart, the corresponding Christoffel symbols satisfy equality $\Gamma_{ij}^k = \Gamma_{ji}^k$, for every $i, j, k = 1, \ldots, m$.
[Note. This explains the terminology *symmetric* for a zero torsion connection.]

13. If ∇ is a symmetric connection, then the following ***Bianchi's identity***

$$R(\xi, \eta)\zeta + R(\eta, \zeta)\xi + R(\zeta, \xi)\eta = 0,$$

holds for every $\xi, \eta, \zeta \in \mathcal{X}(M)$.

14. Translate in terms of the components of T and R the first formula of Exercise 7, the first formula of Exercise 8, and the Bianchi's identity.

15. Prove that the difference of two connections on M is a bilinear map on $\mathcal{X}(M) \times \mathcal{X}(M)$, with respect to $\mathcal{C}^\infty(M, \mathbb{R})$. Then derive that the set of connections on a manifold M is an affine space with carrier (or modeled on) an appropriate vector space.

16. Let $\mathcal{B} = \{X_1, \ldots, X_m\}$ be a parallelization on a manifold M. Then find a connection having the characteristic property $\nabla_{X_i} X_j = 1/2[X_i, X_j](i, j = 1, \ldots, m)$, and show that $T \equiv 0$. This is known as the ***torsion free connection*** of M relative to P.

17. Let $f\colon M \to M'$ be diffeomorphism. If ∇ is a connection on M, prove that f induces s connection ∇' on M' such that (in the notation of Theorem 3.5.4)

$$f_*\left(\nabla_\xi \eta\right) = \nabla'_{f_*(\xi)} f_*(\eta); \qquad \xi, \eta \in \mathcal{X}(M).$$

6.4 Riemannian connections

The final section of this chapter is dealing with the intertwining of connections with Riemannian structures, leading to the so-called Riemannian or Levi-Cività connection. This is a connection of particular importance for the development of Riemannian geometry.

From § 6.1 we recall the following notation: If (M, g) is a Riemannian manifold, then for any vector fields $\xi, \eta \in \mathcal{X}(M)$, we define the function

$$g(\xi,\eta)\colon M \longrightarrow \mathbb{R}\colon x \mapsto g(\xi,\eta)(x) := g_x(\xi_x,\eta_x).$$

As a result, the functions g_{ij}, defined by (6.1.4), are also given also by

$$g_{ij} = g\big(\partial_i,\partial_j\big) = g\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right).$$

Definition 6.4.1. Let (M, g) be a Riemannian manifold. A ***Riemannian*** or ***Levi-Cività connection*** is a connection ∇ on M satisfying the following conditions:

$$\text{(RC. 1)}\qquad T(\xi,\eta)=0,$$

$$\text{(RC. 2)}\qquad g\big(\nabla_\xi\eta,\,\zeta\big)+g\big(\eta,\nabla_\xi\zeta\big)=\xi\big(g(\eta,\zeta)\big),$$

for every $\xi, \eta, \zeta \in \mathcal{X}(M)$.

Said differently, a Riemannian connection is a *zero torsion* or *symmetric* connection [in the terminology of Exercise 6.3.12 (10)], which is *compatible* with the Riemannian structure, in the sense of (RC. 2). The latter condition is also referred to as the ***Ricci identity***.

Theorem 6.4.2. *Any Riemannian manifold (M, g) admits a unique Riemannian connection.*

Proof. The existence and uniqueness of the Riemannian connection will be ensured from the interpretation of (RC. 1) and (RC. 2) in local terms. To this end, we consider an arbitrary chart (U, ϕ) of M and the induced canonical basis $\{\partial_i\}_{1\leq i\leq m}$ $(m = \dim(M))$ of $\mathcal{X}(U)$. Then, by Exercise 6.3.12 (12), $\Gamma^m_{ij} = \Gamma^m_{ji}$. Next, applying (RC. 2) to the basic vector fields, we obtain:

$$g\big(\nabla_{\partial_i}\partial_j,\partial_k\big)+g\big(\partial_j,\nabla_{\partial_i}\partial_k\big)=\partial_i(g(\partial_j,\partial_k)).$$

Inserting the Christoffel symbols, and taking into account the symmetry of g and ∇, as well as (6.3.4) and the aforementioned expression of g_{ij}'s, the preceding equality successively implies:

$$g\left(\sum_{l=1}^m \Gamma^l_{ij}\partial_l,\partial_k\right)+g\left(\partial_j,\sum_{l=1}^m \Gamma^l_{ik}\partial_l\right)=\partial_i\big(g(\partial_j,\partial_k)\big)\;\Rightarrow$$

$$\sum_{l=1}^m \Gamma^l_{ij}\,g\big(\partial_l,\partial_k\big)+\sum_{l=1}^m \Gamma^l_{ik}\,g\big(\partial_j,\partial_l\big)=\partial_i\big(g(\partial_j,\partial_k)\big)\;\Rightarrow$$

$$\sum_{l=1}^m \Gamma^l_{ij}\,g_{lk}+\sum_{l} \Gamma^l_{ik}\,g_{jl}=\partial_i(g_{jk})\;\Rightarrow$$

$$\sum_{l=1}^m \Gamma^l_{ij}\,g_{lk}+\sum_{l=1}^m \Gamma^l_{ik}\,g_{lj}=\partial_i(g_{jk}), \tag{6.4.1}$$

and analogously, by a cyclic permutation of $\{i, j, k\}$,

$$\sum_{l=1}^{m} \Gamma_{jk}^{l}\, g_{li} + \sum_{l=1}^{m} \Gamma_{ji}^{l}\, g_{lk} = \partial_j(g_{ki}), \tag{6.4.2}$$

$$\sum_{l=1}^{m} \Gamma_{ki}^{l}\, g_{lj} + \sum_{l=1}^{m} \Gamma_{kj}^{l}\, g_{li} = \partial_k(g_{ij}). \tag{6.4.3}$$

Adding (6.4.1), (6.4.2) and subtracting (6.4.3), we obtain

$$\sum_{l=1}^{m} \Gamma_{ij}^{l}\, g_{lk} = \frac{1}{2}\big(\partial_i(g_{jk}) + \partial_j(g_{ki}) + \partial_k(g_{ij})\big). \tag{6.4.4}$$

If we fix i, j and vary $k \in \{1, \ldots, m\}$, we obtain an ordinary algebraic system from which we determine the symbols Γ_{ij}^{k}, by multiplying with the invertible (nonsingular) matrix (g_{ij}) of g [see Exercise 6.1.5 (1)].

Let us clarify the procedure just described: Denoting, for convenience, the right-hand side of (6.4.4) by A_k ($k \in \{1, \ldots, m\}$), we obtain the algebraic system

$$\begin{gathered}
\sum_{l=1}^{m} \Gamma_{ij}^{l}\, g_{l1} = A_1 \\
\sum_{l=1}^{m} \Gamma_{ij}^{l}\, g_{l2} = A_2 \\
\cdots\cdots\cdots\cdots \\
\sum_{l=1}^{m} \Gamma_{ij}^{l}\, g_{lm} = A_m
\end{gathered}$$

or, in matrix notation,

$$\begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1m} \\ g_{21} & g_{22} & \cdots & g_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ g_{m1} & g_{m2} & \cdots & g_{mm} \end{pmatrix} \begin{pmatrix} \Gamma_{ij}^{1} \\ \Gamma_{ij}^{2} \\ \cdots \\ \Gamma_{ij}^{m} \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \\ \cdots \\ A_m \end{pmatrix}.$$

Multiplying the preceding by the inverse of (g_{ij}), denoted simply by

$$(\overline{g}_{ij}) = \begin{pmatrix} \overline{g}_{11} & \overline{g}_{12} & \cdots & \overline{g}_{1m} \\ \overline{g}_{21} & \overline{g}_{22} & \cdots & \overline{g}_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ \overline{g}_{m1} & \overline{g}_{m2} & \cdots & \overline{g}_{mm} \end{pmatrix},$$

we immediately obtain the equations

$$\Gamma_{ij}^{1} = \sum_{l=1}^{m} \overline{g}_{1l}\, A_l = \frac{1}{2} \sum_{l=1}^{m} \overline{g}_{1l}\big(\partial_i(g_{jl}) + \partial_j(g_{li}) + \partial_l(g_{ij})\big)$$

$$\Gamma_{ij}^{2} = \sum_{l=1}^{m} \overline{g}_{2l}\, A_l = \frac{1}{2}\sum_{l=1}^{m} \overline{g}_{2l}\big(\partial_i(g_{jl}) + \partial_j(g_{li}) + \partial_l(g_{ij})\big)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\Gamma_{ij}^{m} = \sum_{l=1}^{m} \overline{g}_{ml}\, A_l = \frac{1}{2}\sum_{l=1}^{m} \overline{g}_{ml}\big(\partial_i(g_{jl}) + \partial_j(g_{li}) + \partial_l(g_{ij})\big)$$

and similarly for every other pair of lower indices (i,j). Therefore, for any $i,j,k \in \{1,\ldots,m\}$,

$$\boxed{\Gamma_{ij}^{k} = \frac{1}{2}\sum_{l=1}^{m} \overline{g}_{kl}\big(\partial_i(g_{jl}) + \partial_j(g_{li}) + \partial_l(g_{ij})\big)} \tag{6.4.5}$$

The preceding equality shows that

> *The Christoffel symbols of a Riemannian connection over a chart (U,ϕ) are completely determined by the corresponding basic vector fields $\{\partial_i\}$ and the Riemannian structure expressed by its matrix (g_{ij}), over the same chart.*

Equality (6.4.5) essentially proves the assertion of the statement. More precisely:

i) The uniqueness of the Riemannian connection is almost obvious. Indeed, assume that there are two Riemannian connections ∇ and $\widetilde{\nabla}$ on M. Then the corresponding Christoffel symbols, over the same chart (U,ϕ), necessarily satisfy (6.4.5), thus $\Gamma_{ij}^{k} = \widetilde{\Gamma}_{ij}^{k}$, and similarly for the symbols over every other chart. Since the connections are completely determined by the family of their Christoffel symbols, over all of the charts of M, it follows that $\nabla = \widetilde{\nabla}$.

ii) The existence of a Riemannian connection is established as follows: Given a chart (U,ϕ), we define a (local) connection ∇^U on U by taking as its Christoffel symbols those of (6.4.5) (see also Proposition 6.3.9). This is a symmetric connection since $\Gamma_{ij}^{k} = \Gamma_{ij}^{k}$. We verify that ∇^U satisfies (RC. 2) by reversing the procedure leading from (RC. 2) to (6.4.5), as in the first part of the proof. Thus ∇^U is a Riemannian connection on U. From the collection of all these local connections, obtained when we vary (U,ϕ) in the maximal atlas of M, we define a (global) connection ∇ on M by setting

$$\big(\nabla_\xi \eta\big)\big|_U := \nabla^U_{\xi|_U}\eta|_U,$$

for every $\xi,\eta \in \mathcal{X}(M)$. This is indeed a well-defined connection, for if we consider ∇^U and ∇^V, with $U \cap V \neq \emptyset$, then their restrictions on $U \cap V$ are two *Riemannian connections* on the same manifold, thus they coincide. As a result, ∇^U and ∇^V coincide on $U \cap V$, and ∇ is well-defined. Finally ∇ is a Riemannian connection since it is built from such connections. □

Exercises 6.4.3.

1. Verify that ∇^U (in the uniqueness part of Theorem 6.4.2) satisfies (RC. 2).

2. Let R be the curvature of the Levi-Cività connection ∇ on a Riemannian manifold (M, g). Then prove equality

$$g\big(R(\xi, \eta)\zeta, \chi\big) + g\big(R(\eta, \zeta)\xi, \chi\big) + g\big(R(\zeta, \xi)\eta, \chi\big) = 0,$$

for every $\xi, \eta, \zeta, \chi \in \mathcal{X}(M)$.

3. Let $\mathbb{R}^n$ be endowed with the standard Riemannian structure $\widetilde{g}$ defined in Example 6.1.2 (A). Motivated by Example 6.3.2 (A), define the Riemannian (or Levi-Cività) connection on $\mathbb{R}^n$.

4. Under what condition is the connection of Example 6.3.2 (A) the Riemannian connection over a chart with the Riemannian structure of Example 6.1.2 (B)?

5. Let $f\colon (M, g) \to (M', g')$ be an isometry between two Riemannian manifolds [see Exercise 6.2.9 (5)], and let ∇ be the Levi-Cività connection of M. Prove that the connection ∇', induced by f as in Exercise 6.3.12 (17), is the Levi-Cività connection of M'.

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List of symbols

The list contains the main symbols, with a fixed meaning, together with a brief description and the page of their first appearance. For the sake of simplicity, we use the following abbreviations:

s.t. = such that; w.r.t. = with respect to

Chapter 1

S^2	the unit sphere in $\mathbb{R}^3$	2
S_z^+	the north hemisphere	2
D_z	the unit disc	2
(U, ϕ),	a typical chart	4
$x_i = \mathrm{pr}_i \circ \phi$	coordinate functions of (U, ϕ)	5
u_i	the same as pr_i	5
$(U, x^1, \dots, x^m)$	a coordinate chart	5
$\psi \circ \phi^{-1}$	transition function of charts (U, ϕ) and (V, ψ)	6
$\mathcal{C}^k$	differentiability of class k	5
$\mathcal{A}$, $\mathcal{B}$	atlases	7
$\mathfrak{A}_m^k(M)$	set of m-dimensional C^k-atlases on M	7
$\mathcal{A} \preceq \mathcal{B}$	partial order in $\mathfrak{A}_m^k(M)$	8
$\mathcal{A} \overset{k}{\sim} \mathcal{B}$	$\mathcal{C}^k$-compatible atlases	10
$\mathcal{A}'_k \in \mathfrak{A}_m^k(M)$	the maximal atlas of $\mathcal{A} \in \mathfrak{A}_m^k(M)$	8
$\mathcal{A}'$	the maximal atlas of $\mathcal{A} \in \mathfrak{A}_m^\infty(M)$	12
$M \equiv (M, \mathcal{A})$	a (smooth) manifold	11
$\dim(M) = m$	the dimension of a manifold M	11
$\mathcal{M}_{m\times n}(\mathbb{R})$	the space of $m \times n$ real matrices	13
$\mathcal{M}_n(\mathbb{R})$	the set of matrices $\mathcal{M}_{n\times n}(\mathbb{R})$	13
$\mathcal{L}(\mathbb{E}, \mathbb{F})$	the space of linear maps between $\mathbb{E}$, $\mathbb{F}$	13

S^1	the unit circle	16
$\mathbb{R}^3_*$	shorthand notation of $\mathbb{R}^3 \setminus \{(0,0,0)\}$	17
$\mathbb{P}_2(\mathbb{R})$	the projective plane	17
$\mathbb{P}_n(\mathbb{R})$	the n-dimensional projective space	18
T^n	the n-dimensional torus	20
Γ_f	the graph of a function $f\colon \mathbb{R}^2 \to \mathbb{R}$	21
$\mathcal{T}_{\mathcal{A}}$	the topology induced by $\mathcal{A}$	22
det	the determinant map	29
$\mathrm{GL}(n,\mathbb{R})$	the general linear group	29
$\mathcal{L}is(\mathbb{R}^n)$	the group of linear automorphisms of $\mathbb{R}^n$	29
$End(\mathbb{R}^n)$	the space of linear transformations of $\mathbb{R}^n$	29
Ad	the adjoint representation of $\mathrm{GL}(n,\mathbb{R})$	29
$\mathcal{C}^r(M,N)$	the set of $\mathcal{C}^r$-differentiable maps between the manifolds M, N	34
$i\colon A \hookrightarrow M$	natural inclusion or canonical injection	37
$p_M\colon M \times N \to M$	projection (to the 1st factor M)	38
$\mathcal{C}^\infty_x(M,\mathbb{R})$	the set of locally smooth functions at $x \in M$	40
$\mathcal{C}^\infty(M,\mathbb{R})$	the algebra of smooth maps from M to $\mathbb{R}$	40
$\mathrm{Diff}(M)$	group of diffeomorphisms of M	43

Chapter 2

$\alpha\colon J \to M$	curve in M	48
$\alpha \sim_x \beta$	tangent curves	48
T_xM	tangent space at x	49
$u = [(\alpha, x)]$	tangent vector at x	49
$\overline{\phi}\colon T_xM \to \mathbb{R}^m$	bijection induced by (U,ϕ)	49
$\left(\frac{\partial}{\partial x_i}\Big\|_x\right)$	the canonical basis of T_xM, w.r.t.. (U,ϕ)	53
$\frac{d}{dt}\Big\|_s$	the canonical basis of $T_s\mathbb{R}$	53
T_xf	the differential of f at x	55
$f_{*,x}$	another symbol for T_xf	55
d_xf	yet another symbol for T_xf (see also p. 296)	55
Φ_A	$\Phi_A\colon T_xA \xrightarrow{\simeq} T_xM$	58
$\Phi_\times$	$\Phi\colon T_{(x,y)}(M \times N) \xrightarrow{\simeq} T_xM \times T_yN$	59
$M_f = J_xf$	the Jacobian matrix of f at x	61
$\dot{\alpha}(t)$	the tangent vector of α at t	63
$\mathcal{N}_x$	the set of open neighborhoods of $x \in M$	68
$f \overset{x}{\sim} g$	equivalent functions at x	68

$[f]_x$	the germ of f at x	68
$\boldsymbol{\mathcal{C}}_x^\infty(M)$	the algebra of (smooth) germs of M at x	68
δ	a derivation of $\boldsymbol{\mathcal{C}}_x^\infty(M)$	69
$\mathcal{D}_x(M)$	the linear space of all derivations of $\boldsymbol{\mathcal{C}}_x^\infty(M)$	69
δ_u	the derivation of $\boldsymbol{\mathcal{C}}_x^\infty(M)$ induced by $u \in T_xM$	69
$u(f)$	a simplified form of $\delta_u([f]_x$	70
Δ_x	the isomorphism $T_xM \to D_x(M)$	74
$\frac{\partial f}{\partial x_i}\Big\vert_x)$	simplified form of $\left(\frac{\partial}{\partial x_i}\Big\vert_x\right)(f)$	71
$J_{\phi(x)}\left(\frac{f^1,\dots,f^n}{x^1,\dots,x^m}\right)$	other notation of the Jacobian matrix J_xf (see also p. 61)	72
$\det(J_xf) \equiv \frac{\partial(f^1,\dots,f^n)}{\partial(x^1,\dots,x^n)}$	the Jacobian determinant of J_xf	72
$\overline{\mathcal{D}}_x(M)$	the linear space of all derivations of $\mathcal{C}^\infty(M,\mathbb{R})$ at x	80
δ_x	a derivation of $\mathcal{C}^\infty(M,\mathbb{R})$ at x	80
$\mathcal{D}\colon \mathcal{D}_x(M) \to \overline{\mathcal{D}}_x(M)$	the linear isomorphism between the derivations involved	82
TM	the tangent bundle of M	82
$\pi\colon TM \to M$	projection of TM	82
$(\pi^{-1}(U),\Phi)$	the chart of the tangent bundle induced by (U,ϕ)	82

Chapter 3

$\xi\colon M \to TM$	vector field	91
$\mathcal{X}(M)$	the set of smooth vector fields on M	97
ξ_x	equivalent form of $\xi(x)$	92
$\Omega \equiv 0$	the zero vector field	93
$\frac{\partial}{\partial x^i}$	the basic vector field defined by (U,ϕ)	93
$\frac{d}{dt}$	the basic vector field of $\mathbb{R}$	94
ξ_i	coordinates of a vector field	95
$\xi(f)$	the map $x \mapsto \xi(f)(x) := \xi_x(f)$	94
ξ_ϕ	principal part of ξ	96
$\xi+\eta\,,\lambda\xi$	sum and scalar multiplication in $\mathcal{X}(M)$	97
$f\xi$	multiplication of ξ by $f \in \mathcal{C}^\infty(M,\mathbb{R})$	97
δ_ξ	a derivation of $\mathcal{C}^\infty(M,\mathbb{R})$ induced by ξ	104
$\mathcal{D}(M)$	the set of all (operator) derivations of $\mathcal{C}^\infty(M,\mathbb{R})$	104
$[\xi,\eta]$	the Lie bracket of ξ, η	107
$\mathcal{X}(f)$	the set of smooth vector fields along f	110

$\dot{\alpha}$	the field of velocities of a curve α	110
$\alpha_i := x_i \circ \alpha$	the coordinates a curve α	115
f_*	the map given by $f_*(\xi) = Tf \circ \xi \circ f^{-1}$	113
Θ	the (l0cal) flow $J \times \widetilde{U} \to \widetilde{U}$ of a differential system	117
$\theta_U : J_0 \times U_0 \longrightarrow U$	the local flow of a vector field	117
$J(x)$	the domain of α_x	118
α_x	the maximal integral curve trough x	118
μ_s	the (right) translation of $\mathbb{R}$ by s	119
$\mathfrak{D}(\xi)$	the domain of the flow of $\xi \in \mathcal{X}(M)$	125
θ	the flow $\theta\colon \mathfrak{D}(\xi) \to M$	125
$J_*(x_0)$	set of $t \in J(x_0)$ such that θ is smooth at (t, x_0)	126
$\mathfrak{D}_t(\xi)$	the set $\{x \in M \,:\, (t,x) \in \mathfrak{D}(\xi)\}$	128
$L_\xi(\eta)$	the Lie derivative of η along ξ	134

Chapter 4

$\boldsymbol{\gamma}\colon G \times G \to G$	the multiplication/product of a Lie group	145
$\boldsymbol{\alpha}\colon G \to G$	the inversion map of a Lie group	145
L_g	the left translation by $g \in G$	146
R_g	the right translation by $g \in G$	146
$U \cdot V$	the set $\{xy \,\vert\, x \in U \subseteq, \, y \in V \subseteq G\}$	146
gU	$gU := L_g(U)$	146
Ug	$Ug := R_g(U)$	146
U^{-1}	$U^{-1} := \{x^{-1} \,\vert\, x \in U\} = \alpha(U)$	148
$\mathcal{L}(G)$	the set of left invariant vector fields of G	154
$\mathfrak{g}$	another notation for $\mathcal{L}(G)$	154
$\mathcal{R}(G)$	the set of right invariant vector fields of G	154
$\boldsymbol{h}$	the isomorphism $\mathfrak{g} \to T_eG$	155
ξ^v	the left invariant vector field determined by a vector $v \in T_eG$	156
$\mathfrak{gl}(n, \mathbb{R})$	the Lie algebra of $GL(n, \mathbb{R})$	158
C_{ij}^k	the structural constants of a Lie group G	162
$\mathcal{H}om(\mathbb{R}, G)$	the set of one-parameter subgroups of G	163
$\mathcal{F}\colon \mathfrak{g} \to \mathcal{H}om(\mathbb{R}, G)$	the map assigning to ξ its one-parameter subgroup of G	166
α_ξ	the one-parameter subgroup $\mathcal{F}(\xi)$	166
exp	the exponential map $\exp\colon \mathfrak{g} \to G$	168
$\widetilde{\exp}$	the map $\widetilde{\exp}\colon T_eG \longrightarrow G$ (another form of exponential)	168
(N_e, ν)	the normal chart at $e \in G$	172

$I_g)$	the inner automorphism of G	177
$\mathrm{Aut}(T_eG)$	the group of linear automorphisms of T_eG	177
$\mathrm{Aut}(\mathfrak{g})$	the group of linear automorphisms of $\mathfrak{g}$	177
Ad	the adjoint representation of G	177
ad	the adjoint representation of $\mathfrak{g}$	180
$\delta\colon M \times G \to M$	an action of G on M	182
$\rho_g = \delta_g\colon M \to M$	the right translation of M by $g \in G$	182
ξ^*	the Killing vector field corresponding to $\xi \in \mathfrak{g}$	182

Chapter 5

$\mathcal{A}_X$	the atlas induced on $X \subset M$ by $(M, \mathcal{A})$	191
$p_{1,0}$	the projection $: \mathbb{R}^k \times 0 \to \mathbb{R}^k$	191
Γ_α	the graph of a curve $\alpha\colon \mathbb{R} \supset J \to \mathbb{R}^n$	199
Γ_f	the graph of a map $f\colon M \to N$	200
$\mathrm{GL}^+(n, \mathbb{R})$	the subgroup of $\mathrm{GL}(n, \mathbb{R})$ consisting of matrices with positive determinant	222
$\mathrm{SL}(n, \mathbb{R})$	the special linear group	222
$\mathrm{S}(n, \mathbb{R})$	the vector space of $n \times n$ symmetric matrices	223
A^t	the transpose of a matrix A	223
$\mathrm{O}(n)$	the orthogonal group	223
$\mathfrak{sl}(n, \mathbb{R})$	the lie algebra of $\mathrm{SL}(n, \mathbb{R})$	225
$\mathfrak{o}(n)$	the Lie algebra of $\mathrm{O}(n)$	225
$\mathrm{SO}(n)$	the special orthogonal group	226

Chapter 6

$\mathcal{J}(T_xM)$	the set of all inner products of T_xM	228
$\mathcal{J}(M)$	the set $\bigcup_{x \in M} \mathcal{J}(T_xM)$	228
$g\colon M \to \mathcal{J}(M)$	a Riemannian structure on M	228
$G_{\xi,\eta}\colon M \to \mathbb{R}$	the map with $G_{\xi,\eta}(x) := g(x)\big(\xi(x), \eta(x)\big)$	228
$g_x := g(x)$	the value of Riemannian structure at x	228
$\alpha * \beta$	composition of curves	235
γ^{-1}	inverse curve	235
(M, g)	a Riemannian manifold	228
g_{ij}	the function $g_{ij}(x) = g_x\big(\frac{\partial}{\partial x_i}\big\|_x, \frac{\partial}{\partial x_j}\big\|_x\big)$	232
$\\|u\\|_g$	the norm of $u \in T_xM$ w.r.t. g	237
$L_g(\gamma)$	the length of a (piecewise) smooth curve w.r.t. to g	237
$\Omega(x, y)$	the set of piecewise smooth curves from x to y	237
$B_g(x, r)$	the r-ball centered at x	239

$\mathcal{T}_g$	the topology induced by d_g	239
$\bar{B}(0,\varepsilon)$	the closed ball $\{h \in \mathbb{R}^m : \|h\| \leq \varepsilon\}$	239
∇	a connection or covariant derivative	
∂_i	abbreviated form of $\frac{\partial}{\partial x_i}$	244
$\nabla_\xi \eta\big\|_A$	the restriction of ∇ on open $A \subset M$	245
$\nabla = \nabla_A$	the connection induced by ∇ on open $A \subset M$	248
Γ_{ij}^k	Christoffel symbols	249
T	the torsion of a connection	251
R	the curvature of a connection	251
$g(\xi,\eta)$	the function $g(\xi,\eta)(x) := g_x(\xi_x,\eta_x)$	254

Chapter 7

$\mathcal{L}_p(\mathbb{E}_1,\ldots,\mathbb{E}_p;\mathbb{F})$	the space of p-multilinear maps	260
$\mathcal{L}_p(\mathbb{E},\mathbb{F})$	the above space for $\mathbb{E}_i = \mathbb{E}$	260
$\mathcal{A}_p(\mathbb{E},\mathbb{F})$	the space of p-antisymmetric maps	260
$f \wedge g$	the wedge product of two linear maps	262
S_p	the set of permutations of $\{1,\ldots,p\}$	261
$\varepsilon(\sigma) = \mathrm{sgn}(\sigma) = \pm 1$	the signature of a permutation σ	261
$\widehat{h}$	the hat (caret) indicates omission of h	266
$(e_1^*,\ldots,e_n^*)$	the dual basis of $(e_1,\ldots,e_n)$	266
$u^i = \mathrm{pr}_i : \mathbb{E} \to \mathbb{R}$	the i-th projection ($\mathbb{E}$ finite-dimensional)	266
$a_{i_1\ldots i_p}$	$a_{i_1\ldots i_p} = f\left(e_{i_1},\ldots,e_{i_p}\right)$	267
$\Omega^p(A)$	the set of differential p-forms on an open set $A \subseteq \mathbb{R}^n$	272
du^i	the differential of u^i equal to Du^i	275
$\mathrm{curl}(\xi)$	the curl of a vector field ξ	281
$\mathrm{div}(\xi)$	the divergence of a vector field ξ	282
$\mathrm{grad}(f)$	the gradient of $f : \mathbb{R}^3 \to \mathbb{R}$	282
$d\omega$	the differential of ω	276
$[x_0, x]$	the line segment from x_0 to x	284
$Z^p(A)$	the space of cocycles	290
$B^p(A)$	the space of coboundaries	290
$\mathcal{A}_p(T_xM,\mathbb{R})$	the space of antisymmetric p-linear forms on T_xM	292
$\mathcal{A}_p(M,\mathbb{R})$	the set of p-linear forms on M	292
π_p	the projection of $\mathcal{A}_p(M,\mathbb{R})$ to M	292
$(\pi_p^{-1}(U),\Phi_p)$	chart of $\mathcal{A}_p(M,\mathbb{R})$	293
T_x^*M	the cotangent space of M at $x \in M$	295

$\Omega^p(M)$	the set of differential p-forms on M	295
$d_x f$	the map $\overline{\mathrm{id}_{\mathbb{R}^n}} \circ T_x f$ (see also p. 55)	296
df	the map $M \longrightarrow T * M$ with $(df)(x) := (df)_x \equiv d_x f$	296
$df(\xi)$	the map defined by $[df(\xi)](x) = d_x f(\xi_x)$	297
ω_ϕ	the principal part of ω	298
$G^p_U \equiv G_U$	the map $\Omega^p(U) \to \Omega^p(\phi(U))$	298
$\omega(\xi_1, \dots, \xi_p)$	the function $x \mapsto \omega_x(\xi_1(x), \dots, \xi_p(x))$	299
$\{dx_i \,\vert\, i = 1, \dots, n\}$	the natural basis for $\Omega^p(U)$	301
$\omega \wedge \theta$	the wedge product in Ω^*	301
$d_U \colon \Omega^p(U) \to \Omega^{p+1}(U)$	the exterior differential over $U \subseteq M$	303
$d \colon \Omega^p(M) \to \Omega^p(M)$	exterior differentiation on M	307
****	Insert the symbols for global d, cocycles,	
	coboundaries and cohomology groups	
$\widehat{\xi^i}$	omitted vector field	
$f^*\omega$	the pullback of a differential form	313
$\Omega^p_L(G)$	the space of left invariant forms on G	317
$\Omega^p_R(G)$	the space of right invariant forms on G	317
$\boldsymbol{h}_1 \colon \Omega^1_L(G) \to (T_e G)^*$	the map $\omega \to \omega_e$ 319	
$L_\xi \omega$	the Lie derivative of ω along ξ	322
$i_\xi \omega$	the interior product or contraction of ω by ξ	328
$\xi \rfloor \omega$	another symbol for the interior product	328
$\widehat{\omega^i}$	omitted form	329

Appendix A

$\mathcal{T}$	topology	337
$(X, \mathcal{T})$	a topological space	337
$\mathcal{P}(X)$	the power set of X	338
$\mathcal{B}$	a basis for topology	338
$\mathcal{B}_x \subset \mathcal{N}_x$	a basis of neighborhoods of x	339
(X, d)	a metric space	340
$B_d(x, r)$	the open ball centered at x with radius r	340
$\bar{B}_d(x, r)$	the closed ball centered at x with radius r	341
$S_d(x, r)$	the sphere centered at x with radius r	341
$\mathcal{T}\vert_Y$	the relative topology on Y	343
$i \colon Y \hookrightarrow X$	the canonical injection or inclusion	345
$\mathcal{B}_{X \times Y}$	a basis for the product topology on $X \times Y$	347
$\mathcal{T}_{X \times Y}$	the product topology on $X \times Y$	347

$\mathcal{T}_\times$	the product topology on $\prod_{i\in I} X_i$	348
$\mathcal{B}_\times$	a basis for $\mathcal{T}_\times$	348
Δ_X	the diagonal of X	352
$[x]$	an equivalence class of X	353
$X/\sim$	a quotient set	353
$\mathcal{T}_\sim$	quotient topology	353
$X/\sim \equiv (X/\sim, \mathcal{T}_\sim)$	a quotient space	353
$\kappa\colon X \longrightarrow X/\sim$	the canonical (natural) projection	353
$\Gamma_\sim$	the graph of an equivalence relation	354
C_x	the connected component of x	360

Appendix B

$Df(a)$	the differential (derivative) of f at a	363
ev	the evaluation map	365
$comp$	the composition map $comp\,(f,g) := g \circ f$	365
$\frac{\partial f}{\partial x_i}\|_a$ the	i-th partial derivative	367
Df	the (total) differential of f	369
$D^2 f(a)$	the second derivative of f at a	369
$D^2 f$	the second derivative of f	369
$D^k f(a)$	the k-th derivative of f at a	369
$\left.\frac{\partial^2 f_j}{\partial x_k \partial x_i}\right\|_a$	the second partial derivative	370

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Answers to Exercises

Chapter 1

1.2.14 (2). Let (U, ϕ) be an n-dimensional chart. Then $\phi(U \cap U) = \phi(U)$ is open in $\mathbb{R}^n$ and $\phi \circ \phi^{-1} = \mathrm{id}_{\phi(U)}$ is smooth.

1.2.14 (3). As the cartesian product of n-copies of the circle, T^n is an n-dimensional smooth manifold.

1.2.14 (11). Let (U, ϕ) a chart belonging to the maximal atlas $\mathcal{A} = \{(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})\}'$. Then, by the compatibility of (U, ϕ) and $(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})$, the sets $\phi(U) = \phi(U \cap \mathbb{R}^n)$ and $U = \mathrm{id}_{\mathbb{R}^n}(U \cap \mathbb{R}^n)$ are open subsets of $\mathbb{R}^n$ and

$$\phi = \phi \circ \mathrm{id}_{\mathbb{R}^n}^{-1} \colon U \longrightarrow \phi(U), \quad \phi^{-1} = \mathrm{id}_{\mathbb{R}^n}^{-1} \circ \phi^{-1} \colon \phi(U) \longrightarrow U$$

are smooth maps. Therefore, $(U, \phi) \in \mathcal{A}$ if and only if U (and necessarily $\phi(U)$) is an open subset of $\mathbb{R}^n$ and ϕ is a smooth map with a smooth inverse, i.e. $\phi \colon U \to \phi(U)$ is a diffeomorphism (in $\mathbb{R}^n$).

1.3.23(2). Apply Corollary 1.3.20.

1.3.23(3). If $\mathrm{GL}(n, \mathbb{R})$ was compact, then $\mathbb{R}_* = \det(\mathrm{GL}(n, \mathbb{R}))$ would be compact (i.e. closed and bounded), which is absurd.

1.3.23(8). Setting $J := (0, 2\pi)$, the reader may be convinced that $E = \gamma(J)$, either using a program like Mathlab and Mathematica, or by evaluating γ at a number of points in J such as in the following table. They are quite useful in sketching a picture and will help us in our discussion. To accommodate the table correctly within the limits of the page, we write $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ and place the coordinates on separate rows as indicated. Thus, for instance, $\gamma(3\pi/4) = (-1, \sqrt{2}/2)$ etc.

t	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
$\gamma_1(t)$	0	1	0	-1	0	1	0	-1	0
$\gamma_2(t)$	0	$\sqrt{2}/2$	1	$\sqrt{2}/2$	0	$-\sqrt{2}/2$	-1	$-\sqrt{2}/2$	0

The image of the curve starts at a point approaching the origin of $\mathbb{R}^2$ without touching it, then it goes upwards to $(1, \sqrt{2}/2)$, then to $(1, 0)$. Next, continuing downwards, it reaches $(-1, \sqrt{2}/2)$, $(1, -\sqrt{2}/2)$, and, by crossing this time the origin,

2.3.13 (7) Applying (2.3.25) we find that:

$$\dot{\alpha}(t)(f) = T_t\alpha\left(\left.\frac{d}{dt}\right|_t\right)(f) = \left.\frac{d}{dt}\right|_t \big((f\circ\alpha)(t)\big) = (f\circ\alpha)'(t).$$

Chapter 3

3.1.13 (1) By the discussion of Example 3.1.3 (C), $\overline{\xi} = \overline{\mathrm{id}}_{\mathbb{R}_n} \circ \xi$. This immediately shows that the map $F\colon \mathcal{X}(A) \to \mathcal{C}^\infty(A,\mathbb{R})$, with $F(\xi) := \overline{\xi}$, is a bijection. Moreover, the linearity of $\overline{\mathrm{id}}_{\mathbb{R}_n}$ implies that $\overline{\lambda\xi + \mu\eta} = \lambda\overline{\xi} + \mu\overline{\eta}$, for every $\xi, \eta \in \mathcal{X}(A)$ and $\lambda, \mu \in \mathbb{R}$, whence it follows that F is a linear isomorphism.

3.1.13 (2) In virtue of Examples 3.1.8 (C) and **??**, the manifolds $\mathbb{R}^n$, A, S^1 are parallelizable, thus their tangent bundles are trivial. We obtain the following trivializations, respectively:

$\Phi\colon T\mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$, with $\Phi(u) = \big(\pi(u), \overline{\mathrm{id}}_{R^n}(u)\big) = \big(x; \lambda^1(x), \dots, \lambda^n(x)\big)$, if $\pi(u) = x$ and $u = \sum_{i=1}^n \lambda^i(x)\frac{\partial}{\partial x}|_x$, where the functions $\lambda^i \in \mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R})$ are determined by the parallelization $\big\{\frac{\partial}{\partial x^1}, \dots \frac{\partial}{\partial x^n}\big\}$ induced by the basic vector fields associated with the global chart $(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})$.

$\Phi_A\colon TA \to A \times \mathbb{R}^n$, defined analogously, with respect to the global chart (A, id_A), and the parallelization $\big\{\frac{\partial}{\partial x^1}\big|_A, \dots \frac{\partial}{\partial x^n}\big|_A\big\}$.

$\Phi_1\colon TS^1 \to S^1 \times \mathbb{R}^2$, given by $\Phi_1(u) = (x, \lambda(x))$, if $\pi(u) = x$, and $u = \lambda(x)\xi_x$, where ξ is the vector field of S^1 defined in Example **??**, and $\lambda \in \mathcal{C}^\infty(S^1,\mathbb{R})$.

3.1.13 (3) Let M and N be parallelizable manifolds, with corresponding parallelizations $\{X_i \mid i = 1,\dots,m\}$, and $\{Y_j \mid j = 1,\dots,n\}$. One way to check the parallelizability of $M \times N$ is to prove that $T(M \times N) \equiv TM \times TN$ is trivial. In this respect, we denote by $\Phi\colon TM \to M \times \mathbb{R}^m$ and $\Psi\colon TN \to N \times \mathbb{R}^n$ the corresponding trivializations, and define the map $F\colon TM \times TN\colon \to M \times N \times \mathbb{R}^m \times \mathbb{R}^n$ by setting

$$F(u,v) := \big(\pi_M(u), \pi_N(v), \mathrm{pr}_2(\Phi(u)), \mathrm{pr}_2(\Psi(v))\big),$$

where π_M, π_N are the respective projections of the tangent bundles involved.

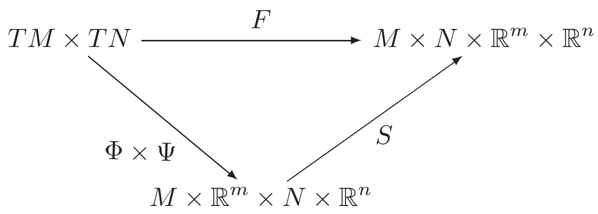
It is a trivial matter to check that F is an injection. It is also a surjection because, given any element$V = \big(x, y, \lambda^1, \dots, \lambda^m, \mu^1, \dots, \mu^n\big) \in M \times N \times \mathbb{R}^m \times R^n$, we immediately see that the tangent vectors

$$u := \sum_{i=1}^m \lambda^i X_i(x) \in T_xM \quad \text{and} \quad v := \sum_{j=1}^m \lambda^j Y_j(y) \in T_yN$$

yield $F(u,v) = V$.

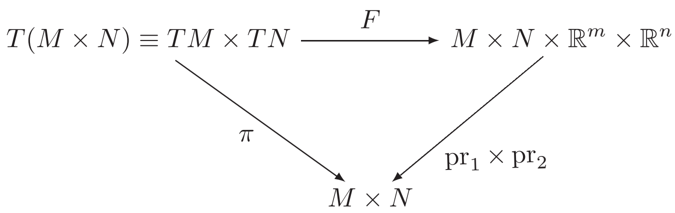
On the other hand, the linearity of $F|_{T_xM\times T_yN} : T_xM \times T_yN \to \mathbb{R}^m \times \mathbb{R}^n$, with $F(u,v) = \big(\mathrm{pr}_2(\Phi(u)), \mathrm{pr}_2(\Psi(v))\big)$, is an obvious consequence of the linearity of both $\Phi|_{T_xM} : T_xM \to \mathbb{R}^m$ and $\Psi|_{T_yN} : T_yN \to \mathbb{R}^n$.

Furthermore, F is a diffeomorphism, as a result of the following commutative diagram



where S is an obvious (diffeomorphism of) symmetries.

Finally, the following commutative diagram completes the requirements of a trivialization, as discussed in Example 3.1.8 (C),



where $\pi \equiv \pi_M \times \pi_N$.

3.1.13 (4)

By the general rule $\xi|_U = \sum_{i=1}^m \xi(x^i)\frac{\partial}{\partial x^i}$, over a chart $(U, \phi) \equiv (U; x^1, \ldots, x^m)$, where $m = \dim M$ [see also equalities (3.1.4)–(3.1.6)], we have:

$$\frac{\partial}{\partial y^j} = \sum_{i=1}^m \frac{\partial}{\partial y^j}(x^i)\frac{\partial}{\partial x^i} = \sum_{i=1}^m \frac{\partial x^i}{\partial y^j}\frac{\partial}{\partial x^i}$$

$$\frac{\partial}{\partial x^i} = \sum_{j=1}^n m\frac{\partial}{\partial x^i}(y^j)\frac{\partial}{\partial y^j} = \sum_{i=1}^m \frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j}.$$

3.1.13 (5) By assumption,

$$\xi|_U = \sum_{i=1}^m \xi^i \frac{\partial}{\partial x^i} = \sum_{j=1}^m \overline{\xi}^j \frac{\partial}{\partial y^j}. \tag{1}$$

Also, by Exercise 3.1.13 (4),

$$\sum_{j=1}^m \overline{\xi}^j \frac{\partial}{\partial y^j} = \sum_{j=1}^m \overline{\xi}^j \left(\sum_{i=1}^m \frac{\partial x^i}{\partial y^j}\frac{\partial}{\partial x^i} \right) = \sum_{i=1}^m \left(\sum_{j=1}^m \overline{\xi}^j \frac{\partial x^i}{\partial y^j} \right) \frac{\partial}{\partial x^i}.$$

From (1) and the preceding equalities we conclude that $\xi^i = \sum_{j=1}^m \overline{\xi}^j \dfrac{\partial x^i}{\partial y^j}$. The second formula is proved analogously.

3.1.13 (6) We imitate the procedure followed in the converse part of the proof of Proposition 3.1.10, for arbitrary ξ^0 and U.

3.2.5 (1) Let $\delta, \delta' \in \mathcal{D}(M)$ and $\alpha \in \mathcal{C}^\infty(M, \mathbb{R})$. We define the operations $\delta + \delta'$ and $\alpha \cdot \delta$, by setting

$$(\delta + \delta')(f) := \delta(f) + \delta'(f), \ \ (\alpha \cdot \delta)(f) := \alpha\, \delta(f),$$

for every $f \in \mathcal{C}^\infty(M, \mathbb{R})$. It is routinely checked that $\delta + \delta'$ and $\alpha \cdot \delta$ belong to $\mathcal{D}(M)$. As an example, we show that $\alpha \cdot \delta$ satisfies the Leibniz condition. Indeed, for every $f, g \in \mathcal{C}^\infty(M, \mathbb{R})$,

$$\begin{aligned}(\alpha \cdot \delta)(fg) &= \alpha\, \delta(fg) = \alpha\left(\delta(f)g + f\delta(g)\right) \\ &= \alpha\big(\delta(f)g\big) + \alpha\big(f\delta(g)\big) = (\alpha \cdot \delta)(f)\, g + f\, (\alpha \cdot \delta)(g),\end{aligned}$$

as $\alpha f = f\alpha$ in the commutative algebra $\mathcal{C}^\infty(M, \mathbb{R})$.

3.2.5 (2) By the notation of Example 2.3.8 (B) and equality (2.3.12),

$$\left.\frac{\partial}{\partial x^i}\right|_x (x^j) = \left.\frac{\partial x^j}{\partial x^i}\right|_x = \frac{\partial x^j}{\partial x^i}(x) = \delta_{ij},$$

for every $x \in U$. Therefore, $\dfrac{\partial x^j}{\partial x^i} = \delta_{ij}$.

3.2.5 (3) In virtue of Example 2.3.8 (A), we have that $\xi(c)(x) = \xi_x(c) = 0$, for every $x \in M$.

Chapter 4

4.1.11 (3) Direct application (2.2.5).

4.1.11 (4) For every $(a, b, c) \in G \times G \times G$,

$$\begin{aligned}[\boldsymbol{\gamma} \circ (\boldsymbol{\gamma} \times \mathrm{id}_G)](a, b, c) &= [\boldsymbol{\gamma} \circ (\boldsymbol{\gamma} \times \mathrm{id}_G)]((a, b), c) = \boldsymbol{\gamma}(ab, c) = (ab)c, \\ [\boldsymbol{\gamma} \circ (\mathrm{id}_G \times \boldsymbol{\gamma})](a, b, c) &= [\boldsymbol{\gamma} \circ (\mathrm{id}_G \times \boldsymbol{\gamma})](a, (b, c)) = \boldsymbol{\gamma}(a, bc) = a(bc).\end{aligned}$$

The commutativity of G yields the result.

4.1.11 (5) Direct consequence of the third equality of Exercise 4.1.11 (2). Furthermore, $R_e = L_e = \mathrm{id}_G$ implies that $T_e\boldsymbol{\alpha}(v) = -T_e\, \mathrm{id}_G = -\,\mathrm{id}_{T_eG} = -v$, for every $v \in T_eG$.

4.1.11 (6) Let $u \in TG$ be a fixed tangent vector, with $\pi(u) = x$ ($\pi\colon TG \to G$ is the projection). Let any $v \in TG$, with $\pi(v) = y$. Then, Example 4.1.10 and (4.1.3) imply

$$\overline{L}_u(v) = u \cdot v := T\boldsymbol{\gamma}(u, v) = T_{(x,y)}\boldsymbol{\gamma}(u, v) = T_xR_y(u),$$

because L_x is constant and $T_yL_x = 0$.

Similarly, we find that $\overline{R}_u(v) = v \cdot u = T_xL_y(u)$.

4.1.11 (7) First observe that, due to the commutativity of G, $L_g = R_g$, for every $g \in G$. Thus, for any $u \in T_xG$ and $v \in T_yG$ [again by Example 4.1.10 and (4.1.3)],

$$\begin{aligned}u \cdot v &= T_{(x,y)}\boldsymbol{\gamma}(u, v) = T_xR_y(u) + T_yL_x(v) = \\ &= T_xL_y(u) + T_yR_x(v) = T_{(y,x)}\boldsymbol{\gamma}(v, u) = v \cdot u.\end{aligned}$$

4.1.11 (8) For every $x, y \in G$,

$$\boldsymbol{\alpha}(xy) = (xy)^{-1} = y^{-1}x^{-1} = x^{-1}y^{-1} = \boldsymbol{\alpha}(x)\boldsymbol{\alpha}(y).$$

4.1.11 (9) $G \times H$ is a smooth manifold. The product $\boldsymbol{\gamma}\colon (G \times H) \times (G \times H) \to G \times H$ is defined by setting $\boldsymbol{\gamma}\big((a,b),(a',b')\big) := (a \cdot a', b \cdot b')$, where the dots represent the multiplications in G and H. This is clearly a smooth map. More formally, this can be seen by writing $\boldsymbol{\gamma}$ as

$$\boldsymbol{\gamma} = \big(\boldsymbol{\gamma}_G \circ (\mathrm{pr}_1, \mathrm{pr}_3), \boldsymbol{\gamma}_H \circ (\mathrm{pr}_2, \mathrm{pr}_4)\big),$$

where, obviously, $\boldsymbol{\gamma}_G$ and $\boldsymbol{\gamma}_H$ are the multiplications of G and H, respectively, and pr_i, $i = 1, \ldots, 4$, are the projections of $G \times H \times G \times H$ to the corresponding factors. Similarly, $\boldsymbol{\alpha}(a,b) = \big(a^{-1}, b^{-1}\big)$, or $\boldsymbol{\alpha} = \big(\boldsymbol{\alpha}_G \circ \mathrm{pr}_1, \boldsymbol{\alpha}_H \circ \mathrm{pr}_2\big)$, showing the smoothness of the inverse map.

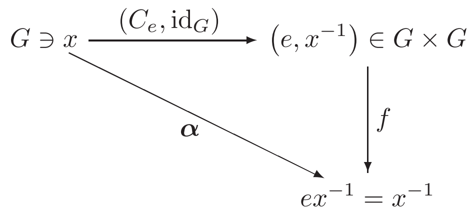
4.1.11 (10) It is a trivial fact that if G and H are abelian groups, then so is $G \times H$. By the definition of its multiplication, S^1 is an abelian Lie group. Therefore, the previous remarks, together with the preceding exercise, prove that $T^n = S^1 \times S^1$ (n factors) is an abelian Lie group.

4.1.11 (11) If G is a Lie group as in Definition 4.1.1, then

$$f(x,y) = xy^{-1} = \boldsymbol{\gamma}\left(x, y^{-1}\right) = \boldsymbol{\gamma}\left(x, \boldsymbol{\alpha}(y)\right), \quad \forall\, (x,y) \in G \times G;$$

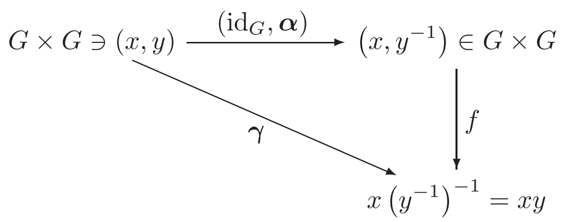
that is, $f = \boldsymbol{\gamma} \circ (\mathrm{id}_G \times \boldsymbol{\alpha})$, whence the smoothness of f.

Conversely, assuming that f is smooth, we first see that the following diagram is commutative:



which shows that $\boldsymbol{\alpha} = f \circ (C_e, \mathrm{id}_G)$ is smooth. Obviously, C_e denotes the constant map $C_e(x) = e$, for every $x \in G$.

Similarly, $\boldsymbol{\gamma}$ is smooth because is written as $\boldsymbol{\gamma} = f \circ (\mathrm{id}_G, \boldsymbol{\alpha})$, shown also in the next diagram:



4.2.12 (1) Let $\mathbb{A}$ be an arbitrary Lie algebra. If $\mathbb{A}$ is abelian, then $[a,b] = [b,a]$, for every $a, b \in \mathbb{A}$. But the bracket is antisymmetric, i.e. $[b,a] = -[a,b]$. Therefore, $[a,b] = 0$, for every $a, b \in \mathbb{A}$.

4.2.12 (2) It is clear that $\mathbb{R} \equiv (\mathbb{R}, +)$ is an (abelian) group. Any $\xi \in \mathcal{X}(\mathbb{R})$ has the form $\xi = f d/dx$. If, for a fixed $s \in \mathbb{R}$, $L_s \colon \mathbb{R} \to \mathbb{R}$ denotes the left translation, with $L_s(t) := s + t$, an $\xi \in \mathcal{X}(\mathbb{R})$ belongs to the Lie algebra $\mathcal{L}(\mathbb{R})$ if and only if it is left-invariant; equivalently, $\xi(s) = T_o L_s(\xi(0))$, for every $s \in \mathbb{R}$. Since

$$\begin{aligned}\xi(s) = T_0 L_s(\xi(0)) = T_0 L_s \left(f(0) \frac{d}{dt}\bigg|_0 \right) &= f(0) T_0 L_s \left(\frac{d}{dt}\bigg|_0 \right) \\ = f(0) L_s'(0) \left.\frac{d}{dt}\right|_{L_s(0)} &= f(0) \frac{d}{dt}\bigg|_s .\end{aligned}$$

Setting $f(0) = \lambda$, we see that

$$\xi \in \mathcal{L}(\mathbb{R}) \quad \Longrightarrow \quad \xi = \lambda \frac{d}{dt}.$$

Conversely, for an arbitrary $\lambda \in \mathbb{R}$, $\xi = \lambda d/dt$ determines a vector field on $\mathbb{R}$. It is left-invariant because, for every $t \in \mathbb{R}$,

$$\begin{aligned}T_t L_s(\xi(t)) = T_t L_s \left(\lambda \frac{d}{dt}\bigg|_s \right) &= \lambda\, T_t L_s \left(\frac{d}{dt}\bigg|_s \right) \\ &= \lambda \left.\frac{d}{dt}\right|_{s+t} = \xi(s+t) = \xi\big(L_s(t)\big);\end{aligned}$$

that is, $\xi = \lambda d/dt \;\Rightarrow\; \xi \in \mathcal{L}(\mathbb{R})$. The previous arguments show that

$$\mathcal{L}(\mathbb{R}) \equiv \mathcal{L}(\mathbb{R}, +) = \left\{ \lambda \frac{d}{dt} \;\middle|\; \lambda \in \mathbb{R} \right\}.$$

4.2.12 (3) A left translation $L_a \colon \mathbb{R}^n \to \mathbb{R}^n$, for $a \in \mathbb{R}^n$, is given by $L_a(h) = a + h$, for every $h \in \mathbb{R}^n$. Then

$$\begin{aligned}T_0 L_a \left(\frac{\partial}{\partial x^i}\bigg|_0 \right) &= \sum_{j=1}^n \left[T_0 L_a \left(\frac{\partial}{\partial x^i}\bigg|_0 \right) \right] (x^j) \cdot \left(\frac{\partial}{\partial x^j}\bigg|_a \right) \\ &= \sum_{j=1}^n \frac{\partial}{\partial x^i}\bigg|_0 (x^j \circ L_a) \cdot \frac{\partial}{\partial x^j}\bigg|_a .\end{aligned}$$

Since $x^j \circ L_a = a^j + x^j$ and a^j is constant, it follows that

$$\frac{\partial}{\partial x^i}\bigg|_0 \big(x^j \circ L_a\big) = \frac{\partial x^j}{\partial x^i}\bigg|_0 = \delta_{ij};$$

hence,

$$T_0 L_a \left(\frac{\partial}{\partial x^i}\bigg|_0 \right) = \frac{\partial}{\partial x^i}\bigg|_a . \tag{1}$$

Therefore, $\xi \in \mathcal{L}(\mathbb{R}^n)$ implies that

$$\begin{aligned}\xi_a = T_0 L_a(\xi_0) &= T_0 L_a \left(\sum_{i=1}^n \xi_0^i \frac{\partial}{\partial x^i}\bigg|_0 \right) \\ &= \sum_{i=1}^n \xi_0^i\, T_0 L_a \left(\frac{\partial}{\partial x^i}\bigg|_0 \right) = \sum_{i=1}^n \xi_0^i \frac{\partial}{\partial x^i}\bigg|_a .\end{aligned}$$

Setting $v^i = \xi_0^i$, we obtain a vector $\big(v^1, \ldots, v^n\big) \in \mathbb{R}^n$ such that

$$\xi_a = \sum_{i=1}^{n} v^i \left.\frac{\partial}{\partial x^i}\right|_a , \tag{2}$$

for every $a \in \mathbb{R}^n$. In conclusion, $\xi \in \mathcal{L}(\mathbb{R}^n)$ implies that ξ is written as in (2).

Conversely, let $v = \big(v^1, \ldots, v^n\big)$ be any vector in $\mathbb{R}^n$, and define a vector field ξ by (2). It is clear that ξ is indeed a smooth vector field on $\mathbb{R}^n$. It remain to show that ξ is left-invariant: For any $a, h \in \mathbb{R}^n$,

$$\begin{aligned}
\xi_{L_a(h)} = \xi_{a+h} &= \sum_{i=1}^{n} v^i \left.\frac{\partial}{\partial x^i}\right|_{a+h} \\
[\text{by } (1)] \qquad &= \sum_{i=1}^{n} v^i\, T_0 L_{a+h} \left(\left.\frac{\partial}{\partial x^i}\right|_0 \right) \\
&= \sum_{i=1}^{n} v^i\, T_0 \left(L_a \circ L_h\right) \left(\left.\frac{\partial}{\partial x^i}\right|_0 \right) \\
&= \sum_{i=1}^{n} v^i\, T_h L_a \left(T_0 L_h \left(\left.\frac{\partial}{\partial x^i}\right|_0 \right)\right) \\
&= T_h L_a \left(v^i \left.\frac{\partial}{\partial x^i}\right|_h \right) = T_h L_a(\xi_h);
\end{aligned}$$

that is, $\xi \in \mathcal{L}(\mathbb{R}^n) \equiv \mathcal{L}(\mathbb{R}^n, +)$. Hence, we obtain the correspondence of the exercise.

4.2.12 (4) We define $\widetilde{f}$ by setting $\widetilde{f} := \boldsymbol{h}_H \circ T_e f \circ \boldsymbol{h}_G^{-1}$, as in the next diagram:

$$\begin{array}{ccc}
T_e G & \xrightarrow{\;T_e f\;} & T_e H \\
\uparrow{\scriptstyle \boldsymbol{h}_G} & & \downarrow{\scriptstyle \boldsymbol{h}_H} \\
\mathfrak{g} & \overset{}{\underset{\widetilde{f}}{\dashrightarrow}} & \mathfrak{h}
\end{array}$$

where, for convenience, we denote by e the unit of both G and H.

We check that $\xi \in \mathfrak{g}$ and $\widetilde{\xi} := \widetilde{f}(\xi) \in \mathfrak{h}$ are f-related; that is, the following diagram is commutative:

$$\begin{array}{ccc}
TG & \xrightarrow{\;Tf\;} & TH \\
\uparrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \widetilde{\xi}} \\
G & \xrightarrow[\;f\;]{} & H
\end{array}$$

Indeed, for every $g \in G$, first observe that $f \circ L_g = L'_{f(g)} \circ f$, where L'_h, $h \in H$, denote left translations in H; hence,

$$\begin{aligned}
(Tf \circ \xi)(g) &= T_g f\big(T_e L_g(\xi_e)\big) = T_e(f \circ L_g)(\xi_e) \\
&= T_e\left(L'_{f(g)} \circ f\right)(\xi_e) = T_e L'_{f(g)}\big(T_e f(\xi_e)\big) \\
\text{[by the first diagram]} \quad &= T_e L'_{f(g)}\left(\left(\boldsymbol{h}_H \circ \widetilde{f} \circ \boldsymbol{h}_G^{-1}\right)(\xi_e)\right) \\
&= T_e L'_{f(g)}\left(\boldsymbol{h}_H\big(\widetilde{\xi}\big)\right) = T_e L'_{f(g)}\big(\widetilde{\xi}_e\big) \\
&= \widetilde{\xi}(f(g)) = (\widetilde{\xi} \circ f)(g),
\end{aligned}$$

which, as desired, verifies the commutativity of the second diagram.

Since $\widetilde{f}$ is linear, to complete the exercise it remains to show that $\widetilde{f}$ preserves the bracket, i.e. $\widetilde{f}\big([\xi, \xi']\big) = \big[\widetilde{f}(\xi), \widetilde{f}(\xi')\big]$, for every $\xi, \xi' \in \mathfrak{g}$. As a matter of fact, because ξ and ξ' are f-related with $\widetilde{\xi}$ and $\widetilde{\xi}'$, respectively, then $[\xi, \xi']$ and $\big[\widetilde{\xi}, \widetilde{\xi}'\big]$ are also f-related (see Corollary 3.5.3); thus, in particular,

$$T_e f\big([\xi, \xi']_e\big) = \big[\widetilde{\xi}, \widetilde{\xi}'\big]_e,$$

and, consequently,

$$\begin{aligned}
\widetilde{f}\big([\xi, \xi']\big) &= \boldsymbol{h}_H^{-1}\big(T_e f\big([\xi, \xi']_e\big)\big) \\
&= \boldsymbol{h}_H^{-1}\big(\big[\widetilde{\xi}, \widetilde{\xi}'\big]_e\big) = \left[\widetilde{\xi}, \widetilde{\xi}'\right] \\
&= \big[\widetilde{f}(\xi), \widetilde{f}(\xi')\big].
\end{aligned}$$

4.2.12 (5) Property (i) is an immediate consequence of $T_e \,\mathrm{id}_G = \mathrm{id}_{T_e G}$. Next, for every $\xi \in \mathfrak{g}$, we have:

$$\begin{aligned}
\left(\widetilde{g \circ f}\right)(\xi) &= \big(\boldsymbol{h}_K^{-1} \circ T_e(g \circ f) \circ \boldsymbol{h}_H\big)(\xi) \\
&= \big(\boldsymbol{h}_K^{-1} \circ T_e g \circ \boldsymbol{h}_H \circ \boldsymbol{h}_H^{-1} \circ T_e f \circ \boldsymbol{h}_G\big)(\xi) \\
&= \widetilde{g}\left(\widetilde{f}(\xi)\right) = \left(\widetilde{g} \circ \widetilde{f}\right)(\xi),
\end{aligned}$$

whence (ii). Finally, (iii) is a direct consequence of (i) and (ii).

4.2.12 (6) By Exercise 4.1.11 (8), the inversion map $\boldsymbol{\alpha}\colon G \to G$ is an isomorphism of Lie groups. It induces the isomorphism of Lie algebras $\widetilde{\boldsymbol{\alpha}}\colon \mathfrak{g} \to \mathfrak{g}$, as a consequence of Exercises 4.2.12 (4) and (5). By the very construction of $\widetilde{\boldsymbol{\alpha}}$,

$$\begin{aligned}
\widetilde{\boldsymbol{\alpha}}(\xi) &= \big(\boldsymbol{h}_G^{-1} \circ T_e\boldsymbol{\alpha} \circ \boldsymbol{h}_G\big)(\xi) \\
&= \boldsymbol{h}_G^{-1}\big(T_e\boldsymbol{\alpha}(\xi_e)\big) \\
\text{[by (4.1.4)]} \quad &= \boldsymbol{h}_G^{-1}(-\xi_e) = -\xi;
\end{aligned}$$

that is,

$$\text{(1)} \qquad \widetilde{\boldsymbol{\alpha}}(\xi) = -\xi, \qquad \forall\, \xi \in \mathfrak{g}.$$