MODULES OF FINITE GORENSTEIN FLAT DIMENSION AND APPROXIMATIONS

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Abstract. We study approximations of modules of finite Gorenstein flat dimension by (projectively coresolved) Gorenstein flat modules and modules of finite flat dimension. These approximations determine the Gorenstein flat dimension and lead to descriptions of the corresponding relative homological dimensions, for such modules, in more classical terms. We also describe two hereditary Hovey triples on the category of modules of finite Gorenstein flat dimension, whose associated exact structures have homotopy categories equivalent to the stable category of projectively coresolved Gorenstein flat modules and the stable category of cotorsion Gorenstein flat modules, respectively.

Contents

0. Introduction

The concept of G-dimension for commutative Noetherian rings that was introduced by Auslander and Bridger in [1] has been extended to modules over any ring R through the notion of a Gorenstein projective module by Enochs and Jenda in [13]. The class of Gorenstein injective modules was also defined in that paper, whereas Gorenstein flat modules were introduced in [15]. The relative homological dimensions based on these modules were defined by Holm [19] in the standard way, by considering resolutions (or coresolutions) by modules in the given class. Gorenstein homological algebra has developed rapidly and found interesting applications in representation theory, algebraic geometry and cohomological group theory.

Many useful properties of the notion of Gorenstein projective dimension are consequences of the fact that the class $\mathtt{GProj}(R)$ of Gorenstein projective modules contains all projective modules and is closed under extensions, kernels of epimorphisms and direct summands. These properties of $\mathtt{GProj}(R)$ were established by Holm in [loc.cit.], where the analogous (dual) properties of Gorenstein injective modules are also proved. It follows that modules of finite Gorenstein projective dimension admit approximations by Gorenstein projective modules and

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modules of finite projective dimension; cf. [10]. Analogous approximations exist for modules of finite Gorenstein injective dimension. As the closure of the class $\mathtt{GFlat}(R)$ of Gorenstein flat modules under extensions was not known until recently, many properties of Gorenstein flat modules and modules of finite Gorenstein flat dimension were only available under additional assumptions on the ground ring. For example, Holm characterized in [19] the Gorenstein flat dimension of modules in terms of the vanishing of certain Tor-groups, in the case where the ring R is right coherent. Bennis realized the importance of knowing that $\mathtt{GFlat}(R)$ is closed under extensions and termed the rings for which this is true as being GF-closed; cf. [3]. For modules over a GF-closed ring, he showed that the finiteness of the Gorenstein flat dimension can be characterized in terms of the vanishing of Tor-groups, extending the above mentioned result by Holm. It was finally proved by Saroch and Stovicek in [22] that the class $\mathtt{GFlat}(R)$ is closed under extensions over any ring, so that all rings are GF-closed; in fact, $\big(\mathtt{GFlat}(R), \mathtt{GFlat}(R)^\perp\big)$ is a complete hereditery cotorsion pair in the category of modules. Saroch and Stovicek have also introduced in [loc.cit.] a certain subclass of $\mathtt{GFlat}(R)$, formed by the so-called projectively coresolved Gorentein flat modules (PGF-modules, for short), and showed that these modules are Gorenstein projective.[1] They showed that the class $\mathtt{PGF}(R)$ of these modules forms the left hand side of another complete hereditary cotorsion pair $\big(\mathtt{PGF}(R), \mathtt{PGF}(R)^\perp\big)$ in the category of modules.

In this paper, we study the class $\overline{\mathtt{GFlat}}(R)$ of modules of finite Gorenstein flat dimension and obtain approximations of modules therein (i) by PGF-modules and modules of finite flat dimension and (ii) by Gorenstein flat modules and cotorsion modules of finite flat dimension. It turns out that these approximations are essentially obtained by restricting the approximations resulting from the complete cotorsion pairs $\big(\mathtt{PGF}(R), \mathtt{PGF}(R)^\perp\big)$ and $\big(\mathtt{GFlat}(R), \mathtt{GFlat}(R)^\perp\big)$ to the exact subcategory $\overline{\mathtt{GFlat}}(R)$ of the module category. These approximations determine the Gorenstein flat dimension $\mathrm{Gfd}_R M$ of any module M. The following result illustrates this assertion, using the approximations that arise from the fact that both cotorsion pairs have enough injectives; it is extracted from parts of Theorem 2.1, Proposition 3.9, Theorem 4.2 and Proposition 5.7.

Theorem. *Let M be a module and consider an exact sequence $0 \longrightarrow M \longrightarrow K \longrightarrow L \longrightarrow 0$, where either:*
(i) $L \in \mathtt{GFlat}(R)$ and $K \in \mathtt{GFlat}(R)^\perp$ or
(ii) $L \in \mathtt{PGF}(R)$ and $K \in \mathtt{PGF}(R)^\perp$.
Then, $Gfd_R M = fd_R K$; in particular, M has finite Gorenstein flat dimension if and only if K has finite flat dimension.

As a consequence of the existence of these approximations for modules in $\overline{\mathtt{GFlat}}(R)$, we obtain characterizations of the numerical values of the PGF-dimension (i.e. of the relative homological dimension based on the class of PGF-modules [11]) and the Gorenstein flat dimension, in terms of the vanishing of certain Ext-groups. We note that these characterizations are analogous to the corresponding characterization of the Gorenstein projective dimension for modules of finite Gorenstein projective dimension. We also note that, in the case of the Gorenstein flat dimension, this characterization has been already established by Christensen et al. in [7].

[1]It is not known whether Gorenstein projective modules are always Gorenstein flat, even though Holm has proved in [19] that this is the case if the ring R is right coherent and there is an upper bound on the projective dimension of all modules that have finite projective dimension.

It is easily seen that the exact categories $\mathtt{PGF}(R)$ and $\mathtt{Cot}(R) \cap \mathtt{GFlat}(R)$ of PGF-modules and cotorsion Gorenstein flat modules respectively are both Frobenius. The projective modules are the projective-injective objects of the former of these categories, whereas the projective-injective objects of the latter are the flat cotorsion modules. The proof of this claim for $\mathtt{PGF}(R)$ is essentially identical to the proof of the corresponding claim for the class of Gorenstein projective modules, which can be found in [12]. The claim for $\mathtt{Cot}(R) \cap \mathtt{GFlat}(R)$ was noted by Gillespie in [17], in the special case where the ground ring is right coherent. We point out that cotorsion Gorenstein flat modules are particular examples of the Gorenstein flat cotorsion modules introduced in [8]. In the sequel, we describe two hereditary Hovey triples

$$\big(\mathtt{PGF}(R), \overline{\mathtt{Flat}}(R), \overline{\mathtt{GFlat}}(R)\big) \quad \text{and} \quad \big(\mathtt{GFlat}(R), \overline{\mathtt{Flat}}(R), \mathtt{Cot}(R) \cap \overline{\mathtt{GFlat}}(R)\big)$$

in the exact category $\overline{\mathtt{GFlat}}(R)$; here, we denote by $\overline{\mathtt{Flat}}(R)$ the class of all modules of finite flat dimension. The exact model structures associated with these Hovey triples have homotopy categories which are equivalent to the stable categories of the Frobenious exact categories $\mathtt{PGF}(R)$ and $\mathtt{Cot}(R) \cap \mathtt{GFlat}(R)$, respectively. In order to realize these stable categories as the homotopy categories of model structures, it is therefore sufficient to work on the subcategory $\overline{\mathtt{GFlat}}(R)$.

Notations and terminology. We work over a fixed unital associative ring R and, unless otherwise specified, all modules are assumed to be left R-modules. We denote by R^{op} the opposite ring of R and do not distinguish between right R-modules and left R^{op}-modules. The classes of projective, flat and cotorsion modules are denoted by $\mathtt{Proj}(R)$, $\mathtt{Flat}(R)$ and $\mathtt{Cot}(R)$ respectively. We say that a class $\mathcal{C}$ of modules is projectively resolving if $\mathtt{Proj}(R) \subseteq \mathcal{C}$ and $\mathcal{C}$ is closed under extensions and kernels of epimorphisms.

1. Preliminaries

In this section, we collect certain basic notions and preliminary results that will be used in the sequel. These involve the concept of a Hovey triple in exact additive categories, the basics on Gorenstein flat and PGF-modules and the notion of relative injectivity of linear maps with respect to a class of modules.

I. Cotorsion pairs and Hovey triples. Let $\mathcal{A}$ be an exact additive category, in the sense of [6]. Then, the Ext^1-pairing induces an orthogonality relation between objects therein. If $\mathcal{B}$ is a class of objects in $\mathcal{A}$, then the left orthogonal ${}^{\perp}\mathcal{B}$ of $\mathcal{B}$ is the class consisting of those objects $X \in \mathcal{A}$, which are such that $\mathrm{Ext}^1_{\mathcal{A}}(X, B) = 0$ for all $B \in \mathcal{B}$. Analogously, the right orthogonal $\mathcal{B}^{\perp}$ of $\mathcal{B}$ is the class consisting of those objects $Y \in \mathcal{A}$, which are such that $\mathrm{Ext}^1_{\mathcal{A}}(B, Y) = 0$ for all $B \in \mathcal{B}$. If $\mathcal{C}, \mathcal{D}$ are two classes of objects in $\mathcal{A}$, then the pair $(\mathcal{C}, \mathcal{D})$ is said to be a cotorsion pair in $\mathcal{A}$ if $\mathcal{C} = {}^{\perp}\mathcal{D}$ and $\mathcal{C}^{\perp} = \mathcal{D}$; cf. [14]. The kernel of the cotorsion pair is the class $\mathcal{C} \cap \mathcal{D}$. The cotorsion pair is called hereditary if $\mathrm{Ext}^i_{\mathcal{A}}(C, D) = 0$ for all $i > 0$ and all objects $C \in \mathcal{C}$ and $D \in \mathcal{D}$. The cotorsion pair is complete if for any object $A \in \mathcal{A}$ there exist short exact sequences (conflations)

$$0 \longrightarrow D \longrightarrow C \longrightarrow A \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow A \longrightarrow D' \longrightarrow C' \longrightarrow 0,$$

where $C, C' \in \mathcal{C}$ and $D, D' \in \mathcal{D}$. An example of a complete hereditary cotorsion pair in the exact category of all modules is provided by the pair $(\mathtt{Flat}(R), \mathtt{Cot}(R))$; cf. [5].

A full subcetagory $\mathcal{A}_0 \subseteq \mathcal{A}$ is called exact if it has the 2-out-of-3 property for short exact sequences (conflations): If we are given a short exact sequence in $\mathcal{A}$ and two of the three objects involved are contained in $\mathcal{A}_0$, then the third object is also contained in $\mathcal{A}_0$. If $\mathcal{A}_0$ is such an exact subcategory and $(\mathcal{C},\mathcal{D})$ is a complete cotorsion pair in $\mathcal{A}$ with $\mathcal{C} \subseteq \mathcal{A}_0$, then the pair $(\mathcal{C},\mathcal{A}_0 \cap \mathcal{D})$ is easily seen to be a complete cotorsion pair in $\mathcal{A}_0$; it is the restriction of the original cotorsion pair to $\mathcal{A}_0$.

A Hovey triple on $\mathcal{A}$ is a triple $(\mathcal{C},\mathcal{W},\mathcal{F})$ of subclasses of $\mathcal{A}$, which are such that the pairs $(\mathcal{C},\mathcal{W}\cap\mathcal{F})$ and $(\mathcal{C}\cap\mathcal{W},\mathcal{F})$ are complete cotorsion pairs and the class $\mathcal{W}$ is thick (i.e. it is closed under direct summands and satisfies the 2-out-of-3 property for short exact sequences in $\mathcal{A}$). Based on the work of Hovey [20], Gillespie has shown that there is a bijection between Hovey triples on an idempotent complete exact category $\mathcal{A}$ and the so-called exact model structures on $\mathcal{A}$; cf. [16, Theorem 3.3]. In the context of Gillespie's bijection, it is proved in [16, Proposition 5.2] that for an exact model structure on $\mathcal{A}$, whose associated complete cotorsion pairs are both hereditary, the exact category $\mathcal{C}\cap\mathcal{F}$ is Frobenius with projective-injective objects equal to $\mathcal{C}\cap\mathcal{W}\cap\mathcal{F}$. A result of Happel [18] implies that the stable category of $\mathcal{C}\cap\mathcal{F}$ modulo its projective-objective objects is triangulated. Then, as shown in [16, Proposition 4.4 and Corollary 4.8], the homotopy category of the given exact model structure is triangulated equivalent to the stable category of the Frobenius exact category $\mathcal{C}\cap\mathcal{F}$.

II. Basics on modules of finite Gorenstein flat dimension. Gorentein flat modules were defined in [15] as the syzygy modules of those acyclic complexes of flat modules that remain acyclic after applying the functor $I \otimes \_$ for any injective right module I. It follows that the abelian groups $\operatorname{Tor}_i^R(I,L)$ are trivial if $i>0$ for any injective right module I and any Gorenstein flat module L. It is clear that the class $\texttt{GFlat}(R)$ of Gorenstein flat modules contains all flat modules and is closed under direct sums. As shown in [22], the class $\texttt{GFlat}(R)$ is also projectively resolving and closed under direct summands. In fact, $\big(\texttt{GFlat}(R),\texttt{GFlat}(R)^{\perp}\big)$ is a complete hereditary cotorsion pair in the category of modules, whose kernel coincides with the class $\texttt{Cot}(R)\cap\texttt{Flat}(R)$ of flat cotorsion modules; cf. [22, Corollary 4.12].

The Gorenstein flat dimension of a module M was defined by Holm in [19] by the standard method, using resolutions by Gorenstein flat modules. Indeed, let M be a module and n a non-negative integer. If

$$0 \longrightarrow K \longrightarrow L_{n-1} \longrightarrow \cdots \longrightarrow L_0 \longrightarrow M \longrightarrow 0,$$

$$0 \longrightarrow K' \longrightarrow L'_{n-1} \longrightarrow \cdots \longrightarrow L'_0 \longrightarrow M \longrightarrow 0$$

are two exact sequences of modules with $L_0,\ldots,L_{n-1},L'_0,\ldots,L'_{n-1} \in \texttt{GFlat}(R)$, then $K \in \texttt{GFlat}(R)$ if and only if $K' \in \texttt{GFlat}(R)$. This follows from [1, Lemma 3.12], since the class $\texttt{GFlat}(R)$ is projectively resolving, closed under direct sums and direct summands; see also [3, Lemma 2.9]. The following result is an immediate consequence of this remark.

Corollary 1.1. *The following conditions are equivalent for a module M and a non-negative integer n:*

(i) There exists an exact sequence of modules

$$0 \longrightarrow L_n \longrightarrow L_{n-1} \longrightarrow \cdots \longrightarrow L_0 \longrightarrow M \longrightarrow 0,$$

with $L_0,\ldots,L_{n-1},L_n \in \texttt{GFlat}(R)$.

(ii) For any exact sequence of modules

$$0 \longrightarrow K \longrightarrow L_{n-1} \longrightarrow \cdots \longrightarrow L_0 \longrightarrow M \longrightarrow 0$$

with $L_0,\ldots,L_{n-1} \in \texttt{GFlat}(R)$, we also have $K \in \texttt{GFlat}(R)$. $\square$

If the equivalent conditions in the above Corollary are satisfied, we say that M has a Gorenstein flat resolution of length n. The Gorenstein flat dimension $\mathrm{Gfd}_R M$ of M is the shortest length of a Gorenstein flat resolution of M. Of course, if M has no Gorenstein flat resolution of finite length, then we write $\mathrm{Gfd}_R M = \infty$. Since Gorenstein flat modules annihilate the functors $\mathrm{Tor}_i^R(I, \_)$ for any $i > 0$ and any injective right module I, a dimension shifting argument shows that $\mathrm{Tor}_i^R(I, M) = 0$ for any $i > \mathrm{Gfd}_R M$ and any injective right module I. Any flat module is Gorenstein flat and hence we always have $\mathrm{Gfd}_R M \leq \mathrm{fd}_R M$. In fact, if M has finite flat dimension, then we have an equality $\mathrm{Gfd}_R M = \mathrm{fd}_R M$; this is proved in [4, Theorem 2.2]. In particular, any Gorenstein flat module of finite flat dimension is necessarily flat.

For future reference, we record some basic properties of the class $\overline{\mathtt{GFlat}}(R)$ of all modules of finite Gorenstein flat dimension.

Proposition 1.2. *Let $(M_i)_i$ be a family of modules and $M = \bigoplus_i M_i$ the corresponding direct sum. Then, $Gfd_R M = \sup_i Gfd_R M_i$. In particular, the class $\overline{\mathtt{GFlat}}(R)$ is closed under finite direct sums and direct summands.*

Proof. In view of Corollary 1.1 above, this result is an immediate consequence of the fact that the class $\mathtt{GFlat}(R)$ is closed under direct sums and direct summands. □

Proposition 1.3. *Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be a short exact sequence of modules. Then:*
(i) $Gfd_R M \leq \max\{Gfd_R M', Gfd_R M''\}$,
(ii) $Gfd_R M' \leq \max\{Gfd_R M, Gfd_R M''\}$,
(iii) $Gfd_R M'' \leq 1 + \max\{Gfd_R M', Gfd_R M\}$.
In particular, the class $\overline{\mathtt{GFlat}}(R)$ has the 2-out-of-3 property for short exact sequences.

Proof. Consider two projective resolutions $P'_* \longrightarrow M' \longrightarrow 0$ and $P''_* \longrightarrow M'' \longrightarrow 0$ of M' and M'' respectively. Then, we may construct by the standard step-by-step process a projective resolution $P_* \longrightarrow M \longrightarrow 0$ of M, such that $P_i = P'_i \oplus P''_i$ and the corresponding syzygy module $\Omega_i M$ is an extension of $\Omega_i M''$ by $\Omega_i M'$ for all i. Then, the inequality in (i) (resp. in (ii)) follows by invoking Corollary 1.1 and the closure of $\mathtt{GFlat}(R)$ under extensions (resp. under kernels of epimorphisms).

In order to prove (iii), we fix a short exact sequence

$$0 \longrightarrow K \longrightarrow P \xrightarrow{p} M'' \longrightarrow 0, \tag{1}$$

where P is a projective module, and consider the pullback of the short exact sequence given in the statement of the Proposition along p

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & K \arrow[r, equal] \arrow[d] & K \arrow[d] & \\
0 \arrow[r] & M' \arrow[r] \arrow[d, equal] & N \arrow[r] \arrow[d] & P \arrow[r] \arrow[d, "p"'] & 0 \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Since P is projective, the horizontal short exact sequence in the middle of the diagram splits and hence $N \simeq P \oplus M'$. We now invoke Proposition 1.2 and conclude that $\mathrm{Gfd}_R N = \mathrm{Gfd}_R M'$.

Then, the vertical short exact sequence in the middle of the diagram and assertion (ii) above show that

$$\mathrm{Gfd}_R K \leq \max\{\mathrm{Gfd}_R N, \mathrm{Gfd}_R M\} = \max\{\mathrm{Gfd}_R M', \mathrm{Gfd}_R M\}.$$

Since we may splice any Gorenstein flat resolution of K of length $\mathrm{Gfd}_R K$ with the short exact sequence (1) and obtain a Gorenstein flat resolution of M'' of length $1 + \mathrm{Gfd}_R K$, it follows that

$$\mathrm{Gfd}_R M'' \leq 1 + \mathrm{Gfd}_R K \leq 1 + \max\{\mathrm{Gfd}_R M', \mathrm{Gfd}_R M\},$$

as needed. □

III. Projectively coresolved Gorenstein flat modules. A variant of the Gorenstein flat modules, the so-called projectively coresolved Gorensein flat modules (PGF-modules), were introduced by Saroch and Stovicek. The PGF-modules are the syzygy modules of those acyclic complexes of projective modules that remain acyclic after applying the functor $I \otimes \_$ for any injective right module I. It is clear that PGF-modules are Gorenstein flat. As shown in [22, Theorem 4.9], the class $\texttt{PGF}(R)$ of PGF-modules is also projectively resolving, closed under direct sums and direct summands. In fact, $\big(\texttt{PGF}(R), \texttt{PGF}(R)^{\perp}\big)$ is a complete hereditary cotorsion pair in the category of modules, whose kernel coincides with the class $\texttt{Proj}(R)$ of projective modules. Moreover, the right orthogonal $\texttt{PGF}(R)^{\perp}$ is thick (i.e. it is closed under direct summands and satisfies the 2-out-of-3 property for short exact sequences). The inclusion $\texttt{PGF}(R) \subseteq \texttt{GFlat}(R)$ induces an inclusion $\texttt{GFlat}(R)^{\perp} \subseteq \texttt{PGF}(R)^{\perp}$. It turns out that a module N is contained in $\texttt{GFlat}(R)^{\perp}$ if and only if N is a cotorsion module contained in $\texttt{PGF}(R)^{\perp}$, i.e. $\texttt{GFlat}(R)^{\perp} = \texttt{Cot}(R) \cap \texttt{GFlat}(R)^{\perp}$; cf. [22, Theorem 4.12]. The homological dimension theory, which is based on the class $\texttt{PGF}(R)$, is studied in [11].

IV. Relative injectivity of linear maps. Let $\mathcal{C}$ be a class of modules. Following [22], we say that a linear map $f : M \longrightarrow N$ is $\mathcal{C}$-injective provided that any linear map $M \longrightarrow C$ factors through f for any module $C \in \mathcal{C}$. If f is injective and $\operatorname{coker} f \in {}^{\perp}\mathcal{C}$, then f is $\mathcal{C}$-injective. On the other hand, if f is injective, $\mathcal{C}$-injective and $N \in {}^{\perp}\mathcal{C}$, then $\operatorname{coker} f \in {}^{\perp}\mathcal{C}$. It is easily seen that the notion of $\mathcal{C}$-injectivity has the following three properties with respect to a composable pair of morphisms $f : M \longrightarrow N$ and $g : N \longrightarrow L$:

(i) If f, g are $\mathcal{C}$-injective, then the composition $g \circ f$ is $\mathcal{C}$-injective.

(ii) If the composition $g \circ f$ is $\mathcal{C}$-injective, then f is $\mathcal{C}$-injective

(iii) If f is surjective and the composition $g \circ f$ is $\mathcal{C}$-injective, then g is $\mathcal{C}$-injective.

In the sequel, we shall mostly use this notion of relative injectivity for the class $D^2(\texttt{Proj}(R))$, which is obtained from the class $\texttt{Proj}(R)$ of projective modules by applying twice the Pontryagin duality functor D to it. We recall that the functor D maps any left (resp. right) module M onto the right (resp. left) module $DM = \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$. The class $D^2(\texttt{Flat}(R))$ is defined analogously.

Lemma 1.4. *The following conditions are equivalent for a module M:*

(i) $M \in {}^{\perp}D^2(\texttt{Flat}(R))$,

(ii) $M \in {}^{\perp}D^2(\texttt{Proj}(R))$,

(iii) $\mathrm{Tor}_1^R(I, M) = 0$ for any injective right module I.

Proof. (i)→(ii): The inclusion $\texttt{Proj}(R) \subseteq \texttt{Flat}(R)$ implies that $D^2(\texttt{Proj}(R)) \subseteq D^2(\texttt{Flat}(R))$ and hence ${}^{\perp}D^2(\texttt{Flat}(R)) \subseteq {}^{\perp}D^2(\texttt{Proj}(R))$.

(ii)→(iii): The right module DR is an injective cogenerator of the category of right modules and hence any injective right module is a direct summand of a direct product of copies of DR.

Therefore, it suffices to consider the case where I is a direct power of DR. For any set Λ we have $(DR)^\Lambda = DP$, where $P = R^{(\Lambda)}$ is the free module with basis Λ. Then, the standard Hom-tensor adjunction and our assumption on M imply that $D\mathrm{Tor}_1^R(DP, M) = \mathrm{Ext}_R^1(M, D^2P) = 0$. Since the abelian group $\mathbb{Q}/\mathbb{Z}$ cogenerates the category of abelian groups, it follows that $\mathrm{Tor}_1^R(DP, M) = 0$, as needed. □

(iii)→(i): If F is a flat module, then Lambek's criterion [21] implies that the right module DF is injective and hence we have $\mathrm{Ext}_R^1(M, D^2F) = D\mathrm{Tor}_1^R(DF, M) = D0 = 0$. □

Corollary 1.5. *$D^2(\mathtt{Proj}(R)) \subseteq \mathtt{GFlat}(R)^\perp \subseteq \mathtt{PGF}(R)^\perp$.*

Proof. In view of Lemma 1.4, we have $\mathtt{Gflat}(R) \subseteq {}^\perp D^2(\mathtt{Proj}(R))$. As we noted in III above, we also have $\mathtt{GFlat}(R)^\perp \subseteq \mathtt{PGF}(R)^\perp$. □

If M is a module with $\mathrm{fd}_R M = n$, then $\mathrm{Tor}_n^R(I, M) \neq 0$ for a suitable injective right module I. Indeed, if A is a right module with $\mathrm{Tor}_n^R(A, M) \neq 0$ and I is any injective right module containing A, then the triviality of the group $\mathrm{Tor}_{n+1}^R(I/A, M)$ implies that $\mathrm{Tor}_n^R(I, M) \neq 0$.

Lemma 1.6. *Let $0 \longrightarrow M' \xrightarrow{f} M \longrightarrow M'' \longrightarrow 0$ be a short exact sequence of modules.*

(i) If $\mathrm{Tor}_1^R(I, M'') = 0$ for any injective right module I, then f is $D^2(\mathtt{Proj}(R))$-injective.

(ii) If $M \in {}^\perp D^2(\mathtt{Proj}(R))$ and f is $D^2(\mathtt{Proj}(R))$-injective, then $M'' \in {}^\perp D^2(\mathtt{Proj}(R))$.

(iii) If $M \in {}^\perp D^2(\mathtt{Proj}(R))$, f is $D^2(\mathtt{Proj}(R))$-injective and $\mathrm{fd}_R M'' \leq 1$, then M'' is flat.

Proof. (i) This follows from Lemma 1.4, which implies that $M'' \in {}^\perp D^2(\mathtt{Proj}(R))$.

(ii) This is clear.

(iii) If $\mathrm{fd}_R M'' = 1$, then $\mathrm{Tor}_1^R(I, M'') \neq 0$ for some injective right module I. Then, Lemma 1.4 implies that M'' is not contained in ${}^\perp D^2(\mathtt{Proj}(R))$, contradicting (ii) above. □

2. Approximations by PGF-modules

In this section, we show that modules of finite Gorenstein flat dimension can be approximated by PGF-modules and modules of finite flat dimension, in a way that generalizes the exact sequences obtained in [22, Theorem 4.11].

Theorem 2.1. *The following conditions are equivalent for a module M and an integer $n \geq 0$:*

(i) $\mathrm{Gfd}_R M = n$.

(ii) There exists a short exact sequence

$$0 \longrightarrow M \longrightarrow K \longrightarrow G \longrightarrow 0,$$

where G is a PGF-module and $\mathrm{fd}_R K = n$.

(iii) There exists a short exact sequence

$$0 \longrightarrow K \xrightarrow{f} G \longrightarrow M \longrightarrow 0,$$

where G is a PGF-module and

(a) if $n > 1$, then $\mathrm{fd}_R K = n - 1$,

(b) if $n = 1$, then K is flat and f is not $D^2(\mathtt{Proj}(R))$-injective,

(c) if $n = 0$, then K is flat and f is $D^2(\mathtt{Proj}(R))$-injective.

(iv) There exists a projective module P, such that the direct sum $M' = M \oplus P$ fits into an exact sequence

$$0 \longrightarrow G \xrightarrow{f} M' \longrightarrow K \longrightarrow 0,$$

where G is a PGF-module, $\mathrm{fd}_R K = n$ and f is $\mathtt{PGF}(R)^\perp$-injective.

(v) There exists a Gorenstein flat module P, such that the direct sum $M' = M \oplus P$ fits into an exact sequence

$$0 \longrightarrow G \xrightarrow{f} M' \longrightarrow K \longrightarrow 0,$$

where G is a PGF-module and $fd_R K = n$. If $n = 1$, we also require f to be $D^2(\mathtt{Proj}(R))$-injective.

Proof. (i)→(ii): We use induction on n. If $n = 0$, this follows from [22, Theorem 4.11(4)]. Assume that $n \geq 1$ and the result is known for modules of Gorenstein flat dimension $< n$. We consider a short exact sequence

$$0 \longrightarrow N \longrightarrow L \longrightarrow M \longrightarrow 0,$$

where L is Gorenstein flat and $\mathrm{Gfd}_R N = n - 1$, and invoke the induction hypothesis to find a short exact sequence

$$0 \longrightarrow N \longrightarrow K \longrightarrow G \longrightarrow 0,$$

where G is a PGF-module and $\mathrm{fd}_R K = n - 1$. We now form the pushout of that short exact sequence along the monomorphisms $N \longrightarrow L$ and obtain the commutative diagram with exacts rows and columns

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & N & \longrightarrow & L & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & K & \longrightarrow & L' & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & G & = & G & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

Since both L and G are Gorenstein flat, the closure of $\mathtt{GFlat}(R)$ under extensions implies that L' is also Gorenstein flat. Therefore, there exists a short exact sequence

$$0 \longrightarrow L' \longrightarrow F \longrightarrow G' \longrightarrow 0,$$

where F is flat and $G' \in \mathtt{PGF}(R)$. Pushing out that short exact sequence along the epimorphism $L' \longrightarrow M$, we obtain the commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & L' & \longrightarrow & M & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & F & \longrightarrow & K' & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & G' & = & G' & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

Then, the rightmost vertical exact sequence is of the required type. Indeed, the horizontal exact sequence in the middle shows that $\mathrm{fd}_R K' \leq n$. In fact, the latter inequality cannot be strict, since otherwise we would have $\mathrm{Gfd}_R K' \leq \mathrm{fd}_R K' \leq n - 1$ and the rightmost vertical exact sequence would imply that $\mathrm{Gfd}_R M \leq \max\{\mathrm{Gfd}_R K', \mathrm{Gfd}_R G'\} \leq n - 1$; cf. Proposition 1.3(ii).

(ii)→(iii): We fix a short exact sequence as in (ii) and note that the module K fits into a short exact sequence

$$0 \longrightarrow K' \longrightarrow P \longrightarrow K \longrightarrow 0,$$

where P is projective and $\mathrm{fd}_R K' = n-1$ (if $n=0$, then K' is also flat). The pullback of that short exact sequence along the monomorphism $M \longrightarrow K$ induces a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K' & = & K' & & & & \\
 & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \jmath} & & & & \\
0 & \longrightarrow & G' & \xrightarrow{\imath} & P & \longrightarrow & G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & K & \longrightarrow & G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since both P and G are PGF-modules, the closure of $\mathtt{PGF}(R)$ under kernels of epimorphisms shows that G' is also a PGF-module. Being a PGF-module, G is Gorenstein flat and hence Lemma 1.6(i) implies that $\imath$ is $D^2(\mathtt{Proj}(R))$-injective. It follows that $\jmath = \imath \circ f$ is $D^2(\mathtt{Proj}(R))$-injective if and only if f is $D^2(\mathtt{Proj}(R))$-injective.

If $n=1$, then $\mathrm{fd}_R K = 1$. Since the projective module P is obviously contained in the left orthogonal ${}^{\perp}D^2(\mathtt{Proj}(R))$, Lemma 1.6(iii) implies that $\jmath$ is not $D^2(\mathtt{Proj}(R))$-injective and hence f is not $D^2(\mathtt{Proj}(R))$-injective either. If $n=0$, then K is flat and hence Lemma 1.6(i) implies that $\jmath$ is $D^2(\mathtt{Proj}(R))$-injective. It follows that f is $D^2(\mathtt{Proj}(R))$-injective as well.

(iii)→(iv): We fix a short exact sequence as in (iii) and note that the PGF-module G fits into a short exact sequence

$$0 \longrightarrow G \xrightarrow{\imath} P \longrightarrow G' \longrightarrow 0,$$

where P is projective and $G' \in \mathtt{PGF}(R)$. Then, the pushout of the latter short exact sequence along the epimorphism $G \longrightarrow M$ induces a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K & = & K & & & & \\
 & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \jmath} & & & & \\
0 & \longrightarrow & G & \xrightarrow{\imath} & P & \longrightarrow & G' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & K' & \longrightarrow & G' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{2}$$

Since G' is a PGF-module, G' is Gorenstein flat and hence $\imath$ is $D^2(\mathtt{Proj}(R))$-injective (cf. Lemma 1.6(i)). It follows that $\jmath = \imath \circ f$ is $D^2(\mathtt{Proj}(R))$-injective if and only if f is $D^2(\mathtt{Proj}(R))$-injective. Now, the definition of the pushout and the injectivity of $\imath$ imply that there is a short exact sequence

$$0 \longrightarrow G \longrightarrow M \oplus P \longrightarrow K' \longrightarrow 0. \tag{3}$$

If $n>1$, the vertical short exact sequence in the middle shows that $\mathrm{Tor}_n^R(\_, K') = \mathrm{Tor}_{n-1}^R(\_, K)$ is non-zero and $\mathrm{Tor}_{n+1}^R(\_, K') = \mathrm{Tor}_n^R(\_, K) = 0$, so that $\mathrm{fd}_R K' = n$. If $n=1$, then K is flat and f is not $D^2(\mathtt{Proj}(R))$-injective. It follows that $\mathrm{fd}_R K' \leq 1$ and $\jmath$ is not $D^2(\mathtt{Proj}(R))$-injective. Then, Lemma 1.6(i) implies that K' is not flat and hence $\mathrm{fd}_R K' = 1$. If $n = 0$, then K is flat and hence $\mathrm{fd}_R K' \leq 1$, as before. Since f is assumed to be $D^2(\mathtt{Proj}(R))$-injective in this case, it follows that $\jmath$ is also $D^2(\mathtt{Proj}(R))$-injective. The projective module P being obviously contained in ${}^\perp D^2(\mathtt{Proj}(R))$, Lemma 1.6(iii) implies that K' is flat.

In order to show that the short exact sequence (3) has the required additional property, we note that for any module $Q \in \mathtt{PGF}(R)^\perp$ the two horizontal short exact sequences in diagram (2) above induce a commutative diagram of abelian groups with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_R(G',Q) & \longrightarrow & \mathrm{Hom}_R(P,Q) & \longrightarrow & \mathrm{Hom}_R(G,Q) & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathrm{Hom}_R(G',Q) & \longrightarrow & \mathrm{Hom}_R(K',Q) & \longrightarrow & \mathrm{Hom}_R(M,Q) & \longrightarrow & 0 \end{array}$$

It follows readily that there is an induced sequence of abelian groups

$$0 \longrightarrow \mathrm{Hom}_R(K',Q) \longrightarrow \mathrm{Hom}_R(M,Q) \oplus \mathrm{Hom}_R(P,Q) \longrightarrow \mathrm{Hom}_R(G,Q) \longrightarrow 0,$$

as needed.

(iv)→(v): This is immediate, since $\mathtt{Proj}(R) \subseteq \mathtt{GFlat}(R)$ and $D^2(\mathtt{Proj}(R)) \subseteq \mathtt{PGF}(R)^\perp$; cf. Corollary 1.5.

(v)→(i): Consider an exact sequence as in (v) and note that Proposition 1.2 implies that $\mathrm{Gfd}_R M' = \mathrm{Gfd}_R M$. Therefore, it suffices to prove that $\mathrm{Gfd}_R M' = n$. Since the PGF-module G is Gorenstein flat and $\mathrm{Gfd}_R K \leq \mathrm{fd}_R K = n$, Proposition 1.3(i) implies that $\mathrm{Gfd}_R M' \leq n$. It remains to show that the latter inequality cannot be strict. Indeed, let us assume that $n \geq 1$ and $\mathrm{Gfd}_R M' \leq n-1$.

If $n = 1$, then M' is Gorenstein flat and hence $M' \in D^2(\mathtt{Proj}(R))^\perp$; cf. Lemma 1.4. Since f is assumed to be $D^2(\mathtt{Proj}(R))$-injective, Lemma 1.6(iii) implies that the equality $\mathrm{fd}_R K = 1$ cannot occur. If $n > 1$, then the PGF-module G is Gorenstein flat and hence the functor $\mathrm{Tor}^R_{n-1}(\_, G)$ vanishes on all injective right modules. Since $\mathrm{Gfd}_R M' \leq n-1$, $\mathrm{Tor}^R_n(\_, M')$ vanishes on all injective right modules as well. It follows that $\mathrm{Tor}^R_n(\_, K)$ also vanishes on all injective right modules. This is a contradiction, since $\mathrm{fd}_R K = n$. □

Remarks 2.2. (i) In the case where $n = 1$, it is necessary to impose some restrictions on the short exact sequences appearing in Theorem 2.1(iii) and (v). Indeed, if P is any non-zero projective module and G is a PGF-module, then the split short exact sequence

$$0 \longrightarrow P \longrightarrow P \oplus G \longrightarrow G \longrightarrow 0$$

is of the type appearing in Theorem 2.1(iii) for $n = 1$, but $\mathrm{Gfd}_R G = 0 \neq 1$. On the other hand, if K is a non-flat module with $\mathrm{pd}_R K = 1$, then a projective resolution of K provides an exact sequence

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow K \longrightarrow 0 \tag{4}$$

of the type appearing in Theorem 2.1(v) for $n = 1$, but $\mathrm{Gfd}_R P_0 = 0 \neq 1$.

(ii) The short exact sequence (4) is of the type appearing in Theorem 2.1(iii) for $n = 0$, but $\mathrm{Gfd}_R K \neq 0$. (Any Gorenstein flat module of finite flat dimension is necessarily flat.) We conclude that it is necessary to impose some additional restriction to the short exact sequence appearing in Theorem 2.1(iii), in the case where $n = 0$, in order to get that the third term of that exact sequence is Gorenstein flat.[2]

[2] The necessity of such an additional restriction was also noted in [22, Theorem 4.11(ii)].

(iii) The proof of the implication (v)$\to$(i) in Theorem 2.1 uses only the weaker assumption that the module G in (v) is Gorenstein flat.

Recall that the (left) finitistic flat dimension $\mathrm{fin.f.dim}\, R$ of the ring is the supremum of the flat dimensions of all modules that have a finite flat dimension. Analogously, the (left) finitistic Gorenstein flat dimension $\mathrm{fin.Gf.dim}\, R$ is the supremum of the Gorenstein flat dimension of all modules that have a finite Gorenstein flat dimension. The next result generalizes [19, Theorem 3.24].

Proposition 2.3. *For any ring R we have an equality $\mathrm{fin.f.dim}\, R = \mathrm{fin.Gf.dim}\, R$.*

Proof. Since the Gorenstein flat dimension is a refinement of the flat dimension, it is clear that $\mathrm{fin.f.dim}\, R \leq \mathrm{fin.Gf.dim}\, R$. In order to prove the reverse inequality, consider a module M of finite Gorenstein flat dimension, say with $\mathrm{Gfd}_R M = n$. Then, Theorem 2.1(ii) implies that there exists a module K with $\mathrm{fd}_R K = n$. It follows that $n \leq \mathrm{fin.f.dim}\, R$. Since $\mathrm{fin.Gf.dim}\, R$ is the supremum of these n's, we conclude that $\mathrm{fin.Gf.dim}\, R \leq \mathrm{fin.f.dim}\, R$, as needed. $\square$

3. The relation to the cotorsion pair $\left(\mathtt{PGF}(R), \mathtt{PGF}(R)^{\perp}\right)$

In this section, we use the approximation sequences in Theorem 2.1 to characterize the PGF-modules and, more generally, the PGF-dimension of modules (which is introduced in [11]) within $\overline{\mathtt{Gflat}}(R)$, in terms of the vanishing of certain Ext-groups. We obtain a hereditary Hovey triple in $\overline{\mathtt{Gflat}}(R)$, such that the homotopy category of the associated exact model structure is triangulated equivalent to the stable category of PGF-modules. It will turn out that the exact sequences in Theorem 2.1 are precisely the approximation sequences of the complete cotorsion pair $\left(\mathtt{PGF}(R), \mathtt{PGF}(R)^{\perp}\right)$ obtained in [22, Theorem 4.9], when applied to modules of finite Gorenstein flat dimension.

It follows from [22, Corollary 4.5] that $\mathrm{Ext}^1_R(M, F) = 0$, whenever M is a PGF-module and F is flat. We elaborate on this result and provide a characterization of PGF-modules among modules in $\overline{\mathtt{GFlat}}(R)$, as an application of Theorem 2.1.

Lemma 3.1. *If $M \in \overline{\mathtt{GFlat}}(R)$, then the following conditions are equivalent:*
(i) $M \in \mathtt{PGF}(R)$,
(ii) $\mathrm{Ext}^i_R(M, F) = 0$ for any $i > 0$ and any flat module F,
(iii) $\mathrm{Ext}^i_R(M, F) = 0$ for any $i > 0$ and any module F of finite flat dimension,
(iv) $\mathrm{Ext}^1_R(M, F) = 0$ for any module F of finite flat dimension.

Proof. (i)$\to$(ii): This follows from [22, Corollary 4.5], since the syzygies of a PGF-module M in any projective resolution of it are PGF-modules and hence $\mathrm{Ext}^i_R(M, F) = \mathrm{Ext}^1_R(\Omega_{i-1} M, F) = 0$ for any $i > 0$ and any flat module F.

The implication (ii)$\to$(iii) follows by induction on the flat dimension of F, whereas the implication (iii)$\to$(iv) is immediate.

(iv)$\to$(i): Since M has finite Gorenstein flat dimension, Theorem 2.1(iii) implies that there exists a short exact sequence

$$0 \longrightarrow K \longrightarrow G \longrightarrow M \longrightarrow 0,$$

where $G \in \mathtt{PGF}(R)$ and $K \in \overline{\mathtt{Flat}}(R)$. Then, our assumption implies that this sequence splits and hence M is a direct summand of the PGF-module G. Since the class $\mathtt{PGF}(R)$ is closed under direct summands, we conclude that M is a PGF-module. $\square$

The next result provides a characterization of the PGF-dimension for modules in $\overline{\mathtt{GFlat}}(R)$ that extends [11, Proposition 3.6] and reduces to Lemma 3.1, in the case where $n = 0$.

Proposition 3.2. *The following conditions are equivalent for a module $M \in \overline{\mathtt{GFlat}}(R)$ and a non-negative integer n:*
(i) PGF-$\dim_R M \leq n$,
(ii) $Ext^i_R(M, F) = 0$ for any $i > n$ and any flat module F,
(iii) $Ext^i_R(M, F) = 0$ for any $i > n$ and any module F of finite flat dimension.
(iv) $Ext^{n+1}_R(M, F) = 0$ for any module F of finite flat dimension.

Proof. (i)→(ii): We consider a PGF-resolution of length n

$$0 \longrightarrow G_n \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

and fix a flat module F. Since the functors $\mathrm{Ext}^j_R(\_, F)$ vanish on PGF-modules for all $j > 0$ (cf. Lemma 3.1), the desired vanishing follows by dimension shifting.
(ii)→(iii): This follows by induction on the flat dimension of the module F.
(iii)→(iv): This is immediate.
(iv)→(i): Let

$$0 \longrightarrow K \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

be an exact sequence, where $G_0, \ldots, G_{n-1}$ are PGF-modules. The modules $M, G_0, \ldots, G_{n-1}$ have finite Gorenstein flat dimension and hence an iterated application of Proposition 1.3(ii) shows that K has finite Gorenstein flat dimension as well. Since the functors $\mathrm{Ext}^j_R(G_i, \_)$ vanish on modules of finite flat dimension for all $j > 0$ and all $i = 0, \ldots, n-1$ (cf. Lemma 3.1), a dimension shifting argument shows that $\mathrm{Ext}^1_R(K, F) = \mathrm{Ext}^{n+1}_R(M, F) = 0$ for any module F of finite flat dimension. Invoking Lemma 3.1 again, we conclude that K is a PGF-module and hence PGF-$\dim_R M \leq n$. □

We note that the triviality of the group $\mathrm{Ext}^1_R(M, F)$, whenever M is a PGF-module and F has finite flat dimension, provides also a characterization of modules of finite flat dimension among all modules of finite Gorenstein flat dimension.

Lemma 3.3. *If $M \in \overline{\mathtt{GFlat}}(R)$, then $M \in \overline{\mathtt{Flat}}(R)$ if and only if $Ext^1_R(G, M) = 0$ for any PGF-module G.*

Proof. As we noted above, the Ext-group is trivial if $M \in \overline{\mathtt{Flat}}(R)$. Conversely, assume that M is a module of finite Gorenstein flat dimension contained in $\mathtt{PGF}(R)^{\perp}$. Invoking Theorem 2.1(ii), we obtain a short exact sequence

$$0 \longrightarrow M \longrightarrow K \longrightarrow G \longrightarrow 0,$$

where $G \in \mathtt{PGF}(R)$ and $K \in \overline{\mathtt{Flat}}(R)$. In view of our assumption on M, this sequence splits and hence M is a direct summand of K. Then, $\mathrm{fd}_R M \leq \mathrm{fd}_R K < \infty$ and hence $M \in \overline{\mathtt{Flat}}(R)$, as needed. □

The (left) Gorenstein weak global dimension $\mathrm{Gwgl.dim}\, R$ of R is defined as the supremum of the Gorenstein flat dimensions of all modules. By considering the Gorenstein flat dimension of direct sums of modules, we can easily deduce that $\mathrm{Gwgl.dim}\, R < \infty$ if and only if $\mathrm{Gfd}_R M < \infty$ for any module M. In order to characterize the finiteness of $\mathrm{Gwgl.dim}\, R$, the relevant homological invariants are $\mathrm{sfli}\, R$, the supremum of the flat lengths (dimensions) of injective modules, and its analogue $\mathrm{sfli}\, R^{op}$ for the opposite ring R^{op}. By considering the flat dimension of products of injective modules, it is easily seen that $\text{sfli}\, R < \infty$ if and only if $\text{fd}_R I < \infty$ for any injective module I. As shown in [9], the following conditions are equivalent:

(i) $\text{Gwgl.dim}\, R < \infty$,

(ii) $\text{Gwgl.dim}\, R^{op} < \infty$,

(iii) the invariants $\text{sfli}\, R$ and $\text{sfli}\, R^{op}$ are finite.

If these conditions are satisfied, then $\text{Gwgl.dim}\, R = \text{Gwgl.dim}\, R^{op} = \text{sfli}\, R = \text{sfli}\, R^{op}$. For rings that satisfy these equivalent conditions, we may describe the class $\texttt{PGF}(R)$ of PGF-modules and its right orthogonal in classical terms.

Corollary 3.4. *Let R be a ring and assume that all injective modules (both left and right) have finite flat dimension. Then,* $\texttt{PGF}(R) = {}^{\perp}\overline{\texttt{Flat}}(R)$ *and* $\texttt{PGF}(R)^{\perp} = \overline{\texttt{Flat}}(R)$.

Proof. Our assumption implies that all modules have finite Gorenstein flat dimension, i.e. that $\overline{\texttt{GFlat}}(R) = R\text{-Mod}$. Then, the two equalities in the statement follow from Lemma 3.1 and Lemma 3.3. □

Remarks 3.5. (i) Since the hypothesis in Corollary 3.4 is left-right symmetric, we also have (under the same assumptions) analogous conclusions for the ring R^{op}, i.e. for the corresponding classes of right modules.

(ii) Since injective modules are obviously contained in the right orthogonal $\texttt{PGF}(R)^{\perp}$ and the same is true for right modules, the hypothesis in Corollary 3.4 is also necessary for the equalities $\texttt{PGF}(R)^{\perp} = \overline{\texttt{Flat}}(R)$ and $\texttt{PGF}(R^{op})^{\perp} = \overline{\texttt{Flat}}(R^{op})$ to hold.

The category $\overline{\texttt{GFlat}}(R)$ of modules of finite Gorenstein flat dimension is an extension closed subcategory of the abelian category of all modules (cf. Proposition 1.3(i)), which is also closed under direct summands (cf. Proposition 1.2). Therefore, $\overline{\texttt{GFlat}}(R)$ is an idempotent complete exact additive category [6].

Proposition 3.6. *The pair* $\big(\texttt{PGF}(R), \overline{\texttt{Flat}}(R)\big)$ *is a complete hereditary cotorsion pair in the exact category* $\overline{\texttt{GFlat}}(R)$.

Proof. Since $\overline{\texttt{GFlat}}(R) \cap \texttt{PGF}(R)^{\perp} = \overline{\texttt{Flat}}(R)$ (cf. Lemma 3.3), the pair $\big(\texttt{PGF}(R), \overline{\texttt{Flat}}(R)\big)$ is the restriction of the cotorsion pair $\big(\texttt{PGF}(R), \texttt{PGF}(R)^{\perp}\big)$ of [22, Theorem 4.9] to $\overline{\texttt{GFlat}}(R)$. □

We note that the class $\overline{\texttt{Flat}}(R)$ of modules of finite flat dimension is thick; it is closed under direct summands and has the 2-out-of-3 property for short exact sequences. We also note that the category $\texttt{PGF}(R)$ of PGF-modules is a Frobenius exact category with projective-injective objects the projective modules. Since $\texttt{PGF}(R) \cap \texttt{PGF}(R)^{\perp} = \texttt{Proj}(R)$, all projective modules are injective objects in $\texttt{PGF}(R)$. The very definition of PGF-modules implies that the exact category $\texttt{PGF}(R)$ has enough injective objects and all of these objects are necessarily projective modules. Of course, all projective modules are projective objects in $\texttt{PGF}(R)$. Since $\texttt{PGF}(R)$ is projectively resolving, it follows that it has enough projective objects and all of these objects are necessarily projective modules. In view of the following result, we may realize the stable category of $\texttt{PGF}(R)$ as the homotopy category of the exact model structure induced by a hereditary Hovey triple in $\overline{\texttt{GFlat}}(R)$.

Theorem 3.7. *The triple* $\big(\texttt{PGF}(R), \overline{\texttt{Flat}}(R), \overline{\texttt{GFlat}}(R)\big)$ *is a hereditary Hovey triple in the idempotent complete exact category* $\overline{\texttt{GFlat}}(R)$. *The homotopy category of the associated exact model structure is equivalent, as a triangulated category, to the stable category of the Frobenius exact category* $\texttt{PGF}(R)$ *modulo its projective-injective objects, i.e. modulo* $\texttt{Proj}(R)$.

Proof. We need to prove that the pairs

$$\big(\mathtt{PGF}(R), \overline{\mathtt{Flat}}(R) \cap \overline{\mathtt{GFlat}}(R)\big) \quad \text{and} \quad \big(\mathtt{PGF}(R) \cap \overline{\mathtt{Flat}}(R), \overline{\mathtt{GFlat}}(R)\big)$$

are complete and hereditary cotorsion pairs in the exact category $\overline{\mathtt{GFlat}}(R)$. We note that $\overline{\mathtt{Flat}}(R) \cap \overline{\mathtt{GFlat}}(R) = \overline{\mathtt{Flat}}(R)$ and hence Proposition 3.6 takes care of the claim for the first of these pairs. Since PGF-modules are Gorenstein flat, it follows that

$$\mathtt{PGF}(R) \cap \overline{\mathtt{Flat}}(R) \subseteq \mathtt{GFlat}(R) \cap \overline{\mathtt{Flat}}(R) = \mathtt{Flat}(R),$$

where the latter equality follows since any Gorenstein flat module of finite flat dimension is necessarily flat. It follows that

$$\mathtt{PGF}(R) \cap \overline{\mathtt{Flat}}(R) \subseteq \mathtt{PGF}(R) \cap \mathtt{Flat}(R) \subseteq \mathtt{PGF}(R) \cap \mathtt{PGF}(R)^{\perp} = \mathtt{Proj}(R).$$

On the other hand, projective modules are contained in both classes $\mathtt{PGF}(R)$ and $\overline{\mathtt{Flat}}(R)$ and hence $\mathtt{PGF}(R) \cap \overline{\mathtt{Flat}}(R) = \mathtt{Proj}(R)$. Thus, the second of the pairs displayed above becomes

$$\big(\mathtt{Proj}(R), \overline{\mathtt{GFlat}}(R)\big) .$$

Of course, $\overline{\mathtt{GFlat}}(R)$ is the right orthogonal of $\mathtt{Proj}(R)$ within $\overline{\mathtt{GFlat}}(R)$. In order to prove that $\mathtt{Proj}(R)$ is the left orthogonal of $\overline{\mathtt{GFlat}}(R)$ within $\overline{\mathtt{GFlat}}(R)$, we let M be a module of finite Gorenstein flat dimension which is also contained in ${}^{\perp}\overline{\mathtt{GFlat}}(R)$ and consider a short exact sequence

$$0 \longrightarrow M' \longrightarrow P \longrightarrow M \longrightarrow 0,$$

where P is projective. Then, Proposition 1.3(ii) implies that M' has also finite Gorenstein flat dimension and hence $\mathrm{Ext}^1_R(M, M') = 0$. In particular, the exact sequence above splits. It follows that M is a direct summand of P and hence M is projective. The cotorsion pair $\big(\mathtt{Proj}(R), \overline{\mathtt{GFlat}}(R)\big)$ in $\overline{\mathtt{GFlat}}(R)$ is hereditary (since all higher Ext's with a projective first argument vanish) and complete (since the class $\overline{\mathtt{GFlat}}(R)$ is projectively resolving). The final statement follows from [16, Proposition 4.4 and Corollary 4.8]. □

We shall now prove that the equality $\overline{\mathtt{GFlat}}(R) \cap \mathtt{PGF}(R)^{\perp} = \overline{\mathtt{Flat}}(R)$ of Lemma 3.3 preserves the filtrations of the categories $\overline{\mathtt{GFlat}}(R)$ and $\overline{\mathtt{Flat}}(R)$ induced by the the Gorenstein flat dimension and the flat dimension, respectively. We recall that $\mathtt{GFlat}(R) \cap \mathtt{PGF}(R)^{\perp} = \mathtt{Flat}(R)$ (cf. [22, Theorem 4.11].

Proposition 3.8. *If $M \in \mathtt{PGF}(R)^{\perp}$, then $Gfd_R M = fd_R M$.*

Proof. Since any flat module is Gorenstein flat, we always have $\mathrm{Gfd}_R M \leq \mathrm{fd}_R M$. In order to prove the reverse inequality, it suffices to assume that $\mathrm{Gfd}_R M = n < \infty$. Then, the truncation of a flat resolution of M provides an exact sequence

$$0 \longrightarrow K \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0,$$

where $F_0, \ldots, F_{n-1}$ are flat modules and $K \in \mathtt{GFlat}(R)$. Since the class $\mathtt{PGF}(R)^{\perp}$ is thick and contains the flat modules, our assumption that $M \in \mathtt{PGF}(R)^{\perp}$ implies that $K \in \mathtt{PGF}(R)^{\perp}$ as well. Therefore, $K \in \mathtt{GFlat}(R) \cap \mathtt{PGF}(R)^{\perp} = \mathtt{Flat}(R)$. We conclude that M admits a flat resolution of length n and hence $\mathrm{fd}_R M \leq n = \mathrm{Gfd}_R M$, as needed. □

We shall conclude this section by showing that the approximation sequences of Theorem 2.1 are precisely the approximation sequences of the complete cotorsion pair $\big(\mathtt{PGF}(R), \mathtt{PGF}(R)^{\perp}\big)$ of [22, Theorem 4.9], applied to modules of finite Gorenstein flat dimension. To that end, we fix a module M and note that the completeness of the cotorsion pair provides two short exact sequences

$$0 \longrightarrow M \longrightarrow N \longrightarrow G \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow N' \longrightarrow G' \longrightarrow M \longrightarrow 0, \tag{5}$$

where $G, G' \in \mathtt{PGF}(R)$ and $N, N' \in \mathtt{PGF}(R)^{\perp}$. We may also consider a short exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow N \longrightarrow 0,$$

where P is projective, and consider its pullback along the monomorphism $M \longrightarrow N$, in order to obtain a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K & = & K & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & G'' & \longrightarrow & P & \longrightarrow & G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since $\mathtt{PGF}(R)$ is closed under kernels of epimorphisms, the horizontal short exact sequence in the middle shows that G'' is a PGF-module. Letting $N'' = N$, we thereby obtain a third short exact sequence

$$0 \longrightarrow G'' \longrightarrow M \oplus P \longrightarrow N'' \longrightarrow 0, \tag{6}$$

where $G'' \in \mathtt{PGF}(R)$, $N'' \in \mathtt{PGF}(R)^{\perp}$ and P is projective.

Proposition 3.9. *Let M be a module and consider the three short exact sequences in (5) and (6), where $G, G', G'' \in \mathtt{PGF}(R)$, $N, N', N'' \in \mathtt{PGF}(R)^{\perp}$ and P is projective.*

(i) If $M \in \overline{\mathtt{GFlat}}(R)$, then $N, N', N'' \in \overline{\mathtt{Flat}}(R)$.

(ii) If one of the modules N, N', N'' is contained in $\overline{\mathtt{Flat}}(R)$, then $M \in \overline{\mathtt{GFlat}}(R)$.

Proof. We note that $\overline{\mathtt{GFlat}}(R)$ is a thick subcategory which contains all Gorenstein flat modules (and hence all PGF-modules).

(i) Assuming that M (and hence $M \oplus P$ for any projective module P) has finite Gorenstein flat dimension, the thickness of $\overline{\mathtt{GFlat}}(R)$ implies that the modules N, N' and N'' have finite Gorenstein flat dimension as well. Then, Proposition 3.8 implies that $N, N', N'' \in \overline{\mathtt{Flat}}(R)$.

(ii) Since $\overline{\mathtt{Flat}}(R) \subseteq \overline{\mathtt{GFlat}}(R)$, the result follows from the thickness of $\overline{\mathtt{GFlat}}(R)$. ☐

Remark 3.10. Using the equality $\mathtt{PGF}(R) \cap \mathtt{PGF}(R)^{\perp} = \mathtt{Proj}(R)$, we may proceed as in the proof of Proposition 3.8 and show that PGF-$\dim_R M = \mathrm{pd}_R M$ for any $M \in \mathtt{PGF}(R)^{\perp}$. Let $\overline{\mathtt{PGF}}(R)$ be the class of modules of finite PGF-dimension. It follows that $\overline{\mathtt{PGF}}(R) \cap \mathtt{PGF}(R)^{\perp} = \overline{\mathtt{Proj}}(R)$ (cf. [11, Proposition 3.8(i)]), with that equality preserving the filtrations of the categories $\overline{\mathtt{PGF}}(R)$ and $\overline{\mathtt{Proj}}(R)$ induced by the PGF-dimension and the projective dimension, respectively. In particular, the cotorsion pair $\big(\mathtt{PGF}(R), \overline{\mathtt{Proj}}(R)\big)$ in the exact category $\overline{\mathtt{PGF}}(R)$ considered in [11, §4] is precisely the restriction of the cotorsion pair $\big(\mathtt{PGF}(R), \mathtt{PGF}(R)^{\perp}\big)$ therein. Moreover, the thickness of the category $\overline{\mathtt{PGF}}(R)$ can be used as in Proposition 3.9, to show that modules in $\overline{\mathtt{PGF}}(R)$ can be characterized by the assertion that one (and hence all) of the modules N, N' and N'' in the exact sequences (5) and (6) be of finite projective dimension.

4. Approximations by Gorenstein flat modules

In this section, we show that modules of finite Gorenstein flat dimension can be approximated by Gorenstein flat modules and cotorsion modules of finite flat dimension, in a way analogous to the exact sequences obtained in Section 2.

As shown in [22, Corollary 4.12], the right orthogonal $\mathtt{GFlat}(R)^{\perp}$ coincides with the class of those cotorsion modules which are contained in $\mathtt{PGF}(R)^{\perp}$, i.e. $\mathtt{GFlat}(R)^{\perp} = \mathtt{Cot}(R) \cap \mathtt{PGF}(R)^{\perp}$. In view of Lemma 3.3, we have an equality $\overline{\mathtt{GFlat}}(R) \cap \mathtt{PGF}(R)^{\perp} = \overline{\mathtt{Flat}}(R)$; it follows that $\overline{\mathtt{GFlat}}(R) \cap \mathtt{GFlat}(R)^{\perp} = \mathtt{Cot}(R) \cap \overline{\mathtt{Flat}}(R)$. Therefore, by restricting the hereditary complete cotorsion pair $\big(\mathtt{GFlat}(R), \mathtt{GFlat}(R)^{\perp}\big)$ established in [22, Corollary 4.12] to the exact category $\overline{\mathtt{GFlat}}(R)$, we obtain the hereditary complete cotorsion pair $\big(\mathtt{GFlat}(R), \mathtt{Cot}(R) \cap \overline{\mathtt{Flat}}(R)\big)$ therein. The result presented in Proposition 4.2 below is well-expected, in view of Theorem 2.1, since the latter cotorsion pair is the supremum of the cotorsion pair $\big(\mathtt{PGF}(R), \overline{\mathtt{Flat}}(R)\big)$ of Proposition 3.6 and the restriction of the cotorsion pair $(\mathtt{Flat}(R), \mathtt{Cot}(R))$ to $\overline{\mathtt{GFlat}}(R)$.

We first state an auxiliary result we need.

Lemma 4.1. *Let L be a Gorenstein flat module. Then, there exists a short exact sequence*

$$0 \longrightarrow L \longrightarrow C \longrightarrow L' \longrightarrow 0,$$

where L' is Gorenstein flat and C is flat cotorsion. If L is a cotorsion Gorenstein flat module, then L' is a cotorsion Gorenstein flat module as well.

Proof. We consider a short exact sequence

$$0 \longrightarrow L \longrightarrow F \longrightarrow L' \longrightarrow 0,$$

where L' is Gorenstein flat and F is flat. Since the cotorsion pair $(\mathtt{Flat}(R), \mathtt{Cot}(R))$ is complete, we may also consider a short exact sequence

$$0 \longrightarrow F \longrightarrow C \longrightarrow F' \longrightarrow 0,$$

where F' is flat and C is cotorsion. Of course, C is then flat cotorsion. Pushing out the latter short exact sequence along the epimorphism $F \longrightarrow L'$, we obtain a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & L & = & L & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & F & \longrightarrow & C & \longrightarrow & F' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & L' & \longrightarrow & L'' & \longrightarrow & F' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since L' and F' are Gorenstein flat, the closure of $\mathtt{GFlat}(R)$ under extensions implies that L'' is Gorenstein flat as well. Then, the vertical exact sequence in the middle has the required properties.

The final claim in the statement follows since the class of cotorsion modules is closed under cokernels of monomorphisms. $\square$

Theorem 4.2. *The following conditions are equivalent for a module M and a non-negative integer n:*

(i) $\mathit{Gfd}_R M = n$.

(ii) There exists a short exact sequence

$$0 \longrightarrow M \longrightarrow C \longrightarrow L \longrightarrow 0,$$

where L is a Gorenstein flat module and C is cotorsion with $\mathit{fd}_R C = n$.

(iii) There exists a short exact sequence

$$0 \longrightarrow C \xrightarrow{f} L \longrightarrow M \longrightarrow 0,$$

where L is a Gorenstein flat module and
(a) if $n > 1$, then C is cotorsion with $\mathit{fd}_R C = n - 1$,
(b) if $n = 1$, then C is flat cotorsion and f is not $D^2(\mathtt{Proj}(R))$-injective,
(c) if $n = 0$, then C is flat cotorsion and f is $D^2(\mathtt{Proj}(R))$-injective.

(iv) There exists a flat cotorsion module F, such that the module $M' = M \oplus F$ fits into an exact sequence

$$0 \longrightarrow L \xrightarrow{f} M' \longrightarrow C \longrightarrow 0,$$

where L is Gorenstein flat, C is cotorsion with $\mathit{fd}_R C = n$ and f is $\mathtt{GFlat}(R)^{\perp}$-injective.

(v) There exists a Gorenstein flat module F, such that the module $M' = M \oplus F$ fits into an exact sequence

$$0 \longrightarrow L \xrightarrow{f} M' \longrightarrow C \longrightarrow 0,$$

where L is Gorenstein flat and C is cotorsion with $\mathit{fd}_R C = n$. If $n = 1$, we also require f to be $D^2(\mathtt{Proj}(R))$-injective.

Proof. (i)$\rightarrow$(ii): In view of Theorem 2.1(ii), there exists a short exact sequence

$$0 \longrightarrow M \longrightarrow K \longrightarrow G \longrightarrow 0,$$

where G is a PGF-module and $\mathrm{fd}_R K = n$. The cotorsion pair $(\mathtt{Flat}(R), \mathtt{Cot}(R))$ is complete and hence there exists an exact sequence

$$0 \longrightarrow K \longrightarrow C \longrightarrow F \longrightarrow 0,$$

where C is cotorsion and F is flat. Then, the functors $\mathrm{Tor}_i^R(\_, K)$ and $\mathrm{Tor}_i^R(\_, C)$ are isomorphic if $i > 0$ and hence $\mathrm{fd}_R C = \mathrm{fd}_R K = n$. Taking the pushout of the latter exact sequence along the epimorphism $K \longrightarrow G$, we obtain a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & K & \longrightarrow & G & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & C & \longrightarrow & L & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & F & = & F & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since G is a PGF-module and F is flat, both of these modules are Gorenstein flat. The class $\mathtt{GFlat}(R)$ is closed under extensions and hence the rightmost vertical exact sequence shows that L is Gorenstein flat as well. Then, the horizontal sequence in the middle is of the required type.

(ii)→(iii): We fix a short exact sequence as in (ii) and use again the completeness of the cotorsion pair $(\texttt{Flat}(R), \texttt{Cot}(R))$ in order to find a short exact sequence

$$0 \longrightarrow C' \longrightarrow F \longrightarrow C \longrightarrow 0,$$

where F is flat and C' cotorsion. Of course, we also have $\mathrm{fd}_R C' = n-1$ (if $n=0$, then C' is also flat). The pullback of that short exact sequence along the monomorphism $M \longrightarrow C$ induces a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & C' & = & C' & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & L' & \longrightarrow & F & \longrightarrow & L & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & M & \longrightarrow & C & \longrightarrow & L & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

Since both F and L are Gorenstein flat, the closure of $\texttt{GFlat}(R)$ under kernels of epimorphisms shows that L' is also Gorenstein flat. Since L is Gorenstein flat, we can show that the leftmost vertical exact sequence is of the required type, as in the proof of the corresponding step in Theorem 2.1.

(iii)→(iv): We fix a short exact sequence as in (iii) and apply Lemma 4.1 to the Gorenstein flat module L, in order to find a short exact sequence

$$0 \longrightarrow L \longrightarrow C' \longrightarrow L' \longrightarrow 0,$$

where L' is Gorenstein flat and C' is flat cotorsion. Then, the pushout of the latter short exact sequence along the epimorphism $L \longrightarrow M$ induces a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & C & = & C & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & L & \longrightarrow & C' & \longrightarrow & L' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & M & \longrightarrow & C'' & \longrightarrow & L' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array} \tag{7}$$

Since both C and C' are cotorsion modules, it follows that C'' is cotorsion as well. Since L' is Gorenstein flat, the horizontal exact sequences in the diagram remain exact after applying the functor $\mathrm{Hom}(\_, Q)$ for any module $Q \in \texttt{GFlat}(R)^{\perp}$. We can now show that the induced short exact sequence

$$0 \longrightarrow L \longrightarrow M \oplus C' \longrightarrow C'' \longrightarrow 0$$

is of the required type, as in the proof of the corresponding step in Theorem 2.1.

(iv)→(v): This is immediate, since flat cotorsion modules are Gorenstein flat and the class $D^2(\texttt{Proj}(R))$ is contained in $\texttt{GFlat}(R)^{\perp}$; cf. Corollary 1.5.

(v)→(i): This follows as in the proof of the corresponding step in Theorem 2.1; cf. Remark 2.2(iii). □

Remark 4.3. As with Theorem 2.1, it is necessary to impose some restrictions on the short exact sequences appearing in Theorem 4.2(iii) and (v), in the case where $n = 1$.[3] The same is also true for the short exact sequence appearing in Theorem 4.2(iii), in the case where $n = 0$.

5. The relation to the cotorsion pair $\big(\mathtt{GFlat}(R), \mathtt{GFlat}(R)^{\perp}\big)$

In this section, we use the approximation sequences in Theorem 4.2 to characterize the Gorenstein flat modules and, more generally, the Gorenstein flat dimension of modules within the class $\overline{\mathtt{Gflat}}(R)$, in terms of the vanishing of certain Ext-groups. We also obtain a hereditary Hovey triple in $\overline{\mathtt{Gflat}}(R)$, such that the homotopy category of the associated exact model structure is triangulated equivalent to the stable category of the Frobenius exact category of cotorsion Gorenstein flat modules.

As we have noted earlier, the very definition of Gorenstein flat modules (and modules of finite Gorenstein flat dimension) implies that these modules have trivial Tor-groups with injective right modules (in degrees exceeding the Gorenstein flat dimension). It was shown by Holm in [19] that, for modules in $\overline{\mathtt{GFlat}}(R)$, the triviality of these Tor-groups characterizes Gorenstein flat modules (and, more generally, the value of their Goresntein flat dimension), provided that the ring is right coherent. The latter assumption on the ring was removed in [3, Theorem 2.8], where it was shown that Holm's characterization is actually valid over any ring, pending the proof of the closure of $\mathtt{GFlat}(R)$ under extensions, that was achieved in [22]. For modules within $\overline{\mathtt{GFlat}}(R)$, we may also characterize the Gorenstein flat modules (and, more generally, the value of their Gorenstein flat dimension), in terms the Ext-functors.

It follows from Lemma 3.1 that any module of finite flat dimension is contained in $\mathtt{PGF}(R)^{\perp}$. Therefore, we conclude that $\mathtt{Cot}(R) \cap \overline{\mathtt{Flat}}(R) \subseteq \mathtt{Cot}(R) \cap \mathtt{PGF}(R)^{\perp} = \mathtt{GFlat}(R)^{\perp}$, i.e. the group $\mathrm{Ext}^1_R(M, C)$ is trivial for any Gorenstein flat module M and any cotorsion module C with $\mathrm{fd}_R C < \infty$. A proof of the next result may be also found in [7, Lemma 5.4].

Lemma 5.1. *If $M \in \overline{\mathtt{GFlat}}(R)$, then the following conditions are equivalent:*

(i) $M \in \mathtt{GFlat}(R)$,
(ii) $Ext^i_R(M, C) = 0$ for any $i > 0$ and any flat cotorsion module C,
(iii) $Ext^i_R(M, C) = 0$ for any $i > 0$ and any cotorsion module C of finite flat dimension,
(iv) $Ext^1_R(M, C) = 0$ for any cotorsion module C of finite flat dimension.

Proof. (i)$\rightarrow$(ii): This follows from the discussion above, since the syzygies of a Gorenstein flat module in any projective resolution of it are Gorenstein flat modules as well and hence the abelian groups $\mathrm{Ext}^i_R(M, C) = \mathrm{Ext}^1_R(\Omega_{i-1}M, C)$ are trivial for any $i > 0$ and any flat cotorsion module C.

The implication (ii)$\rightarrow$(iii) follows by induction on the flat dimension of the cotorsion module C, since such a module admits (by taking successive flat covers) a resolution of finite length by flat cotorsion modules.

(iii)$\rightarrow$(iv): This is immediate.

(iv)$\rightarrow$(i): Since M has finite Gorenstein flat dimension, Theorem 4.2(iii) implies that there exists a short exact sequence

$$0 \longrightarrow C \longrightarrow L \longrightarrow M \longrightarrow 0,$$

[3]As in Remarks 2.2(i) and (ii), this necessity is illustrated by considering the direct sum $C \oplus L$ of a flat cotorsion module C with a Gorenstein flat module L and a flat resolution of a cotorsion module of flat dimension 1.

where L is Gorenstein flat and C is cotorsion with $\mathrm{fd}_R C < \infty$. In view of our assumption on M, this sequence splits and hence M is a direct summand of the Gorenstein flat module L. Since the class $\mathtt{GFlat}(R)$ is closed under direct summands, we conclude that M is Gorenstein flat, as needed. □

The next result provides a characterization of the Gorenstein flat dimension for modules in $\overline{\mathtt{GFlat}}(R)$ in terms of the Ext-functors, that reduces to Lemma 5.1 in the case where $n = 0$. An alternative proof follows by combining [7, Theorem 4.5 and Theorem 5.7].

Proposition 5.2. *The following conditions are equivalent for a module $M \in \overline{\mathtt{GFlat}}(R)$ and a non-negative integer n:*
(i) $\mathrm{Gfd}_R M \leq n$,
(ii) $\mathrm{Ext}^i_R(M, C) = 0$ for any $i > n$ and and any flat cotorsion module C,
(iii) $\mathrm{Ext}^i_R(M, C) = 0$ for any $i > n$ and any cotorsion module C of finite flat dimension,
(iv) $\mathrm{Ext}^{n+1}_R(M, C) = 0$ for any cotorsion module C of finite flat dimension.

Proof. (i)→(ii): We consider a Gorenstein flat resolution of length n

$$0 \longrightarrow L_n \longrightarrow L_{n-1} \longrightarrow \cdots \longrightarrow L_0 \longrightarrow M \longrightarrow 0$$

and fix a flat cotorsion module C. Since the functors $\mathrm{Ext}^j_R(\_, C)$ vanish on Gorenstein flat modules for all $j > 0$ (cf. Lemma 5.1), the desired vanishing follows by dimension shifting.

The implication (ii)→(iii) follows by induction on the flat dimension of the cotorsion module C, since such a module admits (by taking successive flat covers) a resolution of finite length by flat cotorsion modules.

(iii)→(iv): This is immediate.

(iv)→(i): Let

$$0 \longrightarrow K \longrightarrow L_{n-1} \longrightarrow \cdots \longrightarrow L_0 \longrightarrow M \longrightarrow 0$$

be an exact sequence, where $L_0, \ldots, L_{n-1}$ are Gorenstein flat. The modules $M, L_0, \ldots, L_{n-1}$ have finite Gorenstein flat dimension and hence an iterated application of Proposition 1.3(ii) shows that K has finite Gorenstein flat dimension as well. Since the functors $\mathrm{Ext}^j_R(L_i, \_)$ vanish on cotorsion modules of finite flat dimension for all $j > 0$ and all $i = 0, \ldots, n-1$ (cf. Lemma 5.1), a dimension shifting argument shows that $\mathrm{Ext}^1_R(K, C) = \mathrm{Ext}^{n+1}_R(M, C) = 0$ for any cotorsion module C of finite flat dimension. Invoking Lemma 5.1 again, we conclude that K is Gorenstein flat and hence $\mathrm{Gfd}_R M \leq n$. □

As we noted in the beginning of Section 4, the pair $\big(\mathtt{GFlat}(R), \mathtt{Cot}(R) \cap \overline{\mathtt{Flat}}(R)\big)$ is a hereditary complete cotorsion pair in the exact category $\overline{\mathtt{GFlat}}(R)$. In particular, the triviality of the group $\mathrm{Ext}^1_R(M, C)$, whenever M is a Gorenstein flat module and C is cotorsion of finite flat dimension, may also provide a characterization of cotorsion modules of finite flat dimension, if we restrict to modules of finite Gorenstein flat dimension.[4]

Lemma 5.3. *If $M \in \overline{\mathtt{GFlat}}(R)$, then $M \in \mathtt{Cot}(R) \cap \overline{\mathtt{Flat}}(R)$ if and only if $\mathrm{Ext}^1_R(L, M) = 0$ for any Gorenstein flat module L.* □

For rings of finite Gorenstein weak global dimension, we obtain a description of the class of Gorenstein flat modules and its right orthogonal in classical terms.

[4]As with Lemma 3.3, this result may be also proved using the exact sequences in Theorem 4.2(ii).

Corollary 5.4. *Let R be a ring and assume that all injective modules (both left and right) have finite flat dimension. Then,* $\mathtt{GFlat}(R) = {}^{\perp}\big(\mathtt{Cot}(R) \cap \overline{\mathtt{Flat}}(R)\big)$ *and* $\mathtt{GFlat}(R)^{\perp} = \mathtt{Cot}(R) \cap \overline{\mathtt{Flat}}(R)$. □

Remarks 5.5. (i) Since the hypothesis of Corollary 5.4 is left-right symmetric, we have (under the same assumptions) the analogous conclusions for the ring R^{op}, i.e. for the corresponding classes of right modules.

(ii) Since injective modules are obviously contained in the right orthogonal $\mathtt{GFlat}(R)^{\perp}$ and the same is true for right modules, the hypothesis in Corollary 5.4 is also necessary for the equalities $\mathtt{GFlat}(R)^{\perp} = \mathtt{Cot}(R) \cap \overline{\mathtt{Flat}}(R)$ and $\mathtt{GFlat}(R^{op})^{\perp} = \mathtt{Cot}(R^{op}) \cap \overline{\mathtt{Flat}}(R^{op})$ to hold.

We note that the category $\mathtt{Cot}(R) \cap \mathtt{GFlat}(R)$ of cotorsion Gorenstein flat modules is a Frobenius exact category with projective-injective objects the flat cotorsion modules. Indeed, as shown in [22, Corollary 4.12], we have $\mathtt{GFlat}(R) \cap \mathtt{GFlat}(R)^{\perp} = \mathtt{Cot}(R) \cap \mathtt{Flat}(R)$ and hence all flat cotorsion modules are injective objects in $\mathtt{Cot}(R) \cap \mathtt{GFlat}(R)$. Lemma 4.1 implies that the exact category $\mathtt{Cot}(R) \cap \mathtt{GFlat}(R)$ has enough injective objects and all of these objects are necessarily flat cotorsion modules. On the other hand, the flat cotorsion modules are obviously projective objects in $\mathtt{Cot}(R) \cap \mathtt{GFlat}(R)$. Considering flat covers of cotorsion Gorenstein flat modules, it follows that $\mathtt{Cot}(R) \cap \mathtt{GFlat}(R)$ has enough projective objects and all of these objects are necessarily flat cotorsion modules.[5] The following result shows that we may realize the stable category of $\mathtt{Cot}(R) \cap \mathtt{GFlat}(R)$ as the homotopy category of the exact model structure induced by a hereditary Hovey triple in $\overline{\mathtt{GFlat}}(R)$.

Theorem 5.6. *The triple* $\big(\mathtt{GFlat}(R), \overline{\mathtt{Flat}}(R), \mathtt{Cot}(R) \cap \overline{\mathtt{GFlat}}(R)\big)$ *is a hereditary Hovey triple in the idempotent complete exact category* $\overline{\mathtt{GFlat}}(R)$. *The homotopy category of the associated exact model structure is equivalent, as a triangulated category, to the stable category of the Frobenius exact category* $\mathtt{Cot}(R) \cap \mathtt{GFlat}(R)$ *modulo its projective-injective objects, i.e. modulo* $\mathtt{Cot}(R) \cap \mathtt{Flat}(R)$.

Proof. We need to prove that the pairs

$$\big(\mathtt{GFlat}(R), \overline{\mathtt{Flat}}(R) \cap \mathtt{Cot}(R) \cap \overline{\mathtt{GFlat}}(R)\big) \text{ and } \big(\mathtt{GFlat}(R) \cap \overline{\mathtt{Flat}}(R), \mathtt{Cot}(R) \cap \overline{\mathtt{GFlat}}(R)\big)$$

are complete and hereditary cotorsion pairs in the exact category $\overline{\mathtt{GFlat}}(R)$. Since $\overline{\mathtt{Flat}}(R) \subseteq \overline{\mathtt{GFlat}}(R)$, the first one of these pairs is the pair $\big(\mathtt{GFlat}(R), \mathtt{Cot}(R) \cap \overline{\mathtt{Flat}}(R)\big)$ we considered above, i.e. the restriction of the complete hereditary cotorsion pair $\big(\mathtt{GFlat}(R), \mathtt{GFlat}(R)^{\perp}\big)$ to the exact category $\overline{\mathtt{GFlat}}(R)$. On the other hand, any Gorenstein flat module of finite flat dimension is necessarily flat and hence we have an equality $\mathtt{GFlat}(R) \cap \overline{\mathtt{Flat}}(R) = \mathtt{Flat}(R)$. It follows that the second pair is the restriction $\big(\mathtt{Flat}(R), \mathtt{Cot}(R) \cap \overline{\mathtt{GFlat}}(R)\big)$ of the complete hereditary cotorsion pair $(\mathtt{Flat}(R), \mathtt{Cot}(R))$ to the exact category $\overline{\mathtt{GFlat}}(R)$.

The final assertion in the statement of the Theorem follows from [16, Proposition 4.4 and Corollary 4.8]. □

[5] Since $\mathtt{Cot}(R) \cap \mathtt{GFlat}(R)$ has enough projective and injective objects, which coincide with the flat cotorsion modules, it follows that any cotorsion Gorenstein flat module can be realized as a syzygy of an acyclic complex of flat cotorsion modules, that remains acyclic after applying the functor $I \otimes \_$ for any injective right module I. Conversely, the syzygy modules of any acyclic complex of flat cotorsion modules that remains acyclic after applying the functor $I \otimes \_$ for any injective right module I are cotorsion, in view of [2, Theorem 4.1(2)], and Gorenstein flat.

We fix a module M and note that the complete cotorsion pair $\big(\texttt{GFlat}(R), \texttt{GFlat}(R)^{\perp}\big)$ provides two short exact sequences

$$0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow N' \longrightarrow L' \longrightarrow M \longrightarrow 0, \tag{8}$$

where $L, L' \in \texttt{GFlat}(R)$ and $N, N' \in \texttt{GFlat}(R)^{\perp}$. We may also consider a short exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow N \longrightarrow 0,$$

where P is projective and consider its pullback along the monomorphism $M \longrightarrow N$, in order to obtain a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & K & = & K & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & L'' & \longrightarrow & P & \longrightarrow & L & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & L & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

Since $\texttt{GFlat}(R)$ is closed under kernels of epimorphisms, the horizontal short exact sequence in the middle shows that L'' is Gorenstein flat. Letting $N'' = N$, we thereby obtain a third short exact sequence

$$0 \longrightarrow L'' \longrightarrow M \oplus P \longrightarrow N'' \longrightarrow 0, \tag{9}$$

where $L'' \in \texttt{GFlat}(R)$, $N'' \in \texttt{GFlat}(R)^{\perp}$ and P is projective.

Using the thickness of the category $\overline{\texttt{GFlat}}(R)$ and the equality $\overline{\texttt{GFlat}}(R) \cap \texttt{GFlat}(R)^{\perp} = \texttt{Cot}(R) \cap \overline{\texttt{Flat}}(R)$, we may prove as in Proposition 3.9 the following result.

Proposition 5.7. *Let M be a module and consider the three short exact sequences in (8) and (9), where $L, L', L'' \in \texttt{GFlat}(R)$, $N, N', N'' \in \texttt{GFlat}(R)^{\perp}$ and P is projective.*

(i) If $M \in \overline{\texttt{GFlat}}(R)$, then $N, N', N'' \in \texttt{Cot}(R) \cap \overline{\texttt{Flat}}(R)$.

(ii) If one of the modules N, N', N'' is contained in $\overline{\texttt{Flat}}(R)$, then $M \in \overline{\texttt{GFlat}}(R)$. □

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