STRONGLY n-PROJECTIVELY CORESOLVED GORENSTEIN FLAT MODULES

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ABSTRACT. This paper introduces the notion of strongly n-projectively coresolved Gorenstein flat modules (or strongly n-PGF modules, for short), where n is a nonnegative integer. Using this class, a new characterization of modules with finite PGF-dimension is given. Moreover, a stability result for the class of PGF modules, with respect to the very Gorenstein process, is proven.

1. INTRODUCTION

In the study of Gorenstein homological algebra, the strongly Gorenstein projective, strongly Gorenstein injective and strongly Gorenstein flat modules play a central role as they are used to give simple characterizations of the Gorenstein projective, Gorenstein injective and Gorenstein flat modules, respectively [3]. Mahdou and Tamekkante [6] introduced a generalization of the strongly Gorenstein projective, strongly Gorenstein injective and strongly Gorenstein flat modules, namely the strongly n-Gorenstein projective, strongly n-Gorenstein injective and strongly n-Gorenstein flat modules. These modules provide new characterizations of the Gorenstein projective, injective and flat dimensions of modules [6]. Moreover, the strongly 0-Gorenstein projective, strongly 0-Gorenstein injective and strongly 0-Gorenstein flat modules are exactly the strongly Gorenstein projective, strongly Gorenstein injective and strongly Gorenstein flat modules, respectively. Using a special class of modules related to strongly Gorenstein flat modules, Bouchiba and Khaloui [2] proved the stability of the class of Gorenstein flat modules. Sather-Wagstaff, Sharif and White [8] proved the stability of the classes of Gorenstein projective and Gorenstein injective modules. Saroch and Stovicek [7] introduced the class of projectively coresolved Gorenstein flat modules. Dalezios and Emmanouil [4] studied the relative homological dimension based on this class of modules.

In the first part of this paper, inspired by [6], we introduce the notion of strongly n-projectively coresolved Gorenstein flat modules (or strongly n-PGF modules, for short). This class of modules yields a new characterization of modules with finite PGF-dimension. More precisely, in Theorem 3.20, we prove that a module M has $\text{PGF-dim}_R(M) \leqslant n$ if and only if it is a direct summand of a strongly n-PGF module, which is the main result of Section 3.

In the second part of this paper, inspired by [2], we prove the stability of the class of PGF modules, which is the main result of Section 4. In particular, for every

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exact sequence of PGF modules

$$\mathbf{G} = \cdots \to G_1 \to G_0 \to G^0 \to G^1 \to \cdots$$

such that $M \cong \mathrm{Im}(G_0 \to G^0)$ and $H \otimes -$ preserves exactness of $\mathbf{G}$ for every Gorenstein injective module H, the module M is PGF (see Theorem 4.7). A central role in the proof is played by the subcategory consisting of the R-modules M for which there exists a short exact sequence of the form $0 \to M \to G \to M \to 0$, where G is a PGF module, such that $I \otimes -$ preserves exactness of this sequence for every injective module I (see Theorem 4.7).

2. Preliminaries

In this section, we collect certain notions and preliminary results that will be used in the sequel. These involve basic concepts related to Gorenstein homological algebra. Throughout this paper, R is a unital associative ring and all modules are left R-modules.

The notions of Gorenstein projective, Gorenstein injective and Gorenstein flat modules, respectively, were introduced by Holm [5].

Definition 2.1. ([5])

(1) An R-module M is called Gorenstein projective (or G-projective, for short), if there exists an exact sequence of projective modules

$$\mathbf{P} = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

such that $M \cong \mathrm{Im}(P_0 \to P^0)$ and such that $\mathrm{Hom}_R(-, Q)$ preserves exactness of $\mathbf{P}$ whenever Q is a projective module.

(2) An R-module M is called Gorenstein injective (or G-injective, for short), if there exists an exact sequence of injective modules

$$\mathbf{E} = \cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$$

such that $M \cong \mathrm{Im}(E_0 \to E^0)$ and such that $\mathrm{Hom}_R(I, -)$ preserves exactness of $\mathbf{E}$ whenever I is a injective module.

(3) An R-module M is called Gorenstein flat (or G-flat, for short), if there exists an exact sequence of flat modules

$$\mathbf{F} = \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

such that $M \cong \mathrm{Im}(E_0 \to E^0)$ and such that $I \otimes -$ preserves exactness of $\mathbf{E}$ whenever I is a injective module.

Bennis and Mahdou [3] introduced the notions of strongly Gorenstein projective, strongly Gorenstein injective and strongy Gorenstein flat modules, respectively, which are special cases of the Gorenstein projective, Gorenstein injective and Gorenstein flat modules, respectively.

Definition 2.2. ([3])

(1) An R-module M is called strongly Gorenstein projective, if there exists a short exact sequence of the form

$$0 \to M \to P \to M \to 0$$

such that P is a projective module and $\mathrm{Hom}_R(-, Q)$ preserves the exactness of this sequence, whenever Q is a projective module.

(2) An R-module M is called strongly Gorenstein injective, if there exists a short exact sequence of the form
$$0 \to M \to E \to M \to 0$$
such that E is an injective module and $\mathrm{Hom}_R(I, -)$ preserves the exactness of this sequence, whenever I is an injective module.

(3) An R-module M is called stronly Gorenstein flat, if there exists a short exact sequence of the form
$$0 \to M \to F \to M \to 0$$
such that F is an injective module and $I \otimes -$ preserves the exactness of this sequence, whenever I is an injective module.

Strongly Gorenstein projective, strongly Gorenstein injective and strongly Gorenstein flat modules, give a simple characterization of Gorenstein projective, Gorenstein injective and Gorenstein flat modules, respectively, as follows.

Theorem 2.3. ([3, Theorems 2.7 and 3.5])

(1) A module is Gorenstein projective (respectively, injective) if and only if it is a direct summand of a strongly Gorenstein projective (respectively, injective) module.

(2) Every Gorenstein flat module is a direct summand of a strongly Gorenstein flat module.

Mahdou and Tamekkante [6] introduced a generalization of the strongly Gorenstein projective, strongly Gorenstein injective and strongly Gorenstein flat modules, namely the strongly n-Gorenstein projective, strongly n-Gorenstein injective and strongly n-Gorenstein flat modules, respectively.

Definition 2.4. ([6])

(1) An R-module M is called strongly n-Gorenstein projective if there exists a short exact sequence of the form
$$0 \to M \to P \to M \to 0$$
such that $\mathrm{pd}_R(P) \leqslant n$ and $\mathrm{Ext}_R^{n+1}(M, Q) = 0$ for every projective module Q.

(2) An R-module M is called strongly n-Gorenstein injective if there exists a short exact sequence of the form
$$0 \to M \to E \to M \to 0$$
such that $\mathrm{id}_R(E) \leqslant n$ and $\mathrm{Ext}_{n+1}^R(I, M) = 0$ for every injective module I.

(3) An R-module M is called strongly n-Gorenstein flat if there exists a short exact sequence of the form
$$0 \to M \to F \to M \to 0$$
such that $\mathrm{fd}_R(F) \leqslant n$ and $\mathrm{Tor}_{n+1}^R(I, M) = 0$ for every injective module I.

Theorem 2.5. ([6, Theorems 2.6, 2.7 and 3.3])

(1) Let M be an R-module and n be a nonnegative integer. Then $\mathrm{Gpd}_R(M) \leqslant n$ (respectively, $\mathrm{Gid}_R(M) \leqslant n$) if and only if M is a direct summand of a strongly n-Gorenstein projective (respectively, strongly n-Gorenstein injective) module.

(2) Let R be a coherent ring, M be an R-module and n be a nonnegative integer. Then $\mathrm{Gfd}_R(M) \leqslant n$ if and only if M is a direct summand of a strongly n-Gorenstein flat module.

Saroch and Stovicek [7] defined the notion of projectively coresolved Gorenstein flat modules (or PGF modules, for short).

Definition 2.6. *An R-module M is called projectively coresolved Gorenstein flat (or PGF, for short) if there exists an exact sequence of projective modules*

$$\boldsymbol{P} = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots ,$$

such that $M \cong Im(P_0 \to P^0)$ and such that $I \otimes -$ preserves the exactness of $\boldsymbol{P}$ whenever I is a injective module. The exact sequence $\boldsymbol{P}$ is called a complete PGF resolution.

The class of PGF R-modules, denoted by $\mathrm{PGF}(R)$, is closed under extensions, direct sums, direct summands and kernels of epimorphisms [7].

The following proposition gives a characterization of PGF modules.

Proposition 2.7. *For every module M, the following are equivalent:*

(1) *M is PGF.*
(2) *M satisfies the following two conditions:*
   (i) *There exists an exact sequence of the form $0 \to M \to P^0 \to P^1 \to \cdots$, where each P^i is projective, such that $I \otimes -$ preserves exactness of this sequence for every injective module I.*
   (ii) *$Tor_i^R(I, M) = 0$ for every $i > 0$ and every injective module I.*
(3) *M satisfies the following two conditions:*
   (i) *There exists an exact sequence of the form $0 \to M \to P^0 \to P^1 \to \cdots$, where each P^i is projective, such that $I \otimes -$ preserves exactness of this sequence for every injective module I.*
   (ii) *$Tor_i^R(I', M) = 0$ for every $i > 0$ and every module I' with finite injective dimension.*
(4) *There exists a short exact sequence of the form $0 \to M \to P \to G \to 0$, where P is projective and G is a PGF module.*

Proof. (1) ⇒ (2): By definition of PGF modules, there exists an exact sequence of the form

$$\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} P^0 \xrightarrow{d^0} P^1 \xrightarrow{d^1} \cdots ,$$

such that $M \cong \mathrm{Im} d_0$ and such that the sequence

$$\cdots \xrightarrow{1 \otimes d_2} I \otimes P_1 \xrightarrow{1 \otimes d_1} I \otimes P_0 \xrightarrow{1 \otimes d_0} I \otimes P^0 \xrightarrow{1 \otimes d^0} I \otimes P^1 \xrightarrow{1 \otimes d^1} \cdots$$

is exact for every injective module I. Let I be an injective module. Applying the functor $I \otimes -$ on the exact sequence $\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to M \to 0$, we obtain that $I \otimes M = \mathrm{Coker}(1 \otimes d_1) = \mathrm{Ker}(1 \otimes d^0)$. Thus,

$$0 \to I \otimes M \to I \otimes P^0 \xrightarrow{1 \otimes d^0} I \otimes P^1 \xrightarrow{1 \otimes d^1} \cdots$$

is exact for every injective module I. Obviously, $\mathrm{Tor}_i^R(I, M) = 0$ for every $i > 0$ and every injective module I.

(2) ⇒ (1): Consider a projective resolution of M, $\cdots \to P_0 \to P_1 \to M \to 0$. From (2), there exists an exact sequence of the form $0 \to M \to P^0 \to P^1 \to \cdots$, where each P^i is projective, such that $I \otimes -$ preserves exactness of this sequence for

every injective module I. The splicing of the above exact sequences yields the exact sequence

$$\mathbf{P} = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots ,$$

where $M \cong \operatorname{Im}(P_0 \to P^0)$. It remains to prove that the sequence $I \otimes \mathbf{P}$ is exact whenever I is an injective module. Indeed, the homology of the complex $I \otimes \mathbf{P}$ computes the abelian groups $\operatorname{Tor}^R_\star(I, M)$ which are trivial by assumption. Thus, the module M is PGF.

(2) $\Leftrightarrow$ (3): This follows by induction on the injective dimension of I'.

(4) $\Rightarrow$ (2): Let $0 \to M \to P \to G \to 0$ be an exact sequence, where P is projective and G is a PGF module. Since G is PGF, the implication (1) $\Rightarrow$ (2) yields $\operatorname{Tor}^R_i(I, G) = 0$ for every $i > 0$ and every injective module I. Let I be an injective module. Then, the short exact sequence $0 \to M \to P \to G \to 0$ induces a long exact sequence of the form

$$\cdots \to \operatorname{Tor}^R_{i+1}(I, G) \to \operatorname{Tor}^R_i(I, M) \to \operatorname{Tor}^R_i(I, P) \to \cdots ,$$

where $i > 0$, which implies that $\operatorname{Tor}^R_i(I, M) = 0$ for every $i > 0$. Moreover, there exists an exact sequence of the form $0 \to G \to P^0 \to P^1 \to \cdots$, such that $I \otimes -$ preserves the exactness of this sequence. We obtain an exact sequence of the form

$$0 \to M \to P \to P^0 \to P^1 \to \cdots ,$$

such that $I \otimes -$ preserves exactness of this sequence for every injective module I. The implication (1) $\Rightarrow$ (4) is clear. □

3. The strongly n-PGF modules

In this section, we define the notion of strongly PGF modules and prove that a module is PGF if and only if it is a direct summand of a strongly PGF module (see Theorem 3.5). We also define the notion of strongly n-PGF modules and we prove a new characterization of modules with finite PGF-dimension. In particular, we prove that an R module M has $\text{PGF-dim}_R(M) \leqslant n$ if and only if it is a direct summand of a strongly n-PGF module (see Theorem 3.20).

Definition 3.1. *An R-module M is called strongly projectively Gorenstein flat (or strongly PGF, for short), if there exists a short exact sequence of the form*

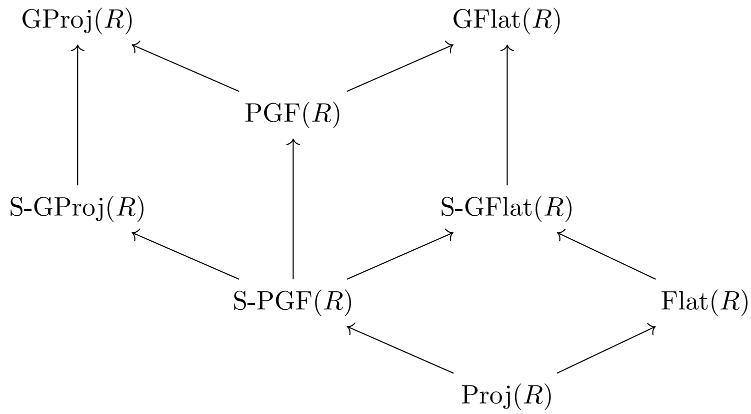
$$0 \to M \to P \to M \to 0$$

such that P is a projective R-module and $I \otimes -$ preserves exactness of this sequence whenever I is a injective module.

Remark 3.2. *Let M be a PGF module. Corollary 4.5 of* [7] *yields $Ext^i_R(M, P) = 0$ for every $i > 0$ and every projective module P. Thus, every strongly PGF module is also strongly Gorenstein projective and strongly Gorenstein flat. We also have $Ext^i_R(M, P') = 0$ for every $i > 0$ and every module P' with finite projective dimension, using induction on $pd_R(P')$.*

We denote by S-PGF(R), S-GProj(R), S-GFlat(R), the classes of strongly PGF, strongly Gorenstein projective and strongly Gorenstein injective modules respectively.

A schematic presentation is given below:



Here, $\mathrm{GProj}(R)$ and $\mathrm{GFlat}(R)$ are the classes of Gorenstein projective and Gorenstein injective modules, respectively. See also Propositions 2.3 and 3.2 of [3].

Definition 3.3. *An R-module M is called strongly n-projectively Gorenstein flat (or strongly n-PGF, for short), if there exists a short exact sequence of the form*

$$0 \to M \to F \to M \to 0$$

such that $pd(F) \leqslant n$ *and* $Tor^R_{n+1}(I, M) = 0$ *for every injective module I.*

Remark. *A direct consequence of the above definition is that the strongly 0-PGF modules are precisely the strongly PGF modules.*

A direct consequence of Definition 3.3 is the following statement.

Corollary 3.4. *Every module with projective dimension less than or equal to n is a strongly n-PGF module.*

Proof. Let M be a module with $\mathrm{pd}(M) \leqslant n$. Consider the short exact sequence

$$0 \to M \to M \oplus M \to M \to 0$$

where $\mathrm{pd}(M \oplus M) \leqslant n$. The inequality $\mathrm{fd}(M) \leqslant \mathrm{pd}(M) \leqslant n$ implies that $\mathrm{Tor}^R_{n+1}(I, M) = 0$ for every injective module I. We conclude that M is a strongly n-PGF module. □

Now we give a new characterization of PGF-modules by means of the notion of strongly PGF-modules.

Theorem 3.5. *A module is PGF if and only if it is a direct summand of a strongly PGF module.*

Proof. We observe that every strongly PGF module is also a PGF module. As the class of PGF modules is closed under direct summands, we conclude that every direct summand of a strongly PGF module is a PGF module.

It remains to prove the other implication. Let M be a PGF module. Then, there exists an exact sequence of projective modules

$$\mathbf{P} = \cdots \to P_1 \xrightarrow{d_1^P} P_0 \xrightarrow{d_0^P} P_{-1} \xrightarrow{d_{-1}^P} P_{-2} \to \cdots$$

such that $M \cong \mathrm{Im}(d_0^P)$ and such that $I \otimes -$ preserves exactness of $\mathbf{P}$ whenever I is a injective module.

For every $n \in \mathbb{Z}$, we denote by $\Sigma^n\mathbf{P}$ the exact sequence obtained from $\mathbf{P}$ by increasing all indices by n: $(\Sigma^n\mathbf{P})_i = P_{i-n}$ and $d_i^{\Sigma^n P} = d_{i-n}^P$ for every $i \in \mathbb{Z}$.

We consider now the exact sequence

$$\bigoplus_{n\in\mathbb{Z}}(\Sigma^n\mathbf{P}) = \cdots \to \bigoplus_{i\in\mathbb{Z}} P_i \xrightarrow{\oplus_{i\in\mathbb{Z}} d_i^P} \bigoplus_{i\in\mathbb{Z}} P_i \xrightarrow{\oplus_{i\in\mathbb{Z}} d_i^P} \bigoplus_{i\in\mathbb{Z}} P_i \to \cdots$$

where $\mathrm{Im}(\bigoplus_{i\in\mathbb{Z}} d_i^P) \cong \bigoplus_{i\in\mathbb{Z}} \mathrm{Im}d_i$ and so M is a direct summand of $\mathrm{Im}(\bigoplus_{i\in\mathbb{Z}} d_i^P)$.

Moreover, from Proposition 20.2 (3) of [1],we obtain the isomorphism of complexes

$$I \otimes (\bigoplus_{n\in\mathbb{Z}}(\Sigma^n\mathbf{P})) \cong \bigoplus_{n\in\mathbb{Z}}(I \otimes \Sigma^n\mathbf{P})$$

which is an exact sequence for every injective module I. Thus, $\mathrm{Im}(\bigoplus_{i\in\mathbb{Z}} d_i^P)$ is a strongly PGF module and M is a direct summand of this strongly PGF module. □

The next result gives a simple characterization of strongly PGF modules.

Proposition 3.6. *For every module M, the following are equivalent:*

(1) *M is strongly PGF.*
(2) *There exists a short exact sequence $0 \to M \to P \to M \to 0$ such that P is a projective R-module and $Tor_1^R(I, M) = 0$ for every injective module I.*
(3) *There exists a short exact sequence $0 \to M \to P \to M \to 0$ such that P is a projective R-module and $Tor_i^R(I, M) = 0$ for every $i > 0$ and every injective module I.*
(4) *There exists a short exact sequence $0 \to M \to P \to M \to 0$ such that P is a projective R-module and $Tor_1^R(I', M) = 0$ for every module I' with finite injective dimension.*
(5) *There exists a short exact sequence $0 \to M \to P \to M \to 0$ such that P is a projective R-module and $Tor_i^R(I', M) = 0$ for every $i > 0$ and every module I' with finite injective dimension.*
(6) *There exists a short exact sequence $0 \to M \to P \to M \to 0$ such that P is a projective R-module and $I' \otimes -$ preserves exactness of this sequence for every module I' with finite injective dimension.*

Proof. This follows immediately from the Definition 3.3 of strongly PGF modules, using standard arguments. □

Proposition 3.6 yields the following corollary about projective modules.

Corollary 3.7. *Every projective module is strongly PGF module.*

Proof. This is an immediate consequence of Corollary 3.4. □

Remark 3.8. *A strongly PGF module is projective if and only if it has finite projective dimension.*

Remark 3.9. *A strongly PGF module is finitely generated if and only if it is finitely presented.*

The next result gives a simple characterization of strongly n-PGF modules.

Proposition 3.10. *For every module M, the following are equivalent:*

(1) *M is strongly n-PGF.*

(2) *There exists a short exact sequence* $0 \to M \to F \to M \to 0$ *such that* $pd_R(F) \leqslant n$ *and* $Tor_i^R(I, M) = 0$ *for every* $i > n$ *and every injective module* I.
(3) *There exists a short exact sequence* $0 \to M \to F \to M \to 0$ *such that* $pd_R(F) \leqslant n$ *and* $Tor_i^R(I', M) = 0$ *for every* $i > n$ *and every module* I' *with finite injective dimension.*
(4) *There exists a short exact sequence* $0 \to M \to F \to M \to 0$ *such that* $pd_R(F) \leqslant n$ *and* $Tor_{n+1}^R(I', M) = 0$ *for every module* I' *with finite injective dimension.*

Proof. (1) ⇒ (2): By the definition of strongly n-PGF modules, there exists a short exact sequence $0 \to M \to F \to M \to 0$ such that $\mathrm{pd}_R(F) \leqslant n$ and $\mathrm{Tor}_{n+1}^R(I, M) = 0$ for every injective module I. Let I be an injective R-module. We will prove that $\mathrm{Tor}_i^R(I, M) = 0$ for every $i > n$. The short exact sequence induces a long exact sequence of the form

$$\cdots \to \mathrm{Tor}_{i+1}^R(I, F) \to \mathrm{Tor}_{i+1}^R(I, M) \to \mathrm{Tor}_i^R(I, M) \to \mathrm{Tor}_i^R(I, F) \to \cdots$$

for every $i > n$. Since $\mathrm{fd}_R(F) \leqslant \mathrm{pd}_R(F) \leqslant n$, we obtain that $\mathrm{Tor}_i^R(I, F) = 0$ for $i > n$. Thus, $\mathrm{Tor}_{i+1}^R(I, M) = \mathrm{Tor}_i^R(I, M)$ for every $i > n$. Since $\mathrm{Tor}_{n+1}^R(I, M) = 0$, we conclude that $\mathrm{Tor}_i^R(I, M) = 0$ for every $i > n$.

(2) ⇒ (3): We assume that there exists a short exact sequence of the form $0 \to M \to F \to M \to 0$, such that $\mathrm{pd}_R(F) \leqslant n$ and $\mathrm{Tor}_i^R(I, M) = 0$ for every $i > n$ and every injective module I. Let I' be a module with finite injective dimension $0 \leqslant \mathrm{id}_R(I') = m < \infty$. We will prove that $\mathrm{Tor}_i^R(I', M) = 0$ for every $i > n$, by induction on m. We consider an injective resolution of I' of length m

$$0 \to I' \to I_0 \to \cdots \to I_m \to 0.$$

The case $m = 0$ is trivial, since I' is injective. Assume that $m > 0$ and let $J = \mathrm{Coker}(I' \to I_0)$ to obtain the short exact sequence

$$0 \to I' \to I_0 \to J \to 0.$$

This short exact sequence induces a long exact sequence of the form

$$\cdots \to \mathrm{Tor}_{i+1}^R(J, M) \to \mathrm{Tor}_i^R(I', M) \to \mathrm{Tor}_i^R(I_0, M) \to \cdots$$

where $i > n$. Since $\mathrm{id}_R(J) \leqslant m - 1$, our induction hypothesis implies that $\mathrm{Tor}_{i+1}^R(J, M) = 0$ for every $i > n$. We also have $\mathrm{Tor}_i^R(I_0, M) = 0$ for every $i > n$, since I_0 is injective. Therefore, the long exact sequence implies that $\mathrm{Tor}_i^R(I', M) = 0$ for every $i > n$.

The implications (3) ⇒ (4) and (4) ⇒ (1) are clear. □

Proposition 3.11. *Let* n *be a nonnegative integer and* $(M_i)_i$ *be a family of strongly n-PGF modules. Then, the direct sum* $M = \bigoplus_i M_i$ *is also a strongly n-PGF module.*

Proof. Since the modules M_i are strongly n-PGF, by definition there exist short exact sequences of the form $0 \to M_i \to F_i \to M_i \to 0$, where $\mathrm{pd}_R(F_i) \leqslant n$ and $\mathrm{Tor}_{n+1}^R(I, M_i) = 0$ for every i and every injective module I. Thus, we obtain a short exact sequence of the form

$$0 \to \bigoplus_i M_i \to \bigoplus_i F_i \to \bigoplus_i M_i \to 0$$

where $\mathrm{pd}_R(\bigoplus_i F_i) = \sup_i\{\mathrm{pd}_R(F_i)\} \leqslant n$ and

$$\mathrm{Tor}^R_{n+1}(I, \bigoplus_i M_i) \cong \bigoplus_i \mathrm{Tor}^R_{n+1}(I, M_i) = 0$$

for every injective module I. Thus, $M = \bigoplus_i M_i$ is strongly n-PGF. □

Dalezios and Emmanouil [4] defined the following notions.

Definition 3.12. ([4, Proposition 2.2]) We say that a module M has a PGF-resolution of length n if there exists an exact sequence of the form

$$0 \to G_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0,$$

where $G_0, \ldots, G_{n-1}, G_n$ are PGF modules.

The PGF-dimension of a module M, denoted by PGF-$\dim_R(M)$, is defined by declaring that PGF-$\dim_R(M) \leqslant n$ if and only if M has a PGF-resolution of length n. In the case where PGF-$\dim_R(M) \leqslant n$ and M has no PGF-resolution of length less than n, we say that M has PGF-dimension equal to n and write PGF-$\dim_R(M) = n$. Finally, we say that M has infinite PGF-dimension and write PGF-$\dim_R(M) = \infty$, if M has no PGF-resolution of finite length.

Throughout the rest of this section, we use the following results concerning PGF-$\dim_R(M)$ from [4].

Proposition 3.13. ([4, Proposition 2.3]) Let $(M_i)_i$ be a family of modules and $M = \bigoplus_i M_i$ be their direct sum. Then, PGF-$\dim_R(M) = \sup_i$PGF-$\dim_R(M_i)$.

Proposition 3.14. ([4, Propositions 2.4 and 3.6])

Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of modules. Then,

(1) PGF-$\dim_R(M) \leqslant \max\{$PGF-$\dim_R(M'),$ PGF-$\dim_R(M'')\}$,
(2) PGF-$\dim_R(M') \leqslant \max\{$PGF-$\dim_R(M),$ PGF-$\dim_R(M'') - 1\}$,
(3) PGF-$\dim_R(M'') \leqslant \max\{$PGF-$\dim_R(M),$ PGF-$\dim_R(M') + 1\}$.

Theorem 3.15. ([4, Theorem 3.4]) Let M be a module and n a nonnegative integer. Then, PGF-$\dim_R(M) = n \geqslant 0$ if and only if there exists a short exact sequence of the form

$$0 \to M \to D \to G \to 0$$

where G is a PGF-module and $\mathrm{pd}_R(D) = n$.

Proposition 3.16. ([4, Corollary 3.7]) Let M be a module such that $\mathrm{pd}_R(M) < \infty$. Then, PGF-$\dim_R(M) = \mathrm{pd}_R(M)$.

We continue with our results of this section.

Proposition 3.17. *Let n be a positive integer and M be a strongly n-PGF module. Then, the following hold:*

(1) *If $0 \to N \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ is an exact sequence where all P_i are projective, then N is strongly PGF and consequently PGF-$\dim_R(M) \leqslant n$.*
(2) *Moreover, if $0 \to M \to F \to M \to 0$ is a short exact sequence where $\mathrm{pd}_R(F) < \infty$, then PGF-$\dim_R(M) = \mathrm{pd}(F)$ and consequently M is strongly k-PGF with $k := \mathrm{pd}_R(F)$.*

Proof. (1) Since M is strongly n-PGF, there exists a short exact sequence of the form $0 \to M \to F \to M \to 0$, such that $\mathrm{pd}_R(F) \leqslant n$ and $\mathrm{Tor}^R_{n+1}(I, M) = 0$ for every injective module I. Since the exact sequence $0 \to N \to P_{n-1} \to \cdots \to P_0 \to M \to 0$

is a truncated projective resolution of M, there is a module Q such that the following diagram is commutative:

$$\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q & \longrightarrow & P_{n-1} \oplus P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_0 \oplus P_0 & \longrightarrow & F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & & & 0 & & 0 & &
\end{array}$$

Since $\mathrm{pd}_R(F) \leqslant n$, it follows that Q is projective module. We observe that $\mathrm{Tor}_1^R(I,N) = \mathrm{Tor}_{n+1}^R(I,M) = 0$ for every injective module I. Thus, by Proposition 3.6 (2), we conclude that the module N is strongly PGF.

(2) We consider a short exact sequence of the form $0 \to M \to F \to M \to 0$, such that $\mathrm{pd}_R(F) = k < \infty$. Consider a truncated projective resolution of the module M of length n

$$0 \to N \to P_{n-1} \to \cdots \to P_0 \to M \to 0.$$

Using (1) which we have already proved, we obtain that N is strongly PGF. By Proposition 3.6 (2), there exists a short exact sequence $0 \to N \to P \to N \to 0$ such that P is projective and $\mathrm{Tor}_i^R(I,N) = 0$ for every $i > 0$ and every injective module I. Then, $\mathrm{Tor}_{n+i}^R(I,M) = \mathrm{Tor}_i^R(I,N) = 0$ for every $i > 0$ and every injective module I. Let I be an injective module. The short exact sequence $0 \to N \to P \to N \to 0$ induces a long exact sequence of the form

$$\cdots \to \mathrm{Tor}_{i+1}^R(I,F) \to \mathrm{Tor}_{i+1}^R(I,M) \to \mathrm{Tor}_i^R(I,M) \to \mathrm{Tor}_i^R(I,F) \to \cdots$$

for every injective module I, where $i > 0$. The inequality $\mathrm{fd}_R(F) \leqslant \mathrm{pd}_R(F) = k$ implies that $\mathrm{Tor}_i^R(I,F) = 0$ whenever $i > k$. The long exact sequence above yields $\mathrm{Tor}_{i+1}^R(I,M) = \mathrm{Tor}_i^R(I,M)$ for every $i > k$. Thus, $\mathrm{Tor}_i^R(I,M) = \mathrm{Tor}_{n+i}^R(I,M) = \mathrm{Tor}_i^R(I,N) = 0$ for every $i > k$. By Proposition 3.10 (2), we conclude that M is strongly k-PGF. It remains to prove that $\text{PGF-dim}_R(M) = k$. By (1) we have $\text{PGF-dim}_R(M) \leqslant k$. Since $\mathrm{pd}_R(F) = k < \infty$, Proposition 3.16 implies that $\text{PGF-dim}_R(F) = \mathrm{pd}_R(F) = k$. Using Proposition 3.14 (1) and the short exact sequence $0 \to M \to F \to M \to 0$, we get the inequality $k = \text{PGF-dim}_R(F) \leqslant \text{PGF-dim}_R(M)$. We conclude that $\text{PGF-dim}_R(M) = k$. □

Proposition 3.18. *Let M be an R-module with finite PGF-dimension and let n be a nonnegative integer such that PGF-dim$_R(M) \leqslant n$. Then, $\mathrm{Tor}_i^R(I',M) = 0$ for every $i > n$ and every module I' with finite injective dimension.*

Proof. Since PGF modules are Gorenstein flat, the PGF dimension bounds the Gorenstein flat dimension. The result is known for modules of finite Gorenstein flat dimension (see [5, Theorem 3.14]). □

Proposition 3.19. *Let M be an R-module with finite PGF dimension. Then, the module M is strongly n-PGF if and only if there exists a short exact sequence of the form $0 \to M \to F \to M \to 0$, where $pd_R(F) \leqslant n$.*

Proof. Let M be a strongly n-PGF module. By definition, there exists a short exact sequence of the form $0 \to M \to F \to M \to 0$, where $\mathrm{pd}_R(F) \leqslant n$.

Conversely, let M be a module such that there exists a short exact sequence of the form $0 \to M \to F \to M \to 0$, where $\mathrm{pd}_R(F) \leqslant n$. By assumption, there exists a nonnegative integer k such that PGF-$\dim_R(M) \leqslant k$. Let I be an injective module. Then, Proposition 3.18 yields $\mathrm{Tor}_i^R(I, M) = 0$ for every $i > k$. The inequality $\mathrm{fd}_R(F) \leqslant \mathrm{pd}_R(F) \leqslant n$ implies that $\mathrm{Tor}_i^R(I, F) = 0$ for every $i > n$. The short exact sequence $0 \to M \to F \to M \to 0$ induces a long exact sequence of the form

$$\cdots \to \mathrm{Tor}_{i+1}^R(I, F) \to \mathrm{Tor}_{i+1}^R(I, M) \to \mathrm{Tor}_i^R(I, M) \to \mathrm{Tor}_i^R(I, F) \to \cdots,$$

where $i > 0$. We obtain that $\mathrm{Tor}_{i+1}^R(I, M) = \mathrm{Tor}_i^R(I, M)$ for every $i > n$. Equivalently, we have $\mathrm{Tor}_{n+1}^R(I, M) = \mathrm{Tor}_{n+i}^R(I, M)$ for every $i > 0$. Thus, letting $n+i > k$, we get $\mathrm{Tor}_{n+1}^R(I, M) = \mathrm{Tor}_{n+i}^R(I, M) = 0$ and the module M is strongly n-PGF. $\square$

Theorem 3.20. *Let M be an R-module and n a nonnegative integer. Then, PGF-$\dim_R(M) \leqslant n$ if and only if M is a direct summand of a strongly n-PGF module.*

Proof. If $n = 0$, then M is a PGF-module and the result holds by Theorem 3.5. We assume now that $0 <$ PGF-$\dim_R(M) \leqslant n$. By Theorem 3.15, there exists a short exact sequence of the form

$$0 \to M \to D \to G^0 \to 0,$$

where G^0 is PGF and $\mathrm{pd}_R(D) =$ PGF-$\dim_R(M) \leqslant n$. We consider now a truncated projective resolution of M of length n

$$0 \to G_{n-1} \to P_{n-1} \to \cdots \to P_0 \to M \to 0,$$

where P_i is projective for every i such that $0 \leqslant i \leqslant n-1$ and G_{n-1} is a PGF module (see [4, Proposition 2.2]). Let $G_0 = \mathrm{Ker}(P_0 \to M)$ and $G_i = \mathrm{Ker}(P_i \to P_{i-1})$ for every $i \geqslant 1$. Then, by Proposition 3.14 (2) and the short exact sequence $0 \to G_0 \to P_0 \to M \to 0$, we have PGF-$\dim_R(G_0) \leqslant n$. Using again Proposition 3.14 (2) and the short exact sequences $0 \to G_i \to P_i \to G_{i-1} \to 0$ where $1 \leqslant i \leqslant n-1$, an inductive argument on i shows that PGF-$\dim_R(G_i) \leqslant n$ for every i such that $0 \leqslant i \leqslant n-1$. We consider now a projective resolution of G_{n-1}

$$\cdots \to P_{n+2} \to P_{n+1} \to P_n \to G_{n-1} \to 0$$

and let $G_i = \mathrm{Im}(P_{i+1} \to P_i)$ for every $i \geqslant n$. Since G_{n-1} is a PGF module and the class of PGF modules is closed under kernels of epimorphisms, the short exact sequence $0 \to G_n \to P_n \to G_{n-1} \to 0$ implies that G_n is also PGF. Using induction on i and the short exact sequences $0 \to G_i \to P_i \to G_{i-1} \to 0$ for $i \geqslant n$, the same argument implies that G_i is a PGF-module for every $i \geqslant n-1$. We conclude that PGF-$\dim_R(G_i) \leqslant n$ for every $i \geqslant 0$.

Since G^0 is a PGF-module, by definition it admits a right projective resolution

$$0 \to G^0 \to P^0 \to P^1 \to P^2 \to \cdots.$$

By the definition of PGF modules, $G^i = \mathrm{Im}(P^{i-1} \to P^i)$ is a PGF module for every $i \geqslant 1$. Thus PGF-$\dim_R(G^i) = 0$ for every $i \geqslant 0$.

To summarize, we have the following short exact sequences

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ 0 & \longrightarrow & G^1 & \longrightarrow & P^1 & \longrightarrow & G^2 & \longrightarrow & 0 \\ 0 & \longrightarrow & G^0 & \longrightarrow & P^0 & \longrightarrow & G^1 & \longrightarrow & 0 \\ 0 & \longrightarrow & M & \longrightarrow & D & \longrightarrow & G^0 & \longrightarrow & 0 \\ 0 & \longrightarrow & G_0 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\ 0 & \longrightarrow & G_1 & \longrightarrow & P_1 & \longrightarrow & G_0 & \longrightarrow & 0 \\ 0 & \longrightarrow & G_2 & \longrightarrow & P_2 & \longrightarrow & G_1 & \longrightarrow & 0 \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

and the direct sum of them yields the short exact sequence $0 \to N \to Q \to N \to 0$, where $N = \bigoplus_{i \geqslant 0} G^i \bigoplus M \bigoplus_{j \geqslant 0} G_j$ and $Q = \bigoplus_{i \geqslant 0} P^i \bigoplus D \bigoplus_{j \geqslant 0} P_j$. Then, we obviously have $\text{pd}_R(Q) = \text{pd}_R(D) \leqslant n$. Using Proposition 3.13 we get

$$\text{PGF-dim}_R(N) = \sup_{i,j \geqslant 0}\{\text{PGF-dim}_R(G^i), \text{PGF-dim}_R(M), \text{PGF-dim}_R(G_j)\} \leqslant n.$$

Thus, Proposition 3.18 yields $\text{Tor}^R_{n+1}(I, N) = 0$ for every injective module I. We conclude that N is strongly n-PGF and M is a direct summand of N.

Conversely, let M be a direct summand of a strongly n-PGF module N. Then, Proposition 3.13 yields $\text{PGF-dim}_R(M) \leqslant \text{PGF-dim}_R(N)$. Since N is strongly n-PGF, Proposition 3.17 (2) implies that $\text{PGF-dim}_R(N) \leqslant n$. We conclude that $\text{PGF-dim}_R(M) \leqslant n$. □

Corollary 3.21. *Every strongly n-PGF module is strongly m-PGF for every positive integer $m > n$.*

Proof. Let M be a strongly n-PGF module. By definition, there exists a short exact sequence of the form $0 \to M \to F \to M \to 0$ where $\text{pd}_R(F) \leqslant n$ and $\text{Tor}^R_{n+1}(I, M) = 0$ for every injective module I. Let $m > n$ be an integer. Then $\text{pd}_R(F) \leqslant m$. Since the module M is strongly n-PGF, Theorem 3.20 implies that $\text{PGF-dim}_R(M) \leqslant n$. Thus, $\text{PGF-dim}_R(M) \leqslant m$ and using Proposition 3.18 we obtain that $\text{Tor}^R_{m+1}(I, M) = 0$ for every injective module I. By definition, M is a strongly m-PGF module for every integer $m > n$. □

Proposition 3.22. *Let M be a strongly n-PGF module, where $n > 0$. Then, there exists a short exact sequence*

$$0 \to K \to N \to M \to 0,$$

where N is a strongly PGF-module and $pd_R(K) = PGF\text{-}dim_R(M) - 1 \leqslant n - 1$.

Proof. Consider a truncated projective resolution of M of length n

$$0 \to G \to P_{n-1} \to \cdots \to P_0 \to M \to 0.$$

Since M is a strongly n-PGF module, Proposition 3.17 (1) implies that the module G is strongly PGF. By the definition of strongly PGF modules, there exists an exact sequence of the form

$$0 \to G \to Q \to \cdots \to Q \to G \to 0,$$

where all kernels are equal to G and Q is a projective module. Since every strongly PGF module is also PGF, Corollary 4.5 of [7] implies that the latter exact sequence remains exact after applying the functor $\text{Hom}_R(-, P)$ for every projective module P. We conclude that there exists a morphism of complexes:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & \cdots & \longrightarrow & Q & \longrightarrow & G & \longrightarrow & 0 \\
 & & {\scriptstyle =}\big\downarrow & & \big\downarrow & & & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & G & \longrightarrow & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

The unlabeled vertical arrows induce a quasi-isomorphism between the corresponding complexes. Hence, we may consider the associated mapping cone

$$0 \to Q \to Q \oplus P_{n-1} \to \cdots \to Q \oplus P_1 \to G \oplus P_0 \xrightarrow{\pi} M \to 0,$$

which is an exact sequence. The modules $Q, Q \oplus P_1, \ldots, Q \oplus P_{n-1}$ are clearly projective and the module $G \oplus P_0$ is a strongly PGF-module by Corollary 3.7 and Proposition 3.11. Now we set $N = G \oplus P_0$ and $K = \mathrm{Ker}(\pi)$. Thus, we obtain the short exact sequence $0 \to K \to N \to M \to 0$ where $\mathrm{pd}_R(K) \leqslant n-1$ and N is a strongly PGF-module.

Since M is strongly n-PGF, Theorem 3.20 yields $k := \text{PGF-dim}_R(M) \leqslant n$ and the same argument implies that $\mathrm{pd}_R(K) = k-1$. If $k = 0$, this is understood to mean $K = 0$. □

Proposition 3.23. *Let $0 \to N \to P \to M \to 0$ be an exact sequence of R-modules, where P is projective and PGF-dim$_R(M) = n < \infty$.*

(1) *If M is strongly PGF, then N is also strongly PGF.*
(2) *If $n \geqslant 1$ and M is strongly n-PGF, then N is strongly $(n-1)$-PGF and PGF-dim$_R(N) = n-1$.*

Proof. (1) Since M is strongly *PGF*, by Proposition 3.6 (3) there exists a short exact sequence of the form $0 \to M \to Q \to M \to 0$, where Q is projective and $\mathrm{Tor}_i^R(I, M) = 0$ for every $i > 0$ and every injective module I. Moreover, there exists a module Q' such that the following diagram is commutative:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & Q' & \longrightarrow & P \oplus P & \longrightarrow & Q & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Since the modules P and Q are projectives, Q' is also projective. Now, let I be an injective module. The short exact sequence $0 \to N \to P \to M \to 0$ induces a long exact sequence of the form

$$\cdots \to \mathrm{Tor}_{i+1}^R(I, M) \to \mathrm{Tor}_i^R(I, N) \to \mathrm{Tor}_i^R(I, P) \to \cdots,$$

where $i > 0$. Then, $\mathrm{Tor}_i^R(I, N) = 0$ for every $i > 0$ and every injective module I. Thus, by Proposition 3.6 (3), N is strongly PGF.

(2) Since M is strongly n-PGF, by Proposition 3.10 (2) there exists a short exact sequence of the form $0 \to M \to F \to M \to 0$, where $\mathrm{pd}_R(F) \leqslant n$ and

$\operatorname{Tor}_i^R(I,M) = 0$ for every $i > n$ and every injective module I. We also have $\text{PGF-dim}_R(M) = n$, thus Proposition 3.17 (2) yields $\text{pd}_R(F) = n$. Moreover, there exists a module F' such that the following diagram is commutative:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F' & \longrightarrow & P \oplus P & \longrightarrow & F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Since the module P is projective, $\text{pd}_R(F') = n-1$. Now, let I be an injective module. The short exact sequence $0 \to N \to P \to M \to 0$ induces a long exact sequence of the form

$$\cdots \to \operatorname{Tor}_{i+1}^R(I,M) \to \operatorname{Tor}_i^R(I,N) \to \operatorname{Tor}_i^R(I,P) \to \cdots,$$

where $i > 0$. Then, $\operatorname{Tor}_i^R(I,N) = 0$ for every $i > n-1$ and every injective module I. Thus, by Proposition 3.10 (2), N is strongly $(n-1)$-PGF. Furthermore, since $\text{PGF-dim}_R(M) = n$, using Proposition 3.6 by [4] and the short exact sequence $0 \to N \to P \to M \to 0$ we get $\text{PGF-dim}_R(N) = n-1$. □

Proposition 3.24. *Let $0 \to N \xrightarrow{\alpha} P \xrightarrow{\beta} M \to 0$ be an exact sequence of R-modules, where $pd_R(P) = n < \infty$ and N is a strongly PGF module. Then, M is strongly $(n+1)$-PGF.*

Proof. Since N is strongly PGF, there exists a short exact sequence of the form $0 \to N \xrightarrow{\gamma} Q \xrightarrow{\delta} N \to 0$, where Q is a projective module and $\operatorname{Ext}_R^1(N,Q') = 0$ for every module Q' with finite projective dimension (see Remark 3.2). Since $\text{pd}_R(P) = n < \infty$, we have the following short exact sequence

$$0 \to \operatorname{Hom}_R(N,P) \xrightarrow{\delta^*} \operatorname{Hom}_R(Q,P) \xrightarrow{\gamma^*} \operatorname{Hom}_R(N,P) \to 0.$$

Thus, there exists a morphism $\epsilon : Q \to P$ such that $\alpha = \epsilon \circ \gamma$.

We consider now the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \xrightarrow{\gamma} & Q & \xrightarrow{\delta} & N & \longrightarrow & 0 \\
 & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\zeta}\downarrow & & {\scriptstyle\alpha}\downarrow & & \\
0 & \longrightarrow & P & \xrightarrow{i} & P \oplus P & \xrightarrow{\pi} & P & \longrightarrow & 0
\end{array}$$

where $\zeta = (\epsilon, \alpha \circ \delta)$ and i and π are the canonical injection and projection respectively. The Snake Lemma yields the following short exact sequence

$$0 \to M \to (P \oplus P)/\operatorname{Im}(\zeta) \to M \to 0.$$

We observe that ζ is a monomorphism, thus $\text{pd}_R((P \oplus P)/\operatorname{Im}(\zeta)) \leqslant n+1$. Since $\text{pd}_R(P) = n < \infty$, using Proposition 3.14 (3), Proposition 3.16 and the short exact sequence $0 \to N \xrightarrow{\alpha} P \xrightarrow{\beta} M \to 0$ we obtain that PGF-$\dim_R(M) \leqslant n+1$. Then, Proposition 3.18 implies that $\mathrm{Tor}^R_{n+2}(I, M) = 0$ for every injective module I. Consequently, the module M is strongly $(n+1)$-PGF. □

Proposition 3.25. *Let $0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} P \to 0$ be an exact sequence of R-modules, where $pd_R(P) = n < \infty$.*

1. *If $n > 0$ and M is strongly PGF, then N is strongly $(n-1)$-PGF.*
2. *If P is projective, then N is strongly PGF if and only if M is strongly PGF.*

Proof. Let M be a strongly PGF module. By Proposition 3.6 (3), there exists a short exact sequence of the form $0 \to M \xrightarrow{\gamma} Q \xrightarrow{\delta} M \to 0$, where Q is a projective module and $\mathrm{Tor}^R_i(I, M) = 0$ for every $i > 0$ and every injective module I. We also have $\mathrm{Ext}^1_R(M, Q') = 0$ for every module Q' with finite projective dimension (see Remark 3.2). Since $\mathrm{pd}_R(P) = n < \infty$, we have the following short exact sequence

$$0 \to \mathrm{Hom}_R(M, P) \xrightarrow{\delta^*} \mathrm{Hom}_R(Q, P) \xrightarrow{\gamma^*} \mathrm{Hom}_R(M, P) \to 0.$$

Thus, there exists a morphism $\epsilon : Q \to P$ such that $\beta = \epsilon \circ \gamma$.

We consider now the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{\ \gamma\ } & Q & \xrightarrow{\ \delta\ } & M & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle\beta} & & \big\downarrow{\scriptstyle\zeta} & & \big\downarrow{\scriptstyle\beta} & & \\
0 & \longrightarrow & P & \xrightarrow{\ i\ } & P \oplus P & \xrightarrow{\ \pi\ } & P & \longrightarrow & 0
\end{array}$$

where $\zeta = (\epsilon, \beta \circ \delta)$ and i and π are the canonical injection and projection respectively. The Snake Lemma yields the following short exact sequence

$$0 \to N \to \mathrm{Ker}(\zeta) \to N \to 0.$$

We observe that ζ is an epimorphism.

(1) If $n > 0$, then $\mathrm{pd}_R(\mathrm{Ker}(\zeta)) = n-1$. Since $\mathrm{pd}_R(P) = n < \infty$, using Proposition 3.14 (2), Proposition 3.16 and the short exact sequence $0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} P \to 0$ we obtain that PGF-$\dim_R(N) \leqslant n-1$. Then, Proposition 3.18 implies that $\mathrm{Tor}^R_n(I, N) = 0$ for every injective module I. Thus, N is strongly $(n-1)$-PGF.

(2) If M is strongly PGF and P is projective, then $\mathrm{Ker}(\zeta)$ is also projective. Let I be an injective module. Then, the short exact sequence $0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} P \to 0$ induces a long exact sequence of the form

$$\cdots \to \mathrm{Tor}^R_{i+1}(I, P) \to \mathrm{Tor}^R_i(I, N) \to \mathrm{Tor}^R_i(I, M) \to \mathrm{Tor}^R_i(I, P) \to \cdots,$$

where $i > 0$. Since P is projective, $\mathrm{Tor}^R_i(I, N) = \mathrm{Tor}^R_i(I, M) = 0$ for every $i > 0$. Thus, Proposition 3.6 (3) implies that N is strongly PGF. For the other implication of (2), we assume that N is strongly PGF. Since P is projective, the short exact sequence $0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} P \to 0$ splits. Consequently, $M = N \oplus P$, which is also a strongly PGF module by Corollary 3.7 and Proposition 3.11. □

Corollary 3.26. *The following are equivalent.*

1. *Every PGF module is strongly PGF.*
2. *Every module M such that PGF-$\dim_R(M) \leqslant 1$ is a strongly 1-PGF module.*

Proof. (1) ⇒ (2) : Let M be a module such that PGF-$\dim_R(M) \leqslant 1$. We consider a truncated projective resolution of M of length 1, $0 \to N \to P \to M \to 0$. Proposition 2.2 of [4] implies that N is PGF. Thus, N is strongly PGF by (1). By Proposition 3.24, M is strongly 1-PGF.

(2) ⇒ (1) : Let M be a PGF module. Then by (2), M is strongly 1-PGF and there exists a short exact sequence of the form $0 \to M \to F \to M \to 0$, where $\mathrm{pd}_R(F) \leqslant 1$. By Proposition 3.14 (1), the module F is also PGF. Since $\mathrm{pd}_R(F) < \infty$, Proposition 3.16 implies that F is projective. Moreover, Proposition 3.18 yields $\mathrm{Tor}_i^R(I, M) = 0$ for every $i > 0$ and every injective module I. Thus, by Proposition 3.6 (3), M is strongly PGF. □

4. The stability of PGF modules

In this section we prove that the class of PGF modules is stable under the very Gorenstein process used to define PGF modules. In particular, for every exact sequence of PGF modules

$$\mathbf{G} = \cdots \to G_1 \to G_0 \to G^0 \to G^1 \to \cdots$$

such that $M \cong \mathrm{Im}(G_0 \to G^0)$ and such that the functor $H \otimes -$ preserves the exactness of $\mathbf{G}$ whenever H is a Gorenstein injective module, we obtain that the module M is PGF.

Throughout this section we use the following notation.

We denote by $\mathrm{PGF}^{(2)}(R)$ (respectively, $\mathrm{PGF}^{(2)}_{\mathcal{I}}(R)$) the subcategory of the R-modules M for which there exists an exact sequence of PGF modules

$$\mathbf{G} = \cdots \to G_1 \to G_0 \to G^0 \to G^1 \to \cdots ,$$

such that $M \cong \mathrm{Im}(G_0 \to G^0)$ and such that $H \otimes -$ (respectively, $I \otimes -$) preserves the exactness of $\mathbf{G}$ whenever H is a Gorenstein injective module (respectively, I is an injective module).

Since every injective module is also Gorenstein injective, we have the following inclusions

$$\mathrm{PGF}(R) \subseteq \mathrm{PGF}^{(2)}(R) \subseteq \mathrm{PGF}^{(2)}_{\mathcal{I}}(R).$$

Also, we denote by S-$\mathrm{PGF}^{(2)}_{\mathcal{I}}(R)$ the subcategory of the R-modules M for which there exists a short exact sequence of the form $0 \to M \to G \to M \to 0$, where G is a PGF module, such that $I \otimes -$ preserves the exactness of this sequence whenever I is an injective module.

Proposition 4.1. *Let $M \in PGF^{(2)}_{\mathcal{I}}(R)$. Then, $Tor_i^R(I', M) = 0$ for every $i > 0$ and every module I' with finite injective dimension.*

Proof. Let $M \in \mathrm{PGF}^{(2)}_{\mathcal{I}}(R)$ and I' be a module with finite injective dimension. By definition, there exists an exact sequence of PGF modules

$$\mathbf{G} = \cdots \to G_1 \to G_0 \to G^0 \to G^1 \to \cdots ,$$

where $M \cong \mathrm{Im}(G_0 \to G^0)$ and the sequence $I \otimes \mathbf{G}$ is exact for every injective module I. Let I' be a module with $\mathrm{id}_R(I') = n < \infty$. We proceed by induction on $n \geqslant 0$. Consider the short exact sequence $0 \to K \to G_0 \to M \to 0$, where $K = \mathrm{Im}(G_1 \to G_0) \in \mathrm{PGF}^{(2)}_{\mathcal{I}}(R)$. Then, for every injective module I' we have the following short exact sequence

$$0 \to I' \otimes K \to I' \otimes G_0 \to I' \otimes M \to 0.$$

Since the module G_0 is PGF, Proposition 2.7 (2) implies that $\operatorname{Tor}_i^R(I', G_0) = 0$ for every $i > 0$. Thus, the short exact sequence $0 \to K \to G_0 \to M \to 0$ induces a long exact sequence of the form

$$\cdots \to \operatorname{Tor}_1^R(I', G_0) \to \operatorname{Tor}_1^R(I', M) \to I' \otimes K \to I' \otimes G_0 \to I' \otimes M \to 0,$$

which implies that $\operatorname{Tor}_1^R(I', M) = 0$. Consequently, we have proved the fact that $\operatorname{Tor}_1^R(I', M) = 0$ for every module $M \in \mathrm{PGF}_{\mathcal{I}}^{(2)}(R)$ and every injective module I'. Moreover, the long exact sequence

$$\cdots \to \operatorname{Tor}_{i+1}^R(I', G_0) \to \operatorname{Tor}_{i+1}^R(I', M) \to \operatorname{Tor}_i^R(I', K) \to \operatorname{Tor}_i^R(I', G_0) \to \cdots,$$

where $i > 0$, yields $\operatorname{Tor}_{i+1}^R(I', M) = \operatorname{Tor}_i^R(I', K)$ for every $i > 0$. Thus, using induction on i and the fact that K lies in $\mathrm{PGF}_{\mathcal{I}}^{(2)}(R)$, we obtain that $\operatorname{Tor}_i^R(I', M) = 0$ for every $i > 0$ and every injective module I'.

We suppose now that $n \geqslant 1$ and consider an injective resolution of I' of length n

$$0 \to I' \to I_0 \to I_1 \to \cdots \to I_n \to 0.$$

Then, the module $J = \operatorname{Im}(I_0 \to I_1)$ has injective dimension at most $n-1$ and our inductive hypothesis implies that $\operatorname{Tor}_i^R(J, M) = 0$ for every $i > 0$. Thus, the short exact sequence $0 \to I' \to I_0 \to J \to 0$ induces a long exact sequence of the form

$$\cdots \to \operatorname{Tor}_{i+1}^R(J, M) \to \operatorname{Tor}_i^R(I', M) \to \operatorname{Tor}_i^R(I_0, M) \to \cdots,$$

where $i > 0$, from which we obtain that $\operatorname{Tor}_i^R(I', M) = 0$ for every $i > 0$. □

The following proposition gives a characterization of the subcategory $\text{S-PGF}_{\mathcal{I}}^{(2)}(R)$.

Proposition 4.2. *For every module M, the following are equivalent:*

1. *$M \in \text{S-PGF}_{\mathcal{I}}^{(2)}(R)$.*
2. *There exists a short exact sequence $0 \to M \to G \to M \to 0$ such that G is a PGF module and $\operatorname{Tor}_1^R(I, M) = 0$ for every injective module I.*
3. *There exists a short exact sequence $0 \to M \to G \to M \to 0$ such that G is a PGF module and $\operatorname{Tor}_i^R(I, M) = 0$ for every $i > 0$ and every injective module I.*
4. *There exists a short exact sequence $0 \to M \to G \to M \to 0$ such that G is a PGF module and $\operatorname{Tor}_1^R(I', M) = 0$ for every module I' with finite injective dimension.*
5. *There exists a short exact sequence $0 \to M \to G \to M \to 0$ such that G is a PGF module and $\operatorname{Tor}_i^R(I', M) = 0$ for every $i > 0$ and every module I' with finite injective dimension.*
6. *There exists a short exact sequence $0 \to M \to G \to M \to 0$ such that G is a PGF module and $I' \otimes -$ preserves exactness of this sequence for every module I' with finite injective dimension.*

Proof. This follows immediately from the definition of the class $\text{S-PGF}_{\mathcal{I}}^{(2)}(R)$ and Proposition 4.1, using standard arguments. □

Proposition 4.3. *Every module in $\mathrm{PGF}_{\mathcal{I}}^{(2)}(R)$ is a direct summand of a module in $\text{S-PGF}_{\mathcal{I}}^{(2)}(R)$.*

Proof. Let M be a module in $\mathrm{PGF}^{(2)}_{\mathcal{I}}(R)$. Then, there exists an exact sequence of PGF modules

$$\mathbf{G} = \cdots \to G_1 \xrightarrow{d_1^G} G_0 \xrightarrow{d_0^G} G_{-1} \xrightarrow{d_{-1}^G} G_{-2} \to \cdots$$

such that $M \cong \mathrm{Im}(d_0^G)$ and such that the sequence $I \otimes \mathbf{G}$ is exact for every injective module I.

For every $n \in \mathbb{Z}$, we denote by $\Sigma^n \mathbf{G}$ the exact sequence obtained from $\mathbf{G}$ by increasing all indices by n: $(\Sigma^n \mathbf{G})_i = G_{i-n}$ and $d_i^{\Sigma^n G} = d_{i-n}^G$ for every $i \in \mathbb{Z}$.

We consider now the exact sequence

$$\bigoplus_{n\in\mathbb{Z}}(\Sigma^n \mathbf{G}) = \cdots \to \bigoplus_{i\in\mathbb{Z}} G_i \xrightarrow{\bigoplus_{i\in\mathbb{Z}} d_i^G} \bigoplus_{i\in\mathbb{Z}} G_i \xrightarrow{\bigoplus_{i\in\mathbb{Z}} d_i^G} \bigoplus_{i\in\mathbb{Z}} G_i \to \cdots$$

where $\mathrm{Im}(\bigoplus_{i\in\mathbb{Z}} d_i^G) \cong \bigoplus_{i\in\mathbb{Z}} \mathrm{Im} d_i^G$ and so M is a direct summand of $\mathrm{Im}(\bigoplus_{i\in\mathbb{Z}} d_i^G)$. Since the class $\mathrm{PGF}(R)$ is closed under direct sums, we obtain that the module $\bigoplus_{i\in\mathbb{Z}} G_i$ is also PGF.

Moreover, by Proposition 20.2 (3) of [1],we have the isomorphism of complexes

$$I \otimes (\bigoplus_{n\in\mathbb{Z}}(\Sigma^n \mathbf{G})) \cong \bigoplus_{n\in\mathbb{Z}}(I \otimes \Sigma^n \mathbf{G})$$

which is an exact sequence for every injective module I. Thus, $\mathrm{Im}(\bigoplus_{i\in\mathbb{Z}} d_i^G)$ lies in $\text{S-PGF}^{(2)}_{\mathcal{I}}(R)$ and M is a direct summand of this module. □

Definition 4.4. *Let M be a module in S-$PGF^{(2)}_{\mathcal{I}}(R)$. We say that a module N is an M-type module if there exists a short exact sequence of the form*

$$0 \to M \to N \to G \to 0,$$

where G is a PGF module.

Proposition 4.5. *Let M be a module in S-$PGF^{(2)}_{\mathcal{I}}(R)$ and N be an M-type module. Then, the following hold.*

1. *$Tor_i^R(I, N) = 0$ for every $i > 0$ and every injective module I.*
2. *There exists an exact sequence of the form $0 \to N \to P \to K \to 0$, where P is projective, K is an M-type module and the functor $I \otimes -$ preserves the exactness of this sequence for every injective module I.*

Proof. (1) Let I be an injective module. Since N is an M-type module, there exists a short exact sequence of the form $0 \to M \to N \to G \to 0$, where G is PGF, which induces a long exact sequence of the form

$$\cdots \to \mathrm{Tor}_i^R(I, M) \to \mathrm{Tor}_i^R(I, N) \to \mathrm{Tor}_i^R(I, G) \to \cdots,$$

where $i > 0$. Since $M \in \text{S-PGF}^{(2)}_{\mathcal{I}}(R)$, by Proposition 4.2 (3), we have $\mathrm{Tor}_i^R(I, M) = 0$ for every $i > 0$. Moreover, Proposition 2.7 (2) yields $\mathrm{Tor}_i^R(I, G) = 0$ for every $i > 0$. Consequently, $\mathrm{Tor}_i^R(I, N) = 0$, for every $i > 0$ and every injective module I.

(2) Since $M \in \text{S-PGF}^{(2)}_{\mathcal{I}}(R)$, there exists a short exact sequence of the form $0 \to M \to G' \to M \to 0$, where G' is a PGF module. Since N is an M-type module, there exists also a short exact sequence of the form $0 \to M \to N \to G \to 0$, where G is a PGF module. Consider the pushout diagram of the above short exact sequences:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & M & \longrightarrow & G' & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{=} & & \\
0 & \longrightarrow & N & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & G & \xrightarrow{=} & G & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since the class PGF(R) is closed under extensions, using the short exact sequence $0 \to G' \to F \to G \to 0$, we obtain that the module F is also PGF. Thus, there exists a short exact sequence of the form $0 \to F \to P \to F' \to 0$, where P is projective and F' is PGF. Consider now the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \\
 & & {=}\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & P & \longrightarrow & K & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & F' & \xrightarrow{=} & F' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since F' is PGF, the module K is an M-type. By (1) we have $\mathrm{Tor}_1^R(I, K) = 0$ for every injective module I. Thus, the sequence $0 \to I \otimes N \to I \otimes P \to I \otimes K \to 0$ is exact for every injective module I. □

Corollary 4.6. *Let M be a module in S-PGF$_{\mathcal{I}}^{(2)}(R)$ and N be an M-type module. Then, N is PGF.*

Proof. Since N is an M-type module, Proposition 4.5 (2) implies that there exists a short exact sequence of the form $0 \to N \to P^0 \to K \to 0$, where P^0 is projective, K is an M-type and the functor $I \otimes -$ preserves the exactness of this sequence for every injective module I. The iteration of this process yields an exact sequence

$$0 \to N \to P^0 \to P^1 \to P^2 \to \cdots ,$$

where P^i is projective for every $i > 0$ and the functor $I \otimes -$ preserves the exactness of this sequence for every injective module I. Using Proposition 4.5 (1), we also have $\mathrm{Tor}_i^R(I, N) = 0$ for every $i > 0$ and every injective module I. Thus, Proposition 2.7 (2) implies that N is PGF. □

Theorem 4.7. *$PGF(R) = PGF^{(2)}(R) = PGF_{\mathcal{I}}^{(2)}(R)$.*

Proof. It suffices to prove that $\mathrm{PGF}_{\mathcal{I}}^{(2)}(R) \subseteq \mathrm{PGF}(R)$. Since the class $\mathrm{PGF}(R)$ is closed under direct summands, by Proposition 4.3 it suffices to prove that $\text{S-}\mathrm{PGF}_{\mathcal{I}}^{(2)}(R) \subseteq \mathrm{PGF}(R)$. Let M be a module in $\text{S-}\mathrm{PGF}_{\mathcal{I}}^{(2)}(R)$. Thus, there exists a short exact sequence of the form $0 \to M \to G \to M \to 0$ such that the module G is PGF. Since G is PGF, there exists a short exact sequence of the form $0 \to G \to P \to G' \to 0$, where P is projective and G' is also a PGF module. Consider the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & G & \longrightarrow & M & \longrightarrow & 0 \\
 & & {\scriptstyle =}\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & P & \longrightarrow & U & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G' & \xrightarrow{=} & G' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Then, U is an M-type module and Corollary 4.6 implies that U is also a PGF module. Since the class $\mathrm{PGF}(R)$ is closed under kernels of epimorphisms, the short exact sequence $0 \to M \to U \to G' \to 0$ implies that M is also PGF. □

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