ON K-ABSOLUTELY PURE COMPLEXES

IOANNIS EMMANOUIL AND ILIAS KAPERONIS

ABSTRACT. In this paper, we examine the class of K-absolutely pure complexes. These are the complexes which are right orthogonal in the homotopy category $\mathbf{K}(R)$ to the acyclic complexes of pure-projective modules. The class $\mathbf{K}$-abspure of these complexes is preenveloping in $\mathbf{K}(R)$; in fact, a Bousfield localization exists for the embedding $\mathbf{K}\text{-abspure} \subseteq \mathbf{K}(R)$ and the quotient $\mathbf{K}(R)/\mathbf{K}\text{-abspure}$ is equivalent to the homotopy category of acyclic complexes of pure-projective modules. We examine the role of K-absolutely pure complexes in the pure derived category $\mathbf{D}_{pure}(R)$ and show that $\mathbf{K}$-abspure is the isomorphic closure of the class of K-injective complexes therein. We explore the relevance of strongly fp-injective modules in the study of K-absolutely pure complexes and characterize the K-absolutely pure complexes of strongly fp-injective modules. Finally, we show that a K-absolutely pure complex of strongly fp-injective modules admits a K-injective complex of injective modules as a $\mathbf{K}(\text{PInj})$-preenvelope, in the case where the ring is left coherent. The notion of K-absolute purity is dual to the notion of K-flatness in the homotopy category, in a way analogous to the duality between (strongly) fp-injectivity and flatness in the module category.

CONTENTS

0. INTRODUCTION

The study of resolutions of unbounded complexes was initiated by Avramov and Foxby [1] and, independently, by Spaltenstein [22]. The use of unbounded complexes has found many applications in representation theory and algebraic geometry. The proper analogues of the concepts of projective and injective modules in the setting of unbounded complexes consist of the notions of K-projective and K-injective complexes: A complex of R-modules X is called K-projective if the complex of abelian groups $\mathrm{Hom}_R(X, C)$ is acyclic for any acyclic complex C. Dually, a complex Y is called K-injective if the complex of abelian groups $\mathrm{Hom}_R(C, Y)$ is acyclic for any acyclic complex C. Denoting by $\mathbf{K}$-proj and $\mathbf{K}$-inj the associated triangulated subcategories of the homotopy category $\mathbf{K}(R)$, we obtain two Bousfield localizing pairs in $\mathbf{K}(R)$, that we refer to as *standard*:

$$(\mathbf{K}\text{-proj}, \mathbf{K}_{ac}(R)) \quad \text{and} \quad (\mathbf{K}_{ac}(R), \mathbf{K}\text{-inj}).$$

Research supported by an HFRI/Greece grant.

Here, $\mathbf{K}_{ac}(R)$ is the triangulated subcategory of acyclic complexes. As a consequence of the existence of these two Bousfield localizing pairs, we obtain the projective and the injective model for the derived category $\mathbf{D}(R) = \mathbf{K}(R)/\mathbf{K}_{ac}(R)$.

The notion of purity is fundamental in the study of various questions in the category of modules. It may be used in order to define the subcategory $\mathbf{K}_{pac}(R) \subseteq \mathbf{K}_{ac}(R)$ of pure acyclic complexes. The projective (resp. injective) objects for the pure exact structure in the category of modules are the pure-projective (resp. pure-injective) modules. As shown in [24, §5], there are two Bousfield localizing pairs analogous to the standard ones:

$$(\mathbf{K}(\text{PProj}), \mathbf{K}_{pac}(R)) \quad \text{and} \quad (\mathbf{K}_{pac}(R), \mathbf{K}(\text{PInj})).$$

Here, $\mathbf{K}(\text{PProj})$ and $\mathbf{K}(\text{PInj})$ are the homotopy categories of pure-projective and pure-injective modules respectively. As a consequence of the existence of these two Bousfield localizing pairs, we obtain the pure-projective and the pure-injective model for the pure derived category $\mathbf{D}_{pure}(R) = \mathbf{K}(R)/\mathbf{K}_{pac}(R)$.

Following Spaltenstein [22], we say that a complex of left R-modules F is K-flat if the complex of abelian groups $L \otimes_R F$ is acyclic for any acyclic complex of right R-modules L. The triangulated subcategory $\mathbf{K}$-flat of K-flat complexes contains $\mathbf{K}$-proj and forms the left hand side of a Bousfield localizing pair $(\mathbf{K}\text{-flat}, \mathbf{K}_{ac}(\text{PInj}))$ in $\mathbf{K}(R)$; cf. [7]. The implications of the existence of this Bousfield localizing pair for the (K-flat) derived category of R are analysed in [10]. We may present schematically the relation of the class $\mathbf{K}$-flat to other classes of complexes, in the form of the following (periodic) diagram of triangulated subcategories of the homotopy category $\mathbf{K}(R)$

$$\begin{array}{ccccccccccccc}
\mathbf{K}_{ac}(R) & & & & \mathbf{K}\text{-flat} & & & & \mathbf{K}(\text{PProj}) & & & & \mathbf{K}_{ac}(R) & & \\
& \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & \\
& & \mathbf{K}_{pac}(R) & & & & \mathbf{K}\text{-proj} & & & & \mathbf{K}_{ac}(\text{PProj}) & & & & \mathbf{K}_{pac}(R)
\end{array}$$

Here, all arrows are inclusions, any category in the bottom row of the diagram is the intersection of the two categories that sit immediately above it and any category in the top row of the diagram is the smallest triangulated subcategory of $\mathbf{K}(R)$ containing the two subcategories that sit immediately below it.

In this paper, we consider those complexes X, for which the complex of abelian groups $\text{Hom}_R(C, X)$ is acyclic for any acyclic complex of pure-projective modules C; we call these complexes K-absolutely pure. The associated triangulated subcategory $\mathbf{K}$-abspure forms the right hand side of a Bousfield localizing pair $(\mathbf{K}_{ac}(\text{PProj}), \mathbf{K}\text{-abspure})$ in $\mathbf{K}(R)$. The notion of K-absolute purity is formally dual to the notion of K-flatness in the homotopy category $\mathbf{K}(R)$, in a way analogous to the duality between absolute purity (fp-injectivity) and flatness in the module category R-Mod. We prove that a K-absolutely pure complex of pure-injective modules is necessarily K-injective. In fact, K-absolutely pure complexes play the role of K-injective complexes in the pure derived category $\mathbf{D}_{pure}(R)$; it turns out that $\mathbf{K}$-abspure is the closure under isomorphisms of $\mathbf{K}$-inj in $\mathbf{D}_{pure}(R)$, whereas K-absolutely pure resolutions are unique up to isomorphism and behave functorially in $\mathbf{D}_{pure}(R)$. We provide examples of K-absolutely pure complexes, by considering the strongly fp-injective modules defined in [17]; these are the modules that annihilate the functors $\text{Ext}^n_R(C, \_)$ for all finitely presented modules C and all $n \geq 1$. We show that a module is strongly fp-injective if and only if the complex consisting of that module in a single degree and zeroes elsewhere is K-absolutely pure. We characterize the acyclic K-absolutely pure complexes of strongly fp-injective modules, in a way that parallels the characterization of the acyclic K-flat complexes of flat modules by Neeman [19]. We also prove that, in the case of a left coherent ring, the $\mathbf{K}(\text{PInj})$-preenvelope of a complex of (strongly) fp-injective modules may be chosen to be a complex of injective modules. This result is analogous to the fact that any (strongly) fp-injective module over a left coherent ring admits a pure-injective preenvelope by an injective module.

The contents of the paper are as follows: In Section 1, we detail some preliminary notions which are used throughout the text. In Section 2, we define K-absolutely pure complexes and present certain basic properties of them, illustrating the dual role played by the class of these complexes in the homotopy category of the ring, compared to the class of K-flat complexes. In the next section, we show that the class of K-absolutely pure complexes is the closure under isomorphisms of the class of K-injective complexes in the pure derived category $\mathbf{D}_{pure}(R)$ and examine the K-absolutely pure resolutions of complexes therein. In Section 4, we study the relevance of the class of strongly fp-injective modules in the study of K-absolutely pure complexes, provide some interesting examples of such complexes and characterize the K-absolutely pure complexes of strongly fp-injective modules, in a way analogous to the description of K-flat complexes of flat modules. Finally, in Section 5, we examine the $\mathbf{K}$(PInj)-preenvelopes of K-absolutely pure complexes of strongly fp-injective modules and show that, in the case of a left coherent ring, these preenvelopes may be chosen to be K-injective complexes consisting of injective modules.

1. Preliminary notions

In this section, we collect certain basic notions and preliminary results that will be used in the sequel. Throughout this paper, R is a fixed unital associative ring. Unless otherwise specified, all modules are left R-modules and all complexes are chain complexes of (left) R-modules.

Purity. The reader may consult [25] and the monograph [14] for a detailed account on this notion. A short exact sequence of modules

$$0 \longrightarrow M' \xrightarrow{\iota} M \xrightarrow{p} M'' \longrightarrow 0$$

is called pure exact if it remains exact upon applying the functor $\mathrm{Hom}_R(P, \_)$ for any finitely presented module P. In that case, we say that ι (resp. p) is a pure monomorphism (resp. a pure epimorphism). An acyclic complex X with kernels $(Z_nX)_n$ is called pure acyclic if the associated short exact sequences of modules

$$0 \longrightarrow Z_nX \longrightarrow X_n \longrightarrow Z_{n-1}X \longrightarrow 0$$

are pure exact for all n.

A module P is called pure-projective if any pure epimorphism p as above induces an epimorphism of abelian groups $p_* : \mathrm{Hom}_R(P, M) \longrightarrow \mathrm{Hom}_R(P, M'')$. All finitely presented modules are pure projective. Since any module is a pure epimorphic image of a suitable direct sum of finitely presented modules, it follows that the pure-projective modules are precisely the direct summands of direct sums of finitely presented modules. Dually, a module I is called pure-injective if any pure monomorphism ι as above induces an epimorphism of abelian groups $\iota^* : \mathrm{Hom}_R(M, I) \longrightarrow \mathrm{Hom}_R(M', I)$. Examples of pure-injective modules can be obtained, by using the Pontryagin duality functor D from the category of left (resp. right) modules to the category of right (resp. left) modules, which is defined by letting $DM = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. For any right module N the left module DN is pure-injective, whereas the canonical embedding $M \longrightarrow D^2M$ is pure for any module M. Consequently, the pure-injective modules are precisely the direct summands of modules of the form DN for a right module N.

A module M is flat if and only if any epimorphism $P \longrightarrow M$ is pure. Dually, the class of absolutely pure modules consists of those modules M, which are such that any monomorphism $M \longrightarrow I$ is pure. An equivalent characterization of absolutely pure modules may be obtained in terms of the class of finitely presented modules: A module M is absolutely pure if and only if $\text{Ext}^1_R(C, M) = 0$ for any finitely presented module C. For that reason, absolutely pure modules are also called fp-injective; cf. [23].

Complexes and Bousfield localization. The homotopy category $\mathbf{K}(R)$ has objects all complexes of modules and morphisms the homotopy classes of chain complex maps between them. For any class $\mathfrak{X}$ of modules, we denote by $C(\mathfrak{X})$ the class of those complexes consisting of modules from $\mathfrak{X}$ in each degree and let $\mathbf{K}(\mathfrak{X})$ be the closure under homotopy equivalence of $C(\mathfrak{X})$ in the full category of complexes.

If X, Y are two complexes, then the triviality of the group $\text{Hom}_{\mathbf{K}}(X, Y)$ is equivalent to the assertion that any chain complex map $X \longrightarrow Y$ is null-homotopic. In that case, we say that X is left orthogonal to Y and Y is right orthogonal to X in $\mathbf{K}(R)$. If $\mathfrak{A}$ is a class of chain complexes, then the left orthogonal of $\mathfrak{A}$ in $\mathbf{K}(R)$ is the class ${}^{\perp}\mathfrak{A}$ consisting of all complexes X, which are left orthogonal to all complexes in $\mathfrak{A}$. The right orthogonal of $\mathfrak{A}$ in $\mathbf{K}(R)$ is the class $\mathfrak{A}^{\perp}$ consisting of all complexes Y, which are right orthogonal to all complexes in $\mathfrak{A}$. A triangulated subcategory of $\mathbf{K}(R)$ is thick if it closed under direct summands. A pair of thick subcategories $(\mathfrak{A}, \mathfrak{B})$ of $\mathbf{K}(R)$ is a Bousfield localizing pair in $\mathbf{K}(R)$ if $\mathfrak{A} \subseteq^{\perp} \mathfrak{B}$ (or, equivalently, $\mathfrak{B} \subseteq \mathfrak{A}^{\perp}$) and for any complex X there exists a distinguished triangle in $\mathbf{K}(R)$

$$A \longrightarrow X \longrightarrow B \longrightarrow SA,$$

where $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$. In that case, the inclusion functor $\mathfrak{A} \hookrightarrow \mathbf{K}(R)$ (resp. $\mathfrak{B} \hookrightarrow \mathbf{K}(R)$) admits a right (resp. left) adjoint. Two comprehensive references for (Bousfield) localization in triangulated categories are [18, Chapter 9] and [15]. A detailed construction of the Verdier quotient of the homotopy category $\mathbf{K}(R)$ by suitable thick subcategories of it, such as the subcategory of (pure) acyclic complexes, is provided in [18, Chapter 2].

If $X = ((X_n)_n, \partial^X)$ and $Y = ((Y_n)_n, \partial^Y)$ are two chain complexes, then the Hom-complex $\text{Hom}_R(X, Y)$ is a complex of abelian groups, which is given in degree n by $\prod_i \text{Hom}_R(X_i, Y_{i+n})$, the group of all homogeneous maps $X \longrightarrow Y$ of degree n. The differential of any n-chain f is the graded commutator $[\partial, f] = \partial^Y f - (-1)^n f \partial^X$. The 0-cycles of the Hom-complex are precisely the chain complex maps $X \longrightarrow Y$, whereas the 0-boundaries are those chain maps which are null-homotopic. It follows that $H_0\text{Hom}_R(X, Y) = \text{Hom}_{\mathbf{K}}(X, Y)$. More generally, for any integer n the homology $H_n\text{Hom}_R(X, Y)$ is equal to the group $\text{Hom}_{\mathbf{K}}(S^n X, Y)$, where $S^n X = X[n]$ is the n-th suspension of X.

Approximations and cotorsion pairs. Let $\mathbf{C}$ be a category and $\mathbf{D} \subseteq \mathbf{C}$ a full subcategory. If $C \in \mathbf{C}$, then a morphism $f : D \longrightarrow C$ in $\mathbf{C}$ is called a $\mathbf{D}$-precover of C if:

(i) $D \in \mathbf{D}$ and

(ii) the induced map $f_* : \text{Hom}_{\mathbf{C}}(D', D) \longrightarrow \text{Hom}_{\mathbf{C}}(D', C)$ is surjective for any $D' \in \mathbf{D}$.

If any object C of $\mathbf{C}$ has a $\mathbf{D}$-precover, we say that the subcategory $\mathbf{D} \subseteq \mathbf{C}$ is precovering. Dually, if $C \in \mathbf{C}$, then a morphism $g : C \longrightarrow D$ in $\mathbf{C}$ is called a $\mathbf{D}$-preenvelope of C if:

(i)' $D \in \mathbf{D}$ and

(ii)' the induced map $g^* : \text{Hom}_{\mathbf{C}}(D, D') \longrightarrow \text{Hom}_{\mathbf{C}}(C, D')$ is surjective for any $D' \in \mathbf{D}$.

If any object C of $\mathbf{C}$ has a $\mathbf{D}$-preenvelope, we say that the subcategory $\mathbf{D} \subseteq \mathbf{C}$ is preenveloping. The reader is referred to [12] for a thorough study of these notions.

We consider an abelian category $\mathbf{A}$ (for example, the category of modules or the category of complexes) and two classes of objects $\mathfrak{C}, \mathfrak{D} \subseteq \mathbf{A}$. The pair $(\mathfrak{C}, \mathfrak{D})$ is a cotorsion theory in

$\mathbf{A}$ (cf. [8]) if $\mathfrak{C}$ is the class consisting of those objects $C \in \mathbf{A}$ for which $\mathrm{Ext}^1_{\mathbf{A}}(C, D) = 0$ for all $D \in \mathfrak{D}$ and $\mathfrak{D}$ is the class consisting of those objects $D \in \mathbf{A}$ for which $\mathrm{Ext}^1_{\mathbf{A}}(C, D) = 0$ for all $C \in \mathfrak{C}$. The cotorsion pair is called hereditary if $\mathfrak{C}$ is closed under kernels of epimorphisms and $\mathfrak{D}$ is closed under cokernels of monomorphisms. We say that the cotorsion pair is complete if for any object $A \in \mathbf{A}$ there exist short exact sequences

$$0 \longrightarrow D \longrightarrow C \longrightarrow A \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow A \longrightarrow D' \longrightarrow C' \longrightarrow 0,$$

where $C, C' \in \mathfrak{C}$ and $D, D' \in \mathfrak{D}$. In that case, the morphism $C \longrightarrow A$ (resp. $A \longrightarrow D'$) is a $\mathfrak{C}$-precover (resp. a $\mathfrak{D}$-preenvelope) of A.

2. Basic properties of K-absolutely pure complexes

In this section, we define K-absolutely pure complexes and present certain basic properties of them, which illustrate the dual role played by the class of these complexes in the homotopy category of the ring, compared to the class of K-flat complexes. In particular, we show that the class of K-absolutely pure complexes is the right hand side of a Bousfield localizing pair in $\mathbf{K}(R)$.

It follows from [7] that a complex X is K-flat if and only if any chain complex map $X \longrightarrow Y$ is null-homotopic for any acyclic complex of pure-injective modules Y, i.e. if and only if X is contained in the left orthogonal ${}^{\perp}\mathbf{K}_{ac}(\mathrm{PInj})$ of the class $\mathbf{K}_{ac}(\mathrm{PInj})$ of acyclic complexes of pure-injective modules in $\mathbf{K}(R)$.

Definition 2.1. *We say that a complex X is K-absolutely pure if any chain map $Y \longrightarrow X$ is null-homotopic for any acyclic complex of pure-projective modules Y. We denote by $\mathbf{K}$-abspure the class of K-absolutely pure complexes; in other words, $\mathbf{K}$-abspure $= \mathbf{K}_{ac}(PProj)^{\perp}$ is the right orthogonal of the class $\mathbf{K}_{ac}(PProj)$ of acyclic complexes of pure-projective modules in $\mathbf{K}(R)$.*

Being a right orthogonal, $\mathbf{K}$-abspure is a thick triangulated subcategory of the homotopy category $\mathbf{K}(R)$.

As shown in [24, §5], the class $\mathbf{K}_{pac}(R)$ of pure acyclic complexes is the right orthogonal $\mathbf{K}(\mathrm{PProj})^{\perp}$ of the homotopy category $\mathbf{K}(\mathrm{PProj})$ of pure-projective modules. It follows readily that $\mathbf{K}_{pac}(R) \subseteq \mathbf{K}$-abspure. The following result is the analogue of the characterization of the pure acyclic complexes as the acyclic K-flat complexes (cf. [5, §1]).

Lemma 2.2. *A complex X is pure acyclic if and only if X is acyclic and K-absolutely pure. In other words, we have an equality $\mathbf{K}_{pac}(R) = \mathbf{K}_{ac}(R) \cap \mathbf{K}$-abspure.*

Proof. Any pure acyclic complex is acyclic and (as noted above) K-absolutely pure. Conversely, assume that X is an acyclic K-absolutely pure complex. The Bousfield localizing pair $(\mathbf{K}(\mathrm{PProj}), \mathbf{K}_{pac}(R))$ provides us with the existence of a distinguished triangle in $\mathbf{K}(R)$

$$Y \xrightarrow{a} X \longrightarrow Z \longrightarrow SY,$$

where Y is a complex of pure projective modules and Z is pure acyclic. The complexes X, Z being acyclic, it follows that the complex Y is also acyclic, i.e. $Y \in \mathbf{K}_{ac}(\mathrm{PProj})$. Since X is K-absolutely pure, a is necessarily represented by a null-homotopic chain map and hence the triangle splits. Then, Z is homotopy equivalent to $X \oplus SY$ and hence $X \oplus SY \in \mathbf{K}_{pac}(R)$. It follows that $X \in \mathbf{K}_{pac}(R)$, as needed. □

We recall that a module is called absolutely pure (fp-injective) if it is a pure submodule of any module containing it. Since any split monomorphism is pure, all injective modules are absolutely pure. Moreover, a pure-injective module is injective if and only if it is absolutely pure. We explore the relation between absolutely pure modules and K-absolutely pure complexes in §4. For the moment, the following result may serve as an indication that the chosen terminology is not unreasonable. The result itself is the analogue of the equivalence between K-flatness and K-projectivity for complexes of pure-projective modules; cf. [6, Corollary 3.4].

Proposition 2.3. *A complex of pure-injective modules is K-absolutely pure if and only if it is K-injective.*

Proof. Since K-injective complexes are right orthogonal to all acyclic complexes, it is clear that $\mathbf{K}$-inj $\subseteq$ $\mathbf{K}$-abspure. Conversely, assume that X is a K-absolutely pure complex consisting of pure-injective modules. The standard Bousfield localizing pair $(\mathbf{K}_{ac}(R), \mathbf{K}\text{-inj})$ provides us with the existence of a distinguished triangle in $\mathbf{K}(R)$

$$Y \xrightarrow{a} X \longrightarrow Z \longrightarrow SY,$$

where Y is acyclic and Z is K-injective. In particular, Z is K-absolutely pure and hence Y is K-absolutely pure as well. We may now invoke Lemma 2.2 and conclude that the complex Y is actually pure acyclic. Since $X \in \mathbf{K}(\text{PInj})$, the morphism a is necessarily represented by a null-homotopic chain map and hence the triangle splits. Then, Z is homotopy equivalent to $X \oplus SY$ and hence $X \oplus SY \in \mathbf{K}$-inj. It follows that $X \in \mathbf{K}$-inj, as needed. □

Remarks 2.4. (i) In view of the existence of the two Bousfield localizing pairs $(\mathbf{K}_{ac}(R), \mathbf{K}\text{-inj})$ and $(\mathbf{K}_{pac}(R), \mathbf{K}(\text{PInj}))$ in $\mathbf{K}(R)$, the inclusion $\mathbf{K}_{pac}(R) \subseteq \mathbf{K}_{ac}(R)$ induces, by taking right orthogonals, an inclusion $\mathbf{K}$-inj $\subseteq \mathbf{K}(\text{PInj})$. Since we also have an inclusion $\mathbf{K}$-inj $\subseteq$ $\mathbf{K}$-abspure, it follows that $\mathbf{K}$-inj $\subseteq$ $\mathbf{K}$-abspure $\cap\, \mathbf{K}(\text{PInj})$. Proposition 2.3 may be restated as the assertion that the latter inclusion is actually an equality, i.e. that $\mathbf{K}$-inj $=$ $\mathbf{K}$-abspure $\cap\, \mathbf{K}(\text{PInj})$.

(ii) We recall that the ring R is called (left) pure semi-simple if any module is pure-injective or, equivalently, if any module is pure-projective. The pure semi-simple rings are precisely the rings over which any module may be expressed as a direct sum of indecomposable modules (cf. [27]). It follows that K-absolute purity and K-injectivity are equivalent notions for complexes over a pure semi-simple ring.

The next result is the analogue in our setting of the fact that the pair $(\mathbf{K}\text{-flat}, \mathbf{K}_{ac}(\text{PInj}))$ is a Bousfield localizing pair in $\mathbf{K}(R)$ (cf. [7, §3]).

Theorem 2.5. *The pair $(\mathbf{K}_{ac}(\mathit{PProj}), \mathbf{K}\text{-}\mathit{abspure})$ is a Bousfield localizing pair in $\mathbf{K}(R)$.*

Proof. Since the direct sum of any family of acyclic complexes of pure-projective modules is an acyclic complex of pure-projective modules, the triangulated subcategory $\mathbf{K}_{ac}(\text{PProj})$ of the homotopy category $\mathbf{K}(R)$ is thick; cf. [18, Remark 3.2.7]. Since $\mathbf{K}$-abspure $= \mathbf{K}_{ac}(\text{PProj})^{\perp}$, it only remains to show the existence of approximation triangles.

To that end, we fix a chain complex X and note that the standard Bousfield localizing pair $(\mathbf{K}_{ac}(R), \mathbf{K}\text{-inj})$ provides us with the existence of a distinguished triangle in $\mathbf{K}(R)$

$$C \xrightarrow{g} X \longrightarrow I \longrightarrow SC,$$

where C is acyclic and I is K-injective. Then, the Bousfield localizing pair $(\mathbf{K}(\text{PProj}), \mathbf{K}_{pac}(R))$ provides us with the existence of another distinguished triangle in $\mathbf{K}(R)$

$$Y \xrightarrow{f} C \longrightarrow Z \longrightarrow SY,$$

where Y is a complex of pure-projective modules and Z is pure acyclic. Since both C and Z are acyclic, it follows that the complex Y is also acyclic, i.e. $Y \in \mathbf{K}_{ac}(\text{PProj})$. We complete the composition $gf : Y \longrightarrow X$ to a distinguished triangle in $\mathbf{K}(R)$

$$Y \xrightarrow{gf} X \longrightarrow W \longrightarrow SY \tag{1}$$

and apply the octahedral axiom to the composable pair of morphisms (f, g)

$$\begin{array}{ccccccc}
Y & \xrightarrow{f} & C & \longrightarrow & Z & \longrightarrow & SY \\
\downarrow 1_X & & \downarrow g & & & & \downarrow 1_X \\
Y & \xrightarrow{gf} & X & \longrightarrow & W & \longrightarrow & SY \\
 & & \downarrow & & & & \downarrow Sf \\
 & & I & \xrightarrow{1_I} & I & \longrightarrow & SC \\
 & & \downarrow & & & & \\
 & & SC & \longrightarrow & SZ & &
\end{array}$$

in order to obtain a distinguished triangle in $\mathbf{K}(R)$

$$Z \longrightarrow W \longrightarrow I \longrightarrow SZ.$$

We note that both complexes Z and I are K-absolutely pure; indeed, Z is pure acyclic and I is K-injective. Since $\mathbf{K}$-abspure is a triangulated subcategory of the homotopy category $\mathbf{K}(R)$, it follows that $W \in \mathbf{K}$-abspure. Then, the distinguished triangle (1) is the approximation triangle we are looking for. □

The following result follows formally from Theorem 2.5 and general facts concerning Bousfield localization (cf. [18, Chapter 9], [15, Proposition 4.9.1]).

Corollary 2.6. *(i) The left orthogonal* ${}^{\perp}\mathbf{K}$*-abspure of* $\mathbf{K}$*-abspure coincides with* $\mathbf{K}_{ac}$*(PProj).*

(ii) The inclusion $\mathbf{K}$*-abspure* $\hookrightarrow \mathbf{K}(R)$ *admits a left adjoint functor and the inclusion* $\mathbf{K}_{ac}$*(PProj)* $\hookrightarrow \mathbf{K}(R)$ *admits a right adjoint functor.*

(iii) Let $\lambda : \mathbf{K}(R) \longrightarrow \mathbf{K}$*-abspure be a left adjoint to the inclusion* $\mathbf{K}$*-abspure* $\hookrightarrow \mathbf{K}(R)$*. Then, for any complex* X *the unit of adjunction morphism* $X \longrightarrow \lambda X$ *is a K-absolutely pure preenvelope of* X*.*

(iv) Let $\mathbf{K}(R)/\mathbf{K}$*-abspure be the Verdier quotient of* $\mathbf{K}(R)$ *by the triangulated subcategory* $\mathbf{K}$*-abspure. Then, the quotient functor* $\pi : \mathbf{K}(R) \longrightarrow \mathbf{K}(R)/\mathbf{K}$*-abspure admits a left adjoint and the composition* $\mathbf{K}_{ac}$*(PProj)* $\hookrightarrow \mathbf{K}(R) \xrightarrow{\pi} \mathbf{K}(R)/\mathbf{K}$*-abspure is a category equivalence. In particular, the quotient* $\mathbf{K}(R)/\mathbf{K}$*-abspure has small Hom-sets.* □

If we identify the Verdier quotient $\mathbf{K}(R)/\mathbf{K}$-abspure with $\mathbf{K}_{ac}$(PProj), by means of the composition $\mathbf{K}_{ac}$(PProj) $\hookrightarrow \mathbf{K}(R) \xrightarrow{\pi} \mathbf{K}(R)/\mathbf{K}$-abspure, then the left adjoint to the quotient functor π is identified with the inclusion functor $\mathbf{K}_{ac}$(PProj) $\hookrightarrow \mathbf{K}(R)$ and, of course, the right adjoint to the latter inclusion functor is identified with the quotient functor π.

We may reformulate the orthogonality assertion made in Corollary 2.6(i), in terms of the following diagrams of triangulated subcategories of the homotopy category $\mathbf{K}(R)$

$$\begin{array}{ccc}
\mathbf{K}_{ac}(\text{PProj}) & \longrightarrow & \mathbf{K}_{ac}(R) \\
\downarrow & & \\
\mathbf{K}(\text{PProj}) & &
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathbf{K}\text{-abspure} & \longleftarrow & \mathbf{K}\text{-inj} \\
\uparrow & & \\
\mathbf{K}_{pac}(R) & &
\end{array}$$

Here, all arrows are inclusions and the left (resp. right) hand side diagram is obtained from the right (resp. left) hand side diagram by taking left (resp. right) Hom-orthogonals. Clearly, $\mathbf{K}_{ac}(\text{PProj}) = \mathbf{K}_{ac}(R) \cap \mathbf{K}(\text{PProj})$ is the biggest triangulated subcategory of the homotopy category, which is contained in both $\mathbf{K}_{ac}(R)$ and $\mathbf{K}$(PProj). We now prove the dual assertion for $\mathbf{K}$-abspure, which provides us with an alternative definition of it. This result is the analogue of the fact that the category of K-flat complexes is the smallest triangulated subcategory of $\mathbf{K}(R)$, which contains all pure acyclic and all K-projective complexes; cf. [7, Proposition 3.6].

Proposition 2.7. *The category* $\mathbf{K}$*-abspure is the smallest triangulated subcategory of* $\mathbf{K}(R)$*, which contains both* $\mathbf{K}_{pac}(R)$ *and* $\mathbf{K}$*-inj.*

Proof. Let $\mathbf{T} \subseteq \mathbf{K}(R)$ be a triangulated subcategory of the homotopy category, containing both $\mathbf{K}_{pac}(R)$ and $\mathbf{K}$-inj, and consider a K-absolutely pure complex X. Then, the Bousfield localizing pair $(\mathbf{K}_{pac}(R), \mathbf{K}(\text{PInj}))$ provides us with the existence of a distinguished triangle in $\mathbf{K}(R)$

$$Y \longrightarrow X \longrightarrow Z \longrightarrow SY, \tag{2}$$

where Y is pure acyclic and Z is a complex of pure-injective modules. Since $\mathbf{K}_{pac}(R) \subseteq \mathbf{T}$, it follows that $Y \in \mathbf{T}$. Since pure acyclic complexes are K-absolutely pure, we conclude that both complexes Y, X are K-absolutely pure; hence, Z is K-absolutely pure as well. We may now invoke Proposition 2.3 and conclude that the complex Z (a K-absolutely pure complex of pure-injective modules) is actually K-injective. Since $\mathbf{K}$-inj $\subseteq \mathbf{T}$, it follows that $Z \in \mathbf{T}$. The distinguished triangle (2) then shows that $X \in \mathbf{T}$ as well. It follows that $\mathbf{K}$-abspure $\subseteq \mathbf{T}$, as needed. □

One may interpret the apparent formal duality between the results established above regarding K-absolutely pure complexes and the analogous results regarding K-flat complexes, in terms of the Pontryagin duality functor D. We recall that the functor D is defined from the category of left (resp. right) modules to the category of right (resp. left) modules, by letting $DM = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. The functor D is extended to the category of complexes in the obvious way.

Proposition 2.8. *Let X be a complex of right modules. Then, X is K-flat if and only if the Pontryagin dual complex DX is K-absolutely pure.*

Proof. We note that the standard Hom-tensor duality and the fact that $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator of the category of abelian groups, imply that for any complex Y the acyclicity of $X \otimes_R Y$ is equivalent to the acyclicity of $\text{Hom}_R(Y, DX)$.

If X is K-flat, then $X \otimes_R Y$ (and hence $\text{Hom}_R(Y, DX)$) is acyclic for any acyclic complex Y. It follows that DX is right orthogonal to any acyclic complex; in particular, DX is right orthogonal to any acyclic complex of pure-projective modules, i.e. $DX \in \mathbf{K}$-abspure.

Conversely, assume that DX is K-absolutely pure. Reversing the argument above, it follows that $X \otimes_R Y$ is acyclic for any acyclic complex of pure-projective modules Y. Then, the K-flatness of X follows from [6, Proposition 2.6], which is itself proved by employing the existence of a right adjoint to the embedding $\mathbf{K}(\text{PProj}) \hookrightarrow \mathbf{K}(R)$. □

Remark 2.9. If X is a complex of right modules, then we may reformulate Proposition 2.8 as the assertion that the complex DX is K-injective if and only if DX is K-absolutely pure (see also Proposition 2.3). This property of complexes is reminiscent of the following well-known property of modules: If M is a right module, then the Pontryagin dual module DM is injective if and only if DM is absolutely pure (fp-injective); indeed, both assertions are equivalent to the flatness of M.

We may present schematically the class $\mathbf{K}$-abspure of K-absolutely pure complexes and its relation to the other classes of complexes that are examined above, in the form of the following (periodic) diagram of triangulated subcategories of $\mathbf{K}(R)$, where all arrows are inclusions

$$\begin{array}{ccccccccc} \mathbf{K}_{ac}(R) & & \mathbf{K}\text{-abspure} & & \mathbf{K}(\mathrm{PInj}) & & \mathbf{K}_{ac}(R) & & \\ & \nwarrow \quad \nearrow & & \nwarrow \quad \nearrow & & \nwarrow \quad \nearrow & & \nwarrow & \\ & \mathbf{K}_{pac}(R) & & \mathbf{K}\text{-inj} & & \mathbf{K}_{ac}(\mathrm{PInj}) & & & \mathbf{K}_{pac}(R) \end{array}$$

This schematic diagram, which is the analogue of the diagram presented in the Introduction involving K-flat complexes and complexes of pure-projective modules, has the following two properties:

(a) Any category in the bottom row of the diagram is the intersection of the two categories that sit immediately above it.
Proof. This follows from Lemma 2.2, Remark 2.4(i) and the definition of $\mathbf{K}_{ac}(\mathrm{PInj})$. □

(b) Any category in the top row of the diagram is the smallest triangulated subcategory of $\mathbf{K}(R)$ containing the two subcategories that sit immediately below it.
Proof. As far as $\mathbf{K}$-abspure is concerned, this follows from Proposition 2.7.

In order to show that $\mathbf{K}(\mathrm{PInj})$ is the smallest triangulated subcategory of $\mathbf{K}(R)$ containing $\mathbf{K}$-inj and $\mathbf{K}_{ac}(\mathrm{PInj})$, we let X be a complex of pure-injective modules. The standard Bousfield localizing pair $(\mathbf{K}_{ac}(R), \mathbf{K}\text{-inj})$ provides us with a quasi-isomorphism $X \longrightarrow I$, where I is K-injective. Since $\mathbf{K}\text{-inj} \subseteq \mathbf{K}(\mathrm{PInj})$ (cf. Remark 2.4(i)), we may assume that I consists of pure-injective modules. Then, the corresponding mapping cone is an acyclic complex of pure-injective modules and hence there exists a distinguished triangle in $\mathbf{K}(R)$

$$Y \longrightarrow X \longrightarrow I \longrightarrow SY,$$

where $Y \in \mathbf{K}_{ac}(\mathrm{PInj})$ and $I \in \mathbf{K}$-inj. It follows that any triangulated subcategory of $\mathbf{K}(R)$ that contains $\mathbf{K}$-inj and $\mathbf{K}_{ac}(\mathrm{PInj})$ must necessarily contain $\mathbf{K}(\mathrm{PInj})$ as well.

Finally, in order to show that $\mathbf{K}_{ac}(R)$ is the smallest triangulated subcategory of $\mathbf{K}(R)$ containing $\mathbf{K}_{ac}(\mathrm{PInj})$ and $\mathbf{K}_{pac}(R)$, we let Z be an acyclic complex and use the Bousfield localizing pair $(\mathbf{K}_{pac}(R), \mathbf{K}(\mathrm{PInj}))$, in order to obtain a distinguished triangle in $\mathbf{K}(R)$

$$W \longrightarrow Z \longrightarrow V \longrightarrow SW,$$

where $W \in \mathbf{K}_{pac}(R)$ and $V \in \mathbf{K}(\mathrm{PInj})$. Since both W and Z are acyclic, the complex V is also acyclic and hence $V \in \mathbf{K}_{ac}(\mathrm{PInj})$. It follows that any triangulated subcategory of $\mathbf{K}(R)$ that contains $\mathbf{K}_{ac}(\mathrm{PInj})$ and $\mathbf{K}_{pac}(R)$ must necessarily contain $\mathbf{K}_{ac}(R)$ as well. □

3. K-absolutely pure complexes and the pure derived category

In this section, we show that K-absolute purity admits a natural interpretation in the pure derived category $\mathbf{D}_{pure}(R) = \mathbf{K}(R)/\mathbf{K}_{pac}(R)$ of the ring. It turns out that $\mathbf{K}$-abspure is the closure of $\mathbf{K}$-inj under isomorphisms in $\mathbf{D}_{pure}(R)$. We also examine the K-absolutely pure resolutions of complexes and show that these resolutions are functorial in $\mathbf{D}_{pure}(R)$.

We study the properties of K-absolutely pure complexes in $\mathbf{D}_{pure}(R)$, using the injective model of the latter category. We consider the Bousfield localizing pair $(\mathbf{K}_{pac}(R), \mathbf{K}(\mathrm{PInj}))$ in $\mathbf{K}(R)$ and note that the inclusion functor $\imath : \mathbf{K}(\mathrm{PInj}) \hookrightarrow \mathbf{K}(R)$ admits a left adjoint $\imath_\lambda$, whose kernel consists precisely of the pure acyclic complexes.

Lemma 3.1. *For any complex X the unit of adjunction morphism $X \longrightarrow \imath_\lambda X$ is represented by a quasi-isomorphism and defines an isomorphism in the pure derived category $\mathbf{D}_{pure}(R)$.*

Proof. The morphism $a : X \longrightarrow \imath_\lambda X$ can be completed to a distinguished triangle in $\mathbf{K}(R)$

$$Z \longrightarrow X \stackrel{a}{\longrightarrow} \imath_\lambda X \longrightarrow SZ,$$

where Z is pure acyclic (and, in particular, acyclic). It follows that a defines an isomorphism in $\mathbf{D}_{pure}(R)$ and any chain map representing a is a quasi-isomorphism. □

Corollary 3.2. *The functor $\imath_\lambda : \mathbf{K}(R) \longrightarrow \mathbf{K}(PInj)$ preserves the acyclicity of complexes, i.e. if X is an acyclic complex, then the complex $\imath_\lambda X$ is acyclic as well.* □

Even though the K-injectivity of a complex requires the complex to be right orthogonal to all acyclic complexes, a weaker conditions is sufficient to guarantee K-injectivity for complexes of pure-injective modules. This result, which is analogous to [6, Proposition 2.5], will be used in a crucial way in the sequel.

Proposition 3.3. *A complex X of pure-injective modules is K-injective if (and only if) the group $Hom_{\mathbf{K}}(Y, X)$ is trivial for any acyclic complex of pure-injective modules Y.*

Proof. By definition, a K-injective complex (of pure-injective modules) is right orthogonal to any acyclic complex; in particular, it is right orthogonal to any acyclic complex of pure-injective modules.

Conversely, assume that X is a complex of pure-injective modules and $\text{Hom}_{\mathbf{K}}(Y, X) = 0$ for any acyclic complex Y of pure-injective modules. We fix an acyclic complex $C \in \mathbf{K}(R)$. Since $X \in \mathbf{K}(\text{PInj})$, the unit of adjunction morphism $C \longrightarrow \imath_\lambda C$ induces an isomorphism of abelian groups

$$\text{Hom}_{\mathbf{K}}(\imath_\lambda C, X) \longrightarrow \text{Hom}_{\mathbf{K}}(C, X).$$

In view of Corollary 3.2, $\imath_\lambda C$ is an acyclic complex of pure-injective modules; hence, our assumption implies that the group $\text{Hom}_{\mathbf{K}}(\imath_\lambda C, X)$ is trivial. It follows that $\text{Hom}_{\mathbf{K}}(C, X) = 0$ as well. Since this is the case for any acyclic complex C, it follows that X is K-injective, as needed. □

We shall also use the following result, which states that K-absolute purity remains invariant under isomorphisms in the pure derived category.

Lemma 3.4. *Let X, Y be two complexes, which are isomorphic in the pure derived category $\mathbf{D}_{pure}(R)$. Then, X is K-absolutely pure if and only if Y is K-absolutely pure.*

Proof. An isomorphism in $\mathbf{D}_{pure}(R) = \mathbf{K}(R)/\mathbf{K}_{pac}(R)$ between X and Y is represented by a diagram of morphisms $X \stackrel{f}{\longleftarrow} Z \stackrel{g}{\longrightarrow} Y$ in $\mathbf{K}(R)$, which fit in distinguished triangles

$$Z \stackrel{f}{\longrightarrow} X \longrightarrow W \longrightarrow SZ \quad \text{and} \quad Z \stackrel{g}{\longrightarrow} Y \longrightarrow W' \longrightarrow SZ$$

in $\mathbf{K}(R)$, where $W, W' \in \mathbf{K}_{pac}(R)$. Since all complexes in $\mathbf{K}_{pac}(R)$ are K-absolutely pure and $\mathbf{K}$-abspure is a triangulated subcategory of $\mathbf{K}(R)$, it follows that X is K-absolutely pure if and only if Z is K-absolutely pure if and only if Y is K-absolutely pure. □

The claim made in Lemma 3.1, namely that any complex X is isomorphic in the pure derived category with the complex of pure-injective modules $\imath_\lambda X$, may be strengthened as follows: If $p : \mathbf{K}(R) \longrightarrow \mathbf{K}(R)/\mathbf{K}_{pac}(R) = \mathbf{D}_{pure}(R)$ denotes the quotient functor, then the composition

$$\mathbf{K}(\text{PInj}) \stackrel{\imath}{\hookrightarrow} \mathbf{K}(R) \stackrel{p}{\longrightarrow} \mathbf{D}_{pure}(R) \tag{3}$$

is an equivalence of categories, with quasi-inverse the functor obtained from the left adjoint $\imath_\lambda : \mathbf{K}(R) \longrightarrow \mathbf{K}(\text{PInj})$ by passage to the quotient. In this way, we may identify the categories $\mathbf{K}(\text{PInj})$ and $\mathbf{D}_{pure}(R)$. Since both functors $\imath$ and $\imath_\lambda$ preserve the acyclicity of complexes (this is obvious for $\imath$ and Corollary 3.2 shows that it is also true for $\imath_\lambda$), it follows that under this identification the acyclic complexes $\mathbf{K}_{ac}(\text{PInj})$ in $\mathbf{K}(\text{PInj})$ are identified with the acyclic complexes $p\mathbf{K}_{ac}(R)$ in $\mathbf{D}_{pure}(R)$.

We denote by $\mathrm{Hom}_{\mathbf{D}_{pure}}(\_\,,\_)$ the Hom-pairing in the pure derived category $\mathbf{D}_{pure}(R)$. Since the composition $p\imath : \mathbf{K}(\mathrm{PInj}) \longrightarrow \mathbf{D}_{pure}(R)$ is an equivalence of categories, for any two complexes X, Y of pure-injective modules, there is an induced isomorphism of abelian groups $\mathrm{Hom}_{\mathbf{K}}(X, Y) \simeq \mathrm{Hom}_{\mathbf{D}_{pure}}(X, Y)$.

Theorem 3.5. *The following conditions are equivalent for a complex X:*
(i) X is K-absolutely pure,
(ii) X is isomorphic in $\mathbf{D}_{pure}(R)$ with a K-injective complex of pure-injective modules,
(iii) X is isomorphic in $\mathbf{D}_{pure}(R)$ with a K-injective complex,
(iv) $Hom_{\mathbf{D}_{pure}}(C, X) = 0$ for all acyclic complexes C and
(v) the complex $\imath_\lambda X$ is K-injective.

Proof. We shall prove that (i)→(ii)→(iii)→(i) and (ii)→(iv)→(v)→(iii). It is obvious that (ii)→(iii), whereas the implication (v)→(iii) is an immediate consequence of Lemma 3.1

(i)→(ii): Assume that X is K-absolutely pure and fix a quasi-isomorphism $f : X \longrightarrow I$, where I is a K-injective complex of pure-injective modules. Then, I is K-absolutely pure and hence the mapping cone $c(f)$ of f is an acyclic K-absolutely pure complex. We now invoke Lemma 2.2 and conclude that $c(f)$ is pure acyclic. The existence of the distinguished triangle

$$X \longrightarrow I \longrightarrow c(f) \longrightarrow SX$$

in $\mathbf{K}(R)$ implies that f induces an isomorphism in $\mathbf{D}_{pure}(R)$.

(iii)→(i): Assume that X is isomorphic in $\mathbf{D}_{pure}(R)$ with a K-injective complex Y. Since Y is, in particular, K-absolutely pure, Lemma 3.4 implies that X is K-absolutely pure as well.

(ii)→(iv): Assume that X is isomorphic in $\mathbf{D}_{pure}(R)$ with a K-injective complex of pure-injective modules Y and fix an acyclic complex C. We have to show that the abelian group $\mathrm{Hom}_{\mathbf{D}_{pure}}(C, X) \simeq \mathrm{Hom}_{\mathbf{D}_{pure}}(C, Y)$ is trivial. The complex C is isomorphic in $\mathbf{D}_{pure}(R)$ with the complex $\imath_\lambda C$, which consists of pure-injective modules and is acyclic (cf. Lemma 3.1 and Corollary 3.2). Thus, replacing C by $\imath_\lambda C$, we reduce the problem to the case where C is an acyclic complex of pure-injective modules. Then, the triviality of the group $\mathrm{Hom}_{\mathbf{D}_{pure}}(C, Y)$ follows since that group is isomorphic with $\mathrm{Hom}_{\mathbf{K}}(C, Y)$; the latter group is trivial, as Y is K-injective and C is acyclic.

(iv)→(v): Assume that $\mathrm{Hom}_{\mathbf{D}_{pure}}(C, X) = 0$ for any acyclic complex C. Since X is isomorphic with $\imath_\lambda X$ in $\mathbf{D}_{pure}(R)$ (cf. Lemma 3.1), it follows that $\mathrm{Hom}_{\mathbf{D}_{pure}}(C, \imath_\lambda X) = 0$ for any acyclic complex C. In particular, for any acyclic complex of pure-injective modules C we have $\mathrm{Hom}_{\mathbf{K}}(C, \imath_\lambda X) \simeq \mathrm{Hom}_{\mathbf{D}_{pure}}(C, \imath_\lambda X) = 0$. Proposition 3.3 now implies that the complex of pure-injective modules $\imath_\lambda X$ is K-injective. □

Any complex X admits a quasi-isomorphism $i : X \longrightarrow I$, where I is a K-injective complex of pure-injective modules. (The argument provided in the proof of the implication (i)→(ii) in Theorem 3.5 above shows that, if X is K-absolutely pure, then i induces an isomorphism in $\mathbf{D}_{pure}(R)$.) Equivalently, any complex X admits a quasi-isomorphism $i : X \longrightarrow I$, where I is a K-absolutely pure complex of pure-injective modules; cf. Proposition 2.3. We conclude this section by analysing the uniqueness and the functoriality of such K-absolutely pure resolutions in $\mathbf{D}_{pure}(R)$.

Remarks 3.6. (i) If I is a K-injective complex and $f : X \longrightarrow Y$ a quasi-isomorphism, then the induced additive map $f^* : \mathrm{Hom}_{\mathbf{K}}(Y, I) \longrightarrow \mathrm{Hom}_{\mathbf{K}}(X, I)$ is bijective. It follows that for any complex X there is a unique, up to homotopy equivalence, quasi-isomorphism $X \longrightarrow I$ from X to a K-injective complex I. It also follows that these K-injective resolutions are functorial in $\mathbf{K}(R)$ (cf. [22]).

(ii) Let X be a complex and fix a quasi-isomorphism $i : X \longrightarrow I$, where I is a K-injective complex. We also consider a quasi-isomorphism $j : X \longrightarrow J$, where J is K-absolutely pure. It follows from (i) that i may be factored, up to homotopy, as the composition $X \stackrel{j}{\longrightarrow} J \stackrel{f}{\longrightarrow} I$, for a suitable chain map f, which is unique up to homotopy. Since both j and $fj \simeq i$ are quasi-isomorphisms, it follows that f is a quasi-isomorphism as well. Then, considering the mapping cone of f as in the proof of the implication (i)→(ii) in Theorem 3.5, it follows that f induces an isomorphism in $\mathbf{D}_{pure}(R)$. Hence, K-absolutely pure resolutions are unique up to isomorphism in $\mathbf{D}_{pure}(R)$.

(iii) Let $f : X \longrightarrow Y$ be a quasi-isomorphism. We choose K-injective complexes I_X, I_Y and quasi-isomorphisms $i_X : X \longrightarrow I_X$ and $i_Y : Y \longrightarrow I_Y$. Then, there exists a, unique up to homotopy, chain map $\phi : I_X \longrightarrow I_Y$ that makes the following diagram homotopy commutative (cf. (i))

$$\begin{array}{ccc} X & \stackrel{f}{\longrightarrow} & Y \\ {\scriptstyle i_X}\downarrow & & \downarrow{\scriptstyle i_Y} \\ I_X & \stackrel{\phi}{\longrightarrow} & I_Y \end{array}$$

The mapping cone $c(\phi)$ of ϕ is K-injective, as both I_X and I_Y are. On the other hand, the homotopy commutative diagram above shows that ϕ is a quasi-isomorphism and hence $c(\phi)$ is acyclic. Being acyclic and K-injective, the complex $c(\phi)$ is contractible.

(iv) If X, Y are two complexes, then a morphism $\xi : X \longrightarrow Y$ in $\mathbf{D}_{pure}(R)$ is the equivalence class of a diagram of morphisms $X \longleftarrow W \longrightarrow Y$ in $\mathbf{K}(R)$, which are themselves represented by chain maps $X \stackrel{f}{\longleftarrow} W \stackrel{g}{\longrightarrow} Y$, such that the mapping cone $c(f)$ of f is pure acyclic. We choose K-injective complexes I_X, I_W, I_Y and quasi-isomorphisms $i_X : X \longrightarrow I_X$, $i_W : W \longrightarrow I_W$ and $i_Y : Y \longrightarrow I_Y$. In view of (i), there exist chain maps $\phi : I_W \longrightarrow I_X$ and $\gamma : I_W \longrightarrow I_Y$, which are unique up to homotopy and make the following diagram homotopy commutative

$$\begin{array}{ccccc} X & \stackrel{f}{\longleftarrow} & W & \stackrel{g}{\longrightarrow} & Y \\ {\scriptstyle i_X}\downarrow & & {\scriptstyle i_W}\downarrow & & {\scriptstyle i_Y}\downarrow \\ I_X & \stackrel{\phi}{\longleftarrow} & I_W & \stackrel{\gamma}{\longrightarrow} & I_Y \end{array}$$

Since f is, in particular, a quasi-isomorphism, it follows from (iii) above that the mapping cone $c(\phi)$ of ϕ is contractible (and hence pure acyclic). Then, the bottom row of the diagram induces a morphism $\eta : I_X \longrightarrow I_Y$ in $\mathbf{D}_{pure}(R)$ that makes the following diagram commutative

$$\begin{array}{ccc} X & \stackrel{\xi}{\longrightarrow} & Y \\ \downarrow & & \downarrow \\ I_X & \stackrel{\eta}{\longrightarrow} & I_Y \end{array}$$

The morphism $\eta \in \mathrm{Hom}_{\mathbf{D}_{pure}}(I_X, I_Y)$ is unique with that property. Indeed, we note that the complexes I_X, I_Y are contained in $\mathbf{K}$-inj $\subseteq \mathbf{K}(\mathrm{PInj}) = \mathbf{K}_{pac}(R)^{\perp}$ and hence the canonical map $\mathrm{Hom}_{\mathbf{K}}(I_X, I_Y) \longrightarrow \mathrm{Hom}_{\mathbf{D}_{pure}}(I_X, I_Y)$ is bijective, whereas $\mathrm{Hom}_{\mathbf{K}}(X, I_Y) \longrightarrow \mathrm{Hom}_{\mathbf{D}_{pure}}(X, I_Y)$ is injective; the latter assertion follows invoking [18, Lemma 2.1.26]. We conclude that K-absolutely pure resolutions are functorial in $\mathbf{D}_{pure}(R)$.

4. Strongly fp-injective modules

In this section, we consider the class Sfpinj of strongly fp-injective modules and examine its relevance to the study of K-absolutely pure complexes. In particular, we use strongly fp-injective modules in order to provide examples of K-absolutely pure complexes. We also describe the class of (acyclic) K-absolutely pure complexes of strongly fp-injective modules, in a way that is analogous to the description of (acyclic) K-flat complexes of flat modules.

We recall that a module M is absolutely pure if it is a pure submodule of any other module containing it. Equivalently, M is absolutely pure if $\mathrm{Ext}^1_R(C,M)=0$ for any finitely presented module C; for that reason, absolutely pure modules are also called fp-injective [23]. It is easily seen that the class Fpinj of fp-injective modules is closed under pure submodules. Hence, any pure submodule of an injective module is fp-injective.

Following [17], we say that a module M is strongly fp-injective if $\mathrm{Ext}^n_R(C,M)=0$ for any $n\geq 1$ and any finitely presented module C. Of course, any injective module is strongly fp-injective and any strongly fp-injective module is fp-injective. It follows that a right module N is flat if and only if the Pontryagin dual module DN is strongly fp-injective. If the ring R is left coherent, then we may choose the syzygy modules of any finitely presented module to be finitely presented as well: it follows that, in this case, any fp-injective module is strongly fp-injective.

Lemma 4.1. *A module M is strongly fp-injective if and only if the cosyzygy modules $(\Sigma^n M)_{n\geq 0}$ of M in any injective resolution of it are fp-injective.*

Proof. This is clear since $\mathrm{Ext}^n_R(\_,M)=\mathrm{Ext}^1_R(\_,\Sigma^{n-1}M)$ for all $n\geq 1$. □

The relevance of the class Sfpinj of strongly fp-injective modules to the study of K-absolutely pure complexes stems from the following result.

Proposition 4.2. *A module M is strongly fp-injective if and only if the complex $M[0]$ (consisting of M in degree 0 and zeroes elsewhere) is K-absolutely pure.*

Proof. We consider an injective resolution

$$0\longrightarrow M\stackrel{\eta}{\longrightarrow} I^0\longrightarrow I^1\longrightarrow\dots$$

and the truncated complex

$$0\longrightarrow I^0\longrightarrow I^1\longrightarrow I^2\longrightarrow\dots$$

We denote these complexes by X and I respectively. The chain map $\eta: M[0]\longrightarrow I$ induces a distinguished triangle in $\mathbf{K}(R)$

$$S^{-1}I\longrightarrow S^{-1}X\longrightarrow M[0]\longrightarrow I.$$

We note that I, being a left-bounded complex of injective modules, is K-injective (and hence K-absolutely pure). It follows that the complex $M[0]$ is K-absolutely pure if and only if this is the case for X. In view of Lemma 2.2 and the acyclicity of X, the latter condition is equivalent to the assertion that X is pure acyclic. Therefore, we only have to show that the module M is strongly fp-injective if and only if X is pure acyclic.

If M is strongly fp-injective, then the cosyzygy modules $\Sigma^n M$ are fp-injective for all $n\geq 0$ (cf. Lemma 4.1). Then, the short exact sequences

$$0\longrightarrow \Sigma^n M\longrightarrow I^n\longrightarrow \Sigma^{n+1}M\longrightarrow 0$$

are pure for all $n\geq 0$ and hence X is pure acyclic. Conversely, if X is pure acyclic, then the short exact sequences above are all pure. It follows that $\Sigma^n M$, being a pure submodule of the injective module I^n, is fp-injective for all $n\geq 0$. Invoking Lemma 4.1 again, we conclude that M is strongly fp-injective. □

We can obtain some more interesting examples of K-absolutely pure complexes, by using the following simple lemma.

Lemma 4.3. *Let X, Y be two chain complexes and assume that:*
(i) the complex of abelian groups $Hom_R(X, Y_n)$ is acyclic for all n and
(ii) the group $Hom_R(X_n, Y_n)$ is trivial for all $n \gg 0$.
Then, any chain map $f : X \longrightarrow Y$ is null-homotopic.

Proof. We denote by ∂^X and ∂^Y the differentials of the complexes X and Y respectively. For any chain map $f : X \longrightarrow Y$ we construct linear maps $\Sigma_n : X_n \longrightarrow Y_{n+1}$, such that $f_n = \partial^Y \Sigma_n + \Sigma_{n-1}\partial^X$ for all n. We define $\Sigma_n = 0$ for all $n \gg 0$ and proceed by descending induction on n. We assume therefore that n is an integer and the construction of the Σ_i's has been performed for all $i \geq n$. Then, the linear map $f_n - \partial^Y \Sigma_n : X_n \longrightarrow Y_n$ is an n-cycle of the complex $\mathrm{Hom}_R(X, Y_n)$, since

$$\begin{aligned} (f_n - \partial^Y \Sigma_n)\partial^X &= f_n\partial^X - \partial^Y \Sigma_n \partial^X \\ &= f_n\partial^X - \partial^Y (f_{n+1} - \partial^Y \Sigma_{n+1}) \\ &= f_n\partial^X - \partial^Y f_{n+1} + \partial^Y \partial^Y \Sigma_{n+1} \\ &= f_n\partial^X - \partial^Y f_{n+1} \\ &= 0. \end{aligned}$$

In view of assumption (i), there exists a linear map $\Sigma_{n-1} : X_{n-1} \longrightarrow Y_n$, such that $f_n - \partial^Y \Sigma_n = \Sigma_{n-1}\partial^X$ and hence the inductive step of the proof is complete. □

Corollary 4.4. *Any left-bounded complex of strongly fp-injective modules is K-absolutely pure.*

Proof. Let X be a left-bounded complex of strongly fp-injective modules. We have to show that any chain map $P \longrightarrow X$ from any acyclic complex of pure-projective modules P to X is null-homotopic. In view of Proposition 4.2, the complex $\mathrm{Hom}_R(P, X_n)$ is acyclic for all n. On the other hand, we also have $\mathrm{Hom}_R(P_n, X_n) = 0$ for all $n \gg 0$, since the complex X is left-bounded. Therefore, the result follows by invoking Lemma 4.3. □

Our next goal is to describe the full class of K-absolutely pure complexes of strongly fp-injective modules. To that end, we note (cf. [17]) that the class Sfpinj of strongly fp-injective modules has the following properties:

(i) It is closed under products, extensions, cokernels of monomorphisms and kernels of pure epimorphisms.

(ii) There is a set S of modules, which may be used in order to describe Sfpinj as the kernel of the functor $\bigoplus_{C \in S} \mathrm{Ext}^1_R(C, \_)$.[1]

As noted in [17, Theorem 3.4] these properties imply the existence of a complete hereditary cotorsion pair $(\mathfrak{C}, \mathrm{Sfpinj})$ in the category of modules.[2] Following Gillespie [9], we now define the following four classes of complexes:

(a) The class $\widetilde{\mathfrak{C}}$ consists of those acyclic complexes whose kernels are contained in $\mathfrak{C}$.

(b) The class $\widetilde{\mathrm{Sfpinj}}$ consists of those acyclic complexes whose kernels are strongly fp-injective.

[1] A set S with that property is the set containing the syzygies $\Omega_n C$, $n \geq 0$, where C runs through a set of representatives of the isomorphism classes of all finitely presented modules.

[2] It follows from [12, Corollary 3.2.4] that the class $\mathfrak{C}$ consists precisely of the direct summands of those modules that admit continuous ascending filtrations with successive quotients contained in S.

(c) The class $dg\,\widetilde{\mathfrak{C}}$ consists of those complexes of $\mathfrak{C}$-modules, which are left orthogonal to any complex in $\widetilde{\text{Sfpinj}}$. In other words, $X \in dg\,\widetilde{\mathfrak{C}}$ if and only if X is a complex of $\mathfrak{C}$-modules and any chain map $X \longrightarrow Y$ is null-homotopic for any $Y \in \widetilde{\text{Sfpinj}}$.

(d) The class $dg\,\widetilde{\text{Sfpinj}}$ consists of those complexes of strongly fp-injective modules, which are right orthogonal to any complex in $\widetilde{\mathfrak{C}}$. In other words, $Y \in dg\,\widetilde{\text{Sfpinj}}$ if and only Y is a complex of strongly fp-injective modules and any chain map $X \longrightarrow Y$ is null-homotopic for any $X \in \widetilde{\mathfrak{C}}$.

Using the above notation, it follows from [9] (see also [26]) that:

(i) The pairs $\left(\widetilde{\mathfrak{C}}, dg\,\widetilde{\text{Sfpinj}}\right)$ and $\left(dg\,\widetilde{\mathfrak{C}}, \widetilde{\text{Sfpinj}}\right)$ are complete cotorsion pairs in the category of chain complexes.

(ii) If we denote by $C_{ac}(R)$ the class of acyclic chain complexes, then $C_{ac}(R) \cap dg\,\widetilde{\mathfrak{C}} = \widetilde{\mathfrak{C}}$ and $C_{ac}(R) \cap dg\,\widetilde{\text{Sfpinj}} = \widetilde{\text{Sfpinj}}$.

In order to characterize the class of K-absolutely pure complexes of strongly fp-injective modules, we shall use the following auxiliary result.

Lemma 4.5. *(i) There are equalities $C_{ac}(R) \cap dg\,\widetilde{\mathit{Sfpinj}} = \widetilde{\mathit{Sfpinj}} = C_{pac}(R) \cap C(\mathit{Sfpinj})$.*

(ii) Any complex of pure projective modules is contained in $dg\,\widetilde{\mathfrak{C}}$.

(iii) Any acyclic complex of pure-projective modules is contained in $\widetilde{\mathfrak{C}}$.

Proof. (i) As we noted above, the first equality follows from [9, Theorem 3.12]. Hence, it only remains to prove the second equality. If X is an acyclic complex with strongly fp-injective kernels, then the short exact sequences

$$0 \longrightarrow Z_nX \longrightarrow X_n \longrightarrow Z_{n-1}X \longrightarrow 0$$

are necessarily pure and hence X is pure acyclic. Since Sfpinj is closed under extensions, it follows that $X \in C(\text{Sfpinj})$. Conversely, assume that X is a pure acyclic complex of strongly fp-injective modules. Any pure submodule of an fp-injective module is also fp-injective; since each X_n is fp-injective, the purity of the short exact sequences above implies that Z_nX is fp-injective for all n. Then, for any finitely presented module C and any integers n, i with $i \geq 1$ we have $\text{Ext}^i_R(C, Z_nX) = \text{Ext}^1_R(C, Z_{n-i+1}X) = 0$. Here, the first equality follows by dimension shifting since X is a complex of *strongly* fp-injective modules, whereas the second one follows since the kernels of X are fp-injective (as we noted above). We conclude that the kernels Z_nX are strongly fp-injective for all n and hence $X \in \widetilde{\text{Sfpinj}}$, as needed.

(ii) The class $\mathfrak{C}$ contains all finitely presented modules and is closed under direct sums and direct summands. In particular, all pure-projective modules are contained in $\mathfrak{C}$. Hence, in order to show that any complex of pure-projective modules X is contained in $dg\,\widetilde{\mathfrak{C}}$, we only have to show that $\text{Hom}_{\mathbf{K}}(X, C) = 0$ for any $C \in \widetilde{\text{Sfpinj}}$. This is an immediate consequence of (i) above, which implies that such a complex C is necessarily pure acyclic.

(iii) This follows from (ii), since $C_{ac}(R) \cap dg\,\widetilde{\mathfrak{C}} = \widetilde{\mathfrak{C}}$. □

Theorem 4.6. *The complexes in $dg\,\widetilde{\mathit{Sfpinj}}$ are precisely the K-absolutely pure complexes of strongly fp-injective modules.*

Proof. Let X be a complex in $dg\,\widetilde{\text{Sfpinj}}$. Since any acyclic complex of pure-projective modules C is contained in $\widetilde{\mathfrak{C}}$ (cf. Lemma 4.5(iii)), the abelian group $\text{Hom}_{\mathbf{K}}(C, X)$ is trivial. Hence, the complex X is K-absolutely pure.

Conversely, assume that X is a K-absolutely pure complex of strongly fp-injective modules. In order to show that $X \in dg\,\widetilde{\text{Sfpinj}}$, we have to prove that $\text{Hom}_{\mathbf{K}}(C, X) = 0$ for any complex $C \in \widetilde{\mathfrak{C}}$. We fix such a complex C and choose a quasi-isomorphism $f : X \longrightarrow I$, where I is a K-injective complex of injective modules; cf. [1, §1]. Since C is acyclic, the abelian group $\text{Hom}_{\mathbf{K}}(C, I)$ is trivial. We also consider the mapping cone $c(f)$ of f and the distinguished triangle

$$S^{-1}c(f) \longrightarrow X \longrightarrow I \longrightarrow c(f)$$

in $\mathbf{K}(R)$, which is associated with f. The triviality of the group $\text{Hom}_{\mathbf{K}}(C, X)$ will follow if we show that $\text{Hom}_{\mathbf{K}}(C, S^{-1}c(f)) = 0$. As injective modules are strongly fp-injective, the mapping cone $c(f)$ is a complex consisting of strongly fp-injective modules. As K-injective complexes are K-absolutely pure, both complexes X, I are K-absolutely pure. Then, the distinguished triangle above shows that $c(f)$ is K-absolutely pure as well. Since $c(f)$ is also acyclic, Lemma 2.2 implies that $c(f)$ (and hence $S^{-1}c(f)$) is actually pure acyclic. We now invoke Lemma 4.5(i) and conclude that $S^{-1}c(f) \in dg\,\widetilde{\text{Sfpinj}}$. Since $C \in \widetilde{\mathfrak{C}}$, it follows that the abelian group $\text{Hom}_{\mathbf{K}}(C, S^{-1}c(f))$ is trivial, as needed. □

Corollary 4.7. *The complexes in $\widetilde{\text{Sfpinj}}$ are precisely the acyclic K-absolutely pure complexes of strongly fp-injective modules.*

Proof. Since $\widetilde{\text{Sfpinj}} = C_{ac}(R) \cap dg\,\widetilde{\text{Sfpinj}}$, the result follows from Theorem 4.6. □

Remarks 4.8. (i) The fact that any pure acyclic complex of strongly fp-injective modules has necessarily strongly fp-injective kernels (cf. Lemma 4.5(i)) has an interesting consequence: If M is a module, such that there exists a pure exact sequence

$$0 \longrightarrow M \longrightarrow J \longrightarrow M \longrightarrow 0$$

where J is strongly fp-injective, then M is strongly fp-injective as well. (In the terminology introduced in [2], we therefore claim that any pure Sfpinj-periodic module is trivial. This is reminiscent of [2, Proposition 3.8(1)], which states that any pure Inj-periodic module is trivial.) Indeed, splicing the pure short exact sequence above with itself induces a pure acyclic complex

$$\ldots \longrightarrow J \longrightarrow J \longrightarrow J \longrightarrow \ldots$$

whose kernels coincide with M.

(ii) The result stated in Corollary 4.7 is the analogue of Neeman's characterization [19] of the acyclic K-flat complexes of flat modules, as those acyclic complexes of flat modules whose kernels are also flat. In the same way, Theorem 4.6 is the analogue of the description of the K-flat complexes of flat modules, as those complexes of flat modules which are left orthogonal to the acyclic complexes of cotorsion modules (which have necessarily cotorsion kernels, in view of [2, Theorem 4.1(2)]); a proof of this result may be found in [4, Proposition 1.6].

Stovicek proved in [24, Corollary 5.9] that the kernels of an acyclic complex of injective modules are cotorsion.[3] His argument actually shows that, more generally, any acyclic complex of pure-injective modules has cotorsion kernels. We prove an analogue of these results concerning modules in the class $\mathfrak{C}$. We recall that a module M is contained in $\mathfrak{C}$ if and only if the functor $\text{Ext}^1_R(M, \_)$ vanishes on all strongly fp-injective modules.

[3]In fact, he proved that for any acyclic complex of injective modules X with kernels $(Z_nX)_n$ the functor $\text{Ext}^1_R(\_, Z_nX)$ vanishes for all n on modules of finite flat dimension.

Let M be a module and i a positive integer, such that the cozyzygy module $\Sigma^i M$ in some injective resolution of M is strongly fp-injective. Then, the same is true for the i-th cosyzygy module in *any* injective resolution of M. This follows from Schanuel's lemma, since Sfpinj is closed under finite direct sums and direct summands and contains all injective modules. In fact, since Sfpinj is closed under cokernels of monomorphisms, the cosyzygy modules $\Sigma^j M$ are then strongly fp-injective for all $j \geq i$. We denote by $\overline{\text{Sfpinj}}$ the class consisting of those modules M, for which the cosyzygy module $\Sigma^i M$ in some injective resolution of M is strongly fp-injective for all $i \gg 0$. (We could say that such a module M has finite *Sfpinj dimension*).

Corollary 4.9. *Let X be an acyclic complex of pure-projective modules.*

(i) The kernel $Z_n X$ is contained in $\mathfrak{C}$ for all n.

(ii) If, in addition, X consists of projective modules, then $Ext^1_R(Z_n X, M) = 0$ for all n and all modules $M \in \overline{Sfpinj}$.

Proof. (i) Let M be a strongly fp-injective module. Then, Proposition 4.2 implies that the complex $M[0]$ consisting of M in degree 0 and zeroes elsewhere is K-absolutely pure. Since K-absolute purity is invariant under suspension, we conclude that the complex of abelian groups $\text{Hom}_R(X, M)$ is acyclic. Then, the short exact sequence of modules

$$0 \longrightarrow Z_n X \longrightarrow X_n \longrightarrow Z_{n-1} X \longrightarrow 0$$

induces a short exact sequence of abelian groups

$$0 \longrightarrow \text{Hom}_R(Z_{n-1}X, M) \longrightarrow \text{Hom}_R(X_n, M) \longrightarrow \text{Hom}_R(Z_n X, M) \longrightarrow 0$$

for all n. Since any pure-projective module is a direct summand of a direct sum of finitely presented modules, the abelian groups $\text{Ext}^1_R(X_n, M)$ are trivial for all n. It follows readily that $\text{Ext}^1_R(Z_{n-1}X, M) = 0$ for all n. Since this is the case for any strongly fp-injective module M, we conclude that the kernels of X are contained in $\mathfrak{C}$, as needed.

(ii) Let $M \in \overline{\text{Sfpinj}}$ and assume that i is a positive integer, such that the cosyzygy module $\Sigma^i M$ in some injective resolution of M is strongly fp-injective. Then, for all n we have

$$\text{Ext}^1_R(Z_n X, M) = \text{Ext}^{i+1}_R(Z_{n-i}X, M) = \text{Ext}^1_R(Z_{n-i}X, \Sigma^i M) = 0.$$

In the above chain of equalities, the first one follows using dimension-shifting along the complex of projective modules X, the second one follows using dimension-shifting along the given injective resolution of M and the third one follows from (i) above. □

Remarks 4.10. (i) Corollary 4.9(ii) implies that any Gorenstein projective module (cf. [13]) annihilates the functor $\text{Ext}^1_R(\_, M)$ for any $M \in \overline{\text{Sfpinj}}$. In particular, Gorenstein projective modules are contained in $\mathfrak{C}$.

(ii) Stovicek's result [24, Corollary 5.9] has been generalized by Bazzoni et al. in [2, Theorem 4.1(2)], where it is proved that any acyclic complex of cotorsion modules has cotorsion kernels. We do not know whether the corresponding generalization of Corollary 4.9(i) is true, i.e. whether any acyclic complex of $\mathfrak{C}$-modules has kernels in $\mathfrak{C}$.

(iii) Assume that the ring R is left coherent. Then, the class Sfpinj of strongly fp-injective modules coincides with the class Fpinj of fp-injective modules. Therefore, the class $\mathfrak{C}$ consists of those modules M, for which the abelian group $\text{Ext}^1_R(M, J)$ is trivial for any fp-injective module J; such a module M is usually called fp-projective. Hence, [21, Example 4.3] implies that, in the special case of a left coherent ring, any acyclic complex of $\mathfrak{C}$-modules has indeed kernels in $\mathfrak{C}$.

5. Coherence and preenvelopes by complexes of injective modules

In this final section, we examine the $\mathbf{K}(\text{PInj})$-preenvelopes of (K-absolutely pure) complexes of strongly fp-injective modules. In the special case of a left coherent ring, we relate these preenvelopes to (K-injective) complexes of injective modules and show that the left adjoint $\imath_\lambda$ to the inclusion $\imath : \mathbf{K}(\text{PInj}) \hookrightarrow \mathbf{K}(R)$ maps $\mathbf{K}(\text{Sfpinj}) \subseteq \mathbf{K}(R)$ into $\mathbf{K}(\text{Inj}) \subseteq \mathbf{K}(\text{PInj})$.

We therefore consider the left adjoint $\imath_\lambda : \mathbf{K}(R) \longrightarrow \mathbf{K}(\text{PInj})$ to the inclusion $\imath$ and note that for any complex X the unit of adjunction morphism $X \longrightarrow \imath_\lambda X$ is a $\mathbf{K}(\text{PInj})$-prenevelope of X. Indeed, $\imath_\lambda X \in \mathbf{K}(\text{PInj})$ and for any complex $Y \in \mathbf{K}(\text{PInj})$ the induced additive map

$$\text{Hom}_{\mathbf{K}}(\imath_\lambda X, Y) \longrightarrow \text{Hom}_{\mathbf{K}}(X, Y)$$

is bijective (and hence surjective).

We showed in Theorem 3.5 that $\imath_\lambda$ maps K-absolutely pure complexes to K-injective complexes. We wish to examine the extent to which $\imath_\lambda$ maps the homotopy category $\mathbf{K}(\text{Sfpinj})$ of strongly fp-injective modules into the homotopy category $\mathbf{K}(\text{Inj})$ of injective modules. The inclusion functor $\mathbf{K}(\text{Inj}) \hookrightarrow \mathbf{K}(R)$ admits a left adjoint; the reader may find proofs of this fact in [3], [16] and [20]. Following [24, §6.2], we call the complexes in the kernel of that left adjoint coacyclic and denote by $\mathbf{K}_{coac}(R)$ the corresponding triangulated subcategory of $\mathbf{K}(R)$. In other words, we say that a complex X is coacyclic if $\text{Hom}_{\mathbf{K}}(X, I) = 0$ for any complex of injective modules I. By restricting to the homotopy category $\mathbf{K}(\text{Sfpinj})$ of strongly fp-injective modules, we conclude that the inclusion $\jmath : \mathbf{K}(\text{Inj}) \hookrightarrow \mathbf{K}(\text{Sfpinj})$ admits a left adjoint $\jmath_\lambda : \mathbf{K}(\text{Sfpinj}) \longrightarrow \mathbf{K}(\text{Inj})$, whose kernel is the triangulated subcategory of coacyclic complexes of strongly fp-injective modules $\mathbf{K}_{coac}(\text{Sfpinj}) = \mathbf{K}_{coac}(R) \cap \mathbf{K}(\text{Sfpinj})$. Composing $\jmath$ with the quotient functor $q : \mathbf{K}(\text{Sfpinj}) \longrightarrow \mathbf{K}(\text{Sfpinj})/\mathbf{K}_{coac}(\text{Sfpinj})$, we obtain an equivalence of categories

$$\mathbf{K}(\text{Inj}) \stackrel{\jmath}{\hookrightarrow} \mathbf{K}(\text{Sfpinj}) \stackrel{q}{\longrightarrow} \mathbf{K}(\text{Sfpinj})/\mathbf{K}_{coac}(\text{Sfpinj}), \tag{4}$$

whose quasi-inverse is obtained from the left adjoint $\jmath_\lambda$ by passage to the Verdier quotient.

In order to relate diagrams (3) and (4), we need to compare the kernels $\mathbf{K}_{coac}(\text{Sfpinj})$ and $\mathbf{K}_{pac}(R)$ of the left adjoints $\jmath_\lambda$ and $\imath_\lambda$ of the respective inclusion functors $\jmath$ and $\imath$. To that end, we shall assume that the ring R is left coherent, in order to use Proposition 5.1 below. We note that, in the left coherent case, the class Sfpinj of strongly fp-injective modules coincides with the class Fpinj of fp-injective modules. (In fact, the equality of the classes Sfpinj and Fpinj is equivalent to the left coherence of the ring; cf. [17, Theorem 4.2(2)].)

Proposition 5.1. *If the ring R is left coherent, then any coacyclic complex of fp-injective modules is pure acyclic.*

Proof. This is proved in [24, Proposition 6.11], in the more general setting of a locally coherent Grothendieck category. □

Proposition 5.2. *Assume that R is left coherent and consider the embedding $\nu : \mathbf{K}(Fpinj) \hookrightarrow \mathbf{K}(R)$ and its restriction $\nu| : \mathbf{K}(Inj) \hookrightarrow \mathbf{K}(PInj)$.*

(i) There is an embedding $\nu' : \mathbf{K}(Fpinj)/\mathbf{K}_{coac}(Fpinj) \hookrightarrow \mathbf{D}_{pure}(R)$, which fits into the following commutative diagram (whose horizontal compositions are both equivalences)

$$\begin{array}{ccccc}
\mathbf{K}(Inj) & \stackrel{\jmath}{\hookrightarrow} & \mathbf{K}(Fpinj) & \stackrel{q}{\longrightarrow} & \mathbf{K}(Fpinj)/\mathbf{K}_{coac}(Fpinj) \\
{\scriptstyle \nu|}\downarrow & & {\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle \nu'} \\
\mathbf{K}(PInj) & \stackrel{\imath}{\hookrightarrow} & \mathbf{K}(R) & \stackrel{p}{\longrightarrow} & \mathbf{D}_{pure}(R)
\end{array}$$

(ii) The following diagram is commutative up to natural equivalence

$$\begin{array}{ccc} \mathbf{K}(Fpinj) & \xrightarrow{\jmath_\lambda} & \mathbf{K}(Inj) \\ \nu \downarrow & & \downarrow \nu| \\ \mathbf{K}(R) & \xrightarrow{\imath_\lambda} & \mathbf{K}(PInj) \end{array}$$

Proof. (i) Since $\mathbf{K}_{coac}(\text{Fpinj}) \subseteq \mathbf{K}_{pac}(R)$ (cf. Proposition 5.1), any chain map between two complexes of fp-injective modules, whose mapping cone is contained in $\mathbf{K}_{coac}(\text{Fpinj})$, induces an isomorphism in the quotient $\mathbf{K}(R)/\mathbf{K}_{pac}(R) = \mathbf{D}_{pure}(R)$. The functor ν' is then defined by the universal property of the quotient $\mathbf{K}(\text{Fpinj})/\mathbf{K}_{coac}(\text{Fpinj})$. It is full and faithful, since the same is true for $\nu|$ and the compositions $q\jmath$ and $p\imath$ are both category equivalences.

(ii) Let X be a complex of fp-injective modules. Then, the adjunction $\jmath_\lambda \vdash \jmath$ provides us with a distinguished triangle in $\mathbf{K}(\text{Fpinj})$ and hence in $\mathbf{K}(R)$

$$Y \longrightarrow X \longrightarrow \jmath_\lambda X \longrightarrow SY,$$

where $\jmath_\lambda X$ is a complex of injective modules and $Y \in \mathbf{K}_{coac}(\text{Fpinj})$. Since $\mathbf{K}_{coac}(\text{Fpinj}) \subseteq \mathbf{K}_{pac}(R)$, the complex Y is pure acyclic and hence $\text{Hom}_{\mathbf{K}}(Y, W) = 0 = \text{Hom}_{\mathbf{K}}(SY, W)$ for any complex of pure-injective modules W. It follows that the induced map

$$\text{Hom}_{\mathbf{K}}(\jmath_\lambda X, W) \longrightarrow \text{Hom}_{\mathbf{K}}(X, W)$$

is bijective for any complex of pure-injective modules W. Since $\mathbf{K}(\text{Inj}) \subseteq \mathbf{K}(\text{PInj})$, we conclude that the morphism $X \longrightarrow \jmath_\lambda X$ has the universal property of the morphism $X \longrightarrow \imath_\lambda X$, which is obtained by applying the unit of the adjunction $\imath_\lambda \vdash \imath$ to the complex X. It follows that $\jmath_\lambda X$ and $\imath_\lambda X$ are naturally isomorphic in the category $\mathbf{K}(\text{PInj})$, as needed. □

Remarks 5.3. (i) For any module M the canonical map $\nu_M : M \longrightarrow D^2 M$ is easily seen to be a PInj-preenvelope of M.[4] If R is left coherent and M is (strongly) fp-injective, the right module DM is flat (cf. [17, Theorem 4.2(4)]) and hence $D^2 M$ is injective. Therefore, if R is left coherent, any fp-injective module admits a PInj-preenvelope by an injective module.

(ii) If R is left coherent and X is a complex of fp-injective modules, then Proposition 5.2(ii) implies that the complexes $\imath_\lambda X$ and $\jmath_\lambda X$ are homotopy equivalent. In particular, X admits a $\mathbf{K}(\text{PInj})$-preenvelope $X \longrightarrow \imath_\lambda X \simeq \jmath_\lambda X$, whose codomain $\jmath_\lambda X$ is a complex of injective modules. This is the analogue of the result on modules described in (i) above.

Corollary 5.4. *If the ring R is left coherent, then the left adjoint $\jmath_\lambda : \mathbf{K}(Fpinj) \longrightarrow \mathbf{K}(Inj)$ to the embedding $\jmath : \mathbf{K}(Inj) \hookrightarrow \mathbf{K}(Fpinj)$ maps K-absolutely pure complexes of fp-injective modules to K-injective complexes of injective modules.*

Proof. Let X be a K-absolutely pure complex of fp-injective modules. Proposition 5.2(ii) asserts that the complexes $\imath_\lambda X$ and $\jmath_\lambda X$ are homotopy equivalent. Since $\imath_\lambda X$ is K-injective (cf. Theorem 3.5), it follows that $\jmath_\lambda X$ is K-injective as well. □

Addendum. While writing down this paper, we were informed that Gillespie has also studied in [11], working independently, the K-absolutely pure complexes. He showed that, together with the class $C_{ac}(\text{PProj})$ of acyclic complexes of pure-projective modules, they form a complete cotorsion pair in the exact category of complexes, endowed with the pure exact structure. He also showed that the quotient $\mathbf{K}(R)/\mathbf{K}$-abspure is part of a recollement with the derived category $\mathbf{D}(R)$ and the pure derived category $\mathbf{D}_{pure}(R)$. Consequently, Gillespie proves that

---

[4] If J is a pure-injective module, then the pure embedding $\nu_J : J \longrightarrow D^2 J$ admits a left inverse r. We may thus extend any linear map $f : M \longrightarrow J$ to $D^2 M$, by means of the composition $D^2 M \xrightarrow{D^2 f} D^2 J \xrightarrow{r} J$.

$\mathbf{K}_{ac}(\text{PProj}) \simeq \mathbf{K}(R)/\mathbf{K}$-abspure is a compactly generated triangulated category. Apart from the definition of K-absolutely pure complexes and Theorem 2.5, there is no overlap between his work and ours.

Acknowledgments. It is a pleasure to thank G. Dalezios for useful discussions on the results presented in this work.

References

[1] Avramov, L., Foxby, H.-B.: Homological dimensions of unbounded complexes. J. Pure Appl. Algebra **71**, 129-155 (1991)

[2] Bazzoni, S., Cortés-Izurdiaga, M., Estrada, S.: Periodic modules and acyclic complexes. Algebr. Represent. Theory **23**, 1861-1883 (2020)

[3] Bravo, D., Enochs, E., Iacob, A., Jenda, O., Rada, J.: Cotorsion pairs in $C(R\text{-Mod})$. Rocky Mountain J. Math. **42**, 1787-1802 (2012)

[4] Christensen, L.W., Estrada, S., Liang, L., Thompson, P., Wue, D., Yang, G.: A refinement of Gorenstein flat dimension via the flat–cotorsion theory. J. Algebra **567**, 346–370 (2021)

[5] Emmanouil, I.: On pure acyclic complexes. J. Algebra **465**, 190-213 (2016)

[6] Emmanouil, I.: On the relation between K-flatness and K-projectivity. J. Algebra **517**, 320-335 (2019)

[7] Emmanouil, I.: K-flatness and orthogonality in homotopy categories. (to appear in Israel J. Math.)

[8] Enochs, E.E., Jenda, O.M.G.: Relative Homological Algebra. Berlin: Walter de Gruyter 2000

[9] Gillespie, J.: The flat model structure on $Ch(R)$. Trans. Amer. Math. Soc. **356**, 3369-3390 (2004)

[10] Gillespie, J.: K-flat complexes and derived categories. arXiv:2107.13042

[11] Gillespie, J.: The homotopy category of acyclic complexes of pure-projective modules. arXiv:2201.05542

[12] Göbel, R., Trlifaj, J.: Approximations and endomorphism algebras of modules, de Gruyter Expositions in Mathematics **41**, Walter de Gruyter GmbH & Co. KG, Berlin, 2006

[13] Holm, H.: Gorenstein homological dimensions. J. Pure Appl. Algebra **189**, 167-193 (2004)

[14] Jensen, C.U., Lenzing, H.: Model-Theoretic Algebra with Particular Emphasis on Fields, Rings, Modules. Algebra, Logic Appl. **2**, Gordon and Breach 1989

[15] Krause, H.: Localization theory for triangulated categories. In Triangulated categories, vol 375 of London Math. Soc. Lecture Note Ser. **375**, 161–235, Cambridge Univ. Press, Cambridge, 2010

[16] Krause, H.: Approximations and adjoints in homotopy categories. Math. Ann. **353**, 765–781 (2012)

[17] Li, W., Guan, J., Ouyang, B.: Strongly FP-injective modules. Comm. Algebra **45**, 3816-3824 (2017)

[18] Neeman, A.: Triangulated Categories. Annals of Mathematics Studies **148**, Princeton University Press, Princeton, NJ, 2001

[19] Neeman, A.: The homotopy category of flat modules, and Grothendieck duality. Invent. Math. **174**, 255-308 (2008)

[20] Neeman, A.: The homotopy category of injectives. Algebra Number Theory **8**, 429-456 (2014)

[21] Saroch, J., Stovícek, J.: Singular compactness and definability for Σ-cotorsion and Gorenstein modules. Selecta Math. (N.S.) **26**, 2020, Paper No. 23

[22] Spaltenstein, N.: Resolutions of unbounded complexes. Compos. Math. **65**, 121-154 (1988)

[23] Stenström, B.: Coherent rings and FP-injective modules. J. London Math. Soc. **2**, 323–329 (1970)

[24] Stovicek, J.: On purity and applications to coderived and singularity categories. arxiv:1412.1615

[25] Warfield, R.B.: Purity and algebraic compactness for modules. Pacific J. Math. **28**, 699-719 (1969)

[26] Yang, X., Ding, N.: On a question of Gillespie. Forum Math. **27**, 3205–3231 (2015)

[27] Zimmermann-Huisgen, B.: Rings whose right modules are direct sums of indecomposable modules. Proc. Amer. Math. Soc. **77**, 191-197 (1979)

Department of Mathematics, University of Athens, Athens 15784, Greece
E-mail addresses: `emmanoui@math.uoa.gr` and `kaperonn@math.uoa.gr`