HOMOLOGICAL INDEPENDENCE OF INFINITE SYZYGIES

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Abstract. We consider modules that appear as syzygies of acyclic complexes of flat modules and examine a certain condition on pairs of such modules, that generalizes the vanishing of Tate homology. If this condition is satisfied for two modules over a commutative ring, then (a) the tensor product of the two modules is also a syzygy of an acyclic complex of flat modules and (b) the syzygy modules of the tensor product of the corresponding acyclic complexes of flat modules is an acyclic complex of flat modules as well, whose syzygies can be expressed in terms of the syzygies of the factor complexes. We also examine the analogous (dual) case of homomorphism groups of infinite syzygies.

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0. Introduction

In homological algebra, the notion of a (projective, injective or flat) resolution of a module refers to a complex, which is bounded on one side and has certain additional properties. In several applications of the theory, one faces problems that require the use of complexes which are unbounded on both sides. The appropriate analogues of resolutions in this setting were first studied in [11] and [1]. In an ordinary projective resolution of a module M, the syzygies $(\Omega^n M)_n$ appear as kernels of higher differentials. On the other hand, if M is the kernel of an acyclic unbounded complex of projective modules, then not only does M have its own sequence of syzygy modules, but for all positive n the module M is itself the n-th syzygy of a suitable module K_{-n} (and $K_{-n} = \Omega^i K_{-n-i}$ for all $n, i > 0$). This is an algebraic analogue of the notion of an infinite loop space in topology. Modules that appear as kernels of acyclic unbounded complexes have certain unexpected (and perhaps surprising) properties. For example, if the kernels of an acyclic complex of projective modules are all flat, then they are necessarily projective (and hence the complex is contractible); for a proof of that result, the reader is referred to [3, Theorem 2.5], [10, Theorem 8.6] and [5, Proposition 7.6]. As another example, we note that the kernels of any acyclic complex of injective modules are necessarily cotorsion; cf. [12, Corollary 5.9] and [2, Theorem 4.1(2)]. We also note that the kernels of certain unbounded acyclic complexes of projective, injective and flat modules form

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the building blocks of Gorenstein homological algebra; see [7] and [8] for the definitions of Gorenstein projective, Gorenstein injective and Gorenstein flat modules.

It may be of interest to examine the behaviour of modules that appear as kernels of acyclic complexes as above with respect to standard module operations, such as the formation of tensor products and groups of homomorphisms. Assuming that the ground ring R is commutative, we may consider two modules M, N that appear as kernels of acyclic complexes of flat modules X, Y respectively and ask:

Is the tensor product $M \otimes N$ also the kernel of an acyclic complex of flat modules?

There are well-known spectral sequence methods that may be used in order to analyse the acyclicity of the tensor product complex $X \otimes Y$ (cf. [4]). Then, one may also ask:

Assuming that the complex $X \otimes Y$ is acyclic, how can one compute the kernels of its differentials, in terms of the kernels of the differentials of X and Y?

We examine a certain (homological independence) condition on the two complexes X and Y that (a) enables us to answer affirmatively the first question above, using the construction described in [6], (b) implies the acyclicity of the tensor product complex $X \otimes Y$ and (c) enables us to express the kernels of the complex $X \otimes Y$ as extensions, that involve the kernels of the complexes X and Y. These results are presented in Theorem 1.10 and Corollary 2.7. We note that the homological independence condition mentioned above is actually a property characterizing the pair (M, N), i.e. it does not depend on the choice of the acyclic complexes of flat modules X and Y, wherein M and N appear as kernels (cf. Proposition 2.3). If either one of the two modules is Gorenstein projective, then that condition amounts to the vanishing of the Tate homology groups $\widehat{\mathrm{Tor}}_*^R(M, N)$.

An analogous (dual) approach can be followed regarding groups of homomorphisms. Let M, N be two modules and assume that M is a kernel of an acyclic complex of projective modules X and N is a kernel of an acyclic complex of injective modules Y. Then, we describe another homological independence condition on the two complexes X and Y (which is really a property of the pair (M, N) as before and generalizes the vanishing of Tate cohomology) and show that under its presence the following hold: (a) if the ring R is commutative, then the module $\mathrm{Hom}_R(M, N)$ is a kernel of an acyclic complex of injective modules, (b) the Hom complex $\mathrm{Hom}_R(X, Y)$ is acyclic and (c) the kernels of the complex $\mathrm{Hom}_R(X, Y)$ may be expressed as certain extensions involving the kernels of the complexes X and Y. We note that the acyclicity of Hom complexes has been extensively studied, being related to a notion of orthogonality in the homotopy category of the ring; see, for example, [9, §3].

Notations and terminology. Unless otherwise specified, all modules considered in this paper are left modules over a fixed associative unital ring R. If M is a right R-module and N is a (left) R-module, we denote by $M \otimes N$ the tensor product $M \otimes_R N$. If N, N' are two R-modules then we denote by $[N, N']$ the group of homomorphisms $\mathrm{Hom}_R(N, N')$ and adopt a similar notation for the additive maps that are induced by linear maps in the first, the second or both variables.

1. The syzygies of a tensor product of flat complexes

In this section, we examine the tensor product of two acyclic complexes of flat modules. Under certain conditions (which imply that the tensor product complex is acyclic), we compute the syzygy groups of that tensor product complex, in terms of the syzygies of the factors.

I. The tensor product of two differential modules. A differential R-module is a pair (X, ∂_X), where X is a $\mathbb{Z}$-graded R-module and $\partial_X : X \longrightarrow X$ is the differential, i.e. a square zero linear endomorphism of X, which is graded of degree -1. We view X as the direct sum of its homogeneous components and say that the differential module (X, ∂_X) if flat (resp. projective) if this is the case for the underlying R-module X. We denote $B_X = \operatorname{im} \partial_X$ and $Z_X = \ker \partial_X$; these are graded submodules of X. We also let $\imath_X : B_X \hookrightarrow X$ and $\jmath_X : Z_X \hookrightarrow X$ be the inclusion maps and denote by $\theta_X : X \longrightarrow B_X$ the surjective linear map induced by ∂_X (so that $\imath_X \circ \theta_X = \partial_X$). Since $B_X \subseteq Z_X$, we may define the homology module $H_X = Z_X/B_X$. We say that the differential R-module (X, ∂_X) is acyclic if $H_X = 0$, i.e. if $B_X = Z_X$. The differential R-modules are the objects of a category $\Delta(R)$, whose morphisms are the homogeneous (i.e. graded of degree zero) linear maps that commute with the respective differentials.

The tensor product of $\mathbb{Z}$-graded modules induces a bifunctor

$$\Delta(R^{op}) \times \Delta(R) \longrightarrow \Delta(\mathbb{Z}).$$

If (X, ∂_X) is a differential right R-module and (Y, ∂_Y) a differential (left) R-module, then the differential abelian group $(X \otimes Y, \partial_{X\otimes Y})$ is defined as follows: The graded abelian group $X \otimes Y$ consists in degree n of $\bigoplus_{i+j=n} X_i \otimes Y_j$ and the differential $\partial_{X\otimes Y}$ is given by the additive map

$$\partial_X \otimes 1 + 1 \otimes \partial_Y : X \otimes Y \longrightarrow X \otimes Y.$$

We adopt the usual sign convention for the tensor product of two graded maps[1], so that for any homogeneous element $x \in X$ of degree n and any $y \in Y$ the tensor product differential $\partial_{X\otimes Y}$ maps $x \otimes y$ onto $\partial_X(x) \otimes y + (-1)^n x \otimes \partial_Y(y)$.

Lemma 1.1. *Let (X, ∂_X) be a differential right R-module and (Y, ∂_Y) a differential R-module. Then, the additive maps*

$$\jmath_X \otimes \imath_Y : Z_X \otimes B_Y \longrightarrow X \otimes Y \quad \textit{and} \quad \imath_X \otimes \jmath_Y : B_X \otimes Z_Y \longrightarrow X \otimes Y$$

factor through the subgroup $B_{X\otimes Y} \subseteq X \otimes Y$.

Proof. If $x \in Z_X$ is a homogeneous element of degree n and $y \in B_Y$, then $y = \partial_Y(y')$ for some $y' \in Y$ and hence $x \otimes y = x \otimes \partial_Y(y') = (-1)^n \partial_{X\otimes Y}(x \otimes y') \in X \otimes Y$; this proves that $\operatorname{im}(\jmath_X \otimes \imath_Y) \subseteq B_{X\otimes Y}$. Similarly, if $x \in B_X$ and $y \in Z_Y$, then $x = \partial_X(x')$ for some $x' \in X$ and hence $x \otimes y = \partial_X(x') \otimes y = \partial_{X\otimes Y}(x' \otimes y) \in X \otimes Y$; this proves that $\operatorname{im}(\imath_X \otimes \jmath_Y) \subseteq B_{X\otimes Y}$. □

Lemma 1.2. *Let (X, ∂_X) be a differential right R-module, (Y, ∂_Y) a differential R-module and assume that one of them is both acyclic and flat.*

(i) If $Tor_1^R(Z_X, Z_Y) = 0$, then the additive map $\jmath_X \otimes \jmath_Y : Z_X \otimes Z_Y \longrightarrow X \otimes Y$ is injective.

(ii) If $Tor_1^R(B_X, B_Y) = 0$, then the additive map $\imath_X \otimes \imath_Y : B_X \otimes B_Y \longrightarrow X \otimes Y$ is injective.

Proof. We assume that (Y, ∂_Y) is acyclic and Y is flat. (Symmetric arguments to the ones that follow apply if we assume that (X, ∂_X) is acyclic and X is flat.) We note that the surjective linear map $\theta_Y : Y \longrightarrow B_Y$ induces an isomorphism of R-modules $\operatorname{coker} \jmath_Y = Y/Z_Y \simeq B_Y$.

(i) We express the additive map $\jmath_X \otimes \jmath_Y$ as the composition

$$Z_X \otimes Z_Y \xrightarrow{1\otimes \jmath_Y} Z_X \otimes Y \xrightarrow{\jmath_X \otimes 1} X \otimes Y.$$

[1] Whenever $X \xrightarrow{f} X' \xrightarrow{g} X''$ and $Y \xrightarrow{h} Y' \xrightarrow{k} Y''$ are graded linear maps of degrees $\widetilde{f}$, $\widetilde{g}$, $\widetilde{h}$ and $\widetilde{k}$ respectively, then $(g \otimes k) \circ (f \otimes h) = (-1)^{\widetilde{k}\widetilde{f}} (g \circ f) \otimes (k \circ h) : X \otimes Y \longrightarrow X'' \otimes Y''$.

Then, the result follows since both of the additive maps $1 \otimes \jmath_Y$ and $\jmath_X \otimes 1$ are injective: The injectivity of $1 \otimes \jmath_Y$ follows since $\mathrm{Tor}_1^R(Z_X, \mathrm{coker}\, \jmath_Y) \simeq \mathrm{Tor}_1^R(Z_X, B_Y) = \mathrm{Tor}_1^R(Z_X, Z_Y) = 0$, whereas the injectivity of $\jmath_X \otimes 1$ follows from the flatness of Y.

(ii) We express the additive map $\imath_X \otimes \imath_Y$ as the composition

$$B_X \otimes B_Y \xrightarrow{1\otimes\imath_Y} B_X \otimes Y \xrightarrow{\imath_X\otimes 1} X \otimes Y.$$

Then, the result follows since both of the additive maps $1 \otimes \imath_Y$ and $\imath_X \otimes 1$ are injective: The injectivity of $1 \otimes \imath_Y$ follows since $\mathrm{Tor}_1^R(B_X, \mathrm{coker}\, \imath_Y) = \mathrm{Tor}_1^R(B_X, \mathrm{coker}\, \jmath_Y) \simeq \mathrm{Tor}_1^R(B_X, B_Y) = 0$, whereas the injectivity of $\imath_X \otimes 1$ follows from the flatness of Y. □

Corollary 1.3. *Let (X, ∂_X) be a differential right R-module, (Y, ∂_Y) a differential R-module and assume that one of them is acyclic.*

(i) There is a unique homogeneous additive map $\kappa : Z_X \otimes Z_Y \longrightarrow B_{X\otimes Y}$, which is such that the composition $Z_X \otimes Z_Y \xrightarrow{\kappa} B_{X\otimes Y} \xrightarrow{\imath_{X\otimes Y}} X \otimes Y$ is equal to $\jmath_X \otimes \jmath_Y : Z_X \otimes Z_Y \longrightarrow X \otimes Y$. Any elementary tensor $x \otimes y \in Z_X \otimes Z_Y$ is mapped by κ onto $x \otimes y \in B_{X\otimes Y} \subseteq X \otimes Y$.

(ii) If one of the two differential modules is both acyclic and flat and $\mathrm{Tor}_1^R(Z_X, Z_Y) = 0$, then κ is injective.

Proof. Assertion (i) follows from Lemma 1.1. Since both $\imath_{X\otimes Y}$ and the composition $\imath_{X\otimes Y} \circ \kappa = \jmath_X \otimes \jmath_Y$ are homogeneous, we conclude that κ is homogeneous as well. Finally, Lemma 1.2(i) proves assertion (ii). □

We note that for any differential right R-module (X, ∂_X) and any differential R-module (Y, ∂_Y) we have an inclusion

$$Z_{X\otimes Y} = \ker\left(X \otimes Y \xrightarrow{\partial_{X\otimes Y}} X \otimes Y\right) \subseteq \ker\left(X \otimes Y \xrightarrow{\partial_X\otimes\partial_Y} X \otimes Y\right).$$

Indeed, if an element $\xi \in X \otimes Y$ is such that $\partial_{X\otimes Y}\xi = 0$, then $(\partial_X \otimes 1)\xi + (1 \otimes \partial_Y)\xi = 0$ and hence, applying $\partial_X \otimes 1$ to both sides, we conclude that $(\partial_X \otimes \partial_Y)\xi = 0$. Since the additive map $\theta_{X\otimes Y} : X \otimes Y \longrightarrow B_{X\otimes Y}$ induces an isomorphism $(X \otimes Y)/Z_{X\otimes Y} \simeq B_{X\otimes Y}$, it follows that we may factor $\partial_X \otimes \partial_Y : X \otimes Y \longrightarrow X \otimes Y$ as the composition $X \otimes Y \xrightarrow{\theta_{X\otimes Y}} B_{X\otimes Y} \xrightarrow{l} X \otimes Y$ for a unique additive map $l : B_{X\otimes Y} \longrightarrow X \otimes Y$. The map $\theta_{X\otimes Y}$ being surjective, it is clear that $\mathrm{im}\, l = \mathrm{im}\,(\partial_X \otimes \partial_Y)$. Since $\partial_X \otimes \partial_Y$ factors as the composition $X \otimes Y \xrightarrow{\theta_X\otimes\theta_Y} B_X \otimes B_Y \xrightarrow{\imath_X\otimes\imath_Y} X \otimes Y$ and $\theta_X \otimes \theta_Y$ is surjective, we finally conclude that $\mathrm{im}\, l = \mathrm{im}\left(B_X \otimes B_Y \xrightarrow{\imath_X\otimes\imath_Y} X \otimes Y\right)$.

Corollary 1.4. *Let (X, ∂_X) be a differential right R-module, (Y, ∂_Y) a differential R-module and assume that one of them is both acyclic and flat. If $\mathrm{Tor}_1^R(B_X, B_Y) = 0$, then there exists a unique surjective additive map $\lambda : B_{X\otimes Y} \longrightarrow B_X \otimes B_Y$, which is such that the composition*

$$X \otimes Y \xrightarrow{\theta_{X\otimes Y}} B_{X\otimes Y} \xrightarrow{\lambda} B_X \otimes B_Y \xrightarrow{\imath_X\otimes\imath_Y} X \otimes Y$$

coincides with the additive map $\partial_X \otimes \partial_Y : X \otimes Y \longrightarrow X \otimes Y$. The map λ is graded of degree -1; it maps any element $\xi \in B_{X\otimes Y}$ onto $(\partial_X \otimes 1)\xi = -(1 \otimes \partial_Y)\xi \in B_X \otimes B_Y \subseteq X \otimes Y$.

Proof. Since $\imath_X \otimes \imath_Y$ is injective (cf. Lemma 1.2(ii)), the map l in the discussion above can be expressed as the composition $B_{X\otimes Y} \xrightarrow{\lambda} B_X \otimes B_Y \xrightarrow{\imath_X\otimes\imath_Y} X \otimes Y$ for a unique (surjective) additive map λ. The map λ is graded of degree -1, since $\imath_X \otimes \imath_Y$ is homogeneous, $\theta_{X\otimes Y}$ is graded of degree -1 and the composition $(\imath_X \otimes \imath_Y) \circ \lambda \circ \theta_{X\otimes Y} = \partial_X \otimes \partial_Y$ is graded of degree -2. Finally, for any $\xi \in B_{X\otimes Y}$ we may write $\xi = \partial_{X\otimes Y}\eta$ for a suitable $\eta \in X \otimes Y$ and then compute $\lambda(\xi) = (\partial_X \otimes \partial_Y)\eta = (\partial_X \otimes 1)(\partial_X \otimes 1 + 1 \otimes \partial_Y)\eta = (\partial_X \otimes 1)\xi$. Since $\partial_{X\otimes Y}$ vanishes on $B_{X\otimes Y}$, it is clear that $(\partial_X \otimes 1)\xi = -(1\otimes \partial_Y)\xi$. □

In order to compute the kernel of the map λ defined above and obtain conditions that imply the acyclicity of the tensor product of two differential modules, we need the following couple of simple facts.

Lemma 1.5. *Let (X,∂_X) be a differential right R-module, (Y,∂_Y) a differential R-module and assume that one of them is both acyclic and flat. If $\mathrm{Tor}_1^R(B_X,B_Y)=0$, then*

$$\ker\left(X\otimes Y \stackrel{\partial_X\otimes\partial_Y}{\longrightarrow} X\otimes Y\right) = \mathrm{im}\left(Z_X\otimes Y \stackrel{\jmath_X\otimes 1}{\longrightarrow} X\otimes Y\right) + \mathrm{im}\left(X\otimes Z_Y \stackrel{1\otimes\jmath_Y}{\longrightarrow} X\otimes Y\right).$$

Proof. The additive map $\partial_X\otimes\partial_Y$ factors as the composition $X\otimes Y \stackrel{\theta_X\otimes\theta_Y}{\longrightarrow} B_X\otimes B_Y \stackrel{\imath_X\otimes\imath_Y}{\longrightarrow} X\otimes Y$ and $\imath_X\otimes\imath_Y$ is injective (cf. Lemma 1.2(ii)). It follows that $\ker(\partial_X\otimes\partial_Y) = \ker(\theta_X\otimes\theta_Y)$. Since the linear maps θ_X and θ_Y induce isomorphisms $X/Z_X \simeq B_X$ and $Y/Z_Y \simeq B_Y$ respectively, the identification of the subgroup $\ker(\theta_X\otimes\theta_Y)$ as claimed is a well-known consequence of the right exactness of the tensor product. □

Lemma 1.6. *Let (X,∂_X) be a differential right R-module and (Y,∂_Y) a differential R-module, such that $\mathrm{Tor}_1^R(B_X,B_Y)=0$. Then,*

$$\mathrm{im}\left(Z_X\otimes Y \stackrel{\jmath_X\otimes 1}{\longrightarrow} X\otimes Y\right) \cap \mathrm{im}\left(X\otimes Z_Y \stackrel{1\otimes\jmath_Y}{\longrightarrow} X\otimes Y\right) = \mathrm{im}\left(Z_X\otimes Z_Y \stackrel{\jmath_X\otimes\jmath_Y}{\longrightarrow} X\otimes Y\right).$$

Proof. The surjective linear map $\theta_Y : Y \longrightarrow B_Y$ induces an isomorphism $\operatorname{coker}\jmath_Y = Y/Z_Y \simeq B_Y$ and hence the group $\mathrm{Tor}_1^R(B_X, \operatorname{coker}\jmath_Y) \simeq \mathrm{Tor}_1^R(B_X,B_Y)$ is trivial. We therefore conclude that the additive map $1\otimes\jmath_Y : B_X\otimes Z_Y \longrightarrow B_X\otimes Y$ is injective. Then, the result follows by inspecting the commutative diagram

$$\begin{array}{ccccccc}
Z_X\otimes Z_Y & \stackrel{\jmath_X\otimes 1}{\longrightarrow} & X\otimes Z_Y & \stackrel{\theta_X\otimes 1}{\longrightarrow} & B_X\otimes Z_Y & \longrightarrow & 0 \\
\downarrow {\scriptstyle 1\otimes\jmath_Y} & & \downarrow {\scriptstyle 1\otimes\jmath_Y} & & \downarrow {\scriptstyle 1\otimes\jmath_Y} & & \\
Z_X\otimes Y & \stackrel{\jmath_X\otimes 1}{\longrightarrow} & X\otimes Y & \stackrel{\theta_X\otimes 1}{\longrightarrow} & B_X\otimes Y & \longrightarrow & 0
\end{array}$$

whose rows are exact. □

Proposition 1.7. *Let (X,∂_X) be a differential right R-module and (Y,∂_Y) a differential R-module. We assume that one of them is both acyclic and flat and $\mathrm{Tor}_1^R(B_X,B_Y)=0$.*

(i) There exists an exact sequence of graded abelian groups

$$Z_X\otimes Z_Y \stackrel{\kappa}{\longrightarrow} B_{X\otimes Y} \stackrel{\lambda}{\longrightarrow} B_X\otimes B_Y \longrightarrow 0.$$

The map κ is homogeneous and maps any elementary tensor $x\otimes y \in Z_X\otimes Z_Y$ onto $x\otimes y \in B_{X\otimes Y} \subseteq X\otimes Y$, whereas the map λ is graded of degree -1 and maps any element $\xi \in B_{X\otimes Y}$ onto $(\partial_X\otimes 1)\xi = -(1\otimes\partial_Y)\xi \in B_X\otimes B_Y \subseteq X\otimes Y$.

(ii) If we also have $\mathrm{Tor}_1^R(Z_X,Z_Y)=0$, then the map κ in (i) above is injective.

(iii) If both differential modules are acyclic, then $(X\otimes Y, \partial_{X\otimes Y})$ is acyclic as well.

Proof. (i) Since one of the two differential modules is both acyclic and flat and $\mathrm{Tor}_1^R(B_X,B_Y)=0$, the existence of κ and λ (and the surjectivity of λ) follows from Corollary 1.3(i) and Corollary 1.4 respectively. It only remains to show that $\mathrm{im}\,\kappa = \ker\lambda$.

In view of Lemma 1.2(ii), we regard $B_X\otimes B_Y$ as a subgroup of $X\otimes Y$. For any elementary tensor $x\otimes y \in Z_X\otimes Z_Y$ the element $\kappa(x\otimes y) = x\otimes y \in X\otimes Y$ is clearly annihilated by $\partial_X\otimes 1$, whence the inclusion $\mathrm{im}\,\kappa \subseteq \ker\lambda$. In order to prove that $\ker\lambda \subseteq \mathrm{im}\,\kappa$, we let $\xi \in B_{X\otimes Y}$ be an element in $\ker\lambda$ and write $\xi = \partial_{X\otimes Y}\eta$ for some $\eta \in X\otimes Y$. Then, $(\partial_X\otimes\partial_Y)\eta = 0 \in X\otimes Y$

and hence we may write $\eta = \eta_1 + \eta_2$ for suitable elements $\eta_1 \in \operatorname{im}\left(Z_X \otimes Y \stackrel{\jmath_X \otimes 1}{\longrightarrow} X \otimes Y\right)$ and $\eta_2 \in \operatorname{im}\left(X \otimes Z_Y \stackrel{1 \otimes \jmath_Y}{\longrightarrow} X \otimes Y\right)$; cf. Lemma 1.5. We note that the images of the compositions

$$Z_X \otimes Y \stackrel{\jmath_X \otimes 1}{\longrightarrow} X \otimes Y \stackrel{\partial_{X\otimes Y}}{\longrightarrow} X \otimes Y \quad \text{and} \quad X \otimes Z_Y \stackrel{1 \otimes \jmath_Y}{\longrightarrow} X \otimes Y \stackrel{\partial_{X\otimes Y}}{\longrightarrow} X \otimes Y$$

are equal to the images of the additive maps

$$\jmath_X \otimes \imath_Y : Z_X \otimes B_Y \longrightarrow X \otimes Y \quad \text{and} \quad \imath_X \otimes \jmath_Y : B_X \otimes Z_Y \longrightarrow X \otimes Y$$

respectively. Indeed, for any homogeneous element $x \in Z_X$ of degree n and any $y \in Y$ we have $\partial_{X\otimes Y}(x \otimes y) = (-1)^n x \otimes \partial_Y y \in X \otimes Y$, whereas for any $x' \in X$ and $y' \in Z_Y$ we have $\partial_{X\otimes Y}(x' \otimes y') = \partial_X x' \otimes y' \in X \otimes Y$. Since one of the two differential modules is acyclic (and hence $Z_X = B_X$ or $Z_Y = B_Y$), it follows that

$$\xi = \partial_{X\otimes Y}\eta = \partial_{X\otimes Y}\eta_1 + \partial_{X\otimes Y}\eta_2 \in \operatorname{im}(\jmath_X \otimes \imath_Y) + \operatorname{im}(\imath_X \otimes \jmath_Y) = \operatorname{im}\left(Z_X \otimes Z_Y \stackrel{\jmath_X \otimes \jmath_Y}{\longrightarrow} X \otimes Y\right).$$

Therefore, we conclude that $\xi \in \operatorname{im} \kappa$.

(ii) This follows from Corollary 1.3(ii).

(iii) We work under the assumption that Y is flat. (An analogous argument may be used if we assume that X is flat.) We fix $\xi \in Z_{X\otimes Y}$ and note that the equality $\partial_{X\otimes Y}\xi = 0$ implies that $(\partial_X \otimes 1)\xi = -(1 \otimes \partial_Y)\xi$ is an element in the intersection

$$\operatorname{im}\left(B_X \otimes Y \stackrel{\imath_X \otimes 1}{\longrightarrow} X \otimes Y\right) \cap \operatorname{im}\left(X \otimes B_Y \stackrel{1 \otimes \imath_Y}{\longrightarrow} X \otimes Y\right) \subseteq X \otimes Y.$$

In view of our assumptions and Lemma 1.6, we conclude that

$$(\partial_X \otimes 1)\xi = -(1 \otimes \partial_Y)\xi \in \operatorname{im}\left(B_X \otimes B_Y \stackrel{\imath_X \otimes \imath_Y}{\longrightarrow} X \otimes Y\right) = \operatorname{im}\left(X \otimes Y \stackrel{\partial_X \otimes \partial_Y}{\longrightarrow} X \otimes Y\right)$$

and hence $(\partial_X \otimes 1)\xi = -(1 \otimes \partial_Y)\xi = (\partial_X \otimes \partial_Y)\eta$ for a suitable element $\eta \in X \otimes Y$. Then, $(\partial_X \otimes 1)[\xi - (1 \otimes \partial_Y)\eta] = (\partial_X \otimes 1)\xi - (\partial_X \otimes \partial_Y)\eta = 0 \in X \otimes Y$. Since X is acyclic and Y is flat, the kernel of the additive map $\partial_X \otimes 1$ coincides with its image. It follows that $\xi - (1 \otimes \partial_Y)\eta = (\partial_X \otimes 1)\zeta$ for a suitable element $\zeta \in X \otimes Y$ and hence

$$\xi = (1 \otimes \partial_Y)\eta + (\partial_X \otimes 1)\zeta. \tag{1}$$

Applying $1 \otimes \partial_Y$ to both sides of the latter equality, it follows that $(1 \otimes \partial_Y)\xi = -(\partial_X \otimes \partial_Y)\zeta$ and hence $(\partial_X \otimes \partial_Y)\eta = -(1 \otimes \partial_Y)\xi = (\partial_X \otimes \partial_Y)\zeta$. Then, Lemma 1.5 implies that

$$\zeta - \eta \in \ker(\partial_X \otimes \partial_Y) = \operatorname{im}\left(B_X \otimes Y \stackrel{\imath_X \otimes 1}{\longrightarrow} X \otimes Y\right) + \operatorname{im}\left(X \otimes B_Y \stackrel{1 \otimes \imath_Y}{\longrightarrow} X \otimes Y\right).$$

Applying now $\partial_X \otimes 1$, it follows that

$$(\partial_X \otimes 1)\zeta - (\partial_X \otimes 1)\eta \in \operatorname{im}\left(X \otimes B_Y \stackrel{\partial_X \otimes \imath_Y}{\longrightarrow} X \otimes Y\right) = \operatorname{im}\left(X \otimes Y \stackrel{\partial_X \otimes \partial_Y}{\longrightarrow} X \otimes Y\right).$$

Therefore, $(\partial_X \otimes 1)\zeta - (\partial_X \otimes 1)\eta = (\partial_X \otimes \partial_Y)\tau$ for a suitable element $\tau \in X \otimes Y$ and hence $(\partial_X \otimes 1)\zeta = (\partial_X \otimes 1)\eta + (\partial_X \otimes \partial_Y)\tau$. Invoking (1) and the equality $\partial_{X\otimes Y} \circ (1 \otimes \partial_Y) = \partial_X \otimes \partial_Y$, we conclude that $\xi = (1 \otimes \partial_Y)\eta + (\partial_X \otimes 1)\eta + (\partial_X \otimes \partial_Y)\tau = \partial_{X\otimes Y}[\eta + (1 \otimes \partial_Y)\tau] \in B_{X\otimes Y}$. $\square$

Rearranging parts of Proposition 1.7, we may formulate the following result.

Theorem 1.8. *Let (X,∂_X) be a differential right R-module and (Y,∂_Y) a differential R-module. We assume that:*
(i) both differential modules are acyclic,
(ii) one of the two differential modules is flat and
(iii) $Tor_1^R(B_X,B_Y)=0$.
Then, the differential module $X\otimes Y$ is acyclic and there exists a short exact sequence of graded abelian groups

$$0 \longrightarrow B_X\otimes B_Y \xrightarrow{\kappa} B_{X\otimes Y} \xrightarrow{\lambda} B_X\otimes B_Y \longrightarrow 0.$$

The map κ is homogeneous and maps any elementary tensor $x\otimes y\in B_X\otimes B_Y$ onto $x\otimes y\in B_{X\otimes Y}\subseteq X\otimes Y$, whereas the map λ is graded of degree -1 and maps any element $\xi\in B_{X\otimes Y}$ onto $(\partial_X\otimes 1)\xi=-(1\otimes\partial_Y)\xi\in B_X\otimes B_Y\subseteq X\otimes Y$. □

Remark 1.9. Let (X,∂_X) and (Y,∂_Y) be differential modules as in Theorem 1.8. Then, there is an alternative way to describe the additive map λ therein. Indeed, if $p_X : X\longrightarrow X/B_X$ is the quotient map, then $\theta_X : X\longrightarrow B_X$ can be factored as the composition $X\xrightarrow{p_X} X/B_X \xrightarrow{\overline{\theta_X}} B_X$. Since X is acyclic, $\overline{\theta_X}$ is bijective. There is an analogous factorization associated with Y. Then, there is a unique additive map $\lambda' : B_{X\otimes Y}\longrightarrow X/B_X\otimes Y/B_Y$, which when followed by the isomorphism $\overline{\theta_X}\otimes\overline{\theta_Y} : X/B_X\otimes Y/B_Y\longrightarrow B_X\otimes B_Y$ coincides with the map λ. Since λ is graded of degree -1 and $\overline{\theta_X}\otimes\overline{\theta_Y}$ is graded of degree -2, it follows that λ' is graded of degree $+1$. For any $\xi\in X\otimes Y$ the element $\partial_{X\otimes Y}\xi\in B_{X\otimes Y}$ is mapped under λ' onto $(p_X\otimes p_Y)\xi\in X/B_X\otimes Y/B_Y$. Of course, there is an exact sequence of graded abelian groups

$$0 \longrightarrow B_X\otimes B_Y \xrightarrow{\kappa} B_{X\otimes Y} \xrightarrow{\lambda'} X/B_X\otimes Y/B_Y \longrightarrow 0,$$

which is essentially identified with that in Theorem 1.8.

II. Reformulation in the language of chain complexes. Let $X=((X_n)_n,\partial_X)$ be a chain complex of right R-modules and $Y=((Y_n)_n,\partial_Y)$ a chain complex of R-modules. Then, the tensor product complex $X\otimes Y=(((X\otimes Y)_n)_n,\partial_{X\otimes Y})$ is the chain complex of abelian groups with $(X\otimes Y)_n=\bigoplus_{i+j=n}X_i\otimes Y_j$ for all n and differential $\partial_{X\otimes Y}$, which maps any elementary tensor $x_i\otimes y_j\in X_i\otimes Y_j$ onto $\partial_X x_i\otimes y_j+(-1)^i x_i\otimes\partial_Y y_j$. We may reformulate Theorem 1.8, by considering the homogeneous parts of any given degree therein, in order to obtain some information about the boundary groups $(B_n(X\otimes Y))_n$ of the complex $X\otimes Y$, in terms of the boundary groups $(B_nX)_n$ and $(B_nY)_n$ of the complexes X and Y.

Theorem 1.10. *Let X be a chain complex of right R-modules and Y a chain complex of R-modules as above and assume that the following conditions are satisfied:*
(i) both complexes are acyclic,
(ii) one of the two complexes consists of flat modules and
(iii) $Tor_1^R(B_iX,B_jY)=0$ for all i,j.
Then, the tensor product complex $X\otimes Y$ is also acyclic and there exists a short exact sequence of abelian groups

$$0 \longrightarrow \bigoplus\nolimits_{i+j=n}B_iX\otimes B_jY \xrightarrow{\kappa} B_n(X\otimes Y) \xrightarrow{\lambda} \bigoplus\nolimits_{i+j=n-1}B_iX\otimes B_jY \longrightarrow 0$$

for all n. The map κ maps any elementary tensor $x_i\otimes y_j\in B_iX\otimes B_jY$ onto $x_i\otimes y_j\in B_{i+j}(X\otimes Y)\subseteq (X\otimes Y)_{i+j}$, whereas the map λ maps any element $\xi_n\in B_n(X\otimes Y)$ onto $(\partial_X\otimes 1)\xi_n=-(1\otimes\partial_Y)\xi_n\in\bigoplus_{i+j=n-1}B_iX\otimes B_jY\subseteq (X\otimes Y)_{n-1}$. □

Remarks 1.11. (i) As we pointed out in the Introduction, there are several techniques that may be used in order to analyse the acyclicity of the tensor product of two acyclic complexes X and Y. For example, if the modules $(B_i(X))_i$ are flat, then the acyclicity of the tensor product $X \otimes Y$ claimed in Theorem 1.10 follows from Neeman's result [10] (even if the complex Y is not necessarily acyclic). We believe that Theorem 1.10 is of some interest, as it also provides us with a description of the boundary groups of the (acyclic) tensor product complex $X \otimes Y$.

(ii) The short exact sequences in Theorem 1.10 are natural in X and Y. Therefore, if S and T are two rings, X is a complex of (S, R)-bimodules and Y is a complex of (R, T)-bimodules, then these short exact sequences are actually short exact sequences of (S, T)-bimodules. In particular, if the ring R is commutative, then all tensor products involved are R-modules and these short exact sequences are short exact sequences of R-modules.

2. The tensor product of syzygies of flat complexes

In this section, we consider the class of modules that appear as syzygies of acyclic complexes of flat modules and examine a certain homological condition, which implies that (over a commutative ring) the tensor product of two such modules is also a syzygy of an acyclic complex of flat modules.

I. Tor-independence of infinite syzygies. We denote by $\mathfrak{S}^{\infty}_{flat}(R)$ the class of those R-modules that may be expressed as syzygies of acyclic complexes of flat modules. In other words, an R-module M is in $\mathfrak{S}^{\infty}_{flat}(R)$ if and only if there exists an acyclic complex of flat modules X, such that $M = B_0X$ is its 0-th boundary module. The class $\mathfrak{S}^{\infty}_{flat}(R)$ is obviously closed under direct sums. We also consider the corresponding class $\mathfrak{S}^{\infty}_{flat}(R^{op})$ of right R-modules and examine carefully the condition appearing as assumption (iii) in Theorem 1.10.

Lemma 2.1. *Let X be an acyclic chain complex of flat right R-modules and Y an acyclic chain complex of flat R-modules. Then, the following conditions are equivalent:*
(i) $Tor^R_1(B_iX, B_0Y) = 0$ for all i,
(ii) $Tor^R_n(B_iX, B_0Y) = 0$ for all $n \geq 1$ and all i,
(iii) $Tor^R_1(B_iX, B_jY) = 0$ for all i, j and
(iv) $Tor^R_n(B_iX, B_jY) = 0$ for all $n \geq 1$ and all i, j.

Proof. Since $\mathrm{Tor}^R_n(B_iX, \_) = \mathrm{Tor}^R_1(B_{i+n-1}X, \_)$ for all $n \geq 1$ and all i, assertions (i) and (ii) are clearly equivalent. In fact, they are both equivalent to the acyclicity of the complex of abelian groups $X \otimes B_0Y$. For the same reason, assertions (iii) and (iv) are also equivalent; these are precisely the "for all j" versions of the former two assertions. Since (iii)→(i), it only remains to show that (ii)→(iii). To that end, we assume that (ii) holds and fix two integers i, j. If $j \geq 0$, then $\mathrm{Tor}^R_1(B_iX, B_jY) = \mathrm{Tor}^R_{1+j}(B_iX, B_0Y) = 0$. On the other hand, if $j < 0$, then $\mathrm{Tor}^R_1(B_iX, B_jY) = \mathrm{Tor}^R_{1-j}(B_{i+j}X, B_jY) = \mathrm{Tor}^R_1(B_{i+j}X, B_0Y) = 0$. □

Remark 2.2. Note that conditions (iii) and (iv) in Lemma 2.1 remain invariant by replacing the complex Y with any of its suspensions. It follows that, for any integer s, these conditions are also equivalent to the conditions obtained from (i) and (ii) by replacing the module B_0Y therein with B_sY. We also note that conditions (iii) and (iv) are symmetric with respect to the complexes X and Y. Therefore, these conditions are also equivalent to:
(i)' $\mathrm{Tor}^R_1(B_0X, B_jY) = 0$ for all j,
(ii)' $\mathrm{Tor}^R_n(B_0X, B_jY) = 0$ for all $n \geq 1$ and all j.

Proposition 2.3. *The following two conditions are equivalent for a pair of modules (M,N), where $M \in \mathfrak{S}^{\infty}_{flat}(R^{op})$ and $N \in \mathfrak{S}^{\infty}_{flat}(R)$:*

(i) There exists an acyclic complex of flat right R-modules X with $M = B_0X$ and an acyclic complex of flat R-modules Y with $N = B_0Y$, such that the equivalent conditions of Lemma 2.1, as supplemented by Remark 2.2, are satisfied for X and Y.

(ii) If X is any acyclic complex of flat right R-modules with $M = B_0X$ and Y is any acyclic complex of flat R-modules with $N = B_0Y$, then the equivalent conditions of Lemma 2.1, as supplemented by Remark 2.2, are satisfied for X and Y.

If these conditions are satisfied, we say that the modules M and N are Tor-independent.

Proof. Since $M \in \mathfrak{S}^{\infty}_{flat}(R^{op})$ and $N \in \mathfrak{S}^{\infty}_{flat}(R)$, it is clear that (ii)→(i). In order to prove that (i)→(ii), we fix a pair (X,Y) as in (i) and assume that X' (resp. Y') is another acyclic complex of flat right (resp. left) R-modules with $M = B_0X$ (resp. $N = B_0Y$). Since the pair (X,Y) satisfies the list of equivalent conditions in Lemma 2.1, condition (i) therein implies that the pair (X,Y') satisfies these equivalent conditions as well. Then, condition (i)' in Remark 2.2 implies that the pair (X',Y') also satisfies these equivalent conditions. □

In the special case where M or N is Gorenstein projective [7], Tor-independence is equivalent to the vanishing of the Tate homology groups $\widehat{\mathrm{Tor}}^R_*(M,N)$. Indeed, if we assume that M is Gorenstein projective and X is a totally acyclic complex of projective modules admitting M as a kernel, then the homology of the complex $X \otimes N$ is the Tate homology $\widehat{\mathrm{Tor}}^R_*(M,N)$; cf. [6, §2.4].

For any module $M \in \mathfrak{S}^{\infty}_{flat}(R^{op})$ we consider the class $M^{\diamond}$, consisting of those modules $N \in \mathfrak{S}^{\infty}_{flat}(R)$, which are Tor-independent to M. If X is an acyclic complex of flat right R-modules, such that $M = B_0X$, then $M^{\diamond} = \{N \in \mathfrak{S}^{\infty}_{flat}(R) : \text{ the complex } X \otimes N \text{ is acyclic}\}$; this follows from condition (i) in Lemma 2.1.

Lemma 2.4. *Let M be a right R-module in $\mathfrak{S}^{\infty}_{flat}(R^{op})$ and consider the class $M^{\diamond}$ defined above. Then:*

(i) If $(N_i)_i$ is a family of modules in $M^{\diamond}$, then $\bigoplus_i N_i \in M^{\diamond}$.

(ii) If $0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$ is a short exact sequence of modules in $\mathfrak{S}^{\infty}_{flat}(R)$ and two of these modules are contained in $M^{\diamond}$, then so is the third.

Proof. Let X be an acyclic complex of flat right R-modules, such that $M = B_0X$.

(i) Since $\bigoplus_i N_i \in \mathfrak{S}^{\infty}_{flat}(R^{op})$, the result follows from the acyclicity of the complex of abelian groups $X \otimes (\bigoplus_i N_i) = \bigoplus_i (X \otimes N_i)$.

(ii) The long exact sequence in homology, which is induced by the short exact sequence of complexes of abelian groups $0 \longrightarrow X \otimes N' \longrightarrow X \otimes N \longrightarrow X \otimes N'' \longrightarrow 0$ shows that if two of these three complexes are acyclic, then so is the third. □

Remark 2.5. For any $N \in \mathfrak{S}^{\infty}_{flat}(R)$ we may also consider the class ${}^{\diamond}N$, consisting of those modules $M \in \mathfrak{S}^{\infty}_{flat}(R^{op})$, which are Tor-independent to N. If Y is an acyclic complex of flat R-modules, such that $N = B_0Y$, then ${}^{\diamond}N$ consists of those modules $M \in \mathfrak{S}^{\infty}_{flat}(R^{op})$, which are such that the complex $M \otimes Y$ is acyclic; this follows from condition (i)' in Remark 2.2. The analogue of Lemma 2.4 holds for ${}^{\diamond}N$, i.e. ${}^{\diamond}N$ is closed under direct sums and has the 2-out-of-3 property for short exact sequences in $\mathfrak{S}^{\infty}_{flat}(R^{op})$.

II. The pinched tensor product complex. Let X be a chain complex of right R-modules and Y a chain complex of R-modules. We recall the definition of the pinched tensor product complex $X \otimes^{\bowtie} Y$, as defined in [6]. It is the total complex associated with the bicomplex of abelian groups pictured below

$$
\begin{array}{ccccccccccc}
 & & & & & & \vdots & & \vdots & & \\
 & & & & & & \downarrow & & \downarrow & & \\
 & & & & & & X_0 \otimes Y_1 & \longleftarrow & X_1 \otimes Y_1 & \longleftarrow & \cdots \\
 & & & & & & \downarrow & & \downarrow & & \\
\cdots & \longleftarrow & X_{-2} \otimes Y_{-1} & \longleftarrow & X_{-1} \otimes Y_{-1} & \xleftarrow{\delta} & X_0 \otimes Y_0 & \longleftarrow & X_1 \otimes Y_0 & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & & & & & \\
\cdots & \longleftarrow & X_{-2} \otimes Y_{-2} & \longleftarrow & X_{-1} \otimes Y_{-2} & & & & & & \\
 & & \downarrow & & \downarrow & & & & & & \\
 & & \vdots & & \vdots & & & & & &
\end{array}
\tag{2}
$$

Here, the abelian group $X_i \otimes Y_j$ is located in bidegree (i, j) (resp. in bidegree $(i, j+1)$) if $i, j \geq 0$ (resp. if $i, j < 0$). The map δ is the tensor product of the differentials $X_0 \longrightarrow X_{-1}$ and $Y_0 \longrightarrow Y_{-1}$, the other horizontal differentials are induced by the differentials of X and the vertical differentials are, up to a sign, the maps induced by the differentials of Y. (The signs are chosen so that all squares anti-commute.) Let $X_{<0} \subseteq X$ (resp. $Y_{<0} \subseteq Y$) be the subcomplex consisting of X_i (resp. Y_i) in degrees $i < 0$ and 0 in non-negative degrees. Then, setting the differential δ aside, the pinched tensor product complex consists of $X/X_{<0} \otimes Y/Y_{<0}$ in positive degrees (this is the total complex of the fist quadrant part of the bicomplex (2) above) and the suspension $\Sigma(X_{<0} \otimes Y_{<0})$ in negative degrees (this is the total complex of the third quadrant part of the bicomplex (2) above).

Proposition 2.6. *Let X be an acyclic chain complex of flat right R-modules and Y an acyclic chain complex of flat R-modules. Assume that X and Y satisfy the equivalent conditions of Lemma 2.1, as supplemented by Remark 2.2. Then, the pinched tensor product complex $X \otimes^{\bowtie} Y$ is acyclic and $B_{-1}(X \otimes^{\bowtie} Y) = B_{-1}X \otimes B_{-1}Y$.*

Proof. (cf. [6, Theorem 3.5]) Let $M = B_{-1}X$ and factor the differential $X_0 \longrightarrow X_{-1}$ as the composition $X_0 \xrightarrow{\epsilon} M \xrightarrow{\eta} X_{-1}$, where η is the inclusion. We also let $N = B_{-1}Y$ and factor the differential $Y_0 \longrightarrow Y_{-1}$ as the composition $Y_0 \xrightarrow{\epsilon'} N \xrightarrow{\eta'} Y_{-1}$, where η' is the inclusion. Then, the additive map δ is the composition $X_0 \otimes Y_0 \xrightarrow{\epsilon \otimes \epsilon'} M \otimes N \xrightarrow{\eta \otimes \eta'} X_{-1} \otimes Y_{-1}$. The proof follows from the following four assertions:

(i) There is an exact sequence

$$(X_0 \otimes Y_1) \oplus (X_1 \otimes Y_0) \longrightarrow X_0 \otimes Y_0 \xrightarrow{\epsilon \otimes \epsilon'} M \otimes N \longrightarrow 0,$$

where the unlabelled arrow is the differential of the complex $X \otimes^{\bowtie} Y$.

(ii) There is an exact sequence

$$0 \longrightarrow M \otimes N \xrightarrow{\eta \otimes \eta'} X_{-1} \otimes Y_{-1} \longrightarrow (X_{-2} \otimes Y_{-1}) \oplus (X_{-1} \otimes Y_{-2}),$$

where the unlabelled arrow is the differential of the complex $X \otimes^{\bowtie} Y$.

(iii) The complex $X \otimes^{\bowtie} Y$ is acyclic in positive degrees.

(iv) The complex $X \otimes^{\bowtie} Y$ is acyclic in degrees < -1.

Since the linear maps ϵ and ϵ' induce isomorphisms $X_0/\operatorname{im} X_1 \simeq M$ and $Y_0/\operatorname{im} Y_1 \simeq N$, assertion (i) is a well-known consequence of the right exactness of the tensor product. Assertion (ii) follows by inspecting the following commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & M \otimes N & \xrightarrow{\eta\otimes 1} & X_{-1} \otimes N & \longrightarrow & X_{-2} \otimes N \\
 & & \downarrow {\scriptstyle 1\otimes\eta'} & & \downarrow {\scriptstyle 1\otimes\eta'} & & \downarrow {\scriptstyle 1\otimes\eta'} \\
0 & \longrightarrow & M \otimes Y_{-1} & \xrightarrow{\eta\otimes 1} & X_{-1} \otimes Y_{-1} & \longrightarrow & X_{-2} \otimes Y_{-1} \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & M \otimes Y_{-2} & \xrightarrow{\eta\otimes 1} & X_{-1} \otimes Y_{-2} & \longrightarrow & X_{-2} \otimes Y_{-2}
\end{array}$$

whose rows and columns are exact. (We note that the top row is exact, since the complex $X \otimes N$ is acyclic in degrees 0 and -1, whereas the left column is exact, since the complex $M \otimes Y$ is acyclic in degrees 0 and -1.) The module X_i being flat, the i-th column of the bicomplex $X/X_{<0} \otimes Y/Y_{<0}$ is quasi-isomorphic with the abelian group $X_i \otimes N$ sitting in bidegree $(i, 0)$ for all $i \geq 0$ (with the quasi-isomorphism induced by $\epsilon' : Y_0 \longrightarrow N$). Then, a (first quadrant) spectral sequence argument shows that the complex $X/X_{<0} \otimes Y/Y_{<0}$ is quasi-isomorphic with $X/X_{<0} \otimes N$. Assertion (iii) follows since $X/X_{<0} \otimes N$ is acyclic in positive degrees (as this is the case for $X \otimes N$). In the same way, the flatness of X_i shows that the i-th column of the bicomplex $X_{<0} \otimes Y_{<0}$ is quasi-isomorphic with the abelian group $X_i \otimes N$ sitting in bidegree $(i, 0)$ for all $i < 0$ (with the quasi-isomorphism induced by $\eta' : N \longrightarrow Y_{-1}$). Then, a (third quadrant) spectral sequence argument shows that the complex $\Sigma(X_{<0} \otimes Y_{<0})$ is quasi-isomorphic with $X_{<0} \otimes N$. Assertion (iv) follows since $X_{<0} \otimes N$ is acyclic in degrees < -1 (as this is the case for $X \otimes N$). ☐

Corollary 2.7. *Let R be a commutative ring and consider two Tor-independent modules $M, N \in \mathfrak{S}^{\infty}_{flat}(R)$. Then, $M \otimes N \in \mathfrak{S}^{\infty}_{flat}(R)$.*

Proof. Let X, Y be two acyclic complexes of flat R-modules, such that $M = B_{-1}X$ and $N = B_{-1}Y$. Since R is commutative, the pinched tensor product complex $X \otimes^{\bowtie} Y$ is clearly a complex of R-modules. In fact, the class of flat R-modules being closed under tensor products and direct sums, the complex $X \otimes^{\bowtie} Y$ is a complex of flat R-modules. Then, Proposition 2.6 implies that $M \otimes N = B_{-1}(X \otimes^{\bowtie} Y) \in \mathfrak{S}^{\infty}_{flat}(R)$, as needed. ☐

3. Syzygies, complexes and homomorphism groups

In this section, we examine the Hom complex of two acyclic complexes of R-modules. Under certain conditions (which imply that the Hom complex is acyclic), we compute the syzygy groups of that Hom complex, in terms of the syzygies of the two complexes. We also consider the class of modules that appear as syzygies of acyclic complexes of projective or injective modules and examine a homological condition, which implies that (over a commutative ring) the Hom group of two such modules is a syzygy of an acyclic complex of injective modules.

I. The group of homomorphisms of two differential modules. We consider the category $\Delta(R)$ of differential R-modules, as defined in §1.I. We recall that for any object (X, ∂_X) of $\Delta(R)$ the R-module X is the direct sum of its homogeneous components. We now consider a variant of that category, denoted by $\Delta^{\Pi}(R)$, which is defined as follows: The objects of $\Delta^{\Pi}(R)$ are pairs (Y, ∂_Y), where Y is a $\mathbb{Z}$-graded R-module viewed as the *direct product* of its homogeneous components and the differential $\partial_Y : Y \longrightarrow Y$ is a square zero linear endomorphism of Y, which is graded of degree -1. We call (Y, ∂_Y) a Π-differential module. The morphisms of $\Delta^\Pi(R)$ are the homogeneous (i.e. graded of degree zero) linear maps that commute with the respective differentials. We say that the Π-differential module (Y, ∂_Y) if injective if this is the case for the underlying R-module Y. The module of boundaries B_Y, the embedding $\imath_Y : B_Y \hookrightarrow Y$ and the quotient map $p_Y : Y \longrightarrow Y/B_Y$, the module of cycles Z_Y, the embedding $\jmath_Y : Z_Y \hookrightarrow Y$ and the quotient map $q_Y : Y \longrightarrow Y/Z_Y$, the homology module Z_Y/B_Y and the acyclicity of the Π-differential module (Y, ∂_Y) and, finally, the factorization of ∂_Y as the composition $Y \xrightarrow{\theta_Y} B_Y \xrightarrow{\imath_Y} Y$ are defined as in the case of differential modules.

The Hom functor in the category of modules induces a bifunctor

$$\Delta(R) \times \Delta^\Pi(R) \longrightarrow \Delta^\Pi(\mathbb{Z}).$$

If (X, ∂_X) is a differential R-module and (Y, ∂_Y) is a Π-differential R-module, then the Π-differential abelian group $([X, Y], \partial_{[X,Y]})$ consists in degree n of $\prod_{j-i=n}[X_i, Y_j]$ and the differential $\partial_{[X,Y]}$ is given by the additive map

$$[1, \partial_Y] - [\partial_X, 1] : [X, Y] \longrightarrow [X, Y].$$

We adopt the usual sign convention for the maps induced by the contravariant Hom functor[2], so that $\partial_{[X,Y]}$ maps any element $(f_n)_n \in [X, Y]$ onto $(\partial_Y \circ f_{n+1} - (-1)^{n+1} f_{n+1} \circ \partial_X)_n$.

Lemma 3.1. *Let (X, ∂_X) be a differential R-module and (Y, ∂_Y) a Π-differential R-module. Then, the additive maps*

$$[\imath_X, p_Y] : [X, Y] \longrightarrow [B_X, Y/B_Y] \quad \text{and} \quad [\jmath_X, q_Y] : [X, Y] \longrightarrow [Z_X, Y/Z_Y]$$

vanish on the subgroup $Z_{[X,Y]} \subseteq [X, Y]$.

Proof. This is a reformulation of the fact that any element of $Z_{[X,Y]}$ maps boundaries of X to boundaries of Y and cycles of X to cycles of Y. ☐

Lemma 3.2. *Let (X, ∂_X) be a differential R-module, (Y, ∂_Y) a Π-differential R-module and assume that the former is acyclic and projective or else the latter is acyclic and injective.*

(i) If $\text{Ext}^1_R(Z_X, Y/B_Y) = 0$, then the map $[\jmath_X, p_Y] : [X, Y] \longrightarrow [Z_X, Y/B_Y]$ is surjective.

(ii) If $\text{Ext}^1_R(B_X, B_Y) = 0$, then the map $[\imath_X, \theta_Y] : [X, Y] \longrightarrow [B_X, B_Y]$ is surjective.

Proof. We assume that (X, ∂_X) is acyclic and projective. (Similar arguments can be applied if we assume that (Y, ∂_Y) is acyclic and injective.) We note that the surjective linear map $\theta_X : X \longrightarrow B_X$ induces an isomorphism of R-modules $\operatorname{coker} \jmath_X = X/Z_X \simeq B_X$.

(i) We express the additive map $[\jmath_X, p_Y]$ as the composition

$$[X, Y] \xrightarrow{[1,p_Y]} [X, Y/B_Y] \xrightarrow{[\jmath_X,1]} [Z_X, Y/B_Y].$$

The result follows since both of these maps are surjective: The surjectivity of $[1, p_Y]$ follows from the projectivity of X, whereas the surjectivity of $[\jmath_X, 1]$ follows from the triviality of the group $\text{Ext}^1_R(\operatorname{coker} \jmath_X, Y/B_Y) \simeq \text{Ext}^1_R(B_X, Y/B_Y) = \text{Ext}^1_R(Z_X, Y/B_Y)$.

(ii) We express the additive map $[\imath_X, \theta_Y]$ as the composition

$$[X, Y] \xrightarrow{[1,\theta_Y]} [X, B_Y] \xrightarrow{[\imath_X,1]} [B_X, B_Y].$$

The result follows since both of these maps are surjective: The surjectivity of $[1, \theta_Y]$ follows from the projectivity of X, whereas the surjectivity of $[\imath_X, 1]$ follows from the triviality of the

[2]Whenever $X \xrightarrow{f} X' \xrightarrow{g} X''$ and $Y \xrightarrow{h} Y' \xrightarrow{k} Y''$ are graded linear maps of degrees $\widetilde{f}$, $\widetilde{g}$, $\widetilde{h}$ and $\widetilde{k}$ respectively, then $[f, k] \circ [g, h] = (-1)^{\widetilde{g}(\widetilde{f}+\widetilde{k})}[g \circ f, k \circ h] : [X'', Y] \longrightarrow [X, Y'']$.

group $\mathrm{Ext}^1_R(\operatorname{coker}\imath_X, B_Y) = \mathrm{Ext}^1_R(\operatorname{coker}\jmath_X, B_Y) \simeq \mathrm{Ext}^1_R(B_X, B_Y)$. □

Let (Y, ∂_Y) be a Π-differential R-module. Since $B_Y \subseteq Z_Y$, there is a unique linear map $\overline{\partial_Y} : Y/B_Y \longrightarrow Y$ so that $\partial_Y = \overline{\partial_Y} \circ p_Y : Y \longrightarrow Y$.

Corollary 3.3. *Let (X, ∂_X) be a differential R-module, (Y, ∂_Y) a Π-differential R-module and assume that one of them is acyclic.*

(i) There is a unique additive map $\mu : B_{[X,Y]} \longrightarrow [Z_X, Y/B_Y]$, which is graded of degree 1, such that the composition $[X, Y] \stackrel{\theta_{[X,Y]}}{\longrightarrow} B_{[X,Y]} \stackrel{\mu}{\longrightarrow} [Z_X, Y/B_Y]$ provides a factorization of the map $[\jmath_X, p_Y] : [X, Y] \longrightarrow [Z_X, Y/B_Y]$. For any $f \in B_{[X,Y]} \subseteq [X, Y]$ its restriction $f \circ \jmath_X$ to Z_X is equal to the composition $\overline{\partial_Y} \circ \mu(f) \in [Z_X, Y]$.

(ii) Assume that (X, ∂_X) is acyclic and projective or else (Y, ∂_Y) is acyclic and injective. If $\mathrm{Ext}^1_R(Z_X, Y/B_Y) = 0$, then μ is surjective.

Proof. The existence of μ follows from Lemma 3.1. Since $\theta_{[X,Y]}$ is graded of degree -1 and the composition $\mu \circ \theta_{[X,Y]} = [\jmath_X, p_Y]$ is homogeneous, it follows that μ is graded of degree 1. For any $f \in B_{[X,Y]}$ we can write $f = \partial_{[X,Y]} g = [1, \partial_Y]g - [\partial_X, 1]g$ for a suitable element $g \in [X, Y]$ and then compute

$$[\jmath_X, 1]f = [\jmath_X, 1][1, \partial_Y]g - [\jmath_X, 1][\partial_X, 1]g = [\jmath_X, \partial_Y]g = \left[1, \overline{\partial_Y}\right][\jmath_X, p_Y]g = \left[1, \overline{\partial_Y}\right]\mu(f).$$

In the above chain of equalities, the second one follows since $\partial_X \circ \jmath_X = 0$ and the third one since $\partial_Y = \overline{\partial_Y} \circ p_Y$. Assertion (ii) follows from Lemma 3.2(i). □

We note that the factorizations $\partial_X = \imath_X \circ \theta_X$ and $\partial_Y = \imath_Y \circ \theta_Y$ imply that

$$[\partial_X, \partial_Y] = [\imath_X \circ \theta_X, \imath_Y \circ \theta_Y] = [\theta_X, \imath_Y] \circ [\imath_X, \theta_Y] : [X, Y] \longrightarrow [X, Y].$$

It follows that the kernel of the map $[\imath_X, \theta_Y] : [X, Y] \longrightarrow [B_X, B_Y]$ is contained in the kernel of the map $[\partial_X, \partial_Y] : [X, Y] \longrightarrow [X, Y]$. In fact, since $[\theta_X, \imath_Y] : [B_X, B_Y] \longrightarrow [X, Y]$ is injective (as θ_X is surjective and $\imath_Y$ is injective), we conclude that

$$\ker\left([X, Y] \stackrel{[\partial_X, \partial_Y]}{\longrightarrow} [X, Y]\right) = \ker\left([X, Y] \stackrel{[\imath_X, \theta_Y]}{\longrightarrow} [B_X, B_Y]\right).$$

On the other hand, the equalities

$$[\partial_X, \partial_Y] = ([\partial_X, 1] - [1, \partial_Y]) \circ [\partial_X, 1] = -\partial_{[X,Y]} \circ [\partial_X, 1] : [X, Y] \longrightarrow [X, Y] \tag{3}$$

show that $\operatorname{im}[\partial_X, \partial_Y] \subseteq B_{[X,Y]}$.

Corollary 3.4. *Let (X, ∂_X) be a differential R-module, (Y, ∂_Y) a Π-differential R-module and assume that the former is acyclic and projective or else the latter is acyclic and injective. If $\mathrm{Ext}^1_R(B_X, B_Y) = 0$, then there exists a unique injective additive map $\nu : [B_X, B_Y] \longrightarrow B_{[X,Y]}$, which is such that the composition*

$$[X, Y] \stackrel{[\imath_X, \theta_Y]}{\longrightarrow} [B_X, B_Y] \stackrel{\nu}{\longrightarrow} B_{[X,Y]} \stackrel{\imath_{[X,Y]}}{\longrightarrow} [X, Y]$$

provides a factorization of the map $[\partial_X, \partial_Y] : [X, Y] \longrightarrow [X, Y]$. The map ν is graded of degree -1 and for any element $f \in [B_X, B_Y]$ we have $\nu(f) = [\theta_X, \imath_Y]f \in B_{[X,Y]} \subseteq [X, Y]$.

Proof. In view of Lemma 3.2(ii), the map $[\imath_X, \theta_Y] : [X, Y] \longrightarrow [B_X, B_Y]$ is surjective and, as noted above, its kernel is equal to the kernel of the map $[\partial_X, \partial_Y] : [X, Y] \longrightarrow [X, Y]$. We conclude that $[\partial_X, \partial_Y]$ factors as the composition of $[\imath_X, \theta_Y]$ followed by a certain injective map $[B_X, B_Y] \longrightarrow [X, Y]$, whose image is equal to $\operatorname{im}[\partial_X, \partial_Y]$, Since the latter group is contained in

$B_{[X,Y]}$, we obtain a factorization through the subgroup $B_{[X,Y]} \subseteq [X,Y]$, as needed. The map $\imath_{[X,Y]}$ is homogeneous, $[\imath_X, \theta_Y]$ is graded of degree -1 and the composition $\imath_{[X,Y]} \circ \nu \circ [\imath_X, \theta_Y] = [\partial_X, \partial_Y]$ is graded of degree -2; therefore, ν is graded of degree -1. Finally, the factorization $[\partial_X, \partial_Y] = [\theta_X, \imath_Y] \circ [\imath_X, \theta_Y]$ and the fact that $[\imath_X, \theta_Y]$ is surjective, show that ν maps any element $f \in [B_X, B_Y]$ onto $[\theta_X, \imath_Y]f \in B_{[X,Y]} \subseteq [X,Y]$. □

In order to examine the acyclicity of the Π-differential abelian group $[X,Y]$, which is associated with two differential modules as above, we shall use the following result.

Lemma 3.5. *Let (X, ∂_X) be an acyclic differential R-module, (Y, ∂_Y) an acyclic Π-differential R-module and assume that $\mathrm{Ext}^1_R(B_X, B_Y) = 0$. Then,*

$$\ker\left([X,Y] \xrightarrow{[\partial_X,\partial_Y]} [X,Y]\right) = \operatorname{im}\left([B_X,Y] \xrightarrow{[\theta_X,1]} [X,Y]\right) + \operatorname{im}\left([X,B_Y] \xrightarrow{[1,\imath_Y]} [X,Y]\right).$$

Proof. We have noted above that $\ker[\partial_X, \partial_Y] = \ker[\imath_X, \theta_Y]$. Hence, the result follows by inspecting the following commutative diagram, whose rows and columns are exact

$$\begin{array}{ccccccccc}
0 & \longrightarrow & [B_X, B_Y] & \xrightarrow{[1,\imath_Y]} & [B_X, Y] & \xrightarrow{-[1,\theta_Y]} & [B_X, B_Y] & \longrightarrow & 0 \\
 & & \downarrow [\theta_X,1] & & \downarrow [\theta_X,1] & & \downarrow [\theta_X,1] & & \\
0 & \longrightarrow & [X, B_Y] & \xrightarrow{[1,\imath_Y]} & [X, Y] & \xrightarrow{[1,\theta_Y]} & [X, B_Y] & & \\
 & & \downarrow [\imath_X,1] & & \downarrow [\imath_X,1] & & \downarrow [\imath_X,1] & & \\
0 & \longrightarrow & [B_X, B_Y] & \xrightarrow{[1,\imath_Y]} & [B_X, Y] & \xrightarrow{[1,\theta_Y]} & [B_X, B_Y] & &
\end{array}$$

We note that the exactness of the first row follows since $\mathrm{Ext}^1_R(B_X, B_Y) = 0$. □

Proposition 3.6. *Let (X, ∂_X) be a differential R-module, (Y, ∂_Y) a Π-differential R-module and assume that the former is acyclic and projective or else the latter is acyclic and injective. We also assume that $\mathrm{Ext}^1_R(B_X, B_Y) = 0$.*

(i) There exists an exact sequence of graded abelian groups

$$0 \longrightarrow [B_X, B_Y] \xrightarrow{\nu} B_{[X,Y]} \xrightarrow{\mu} [Z_X, Y/B_Y].$$

The map ν is graded of degree -1 and maps any $f \in [B_X, B_Y]$ onto $[\theta_X, \imath_Y]f \in B_{[X,Y]} \subseteq [X,Y]$, i.e. the composition $[X,Y] \xrightarrow{[\imath_X,\theta_Y]} [B_X, B_Y] \xrightarrow{\nu} B_{[X,Y]} \xrightarrow{\imath_{[X,Y]}} [X,Y]$ provides a factorization of $[\partial_X, \partial_Y] : [X,Y] \longrightarrow [X,Y]$. The map μ is graded of degree 1 and the composition $\mu \circ \theta_{[X,Y]}$ coincides with $[\jmath_X, p_Y] : [X,Y] \longrightarrow [Z_X, Y/B_Y]$; for any $g \in B_{[X,Y]} \subseteq [X,Y]$ the restriction $g \circ \jmath_X$ of g to Z_X is equal to the composition $\overline{\partial_Y} \circ \mu(g) \in [Z_X, Y]$.

(ii) If we also have $\mathrm{Ext}^1_R(Z_X, Y/B_Y) = 0$, then the map μ in (i) above is surjective.

(iii) If both (X, ∂_X) and (Y, ∂_Y) are acyclic, then $([X,Y], \partial_{[X,Y]})$ is acyclic as well.

Proof. (i) In view of our assumptions, the existence of μ and ν (and the injectivity of ν) follow from Corollary 3.3(i) and Corollary 3.4 respectively. We have to show that $\operatorname{im}\nu = \ker\mu$. Since $[\imath_X, \theta_Y] : [X,Y] \longrightarrow [B_X, B_Y]$ is surjective (cf. Lemma 3.2(ii)), we have $\operatorname{im}\nu = \operatorname{im}[\partial_X, \partial_Y]$.

In order to prove that $\operatorname{im}\nu \subseteq \ker\mu$, i.e. that $\operatorname{im}[\partial_X, \partial_Y] \subseteq \ker\mu$, we consider an element $f \in \operatorname{im}[\partial_X, \partial_Y]$ and write $f = [\partial_X, \partial_Y]g$ for a suitable $g \in [X,Y]$. As we noted in (3), $[\partial_X, \partial_Y] = -\partial_{[X,Y]} \circ [\partial_X, 1]$; it follows that $f = -\partial_{[X,Y]}[\partial_X, 1]g$ and hence $\mu(f) = -[\jmath_X, p_Y][\partial_X, 1]g = 0$ (as $\partial_X \circ \jmath_X = 0$). Conversely, in order to prove that $\ker\mu \subseteq \operatorname{im}\nu$, consider an element $f \in B_{[X,Y]}$ in the kernel of μ and write $f = \partial_{[X,Y]}g$ for a suitable $g \in [X,Y]$. Then, $[\jmath_X, p_Y]g = \mu(f) = 0 \in [Z_X, Y/B_Y]$ and *a fortiori* $[\imath_X, p_Y]g = 0 \in [B_X, Y/B_Y]$. Since $p_Y \circ \partial_Y = 0$ and $\partial_X = \imath_X \circ \theta_X$, it follows that

$$[1,p_Y]\partial_{[X,Y]}g = [1,p_Y][1,\partial_Y]g - [1,p_Y][\partial_X,1]g = -[\partial_X,p_Y]g = -[\theta_X,1][\imath_X,p_Y]g = 0.$$

Then, $[1,p_Y]f = 0$ and hence we have proved that $\operatorname{im} f \subseteq B_Y$. Since $\partial_X \circ \jmath_X = 0$ and ∂_Y may be factored as the composition $Y \xrightarrow{p_Y} Y/B_Y \xrightarrow{\overline{\partial_Y}} Y$, we compute

$$[\jmath_X,1]\partial_{[X,Y]}g = [\jmath_X,1][1,\partial_Y]g - [\jmath_X,1][\partial_X,1]g = [\jmath_X,\partial_Y]g = \left[1,\overline{\partial_Y}\right][\jmath_X,p_Y]g = 0.$$

Then, $[\jmath_X,1]f = 0$ and hence we have also proved that $Z_X \subseteq \ker f$. It follows that f factors as a composition $X \xrightarrow{\theta_X} B_X \longrightarrow B_Y \xrightarrow{\imath_Y} Y$ and hence $f = [\theta_X,\imath_Y]h = \nu(h) \in \operatorname{im}\nu$ for a suitable $h \in [B_X,B_Y]$, as needed.

(ii) This follows from Corollary 3.3(ii).

(iii) We assume that X is projective. (An analogous argument may be used if we assume that Y is injective.) Let $f \in Z_{[X,Y]}$ and note that the equality $\partial_{[X,Y]}f = 0$ implies that $[1,\partial_Y]f = [\partial_X,1]f \in [X,Y]$ is annihilated by both $[\imath_X,1]$ (since $\partial_X \circ \imath_X = 0$) and $[1,p_Y]$ (since $p_Y \circ \partial_Y = 0$). It follows that the map $[1,\partial_Y]f = [\partial_X,1]f$ vanishes on $B_X = Z_X$ and its image is contained in B_Y. Therefore, that map factors as a composition $X \xrightarrow{\theta_X} B_X \longrightarrow B_Y \xrightarrow{\imath_Y} Y$ and hence $[1,\partial_Y]f = [\partial_X,1]f = [\theta_X,\imath_Y]g$ for a suitable $g \in [B_X,B_Y]$. In view of Corollary 3.2(ii), the map $[\imath_X,\theta_Y] : [X,Y] \longrightarrow [B_X,B_Y]$ is surjective and hence there exists $h \in [X,Y]$ such that $g = [\imath_X,\theta_Y]h$. It follows that

$$[1,\partial_Y]f = [\partial_X,1]f = [\theta_X,\imath_Y]g = [\theta_X,\imath_Y][\imath_X,\theta_Y]h = [\imath_X \circ \theta_X, \imath_Y \circ \theta_Y]h = [\partial_X,\partial_Y]h$$

and hence $[1,\partial_Y](f + [\partial_X,1]h) = [1,\partial_Y]f - [\partial_X,\partial_Y]h = 0$. Since X is projective and (Y,∂_Y) is acyclic, the kernel of the map $[1,\partial_Y]$ coincides with its image; hence, there exists $t \in [X,Y]$, such that $f + [\partial_X,1]h = [1,\partial_Y]t$, i.e. $f = [1,\partial_Y]t - [\partial_X,1]h$. Applying $[\partial_X,1]$ to both sides of this equality, it follows that $[\partial_X,1]f = [\partial_X,\partial_Y]t$ and hence $[\partial_X,\partial_Y]h = [\partial_X,\partial_Y]t$. Then, Lemma 3.5 implies that

$$h - t \in \ker[\partial_X,\partial_Y] = \operatorname{im}\left([B_X,Y] \xrightarrow{[\theta_X,1]} [X,Y]\right) + \operatorname{im}\left([X,B_Y] \xrightarrow{[1,\imath_Y]} [X,Y]\right).$$

Applying the additive map $[1,\partial_Y]$, we conclude that

$$[1,\partial_Y]h - [1,\partial_Y]t \in \operatorname{im}\left([B_X,Y] \xrightarrow{[\theta_X,\partial_Y]} [X,Y]\right) \subseteq \operatorname{im}\left([B_X,B_Y] \xrightarrow{[\theta_X,\imath_Y]} [X,Y]\right).$$

The inclusion above follows since $\partial_Y = \imath_Y \circ \theta_Y$ and hence $[\theta_X,\partial_Y] = [\theta_X,\imath_Y] \circ [1,\theta_Y]$. Using once more the surjectivity of $[\imath_X,\theta_Y] : [X,Y] \longrightarrow [B_X,B_Y]$ (cf. Corollary 3.2(ii)), we may conclude that $[1,\partial_Y]h - [1,\partial_Y]t$ is contained in the image of $[\theta_X,\imath_Y] \circ [\imath_X,\theta_Y] = [\imath_X \circ \theta_X, \imath_Y \circ \theta_Y] = [\partial_X,\partial_Y]$. In other words, there exists $s \in [X,Y]$, such that $[1,\partial_Y]h - [1,\partial_Y]t = [\partial_X,\partial_Y]s$ and hence $[1,\partial_Y]t = [1,\partial_Y]h - [\partial_X,\partial_Y]s$. Since $\partial_{[X,Y]} \circ [\partial_X,1] = -[\partial_X,\partial_Y]$ (cf. (3)), we conclude that

$$f = [1,\partial_Y]t - [\partial_X,1]h = [1,\partial_Y]h - [\partial_X,\partial_Y]s - [\partial_X,1]h = \partial_{[X,Y]}(h + [\partial_X,1]s)$$

and hence $f \in B_{[X,Y]}$, as needed. □

As in §1.I, we can rearrange parts of Proposition 3.6 and state the following result.

Theorem 3.7. *Let (X,∂_X) be a differential R-module and (Y,∂_Y) a Π-differential R-module. We assume that:*

(i) both differential modules are acyclic,

(ii) X is projective or else Y is injective and

(iii) $Ext^1_R(B_X, B_Y) = 0$.
Then, $([X,Y], \partial_{[X,Y]})$ *is acyclic and there exists a short exact sequence of graded abelian groups*

$$0 \longrightarrow [B_X, B_Y] \xrightarrow{\nu} B_{[X,Y]} \xrightarrow{\mu'} [B_X, B_Y] \longrightarrow 0.$$

The map ν *is graded of degree* -1 *and maps any* $f \in [B_X, B_Y]$ *onto* $[\theta_X, \imath_Y]f \in B_{[X,Y]} \subseteq [X,Y]$. *The map* μ' *is homogeneous and maps any* $g \in B_{[X,Y]}$ *onto its restriction* $\mu'(g) \in [B_X, B_Y]$ *(i.e.* $\imath_Y \circ \mu'(g) = g \circ \imath_X$*).*

Proof. We can factor $\overline{\partial_Y} : Y/B_Y \longrightarrow Y$ as the composition $Y/B_Y \xrightarrow{\overline{\theta_Y}} B_Y \xrightarrow{\imath_Y} Y$, where $\overline{\theta_Y}$ is bijective, and define μ' as the composition of the map μ in Proposition 3.6 followed by the isomorphism $[Z_X, Y/B_Y] = [B_X, Y/B_Y] \xrightarrow{[1,\overline{\theta_Y}]} [B_X, B_Y]$. In other words, we let $\mu' = \big[1, \overline{\theta_Y}\big] \circ \mu$, so that $\mu'(g) = \overline{\theta_Y} \circ \mu(g) \in [B_X, B_Y]$ for all $g \in B_{[X,Y]} \subseteq [X,Y]$. For such a g, its restriction $g \circ \imath_X = g \circ \jmath_X$ to $B_X = Z_X$ is equal to $\overline{\partial_Y} \circ \mu(g) = \imath_Y \circ \overline{\theta_Y} \circ \mu(g) = \imath_Y \circ \mu'(g)$. Of course, $\big[1, \overline{\theta_Y}\big]$ being graded of degree -1, the map μ' is homogeneous. □

II. Reformulation in the language of chain complexes. Let $X = ((X_n)_n, \partial_X)$ and $Y = ((Y_n)_n, \partial_Y)$ be two chain complexes of R-modules. Then, the Hom complex $[X,Y] = (([X,Y]_n)_n, \partial_{[X,Y]})$ is the chain complex of abelian groups with $[X,Y]_n = \prod_{j-i=n}[X_i, Y_j]$ for all n and differential $\partial_{[X,Y]}$, which maps any n-chain $f = (f_{i,i+n})_i \in [X,Y]_n$ (where $f_{i,i+n} \in [X_i, Y_{i+n}]$ for all i) onto the $n-1$-chain $g = (g_{i,i+n-1})_i \in [X,Y]_{n-1}$, where $g_{i,i+n-1} = \partial_Y \circ f_{i,i+n} - (-1)^n f_{i-1,i+n-1} \circ \partial_X \in [X_i, Y_{i+n-1}]$ for all i. We may reformulate Theorem 3.7, by considering the homogeneous parts of any given degree therein, in order to obtain some information about the boundary groups $(B_n[X,Y])_n$ of the complex $[X,Y]$, in terms of the boundary groups $(B_nX)_n$ and $(B_nY)_n$ of the complexes X and Y.

Theorem 3.8. *Let* X, Y *be two chain complexes of* R*-modules as above and assume that the following conditions are satisfied:*
(i) both complexes are acyclic,
(ii) X *consists of projective modules or else* Y *consists of injective modules and*
(iii) $Ext^1_R(B_iX, B_jY) = 0$ *for all* i, j.
Then, the Hom complex $[X,Y]$ *is also acyclic and there exists a short exact sequence of abelian groups*

$$0 \longrightarrow \textstyle\prod_i [B_iX, B_{i+n+1}Y] \xrightarrow{\nu} B_n[X,Y] \xrightarrow{\mu'} \prod_i [B_iX, B_{i+n}Y] \longrightarrow 0.$$

for all n. *The map* ν *maps any* $f = (f_{i,i+n+1})_i \in \prod_i [B_iX, B_{i+n+1}Y]$ *onto* $(f'_{i,i+n})_i \in B_n[X,Y] \subseteq [X,Y]_n = \prod_i [X_i, Y_{i+n}]$, *where* $f'_{i,i+n}x_i = (-1)^{n+1} f_{i-1,i+n}\partial_X x_i \in Y_{i+n}$ *for all* $x_i \in X_i$ *and all* i. *The map* μ' *maps any* $g = (g_{i,i+n})_i \in B_n[X,Y] \subseteq [X,Y]_n = \prod_i [X_i, Y_{i+n}]$ *onto* $(g'_{i,i+n})_i \in \prod_i [B_iX, B_{i+n}Y]$, *where* $g'_{i,i+n}x_i = g_{i,i+n}x_i \in B_{i+n}Y \subseteq Y_{i+n}$ *for all* $x_i \in B_iX$ *(i.e.* $g'_{i,i+n} : B_iX \longrightarrow B_{i+n}Y$ *is the restriction of* $g_{i,i+n} : X_i \longrightarrow Y_{i+n}$*) for all* i. □

Remarks 3.9. (i) Under the hypotheses of Theorem 3.8, the acyclicity of the complex $[X,Y]$ in degree 0 means that any chain map $g : X \longrightarrow Y$ is null-homotopic. Furthermore, assigning to such a chain map g the sequence of linear maps which are induced between the respective syzygy modules defines a surjective map μ', whose kernel consists of those chain maps which are given in any degree i by a composition of the form $X_i \xrightarrow{\theta_X} B_{i-1}X \longrightarrow B_iY \hookrightarrow Y_i$. We note that there are many types of hypotheses which are known to imply that all chain maps between two complexes are null-homotopic. We believe that Theorem 3.8 is of some interest, as it also provides us with a description of these chain maps, i.e. of the boundary groups of the (acyclic) Hom complex $[X, Y]$.

(ii) The short exact sequences in Theorem 3.8 are natural in both X and Y. Hence, if S and T are two rings, X is a complex of (R, S)-bimodules and Y is a complex of (R, T)-bimodules, then these short exact sequences are actually short exact sequences of (S, T)-bimodules. In particular, if the ring R is commutative, then all Hom groups involved are R-modules and these short exact sequences are short exact sequences of R-modules.

III. Ext-independence of infinite syzygies. We denote by $\mathfrak{S}^{\infty}_{proj}(R)$ the class of those R-modules that may be expressed as syzygies of acyclic complexes of projective modules. This is a subclass of $\mathfrak{S}^{\infty}_{flat}(R)$, which is closed under direct sums. We also consider the class $\mathfrak{S}^{\infty}_{inj}(R)$ consisting of the syzygies of the acyclic complexes of injective modules; this class is closed under direct products.

We examine the condition appearing as assumption (iii) in Theorem 3.8. The proofs of the following results are omitted, as these are completely analogous to the proofs of Lemma 2.1 (see also Remark 2.2), Proposition 2.3 and Lemma 2.4 respectively.

Lemma 3.10. *Let X be an acyclic complex of projective R-modules and Y an acyclic complex of injective R-modules. Then, the following conditions are equivalent:*

(i) $Ext^1_R(B_iX, B_0Y) = 0$ for all i,

(ii) $Ext^n_R(B_iX, B_0Y) = 0$ for all $n \geq 1$ and all i,

(i)' $Ext^1_R(B_0X, B_jY) = 0$ for all j,

(ii)' $Ext^n_R(B_0X, B_jY) = 0$ for all $n \geq 1$ and all j,

(iii) $Ext^1_R(B_iX, B_jY) = 0$ for all i, j and

(iv) $Ext^n_R(B_iX, B_jY) = 0$ for all $n \geq 1$ and all i, j. □

Proposition 3.11. *The following two conditions are equivalent for a pair of modules (M, N), where $M \in \mathfrak{S}^{\infty}_{proj}(R)$ and $N \in \mathfrak{S}^{\infty}_{inj}(R)$:*

(i) There exists an acyclic complex of projective R-modules X with $M = B_0X$ and an acyclic complex of injective R-modules Y with $N = B_0Y$, such that the equivalent conditions of Lemma 3.10 are satisfied for X and Y.

(ii) If X is any acyclic complex of projective R-modules with $M = B_0X$ and Y is any acyclic complex of injective R-modules with $N = B_0Y$, then the equivalent conditions of Lemma 3.10 are satisfied for X and Y.

If these conditions are satisfied, we say that the modules M and N are Ext-independent. □

In the special case where M is Gorenstein projective or N is Gorenstein injective [7], Ext-independence is equivalent to the vanishing of the Tate cohomology groups $\widehat{\mathrm{Ext}}^*_R(M, N)$. Indeed, if we assume that M is Gorenstein projective and X is a totally acyclic complex of projective modules admitting M as a kernel, then the homology of the complex $[X, N]$ is the Tate cohomology $\widehat{\mathrm{Ext}}^*_R(M, N)$; cf. [6, §4.1].

For any module $M \in \mathfrak{S}^{\infty}_{proj}(R)$ we may consider the class $M^{\square}$, consisting of those modules $N \in \mathfrak{S}^{\infty}_{inj}(R)$, which are Ext-independent to M. If X is an acyclic complex of projective R-modules, such that $M = B_0X$, then $M^{\square} = \{N \in \mathfrak{S}^{\infty}_{inj}(R) : \text{the complex } [X, N] \text{ is acyclic}\}$; this follows from condition (i) in Lemma 3.10.

Lemma 3.12. *Let M be an R-module in $\mathfrak{S}^{\infty}_{proj}(R)$ and consider the class $M^{\square}$ defined above. Then:*

(i) If $(N_i)_i$ is a family of modules in $M^\square$, then $\prod_i N_i \in M^\square$.

(ii) If $0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$ is a short exact sequence of modules in $\mathfrak{S}^\infty_{inj}(R)$ and two of these modules are contained in $M^\square$, then so is the third. $\square$

For any module $N \in \mathfrak{S}^\infty_{inj}(R)$ we may also consider the class ${}^\square N$, consisting of those modules $M \in \mathfrak{S}^\infty_{proj}(R)$, which are Ext-independent to N. If Y is an acyclic complex of injective R-modules, such that $N = B_0Y$, then ${}^\square N$ consists of those modules $M \in \mathfrak{S}^\infty_{proj}(R)$, which are such that the complex $[M, Y]$ is acyclic; this follows from condition (i)' in Lemma 3.10. The class ${}^\square N$ is closed under direct sums and has the 2-out-of-3 property for short exact sequences in $\mathfrak{S}^\infty_{proj}(R)$.

IV. The pinched Hom complex. Let X, Y be two chain complexes of R-modules. We recall the definition of the pinched Hom complex $[X, Y]^{\bowtie}$, as defined in [6]. It is the total complex associated with the bicomplex of abelian groups pictured below

$$
\begin{array}{ccccccccccc}
 & & & & & & & \vdots & & \vdots & \\
 & & & & & & & \downarrow & & \downarrow & \\
 & & & & & & & [X_{-1}, Y_2] & \longleftarrow & [X_{-2}, Y_2] & \longleftarrow \cdots \\
 & & & & & & & \downarrow & & \downarrow & \\
\cdots & \longleftarrow & [X_1, Y_0] & \longleftarrow & [X_0, Y_0] & \xleftarrow{d} & & [X_{-1}, Y_1] & \longleftarrow & [X_{-2}, Y_1] & \longleftarrow \cdots \\
 & & \downarrow & & \downarrow & & & & & & \\
\cdots & \longleftarrow & [X_1, Y_{-1}] & \longleftarrow & [X_0, Y_{-1}] & & & & & & \\
 & & \downarrow & & \downarrow & & & & & & \\
 & & \vdots & & \vdots & & & & & &
\end{array} \tag{4}
$$

Here, the abelian group $[X_{-i}, Y_j]$ is located in bidegree (i, j) (resp. in bidegree $(i, j-1)$) if $i, j \leq 0$ (resp. if $i, j > 0$). The map d is induced by the differentials $X_0 \longrightarrow X_{-1}$ and $Y_1 \longrightarrow Y_0$, the other horizontal differentials are induced by the differentials of X and the vertical differentials are, up to a sign, the maps induced by the differentials of Y. (The signs are chosen so that all squares anti-commute.) Let $X_{<0} \subseteq X$ (resp. $Y_{\leq 0} \subseteq Y$) be the subcomplex consisting of X_i (resp. Y_i) in degrees $i < 0$ (resp. $i \leq 0$) and 0 in non-negative (resp. positive) degrees. Then, setting the differential d aside, the pinched Hom complex consists of $[X/X_{<0}, Y_{\leq 0}]$ in non-positive degrees (this is the total complex of the third quadrant part of the bicomplex (4) above) and the suspension $\Sigma^{-1}[X_{<0}, Y/Y_{\leq 0}]$ in positive degrees (this is the total complex of the first quadrant part of the bicomplex (4) above).

Proposition 3.13. *Let X, Y be two acyclic complexes of R-modules, such that X consists of projective modules and Y consists of injective modules. We assume that X and Y satisfy the equivalent conditions of Lemma 3.10. Then, the pinched Hom complex $[X, Y]^{\bowtie}$ is acyclic and $B_0([X, Y]^{\bowtie}) = [B_{-1}X, B_0Y]$.*

Proof. (cf. [6, Theorem 4.7]) Let $M = B_{-1}X$ and factor the differential $X_0 \longrightarrow X_{-1}$ as the composition $X_0 \xrightarrow{\epsilon} M \xrightarrow{\eta} X_{-1}$, where η is the inclusion. We also let $N = B_0Y$ and factor the differential $Y_1 \longrightarrow Y_0$ as the composition $Y_1 \xrightarrow{\epsilon'} N \xrightarrow{\eta'} Y_0$, where η' is the inclusion. Then, the additive map d in (4) is the composition $[X_{-1}, Y_1] \xrightarrow{[\eta,\epsilon']} [M, N] \xrightarrow{[\epsilon,\eta']} [X_0, Y_0]$. The proof follows from the following four assertions:

(i) There is an exact sequence

$$0 \longrightarrow [M,N] \xrightarrow{[\epsilon,\eta']} [X_0,Y_0] \longrightarrow [X_1,Y_0] \oplus [X_0,Y_{-1}],$$

where the unlabelled arrow is the differential of the complex $[X,Y]^{\bowtie}$.

(ii) There is an exact sequence

$$[X_{-1},Y_2] \oplus [X_{-2},Y_1] \longrightarrow [X_{-1},Y_1] \xrightarrow{[\eta,\epsilon']} [M,N] \longrightarrow 0,$$

where the unlabelled arrow is the differential of the complex $[X,Y]^{\bowtie}$.

(iii) The complex $[X,Y]^{\bowtie}$ is acyclic in negative degrees.

(iv) The complex $[X,Y]^{\bowtie}$ is acyclic in degrees > 1.

The proof of these assertions is omitted; it is completely analogous to the proof of the corresponding assertions in the proof of Proposition 2.6. □

Corollary 3.14. *Let R be a commutative ring and consider two modules $M \in \mathfrak{S}^{\infty}_{proj}(R)$ and $N \in \mathfrak{S}^{\infty}_{inj}(R)$, which are Ext-independent. Then, $[M,N] \in \mathfrak{S}^{\infty}_{inj}(R)$.*

Proof. We consider an acyclic complex of projective R-modules X with $M = B_{-1}X$ and an acyclic complex of injective R-modules Y with $N = B_0Y$. Since R is commutative, the pinched Hom complex $[X,Y]^{\bowtie}$ is clearly a complex of R-modules. In fact, since the R-module $[P,I]$ is injective whenever P is a projective and I is an injective R-module, whereas the class of injective R-modules is closed under finite direct sums, it follows that $[X,Y]^{\bowtie}$ is a complex of injective R-modules. Then, Proposition 3.13 implies that $[M,N] = B_0([X,Y]^{\bowtie}) \in \mathfrak{S}^{\infty}_{inj}(R)$, as needed. □

References

[1] Avramov, L., Foxby, H.-B.: Homological dimensions of unbounded complexes. J. Pure Appl. Algebra **71**, 129-155 (1991)

[2] Bazzoni, S., Cortés-Izurdiaga, M., Estrada, S.: Periodic modules and acyclic complexes. Algebr. Represent. Theory **23**, 1861-1883 (2020)

[3] Benson, D.J., Goodearl, K.R.: Periodic flat modules, and flat modules for finite groups. Pacific J. Math. bf 196, 45-67 (2000)

[4] Cartan, H., Eilenberg, S.: Homological algebra, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1999, With an appendix by David A. Buchsbaum, Reprint of the 1956 original

[5] Christensen, L.W., Holm, H.: The direct limit closure of perfect complexes. J. Pure Appl. Algebra **219**, 449-463 (2015)

[6] Christensen, L.W., Jorgensen, D.A.: Tate (co)homology via pinched complexes. Trans. Amer. Math. Soc. **366**, 667-689 (2014)

[7] Enochs, E.E., Jenda, O.M.G.: Gorenstein injective and projective modules. Math. Z. **220**, 611-633 (1995)

[8] Enochs, E.E., Jenda, O.M.G., Torrecillas, B.: Gorenstein flat modules, Nanjing Daxue Xuebao Shuxue Bannian Kan **10**, 1-9 (1993)

[9] Gillespie, J.: The flat model structure on $Ch(R)$. Trans. Amer. Math. Soc. **356**, 3369-3390 (2004)

[10] Neeman, A.: The homotopy category of flat modules, and Grothendieck duality. Invent. Math. **174**, 255-308 (2008)

[11] Spaltenstein, N.: Resolutions of unbounded complexes. Compos. Math. **65**, 121-154 (1988)

[12] Stovicek, J.: On purity and applications to coderived and singularity categories. arxiv:1412.1615

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