ON THE GORENSTEIN COHOMOLOGICAL DIMENSION OF GROUP EXTENSIONS

IOANNIS EMMANOUIL AND OLYMPIA TALELLI

Abstract. The Gorenstein cohomological dimension is defined for any group G and coincides with the virtual cohomological dimension $\mathrm{vcd}\, G$, whenever the latter is defined and finite. Unlike the virtual cohomological dimension, the Gorenstein cohomological dimension behaves well with respect to extensions, finite graphs of groups and ascending unions. In this paper, we study the Gorenstein cohomological dimension $\mathrm{Gcd}_k G$ of groups G, which are of type FP_∞ over a commutative ring k. We show that if $1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$ is an extension of groups with N of type FP_∞ over a field F and $\mathrm{Gcd}_F Q < \infty$, then $\mathrm{Gcd}_F G = \mathrm{Gcd}_F N + \mathrm{Gcd}_F Q$. We also show that for any group G of type FP_∞ over $\mathbb{Z}$ with $\mathrm{Gcd}_{\mathbb{Z}} G < \infty$, there exists a field F such that $\mathrm{Gcd}_F G = \mathrm{Gcd}_{\mathbb{Z}} G$. This implies, in particular, that if G is a group of type FP_∞ over $\mathbb{Z}$ and G is an extension of N by itself, then $\mathrm{Gcd}_{\mathbb{Z}} G = 2\mathrm{Gcd}_{\mathbb{Z}} N$.

Contents

0. Introduction

If k is a commutative ring and G is a group, then the Gorenstein cohomological dimension $\mathrm{Gcd}_k G$ of G over k is the Gorenstein projective dimension of the trivial kG-module k, i.e. the minimal length of a resolution of k by Gorenstein projective kG-modules. This invariant generalizes the cohomological dimension $\mathrm{cd}_k G$ of G over k, as well as the virtual cohomological dimension $\mathrm{vcd}_k G$ of G over k, whenever the latter is defined; it is known that $\mathrm{Gcd}_k G$ coincides with these, when they are finite. The Gorenstein cohomological dimension of groups over $\mathbb{Z}$ is proposed in [2] to serve as an algebraic invariant, whose finiteness characterizes the groups that admit a finite dimensional model for the classifying space for proper actions. It is related to several numerical invariants that are studied in cohomological group theory: As shown in [12], [17] and [18], for any group G we have a chain of inequalities

$$\mathrm{Gcd}_{\mathbb{Z}} G \leq \mathfrak{F}\mathrm{cd}_{\mathbb{Z}} G \leq \mathrm{cd}_{\mathfrak{F}} G \leq \mathrm{gd}_{\mathfrak{F}} G \leq \max\{3, \mathrm{cd}_{\mathfrak{F}} G\}.$$

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Here, $\mathfrak{F}\mathrm{cd}_{\mathbb{Z}}G$ is the $\mathfrak{F}$-cohomological dimension of G over $\mathbb{Z}$, which is defined in terms of the class $\mathfrak{F}$ of finite subgroups of G. The Bredon cohomological dimension $\mathrm{cd}_{\mathfrak{F}}G$ is defined in terms of the category of contravariant functors from the orbit category $O_{\mathfrak{F}}G$ to the category of abelian groups and $\mathrm{gd}_{\mathfrak{F}}G$ denotes the geometric Bredon dimension, i.e. the minimal dimension of a model for the classifying space for proper actions of G. There are examples of groups for which the inequality $\mathfrak{F}\mathrm{cd}_{\mathbb{Z}}G \leq \mathrm{cd}_{\mathfrak{F}}G$ is strict (cf. [16]), as well as examples of groups with $\mathfrak{F}\mathrm{cd}_{\mathbb{Z}}G = \mathrm{cd}_{\mathfrak{F}}G$ (cf. [1], [7]). As shown in [19], the finiteness of the $\mathfrak{F}$-cohomological dimension of G over $\mathbb{Z}$ implies that $\mathrm{Gcd}_{\mathbb{Z}}G = \mathfrak{F}\mathrm{cd}_{\mathbb{Z}}G$. It follows that the Gorenstein cohomological dimension over $\mathbb{Z}$ coincides for certain classes of groups with the Bredon cohomological dimension.

The invariant Gcd_kG has many properties that are standard for ordinary group cohomology. It is proved in [10] that:
(i) If k has finite global dimension, then the finiteness of Gcd_kG implies that all kG-modules have finite Gorenstein projective dimension.
(ii) If k has finite weak global dimension and $H \subseteq G$ is a subgroup, then $\mathrm{Gcd}_kH \leq \mathrm{Gcd}_kG$.
(iii) If k has finite weak global dimension and $N \trianglelefteq G$ is a normal subgroup, then $\mathrm{Gcd}_kG \leq \mathrm{Gcd}_kN + \mathrm{Gcd}_k(G/N)$.
(iv) If k is any commutative ring, then $\mathrm{Gcd}_kG \leq \mathrm{Gcd}_{\mathbb{Z}}G$.
(v) If k is any commutative ring and G is expressed as the union of a continuous ascending chain of subgroups $(G_\lambda)_\lambda$, then $\mathrm{Gcd}_kG \leq 1 + \sup_\lambda \mathrm{Gcd}_kG_\lambda$.
The class of groups with finite Gorenstein cohomological dimension over $\mathbb{Z}$ is strictly bigger than the class of groups with finite virtual cohomological dimension over $\mathbb{Z}$ and is closed under extensions, free products with amalgamation and HNN extensions; these properties are proved in [2]. All groups of type FP_∞ which are contained in the class $\mathbf{LH}\mathfrak{F}$ defined by Kropholler in [14] have finite Gorenstein cohomological dimension over $\mathbb{Z}$; this is proved in Appendix A.

In this paper, we study more closely property (iii) above. We consider a normal subgroup $N \trianglelefteq G$ and examine conditions, under which the inequality $\mathrm{Gcd}_kG \leq \mathrm{Gcd}_kN + \mathrm{Gcd}_k(G/N)$ is actually an equality. First of all, we note that the latter inequality may be strict, even if N and G/N are groups of type FP_∞ with finite virtual cohomological dimension: There are examples of groups G_1, G_2 of type FP_∞ with finite virtual cohomological dimension, such that $\mathrm{vcd}_{\mathbb{Z}}(G_1 \times G_2) < \mathrm{vcd}_{\mathbb{Z}}G_1 + \mathrm{vcd}_{\mathbb{Z}}G_2$; cf. [8]. Returning to the general case of a normal subgroup $N \trianglelefteq G$ of type FP_∞, we can prove the equality $\mathrm{Gcd}_kG = \mathrm{Gcd}_kN + \mathrm{Gcd}_k(G/N)$, by imposing additional conditions in two (transverse) directions.

In one direction, we may assume that the ring of coefficients is a field and obtain the following result:

Theorem A. *If F is a field and $1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$ is an extension of groups with N of type FP_∞ over F and $Gcd_FQ < \infty$, then $Gcd_FG = Gcd_FN + Gcd_FQ$.*

This result is the analogue in the realm of Gorenstein homological algebra of a classical result by Fel'dman [11] for ordinary group cohomology. The proof of Theorem A is very similar to the proof of Fel'dman's result.

On the other hand, we may assume that the coefficient ring is $\mathbb{Z}$ and consider extensions of a group N of type FP_∞ by itself. We then obtain the following result:

Theorem B. *If $1 \longrightarrow N \longrightarrow G \longrightarrow N \longrightarrow 1$ is an extension of a group N of type FP_∞ by itself, then $Gcd_{\mathbb{Z}}G = 2Gcd_{\mathbb{Z}}N$.*

We note that Theorem B generalizes the main result of [9], which deals with the ordinary cohomological dimension of the direct product $N \times N$, where N is a geometrically finite group. The proof of Theorem B can be reduced to Theorem A, by means of the following result, which is perhaps of independent interest:

Theorem C. *If G is a group of type FP_∞ over the ring $\mathbb{Z}$ of integers with $Gcd_{\mathbb{Z}}G < \infty$, then there exists a field F, such that $Gcd_F G = Gcd_{\mathbb{Z}}G$.*

In fact, we prove a slightly more general version of Theorem C, where the ring $\mathbb{Z}$ of integers is replaced by any principal ideal domain.

The contents of the paper are as follows: In Section 1, we give basic definitions and record some preliminary results that are used throughout the paper. In the following Section, we examine groups of type FP_∞ with finite Gorenstein cohomological dimension and show that these groups admit projective resolutions consisting of finitely generated free modules in each degree. It follows that these groups have characteristic modules of type FP and hence their Gorenstein cohomological dimension is detected by a suitable choice of a field of coefficients; these results are proved in Section 3. In Section 4, we prove the analogue of Fel'dman's result in Gorenstein homological algebra (Theorem B). In the final Section, we apply the previous results and study the Gorenstein cohomological dimension of iterated extensions of a group N of type FP_∞ by itself. In the Appendix, we characterize the finiteness of the Gorenstein cohomological dimension of certain groups, in terms of the associated module of bounded functions.

Terminology. All rings are assumed to be associative and unital and all ring homomorphisms will be unit preserving. Unless otherwise specified, all modules will be left modules.

1. Preliminaries

In this section, we collect certain basic notions and preliminary results that will be used in the sequel. These notions include complete resolutions and the Gorenstein projective dimension of modules over any ring. We also record some related facts concerning the special case of modules over group rings.

I. Gorenstein projective dimension. Let R be a ring. An acyclic complex of projective R-modules

$$\ldots \longrightarrow P_{n+1} \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \ldots,$$

is said to be a complete projective resolution (in the strong sense) if the complex of abelian groups

$$\ldots \longleftarrow \text{Hom}_R(P_{n+1}, P) \longleftarrow \text{Hom}_R(P_n, P) \longleftarrow \text{Hom}_R(P_{n-1}, P) \longleftarrow \ldots$$

is acyclic for any projective R-module P. An R-module M is called Gorenstein projective if it is a syzygy of a complete projective resolution, i.e. if there exists a complete projective resolution as above, such that $M = \text{im}\,(P_n \longrightarrow P_{n-1})$ for some n. Holm's paper [13] is the standard reference in Gorenstein homological algebra. If M is Gorenstein projective, then the groups $\text{Ext}^i_R(M, P)$ vanish when $i \geq 1$ for all projective R-modules P.

The Gorenstein projective dimension $\text{Gpd}_R M$ of an R-module M is the length of a shortest resolution of M by Gorenstein projective modules. (If there is no such resolution of finite length, then we write $\text{Gpd}_R M = \infty$.) If M is an R-module of finite projective dimension, then M has finite Gorenstein projective dimension and $\text{Gpd}_R M = \text{pd}_R M$. In the finite case, the Gorenstein projective dimension admits an alternative, but equivalent, description in terms of complete resolutions. We say that an R-module M admits a complete projective resolution of coincidence index n if there exists a complete projective resolution, which coincides with an ordinary projective resolution of M in degrees $\geq n$. If the Gorenstein projective dimension of M is finite, then $\text{Gpd}_R M$ is actually equal to the minimal n, for which M admits a complete projective resolution of coincidence index n.

II. Group rings. Let k be a commutative ring, G a group and consider the associated group ring $R = kG$. The standard reference for group cohomology is Brown's book [4]. The Hopf algebra structure of kG enriches the theory of kG-modules: Using the diagonal action of the group G, the tensor product $M \otimes_k N$ of two kG-modules is also a kG-module. Moreover, if M is a k-projective kG-module and N is a projective kG-module, then the diagonal kG-module $M \otimes_k N$ is projective as well. We may tensor a projective resolution $\mathbf{P} \longrightarrow N \longrightarrow 0$ of any kG-module N with a k-projective kG-module M and obtain therefore a projective resolution $M \otimes_k \mathbf{P} \longrightarrow M \otimes_k N \longrightarrow 0$ of the diagonal kG-module $M \otimes_k N$. It follows that we then have an inequality $\text{pd}_{kG}(M \otimes_k N) \leq \text{pd}_{kG} N$.

A kG-module M is said to be of type FP_∞ (resp. of type FP) if M admits a projective resolution

$$\ldots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \ldots \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

where the P_n's are finitely generated projective modules (resp. the P_n's are finitely generated projective and vanish for $n \gg 0$). The group G is said to be of type FP_∞ (resp. of type FP) over k if the trivial kG-module k is of type FP_∞ (resp. of type FP). We note that any projective resolution of the trivial kG-module k is k-split. Therefore, if $k \longrightarrow k'$ is a commutative ring homomorphism, then any kG-projective resolution $\mathbf{P} \longrightarrow k \longrightarrow 0$ induces a $k'G$-projective resolution $\mathbf{P} \otimes_k k' \longrightarrow k' \longrightarrow 0$. It follows readily that if G is a group of type FP_∞ (resp. of type FP) over k, then G is also of type FP_∞ (resp. of type FP) over k'. In particular, if G is of type FP_∞ (resp. of type FP) over the ring $\mathbb{Z}$ of integers, then G is of type FP_∞ (resp. of type FP) over any commutative ring k. If G is a geometrically finite group, i.e. if G admits a finite Eilenberg-MacLane space $K(G, 1)$, then the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ admits a resolution of finite length, which consists of finitely generated free $\mathbb{Z}G$-modules; in particular, G is of type FP over $\mathbb{Z}$.

III. Characteristic modules. Let k be a commutative ring and G a group. Then, the Gorenstein cohomological dimension $\text{Gcd}_k G$ of G over k is the Gorenstein projective dimension of the trivial kG-module k. The following notion provides us with a useful characterization of the finiteness of $\text{Gcd}_k G$; cf. [21].

Definition 1.1. *A characteristic module for G over k is a k-projective kG-module A with $pd_{kG} A < \infty$, which admits a k-split kG-linear monomorphism $\iota : k \longrightarrow A$ (where k is regarded as a trivial kG-module).*

A characteristic module for G over k may not always exist and, if it exists, it is certainly not unique. The projective dimension though of any characteristic module for G over k is uniquely determined by the pair (k, G).

Proposition 1.2. *Let k be a commutative ring and G a group. Then:*
(i) If A, B are two characteristic modules for G over k, then $pd_{kG} A = pd_{kG} B$.
(ii) If there exists a characteristic module A for G over k, then G has finite Gorenstein cohomological dimension over k and $Gcd_k G \leq pd_{kG} A$.
(iii) If k has finite weak global dimension and G has finite Gorenstein cohomological dimension over k, then there exists a characteristic module for G over k with $pd_{kG} A = Gcd_k G$.

Proof. Assume that there exists a characteristic module A for G over k. Then, it follows from [10, Proposition 1.4] that $\text{Gpd}_{kG}M \leq \text{pd}_{kG}A$ for any k-projective kG-module M. Since k is a k-projective kG-module, we have $\text{Gcd}_k G = \text{pd}_{kG}k \leq \text{pd}_{kG}A$ and this proves (ii). If B is another characteristic module for G over k, then $\text{pd}_{kG}B < \infty$ and hence $\text{Gpd}_{kG}B = \text{pd}_{kG}B$. Since B is a k-projective kG-module, it follows that $\text{pd}_{kG}B = \text{Gpd}_{kG}B \leq \text{pd}_{kG}A$. Reversing the roles of A and B, we may obtain the opposite inequality and finally conclude that $\text{pd}_{kG}A = \text{pd}_{kG}B$; this proves (i). Finally, assertion (iii) is precisely [10, Corollary 1.3]. □

Corollary 1.3. *Let k be a commutative ring of finite weak global dimension and G a group. Then, G has finite Gorenstein cohomological dimension over k if and only if there exists a characteristic module A for G over k. In that case, we have an equality $\text{pd}_{kG}A = \text{Gcd}_k G$.* □

Remarks 1.4. (i) The assumption in Proposition 1.2(iii) and Corollary 1.3 that the commutative ring k has finite weak global dimension is perhaps unnatural. Tracing back through the arguments in the proof of [10, Corollary 1.3], one needs the following property of the pair (k, G): *Any Gorenstein projective kG-module is k-projective.* The latter property holds for any group G if the ring k has finite weak global dimension.

(ii) If k has finite global dimension and G is a group with finite Gorenstein cohomological dimension over k, then all kG-modules M have finite Gorenstein projective dimension and $\text{Gpd}_{kG}M \leq \text{Gcd}_k G + \text{gl.dim}\, k$ (cf. [10, Corollary 1.6]).

(iii) Let A be a characteristic module for G over k and consider a homomorphism of commutative rings $k \longrightarrow k'$. Then, the $k'G$-module $A \otimes_k k'$ is a characteristic module for G over k'. Indeed, since the kG-module A is k-projective, the $k'G$-module $A \otimes_k k'$ is clearly k'-projective. Moreover, any kG-projective resolution of finite length $\mathbf{P} \longrightarrow A \longrightarrow 0$ is k-split and hence induces a $k'G$-projective resolution of finite length $\mathbf{P} \otimes_k k' \longrightarrow A \otimes_k k' \longrightarrow 0$; it follows that $\text{pd}_{k'G}(A \otimes_k k') < \infty$. Finally, any k-split kG-linear monomorphism $\iota : k \longrightarrow A$ induces a k'-split $k'G$-linear monomorphism $\iota \otimes 1 : k' \longrightarrow A \otimes_k k'$.

In order to find explicit examples of characteristic modules for a group G over a commutative ring k, one may start with the coinduced module $\text{Coind}_1^G k = \text{Hom}_k(kG, k)$, which is naturally identified with the set of all functions from G to k, and then consider the kG-submodule $B(G, k)$ consisting of all functions from G to k whose image is a finite subset of k. This example is analysed in Appendix A, where we examine whether $B(G, k)$ is indeed a characteristic module for G over k for certain groups G.

2. Groups of type FP_∞ with finite Gcd

In this section we show that any group G of type FP_∞ with finite Gorenstein cohomological dimension admits a complete projective resolution by finitely generated free modules. We begin with a couple of preliminary results, whose proof follows from the elegant arguments by Cornick and Kropholler in [5].

Lemma 2.1. *Let k be a commutative ring, G a group and assume that there exists a kG-module A, which admits a k-split kG-linear monomorphism $\iota : k \longrightarrow A$. We also consider a complex of kG-modules*

$$\mathbf{M} : \ \ldots \longrightarrow M_n \longrightarrow M_{n-1} \longrightarrow \ldots$$

and assume that the induced complex of diagonal kG-modules

$$\mathbf{M} \otimes_k A : \ \ldots \longrightarrow M_n \otimes_k A \longrightarrow M_{n-1} \otimes_k A \longrightarrow \ldots$$

is contractible. Then, for any projective kG-module Q the induced complex of abelian groups

$$Hom_{kG}(\mathbf{M}, Q) : \ldots \longleftarrow Hom_{kG}(M_n, Q) \longleftarrow Hom_{kG}(M_{n-1}, Q) \longleftarrow \ldots$$

is acyclic.

Proof. (cf. [5, Lemma 3.5]) Let $\overline{A} = \operatorname{coker} \iota$ and consider the k-split short exact sequence of kG-modules

$$0 \longrightarrow k \stackrel{\iota}{\longrightarrow} A \longrightarrow \overline{A} \longrightarrow 0.$$

We also consider a projective kG-module Q and the induced short exact sequence of kG-modules (with diagonal action)

$$0 \longrightarrow \operatorname{Hom}_k\left(\overline{A}, Q\right) \longrightarrow \operatorname{Hom}_k(A, Q) \stackrel{\iota^*}{\longrightarrow} Q \longrightarrow 0.$$

Since the kG-module Q is projective, the exact sequence above splits and hence Q is a direct summand of $\operatorname{Hom}_k(A, Q)$. The acyclicity of the complex $\operatorname{Hom}_{kG}(\mathbf{M}, Q)$ will therefore follow if we show the acyclicity of the complex

$$\operatorname{Hom}_{kG}(\mathbf{M}, \operatorname{Hom}_k(A, Q)) \simeq \operatorname{Hom}_{kG}(\mathbf{M} \otimes_k A, Q).$$

But the complex $\mathbf{M} \otimes_k A$ is contractible by hypothesis and hence $\operatorname{Hom}_{kG}(\mathbf{M} \otimes_k A, Q)$ is also contractible; in particular, the latter complex is acyclic, as needed. □

Proposition 2.2. *Let k be a commutative ring, G a group and K a finitely generated kG-module. We assume that there exists a k-projective kG-module A, which admits a k-split kG-linear monomorphism $\iota : k \longrightarrow A$ and is such that the diagonal kG-module $K \otimes_k A$ is projective. Then:*
(i) There exists a short exact sequence of kG-modules $0 \longrightarrow K \longrightarrow T \longrightarrow N \longrightarrow 0$, such that T is finitely generated free and the diagonal kG-module $N \otimes_k A$ is projective.
(ii) There exists an exact sequence of kG-modules

$$0 \longrightarrow K \longrightarrow T_{-1} \longrightarrow T_{-2} \longrightarrow \ldots,$$

where T_i is finitely generated free and the image $K_i = im\,(T_i \longrightarrow T_{i-1})$ is such that the diagonal kG-module $K_i \otimes_k A$ is projective for all $i \leq -1$.
(iii) Any projective resolution $\mathbf{P}$ of the kG-module K

$$\ldots \longrightarrow P_m \longrightarrow P_{m-1} \longrightarrow \ldots \longrightarrow P_0 \longrightarrow K \longrightarrow 0$$

may be completed to a complete projective resolution

$$\mathbf{T} : \ldots \longrightarrow P_m \longrightarrow P_{m-1} \longrightarrow \ldots \longrightarrow P_0 \longrightarrow T_{-1} \longrightarrow T_{-2} \longrightarrow \ldots$$

with $K = im\,(P_0 \longrightarrow T_{-1})$, such that T_i is a finitely generated free kG-module for all $i \leq -1$.

Proof. (i) (cf. [5, Lemma 4.1]) Let $\overline{A} = \operatorname{coker} \iota$ and consider the k-split short exact sequence of kG-modules

$$0 \longrightarrow k \stackrel{\iota}{\longrightarrow} A \longrightarrow \overline{A} \longrightarrow 0.$$

Since A is k-projective, it follows that $\overline{A}$ is k-projective as well. We also consider the induced k-split short exact sequence of diagonal kG-modules

$$0 \longrightarrow K \stackrel{1_K \otimes \iota}{\longrightarrow} K \otimes_k A \longrightarrow K \otimes_k \overline{A} \longrightarrow 0. \tag{1}$$

Since the kG-module $\overline{A}$ is k-projective and $K \otimes_k A$ is kG-projective (by assumption), the kG-module $(K \otimes_k \overline{A}) \otimes_k A = (K \otimes_k A) \otimes_k \overline{A}$ is kG-projective as well. It follows that the short exact sequence

$$0 \longrightarrow K \otimes_k A \longrightarrow (K \otimes_k A) \otimes_k A \longrightarrow (K \otimes_k \overline{A}) \otimes_k A \longrightarrow 0, \tag{2}$$

which is obtained from (1) by tensoring with A over k, is split over kG. Since the kG-module $K \otimes_k A$ is projective, there exists a projective kG-module L such that $L \oplus (K \otimes_k A)$ is free. We now consider the k-split short exact sequence

$$0 \longrightarrow K \xrightarrow{\jmath} L \oplus (K \otimes_k A) \longrightarrow L \oplus (K \otimes_k \overline{A}) \longrightarrow 0, \tag{3}$$

which is obtained as the direct sum of (1) and the short exact sequence

$$0 \longrightarrow 0 \longrightarrow L \xrightarrow{1_L} L \longrightarrow 0.$$

The short exact sequence (3) induces upon tensoring with A over k a short exact sequence, which is also split over kG; in fact, the tensored exact sequence is the direct sum of (2) and the short exact sequence

$$0 \longrightarrow 0 \longrightarrow L \otimes_k A \xrightarrow{1_{L\otimes_k A}} L \otimes_k A \longrightarrow 0.$$

Since the kG-module K is finitely generated and $L \oplus (K \otimes_k A)$ is free, there exists a finitely generated free direct summand $T \subseteq L \oplus (K \otimes_k A)$, such that $\operatorname{im} \jmath \subseteq T$. Therefore, letting $N = T/\operatorname{im} \jmath$, we conclude that the short exact sequence of kG-modules

$$0 \longrightarrow K \longrightarrow T \longrightarrow N \longrightarrow 0 \tag{4}$$

is a direct summand of the short exact sequence (3). In particular, the short exact sequence (4) induces upon tensoring with A over k a split short exact sequence of kG-modules

$$0 \longrightarrow K \otimes_k A \longrightarrow T \otimes_k A \longrightarrow N \otimes_k A \longrightarrow 0.$$

Since T is a free kG-module and A is k-projective, the diagonal kG-module $T \otimes_k A$ is projective. Being isomorphic with a direct summand of $T \otimes_k A$, the kG-module $N \otimes_k A$ is then projective as well.

(ii) This follows from (i) using induction. We may construct an exact sequence as in the statement, by splicing together the short exact sequences obtained by a repeated application of (i).

(iii) We may splice any projective resolution $\mathbf{P}$ of K as in the statement with the exact sequence provided in (ii) and obtain an acyclic complex of projective kG-modules

$$\mathbf{T} : \ldots \longrightarrow P_m \longrightarrow P_{m-1} \longrightarrow \ldots \longrightarrow P_0 \longrightarrow T_{-1} \longrightarrow T_{-2} \longrightarrow \ldots$$

with $K = \operatorname{im}(P_0 \longrightarrow T_{-1})$, such that T_i is a finitely generated free kG-module for all $i \leq -1$. Since A is a k-projective kG-module, the induced exact sequence of diagonal kG-modules $\mathbf{T} \otimes_k A$ is acyclic as well. In view of (ii), the images $K_i = \operatorname{im}(T_i \longrightarrow T_{i-1})$ are such that the diagonal kG-modules $K_i \otimes_k A$ are projective for all $i \leq -1$. Since the modules $K_i \otimes_k A$ are precisely the syzygies of $\mathbf{T} \otimes_k A$, it follows that the latter complex is contractible. Invoking Lemma 2.1, we conclude that the complex of abelian groups $\operatorname{Hom}_{kG}(\mathbf{T}, Q)$ is acyclic for any projective kG-module Q. Therefore, $\mathbf{T}$ is a complete projective resolution, as needed. □

Theorem 2.3. *Let k be a commutative ring of finite weak global dimension. We also consider a group G with $Gcd_k G < \infty$ and a k-projective kG-module M of type FP_n, where $n \geq Gcd_k G$. Then, M admits a complete projective resolution $\mathbf{T} = (T_i)_i$ of coincidence index n, such that T_i is a finitely generated free kG-module for all $i \leq n-1$.*

Proof. Since M is of type FP_n, there exists a projective resolution $\mathbf{P}$ of M

$$\ldots \longrightarrow P_m \longrightarrow P_{m-1} \longrightarrow \ldots \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

such that the n-th syzygy module $K_n = \operatorname{im}(P_n \longrightarrow P_{n-1})$ is finitely generated (as a kG-module). Since the commutative ring k has finite weak global dimension and $\mathrm{Gcd}_k G \leq n < \infty$, we may choose a characteristic module for G over k; there exists a k-projective kG-module A with $\mathrm{pd}_{kG}A = \mathrm{Gcd}_k G \leq n$, which admits a k-split kG-linear monomorphism $\iota : k \longrightarrow A$. Since A is k-projective, there is an induced projective resolution of the diagonal kG-module $M \otimes_k A$

$$\ldots \longrightarrow P_m \otimes_k A \longrightarrow P_{m-1} \otimes_k A \longrightarrow \ldots \longrightarrow P_0 \otimes_k A \longrightarrow M \otimes_k A \longrightarrow 0.$$

Since M is k-projective, it follows that $\mathrm{pd}_{kG}(M \otimes_k A) \leq \mathrm{pd}_{kG}A \leq n$. Therefore, the n-th syzygy module of the latter resolution, namely $K_n \otimes_k A = \mathrm{im}\,(P_n \otimes_k A \longrightarrow P_{n-1} \otimes_k A)$, is projective (as a kG-module).

Since K_n is a finitely generated kG-module possessing a projective resolution

$$\ldots \longrightarrow P_m \longrightarrow P_{m-1} \longrightarrow \ldots \longrightarrow P_n \longrightarrow K_n \longrightarrow 0$$

and $K_n \otimes_k A$ is kG-projective, we may apply Proposition 2.2(iii) and conclude that there exists a complete projective resolution

$$\mathbf{T} : \ \ldots \longrightarrow P_m \longrightarrow P_{m-1} \longrightarrow \ldots \longrightarrow P_n \longrightarrow T_{n-1} \longrightarrow T_{n-2} \longrightarrow \ldots,$$

with $K_n = \mathrm{im}\,(P_n \longrightarrow T_{n-1})$, such that T_i is a finitely generated free kG-module for all $i \leq n-1$. Since $\mathbf{T}$ is a complete projective resolution, which coincides with the projective resolution $\mathbf{P}$ of M in degrees $\geq n$, the proof is complete. □

The following result is an immediate consequence of Theorem 2.3 and its proof.

Corollary 2.4. *Let k be a commutative ring of finite weak global dimension, G a group with $\mathrm{Gcd}_k G = n < \infty$ and M a k-projective kG-module of type FP_∞. Then, for any projective resolution $\mathbf{P}$ of M, which consists of finitely generated free kG-modules in each degree, there exists a complete projective resolution $\mathbf{T}$ of M, which consists of finitely generated free kG-modules in each degree and coincides with $\mathbf{P}$ in degrees $\geq n$.*

Proof. Consider a projective resolution $\mathbf{P}$ of M, consisting of finitely generated free kG-modules in each degree. Then, the complete projective resolution $\mathbf{T}$ constructed in the proof of Theorem 2.3 has coincidence index n and consists of finitely generated free kG-modules in each degree: In degrees $\geq n$, this follows since $\mathbf{P}$ consists of finitely generated free kG-modules, whereas in degrees $\leq n-1$ this follows from Theorem 2.3. □

We conclude by recording the special case where $M = k$ is the trivial kG-module; for an alternative proof of the following result, the reader may consult [2, Theorem 4.2(vi)].

Corollary 2.5. *Let k be a commutative ring of finite weak global dimension and G a group of type FP_∞ over k with $\mathrm{Gcd}_k G = n < \infty$. Then, for any projective resolution $\mathbf{P}$ of G over k, which consists of finitely generated free kG-modules in each degree, there exists a complete projective resolution $\mathbf{T}$ of G over k, which consists of finitely generated free kG-modules in each degree and coincides with $\mathbf{P}$ in degrees $\geq n$.* □

3. Characteristic modules for FP_∞ groups with finite Gcd

Using the results of the previous section, we shall now prove that a group of type FP_∞ with finite Gorenstein cohomological dimension has a characteristic module of type FP. As a consequence, it will follow that the Gorenstein cohomological dimension of such groups (at least in the case where $k = \mathbb{Z}$ is the ring of integers) is detected by a suitable choice of a field of coefficients.

Theorem 3.1. *Let k be a commutative ring of finite weak global dimension and G a group of type FP_∞ over k with $\mathrm{Gcd}_k G < \infty$. Then, there exists a characteristic module for G over k of type FP.*

Proof. Let $\mathbf{P}$ be a projective resolution of G over k, which consists of finitely generated free modules in each degree. We also let $\mathrm{Gcd}_k G = n$ and consider a complete projective resolution $\mathbf{T}$ of G over k, which consists of finitely generated free modules in each degree and is such that $T_i = P_i$ for all $i \geq n$; cf. Corollary 2.5. Since the complexes of abelian groups $\mathrm{Hom}_{kG}(\mathbf{T}, P_i)$ are acyclic for $i = 0, 1, \ldots, n-1$, the identity maps $T_i \longrightarrow P_i$, $i \geq n$, extend to a chain map $\tau : \mathbf{T} \longrightarrow \mathbf{P}$

$$\begin{array}{ccccccccccccccc} \ldots & \longrightarrow & T_{n+1} & \longrightarrow & T_n & \longrightarrow & T_{n-1} & \longrightarrow & \ldots & \longrightarrow & T_0 & \longrightarrow & T_{-1} & \longrightarrow & \cdots \\ & & \| & & \| & & {\scriptstyle \tau_{n-1}} \downarrow & & & & {\scriptstyle \tau_0} \downarrow & & {\scriptstyle \tau_{-1}} \downarrow & & \\ \ldots & \longrightarrow & P_{n+1} & \longrightarrow & P_n & \longrightarrow & P_{n-1} & \longrightarrow & \ldots & \longrightarrow & P_0 & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

We may reduce to the case where the linear maps τ_i are surjective in degrees $i < n$. Indeed, for each $i = 0, 1, \ldots, n-1$ we may consider the (contractible) complex $\mathbf{X}_i$, which consists of P_i in degrees $i, i-1$ and 0's elsewhere with differential in degree i given by the identity map of P_i. Let $f_i : \mathbf{X}_i \longrightarrow \mathbf{P}$ be the unique chain map whose component in degree i is the identity map of P_i. Then, the direct sum $\mathbf{T}' = \mathbf{T} \oplus \mathbf{X}_{n-1} \oplus \ldots \oplus \mathbf{X}_1 \oplus \mathbf{X}_0$ is a complete projective resolution of G over k and the chain map $\mathbf{T}' \longrightarrow \mathbf{P}$, which is induced by τ and the f_i's, $i = 0, 1, \ldots, n-1$, is surjective in degrees $\leq n-1$ (and coincides with τ in degrees $\geq n$).

We now consider the (Gorenstein projective) modules

$$K = \mathrm{coker}\,(T_{n+1} \longrightarrow T_n)\,,\ L = \mathrm{coker}\,(T_1 \longrightarrow T_0)\ \text{ and } M = \mathrm{coker}\,(T_0 \longrightarrow T_{-1})\,.$$

The chain map τ induces a commutative diagram

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & K & \longrightarrow & T_{n-1} & \longrightarrow & \ldots & \longrightarrow & T_0 & \longrightarrow & L & \longrightarrow & 0 \\ & & \| & & {\scriptstyle \tau_{n-1}} \downarrow & & & & {\scriptstyle \tau_0} \downarrow & & {\scriptstyle t} \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & P_{n-1} & \longrightarrow & \ldots & \longrightarrow & P_0 & \longrightarrow & k & \longrightarrow & 0 \end{array} \tag{5}$$

with exact rows whose vertical maps are surjective. (Here, $t = \overline{\tau_0}$ is the map induced by τ_0 by passage to the quotients.) We let $N = \ker t$ and $Q_i = \ker \tau_i$ for all $i = 0, 1, \ldots, n-1$. Then, there is an induced exact sequence

$$0 \longrightarrow Q_{n-1} \longrightarrow \ldots \longrightarrow Q_0 \longrightarrow N \longrightarrow 0. \tag{6}$$

Indeed, viewing (5) as a surjective chain map between acyclic complexes, the complex (6) is its kernel (which must therefore be acyclic). Since the P_i's are projective, the surjective linear maps τ_i split and hence Q_i is a direct summand of T_i for all $i = 0, 1, \ldots, n-1$. It follows that Q_i is a finitely generated projective module for all $i = 0, 1, \ldots, n-1$.

Pushing out the short exact sequence

$$0 \longrightarrow L \longrightarrow T_{-1} \longrightarrow M \longrightarrow 0$$

by the linear map $t : L \longrightarrow k$, we obtain a commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & N & = & N & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & L & \longrightarrow & T_{-1} & \longrightarrow & M & \longrightarrow & 0 \\ & & {\scriptstyle t}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & k & \stackrel{\iota}{\longrightarrow} & A & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

with exact rows and columns. Since M is Gorenstein projective as a kG-module, our assumption that k has finite weak global dimension implies that M is k-projective (cf. Remark 1.4(i)). It follows that the kG-linear monomorphism ι splits and A is k-projective. Splicing the exact sequence (6) with the vertical short exact sequence in the middle of the diagram above, we obtain an exact sequence of kG-modules

$$0 \longrightarrow Q_{n-1} \longrightarrow \ldots \longrightarrow Q_0 \longrightarrow T_{-1} \longrightarrow A \longrightarrow 0.$$

Since T_{-1} and the Q_i's are finitely generated projective kG-modules, the kG-module A is of type FP and $\mathrm{pd}_{kG}A \leq n < \infty$. It follows that A is a characteristic module for G over k of type FP, as needed. □

As we have already pointed out in the Introduction, for any commutative coefficient ring k the Gorenstein cohomological dimension of a group over k is bounded by the corresponding dimension over $\mathbb{Z}$; if G is any group, then $\mathrm{Gcd}_k G \leq \mathrm{Gcd}_{\mathbb{Z}} G$. Our next goal is to show that one can always find a field F with $\mathrm{Gcd}_F G = \mathrm{Gcd}_{\mathbb{Z}} G$, provided that G is a group of type FP_∞ with $\mathrm{Gcd}_{\mathbb{Z}} G < \infty$. To that end, we shall adopt a more general point of view and record a few simple auxiliary results. If k is a principal ideal domain (or, more generally, a unique factorization domain), then any non-unit element $a \in k$ may be expressed as a product $a = \prod_{i=1}^n p_i$ of prime (irreducible) elements $p_1, \ldots, p_n$ and this expression is essentially unique. In particular, the number n is uniquely determined by a; we write $n = \lambda(a)$. In this way, $\lambda(a) = 1$ if a is a prime element, $\lambda(a) = 2$ if a is a product of two prime elements, etc.

Lemma 3.2. *Let k be a principal ideal domain and R be a k-algebra.*
(i) Let $C \neq 0$ be a finitely generated R-module, which is torsion as a k-module. Then, there exists a prime element $p \in k$, such that $pC \neq C$.
(ii) Let $f : M \longrightarrow N$ be an R-linear map and assume that N is finitely generated. If f is not surjective, then there exists a k-algebra F, such that F is a field and the induced map $f \otimes 1 : M \otimes_k F \longrightarrow N \otimes_k F$ is not surjective either.

Proof. (i) Let $c_1, \ldots, c_n$ be generators of the R-module C. Since C is torsion as a k-module, there exists an element $a \in k$, such that $ac_i = 0 \in C$ for all $i = 1, \ldots, n$. It follows readily that $aC = 0$. Since $C \neq 0$, the element a is not a unit in k and we may assume that $\lambda(a)$ is minimal in the set $\Lambda = \{\lambda(a') : a' \in k \text{ and } a'C = 0\}$. If $p \in k$ is a prime element dividing a, then we necessarily have $pC \neq C$. Indeed, if $pC = C$ and we write $a = bp$, then we would have $bC = b(pC) = aC = 0$ and $\lambda(b) = \lambda(a) - 1$, contradicting the minimality of $\lambda(a) \in \Lambda$.
(ii) Let $C = \mathrm{coker}\, f$ and consider the exact sequence of R-modules

$$M \stackrel{f}{\longrightarrow} N \longrightarrow C \longrightarrow 0.$$

Since N is finitely generated, it follows that C is finitely generated as well. For any k-algebra F there is an induced exact sequence

$$M \otimes_k F \xrightarrow{f \otimes 1} N \otimes_k F \longrightarrow C \otimes_k F \longrightarrow 0.$$

If C is not torsion as a k-module, then $C \otimes_k K \neq 0$, where K is the field of fractions of k, and we may choose $F = K$. If C is torsion as a k-module, then we may apply (i) and find a prime element p, such that $pC \neq C$. In that case, $C \otimes_k k/pk = C/pC \neq 0$ and we may choose $F = k/pk$. □

In order to apply Lemma 3.2, the following simple general observation will be useful. Let R be a ring and N an R-module with endomorphism ring $\text{End}_R N$. Then, for any R-module M the abelian group $\text{Hom}_R(M, N)$ is endowed with the structure of an $\text{End}_R N$-module, where elements of $\text{End}_R N$ act on linear maps $M \longrightarrow N$ by composition to the left. This $\text{End}_R N$-module structure is clearly natural in M; in other words, any R-linear map $f : M \longrightarrow M'$ induces an additive map $f^* : \text{Hom}_R(M', N) \longrightarrow \text{Hom}_R(M, N)$, which is linear with respect to the $\text{End}_R N$-module structures on the two Hom-groups. In particular, letting $N = R$, we conclude that the abelian group $\text{Hom}_R(M, R)$ admits a natural structure of a right R-module: For any $r \in R$ and any R-linear map $f : M \longrightarrow R$ the R-linear map $f \cdot r : M \longrightarrow R$ is defined by letting $(f \cdot r)(m) = f(m)r \in R$ for any $m \in M$. The right R-module $\text{Hom}_R(R^n, R)$, which is obtained as above by letting $M = R^n$, is naturally identified with the right R-module R^n for any non-negative integer n. As an immediate consequence of this observation, we obtain the following result.

Lemma 3.3. *If M is a finitely generated projective R-module, then the right R-module $Hom_R(M, R)$ defined above is also finitely generated projective.* □

We now let k be a commutative ring, R, S be two k-algebras and consider an R-module N. Then, for any R-module M we consider the additive map

$$\xi_M : \text{Hom}_R(M, N) \otimes_k S \longrightarrow \text{Hom}_{R \otimes_k S}(M \otimes_k S, N \otimes_k S),$$

which is defined by letting $\xi_M(f \otimes s) : M \otimes_k S \longrightarrow N \otimes_k S$ be the $R \otimes_k S$-linear map $m \otimes s' \mapsto f(m) \otimes ss' \in N \otimes_k S$, $m \otimes s' \in M \otimes_k S$, for all $f \in \text{Hom}_R(M, N)$ and $s \in S$. It is clear that ξ_M is natural in M.

Lemma 3.4. *Let R, S be two rings and consider an R-module N. Then:*
(i) If M, M' are two R-modules, then the additive map $\xi_{M \oplus M'}$ is naturally identified with the direct sum $\xi_M \oplus \xi_{M'}$ of the additive maps ξ_M and $\xi_{M'}$.
(ii) The additive map ξ_M is bijective for any finitely generated projective R-module M.

Proof. Assertion (i) is clear, whereas (ii) is obvious in the special case where $M = R$. The general case of a finitely generated projective R-module follows from that special case, in view of (i). □

Proposition 3.5. *Let k be a principal ideal domain, R be a k-algebra and M an R-module of type FP. We assume that both R and M are torsion-free as k-modules. Then, there exists a k-algebra F, such that F is a field and $pd_{R \otimes_k F}(M \otimes_k F) = pd_R M$.*

Proof. Let $\text{pd}_R M = n$ and consider a resolution

$$0 \longrightarrow P_n \xrightarrow{f} P_{n-1} \longrightarrow \ldots \longrightarrow P_0 \longrightarrow M \longrightarrow 0, \tag{7}$$

where P_i is a finitely generated projective R-module for all $i = 0, 1, \ldots, n$. Since $\text{pd}_R M = n$, the monomorphism f does not split and hence $\text{Ext}^n_R(M, P_n) \neq 0$. The R-module P_n being a direct summand of a finitely generated free R-module, we conclude that $\mathrm{Ext}^n_R(M,R) \neq 0$. Hence, the exactness of the sequence

$$\mathrm{Hom}_R(P_{n-1},R) \xrightarrow{f^*} \mathrm{Hom}_R(P_n,R) \longrightarrow \mathrm{Ext}^n_R(M,R) \longrightarrow 0$$

shows that the map f^* is not surjective. As noted above, the groups $\mathrm{Hom}_R(P_{n-1},R)$ and $\mathrm{Hom}_R(P_n,R)$ can be viewed as right R-modules and f^* is then R-linear. Since the right R-module $\mathrm{Hom}_R(P_n,R)$ is finitely generated (cf. Lemma 3.3), we may apply Corollary 3.2 and conclude that there exists a k-algebra F, such that F is a field and the induced map

$$f^* \otimes 1 : \mathrm{Hom}_R(P_{n-1},R) \otimes_k F \longrightarrow \mathrm{Hom}_R(P_n,R) \otimes_k F$$

is not surjective either. In view of Lemma 3.4(ii), the latter map is identified with

$$(8) \qquad (f \otimes 1)^* : \mathrm{Hom}_{R\otimes_k F}(P_{n-1} \otimes_k F, R \otimes_k F) \longrightarrow \mathrm{Hom}_{R\otimes_k F}(P_{n-1} \otimes_k F, R \otimes_k F).$$

Since M and the P_i's are torsion-free as k-modules, (7) induces a projective resolution

$$0 \longrightarrow P_n \otimes_k F \xrightarrow{f\otimes 1} P_{n-1} \otimes_k F \longrightarrow \ldots \longrightarrow P_0 \otimes_k F \longrightarrow M \otimes_k F \longrightarrow 0$$

of $M \otimes_k F$ as an $R \otimes_k F$-module. It follows that the cokernel of the map (8) computes the group $\mathrm{Ext}^n_{R\otimes_k F}(M \otimes_k F, R \otimes_k F)$. We therefore conclude that $\mathrm{Ext}^n_{R\otimes_k F}(M \otimes_k F, R \otimes_k F) \neq 0$ and hence $\mathrm{pd}_{R\otimes_k F}(M \otimes_k F) = n$, as needed. ☐

We shall now apply the previous discussion to the special case that we are really interested in, namely to the study of the projective dimension of characteristic modules for groups of finite Gorenstein cohomological dimension. In the special case where $k = \mathbb{Z}$ is the ring of integers, the following result reduces to Theorem C, as stated in the Introduction.

Theorem 3.6. *Let k be a principal ideal domain and G be a group of type FP_∞ over k with $Gcd_k G < \infty$. Then, there exists a k-algebra F, such that F is a field and $Gcd_F G = Gcd_k G$.*

Proof. Since $\mathrm{gl.dim}\, k \leq 1$, Theorem 3.1 implies that there exists a characteristic module A for G over k of type FP. Then, Proposition 3.5 shows that we can find a k-algebra F, such that F is a field and $\mathrm{pd}_{FG}(A \otimes_k F) = \mathrm{pd}_{kG} A$. Since $A \otimes_k F$ is a characteristic module for G over F (cf. Remark 1.4(iii)), Corollary 1.3 implies that $\mathrm{Gcd}_F G = \mathrm{pd}_{FG}(A \otimes_k F) = \mathrm{pd}_{kG} A = \mathrm{Gcd}_k G$, as needed. ☐

4. Gorenstein cohomological dimension of certain group extensions

Our goal in this section is to formulate and prove the analogue of Fel'dman's computation of the cohomological dimension of certain group extensions [11] in Gorenstein homological algebra. To that end, we shall follow closely the excellent presentation of Fel'dman's result given in Bieri's notes [3].

We fix a commutative ring k and consider an extension of groups

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$

and two kG-modules M, V. The group G acts (diagonally) on the abelian group $\mathrm{Hom}_{kN}(M,V)$. Given $g \in G$ and $f \in \mathrm{Hom}_{kN}(M,V)$, the map $g \cdot f : M \longrightarrow V$ is defined by letting $x \mapsto gf(g^{-1}x) \in V$, $x \in M$. (Since N is normal in G, the map $g \cdot f$ is easily seen to be kN-linear; the diagonal action of G on $\mathrm{Hom}_k(M,V)$ restricts to its subgroup $\mathrm{Hom}_{kN}(M,V)$, which is thereby a kG-submodule of it.) As the action of elements of N is obviously trivial, we actually obtain an action of the quotient group Q on $\mathrm{Hom}_{kN}(M,V)$. The kQ-module structure defined in this way on the abelian group $\mathrm{Hom}_{kN}(M,V)$ is natural in M; any kG-linear map $h : M \longrightarrow M'$

induces an additive map $h^* : \mathrm{Hom}_{kN}(M', V) \longrightarrow \mathrm{Hom}_{kN}(M, V)$, which is kQ-linear. In other words, for any $g \in G$ and $f \in \mathrm{Hom}_{kN}(M', V)$ we have $g \cdot (f \circ h) = (g \cdot f) \circ h \in \mathrm{Hom}_{kN}(M, V)$. We note that the subgroup of Q-invariant elements in $\mathrm{Hom}_{kN}(M, V)$ consists precisely of the kG-linear maps from M to V, i.e. $H^0(Q, \mathrm{Hom}_{kN}(M, V)) = \mathrm{Hom}_{kG}(M, V)$.

Using general principles (cf. the discussion before Lemma 3,3), we can define on the abelian group $\mathrm{Hom}_{kN}(M, kN)$ the structure of a right kN-module. In this way, given $n \in N$ and $f \in \mathrm{Hom}_{kN}(M, kN)$, the map $f \cdot n : M \longrightarrow kN$ is defined by letting $x \mapsto f(x)n \in kN$, $x \in M$. We may extend this right kN-module structure on $\mathrm{Hom}_{kN}(M, kN)$ to a right kG-module structure as follows: Given $g \in G$ and $f \in \mathrm{Hom}_{kN}(M, kN)$, the map $f \cdot g : M \longrightarrow kN$ is defined by letting $x \mapsto g^{-1}f(gx)g \in kN$, $x \in M$. (As before, using the fact that N is normal in G, it is easily seen that the map $f \cdot g$ is well-defined and kN-linear.) This right kG-module structure on the abelian group $\mathrm{Hom}_{kN}(M, kN)$ is natural in M; any kG-linear map $h : M \longrightarrow M'$ induces an additive map $h^* : \mathrm{Hom}_{kN}(M', kN) \longrightarrow \mathrm{Hom}_{kN}(M, kN)$, which is kG-linear. In other words, for any $g \in G$ and $f \in \mathrm{Hom}_{kN}(M', kN)$ we have $(f \circ h) \cdot g = (f \cdot g) \circ h \in \mathrm{Hom}_{kN}(M, kN)$. Using the inversion map of G, we define the structure of a (left) kG-module on $\mathrm{Hom}_{kN}(M, kN)$, by letting any element $g \in G$ act as right multiplication by g^{-1}.

We now consider the diagonal action of G on the tensor product $\mathrm{Hom}_{kN}(M, kN) \otimes_{kN} V$. Given $g \in G$, $f \in \mathrm{Hom}_{kN}(M, kN)$ and $v \in V$, we have $g \cdot (f \otimes v) = (f \cdot g^{-1}) \otimes gv$. Since N is normal in G, it is easily seen that this action is well-defined. Moreover, the elements of N act trivially and hence we obtain an action of the quotient group Q on $\mathrm{Hom}_{kN}(M, kN) \otimes_{kN} V$. The kQ-module structure defined in this way on $\mathrm{Hom}_{kN}(M, kN) \otimes_{kN} V$ is natural in M.

There is a additive map

$$\Phi = \Phi_M : \mathrm{Hom}_{kN}(M, kN) \otimes_{kN} V \longrightarrow \mathrm{Hom}_{kN}(M, V), \tag{9}$$

which is natural in M, given by letting $\Phi(f \otimes v)$ be the map $x \mapsto f(x)v \in V$, $x \in M$, for all $f \in \mathrm{Hom}_{kN}(M, kN)$ and $v \in V$. The map Φ is kG-linear (i.e. kQ-linear) with respect to the module structures defined above; for any $g \in G$, $f \in \mathrm{Hom}_{kN}(M, kN)$ and $v \in V$ we have $\Phi[g \cdot (f \otimes v)] = g \cdot \Phi(f \otimes v) \in \mathrm{Hom}_{kN}(M, V)$.

Remark 4.1 Let k be a commutative ring. We may formally generalize the definition of the additive map Φ as follows: For any group N and any two kN-modules L, U there is an additive map

$$\Psi = \Psi_L : \mathrm{Hom}_{kN}(L, kN) \otimes_{kN} U \longrightarrow \mathrm{Hom}_{kN}(L, U),$$

which is natural in L, given by letting $\Psi(f \otimes u)$ be the map $x \mapsto f(x)u \in U$, $x \in L$, for all $f \in \mathrm{Hom}_{kN}(L, kN)$ and $u \in U$. Here, the right kN-module structure on the abelian group $\mathrm{Hom}_{kN}(L, kN)$ is that defined by the generalities discussed before Lemma 3.3. It is clear that Ψ_L is bijective in the special case where $L = kN$. Moreover, if L, L' are two kN-modules, then the additive map $\Psi_{L \oplus L'}$ is naturally identified with the direct sum $\Psi_L \oplus \Psi_{L'}$ of the additive maps Ψ_L and $\Psi_{L'}$. It follows readily that Ψ_L is bijective if L is any finitely generated projective kN-module.

We now consider a projective resolution $\mathbf{P}$ of the trivial kG-module k

$$\ldots \longrightarrow P_i \longrightarrow P_{i-1} \longrightarrow \ldots \longrightarrow P_0 \longrightarrow k \longrightarrow 0.$$

Since any projective kG-module is kN-projective as well, $\mathbf{P}$ is also a projective resolution of k as a kN-module. Hence, we can (and will) compute the cohomology groups of N using $\mathbf{P}$. The naturality of the kQ-module structure defined above on the abelian groups $\mathrm{Hom}_{kN}(P_i, V)$ shows that $\mathrm{Hom}_{kN}(\mathbf{P}, V)$ is a complex of kQ-modules, In particular, the cohomology groups

$H^i(N,V)$ are kQ-modules as well. Since $H^0(Q, \mathrm{Hom}_{kN}(\mathbf{P},V)) = \mathrm{Hom}_{kG}(\mathbf{P},V)$, it is precisely this kQ-module structure that is involved in the E^2-term of the Lyndon-Hochschild-Serre spectral sequence

$$E^2_{pq} = H^p(Q, H^q(N,V)) \Longrightarrow H^{p+q}(G,V)$$

that computes the cohomology of G with coefficients in V. In an analogous way, the naturality of the kQ-module structure defined above on the abelian groups $\mathrm{Hom}_{kN}(P_i,kN)\otimes_{kN} V$ shows that $\mathrm{Hom}_{kN}(\mathbf{P},kN)\otimes_{kN} V$ is also a complex of kQ-modules, (It follows that its cohomology groups are kQ-modules as well.) The naturality of the kQ-linear maps

$$\Phi_{P_i} : \mathrm{Hom}_{kN}(P_i,kN)\otimes_{kN} V \longrightarrow \mathrm{Hom}_{kN}(P_i,V),$$

defined above, shows that these are the components of a map between cochain complexes of kQ-modules

$$\Phi : \mathrm{Hom}_{kN}(\mathbf{P},kN)\otimes_{kN} V \longrightarrow \mathrm{Hom}_{kN}(\mathbf{P},V). \tag{10}$$

Proposition 4.2. *Let k be a commutative ring and consider an extension of groups*

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1,$$

where N is a group of type FP_∞ over k. We also consider a projective resolution $\mathbf{P}$ of the trivial kG-module k and a kG-module V. Then:
(i) The cochain complex map (10) defined above is a quasi-isomorphism.
(ii) If V is flat as a kN-module, then Φ induces an isomorphism of kQ-modules

$$\phi_i : H^i(N,kN)\otimes_{kN} V \longrightarrow H^i(N,V)$$

for all $i \geq 0$. (Here, the group G acts diagonally on the tensor product.)

Proof. (i) Since the group N is of type FP_∞ over k, there exists a resolution $\mathbf{P}'$ of the trivial kN-module k, which consists of finitely generated projective kN-modules in each degree. We know that there exists a homotopy equivalence $f : \mathbf{P}' \longrightarrow \mathbf{P}$ in the category of cochain complexes of kN-modules. Of course, f induces homotopy equivalences

$$f^* : \mathrm{Hom}_{kN}(\mathbf{P},kN) \longrightarrow \mathrm{Hom}_{kN}(\mathbf{P}',kN) \quad \text{and} \quad f^* : \mathrm{Hom}_{kN}(\mathbf{P},V) \longrightarrow \mathrm{Hom}_{kN}(\mathbf{P}',V)$$

in the category of cochain complexes of abelian groups. The naturality of the right kN-module structure defined on the $\mathrm{Hom}_{kN}(\_,kN)$-groups implies that the former of these two homotopy equivalences is actually a homotopy equivalence in the category of cochain complexes of right kN-modules. Therefore, there is an induced homotopy equivalence

$$f^*\otimes 1 : \mathrm{Hom}_{kN}(\mathbf{P},kN)\otimes_{kN} V \longrightarrow \mathrm{Hom}_{kN}(\mathbf{P}',kN)\otimes_{kN} V$$

in the category of cochain complexes of abelian groups.

In the same way that the natural maps (9) lead to the cochain complex map (10), one may use the natural maps Ψ defined in Remark 4.1 and define a cochain complex map

$$\Psi : \mathrm{Hom}_{kN}(\mathbf{P}',kN)\otimes_{kN} V \longrightarrow \mathrm{Hom}_{kN}(\mathbf{P}',V).$$

Since the chain complex $\mathbf{P}'$ consists of finitely generated projective kN-modules in each degree, the above cochain complex map is bijective.

We now claim that the following diagram of cochain complex maps

$$\begin{array}{ccc} \mathrm{Hom}_{kN}(\mathbf{P},kN)\otimes_{kN} V & \xrightarrow{\Phi} & \mathrm{Hom}_{kN}(\mathbf{P},V) \\ {\scriptstyle f^*\otimes 1}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathrm{Hom}_{kN}(\mathbf{P}',kN)\otimes_{kN} V & \xrightarrow{\Psi} & \mathrm{Hom}_{kN}(\mathbf{P}',V) \end{array} \tag{11}$$

is commutative. In other words, we claim that the following diagram of abelian groups

$$\begin{array}{ccc} \mathrm{Hom}_{kN}(P_i, kN) \otimes_{kN} V & \xrightarrow{\Phi_{P_i}} & \mathrm{Hom}_{kN}(P_i, V) \\ {\scriptstyle f_i^* \otimes 1} \downarrow & & \downarrow {\scriptstyle f_i^*} \\ \mathrm{Hom}_{kN}(P'_i, kN) \otimes_{kN} V & \xrightarrow{\Psi_{P'_i}} & \mathrm{Hom}_{kN}(P'_i, V) \end{array}$$

is commutative for all $i \geq 0$. (Here, we denote by f_i the components of f.) Indeed, since the Φ's are particular instances of the Ψ's, the commutativity of the latter diagram follows from the naturality of Ψ. Since the two vertical maps of diagram (11) are homotopy equivalences and the horizontal map at the bottom is bijective, all of these three cochain complex maps are quasi-isomorphisms. It follows readily that the horizontal cochain complex map at the top of that commutative diagram is also a quasi-isomorphism, as needed

(ii) This is an immediate consequence of (i), since the i-th cohomology groups of the complexes $\mathrm{Hom}_{kN}(\mathbf{P}, kN) \otimes_{kN} V$ and $\mathrm{Hom}_{kN}(\mathbf{P}, V)$ coincide with $H^i(N, kN) \otimes_{kN} V$ and $H^i(N, V)$ respectively. □

Corollary 4.3. *Let k be a commutative ring and consider an extension of groups*

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1,$$

where N is a group of type FP_∞ over k. Then, the cohomology group $H^i(N, kG)$ is isomorphic as a kQ-module with the induced module $\mathrm{Ind}_1^Q H^i(N, kN) = H^i(N, kN) \otimes_k kQ$ for all $i \geq 0$.

Proof. Fix a non-negative integer i. Invoking Proposition 4.2(ii), we know that there exists a kQ-module isomorphism $H^i(N, kG) \simeq H^i(N, kN) \otimes_{kN} kG$, where G acts diagonally on the tensor product. Therefore, the result follows from the existence of a kQ-module isomorphism between the diagonal module $H^i(N, kN) \otimes_{kN} kG$ and the induced module $\mathrm{Ind}_1^Q H^i(N, kN) = H^i(N, kN) \otimes_k kQ$. (If we write $Q = G/N$, then such a kQ-module isomorphism is given by letting $t \otimes g \in H^i(N, kN) \otimes_{kN} kG$ map onto $tg \otimes gN \in H^i(N, kN) \otimes_k kQ$ for all $t \in H^i(N, kN)$ and $g \in G$.) □

Corollary 4.4. *Let k be a commutative ring and consider an extension of groups*

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1,$$

where N is a group of type FP_∞ over k, and a non-negative integer i.
(i) If the k-module $H^i(N, kN)$ is projective, then the kQ-module $H^i(N, kG)$ is projective.
(ii) If the k-module $H^i(N, kN)$ contains a copy of k as a direct summand, then the kQ-module $H^i(N, kG)$ contains a copy of kQ as a direct summand. □

We are now ready to state and prove the Gorenstein homological algebra analogue of Fel'dman's result [11]. As a prelude to one of the assumptions that will be made in the following theorem, we note that if k is a commutative ring and N is a group with $\mathrm{Gcd}_k N = n < \infty$, then the cohomology groups $H^i(N, kN)$ vanish for all $i > n$. Is N is, in addition, of type FP_∞ over k, then $H^n(N, kN) \neq 0$. Indeed, as shown by Holm in [13, Theorem 2.20]), we have $H^n(N, P) \neq 0$ for a suitable projective kN-module P. Since P is a direct summand of a direct sum of copies of kN and the cohomology functor $H^n(N, \_)$ commutes with direct sums (in view of the FP_∞ assumption on N), it follows that the group $H^n(N, kN)$ must be indeed non-trivial.

Theorem 4.5. *Let k be a commutative ring and consider an extension of groups*

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1,$$

such that:
(i) N is a group of type FP_∞ over k,
(ii) both $Gcd_k N$ and $Gcd_k Q$ are finite and
(iii) if $Gcd_k N = n$, then the k-modules $H^i(N, kN)$ are projective for all $i < n$ and $H^n(N, kN)$ contains a copy of k as a k-module direct summand.
Then, we have $Gcd_k G = Gcd_k N + Gcd_k Q$.

Proof. Let $\mathrm{Gcd}_k Q = m$. As shown by Holm in [13, Theorem 2.20], we have an equality

$$m = \sup\{i : H^i(Q, P) \neq 0 \text{ for some projective } kQ\text{-module } P\},$$

i.e. an equality

$$m = \sup\{i : H^i(Q, L) \neq 0 \text{ for some free } kQ\text{-module } L\}.$$

In particular, there exists a cardinal number α, such that the cohomology functor $H^m(Q, \_)$ does not vanish on the free kQ-module L of rank α.

As shown in [10, Proposition 2.9], we always have an inequality $\mathrm{Gcd}_k G \leq n+m$. In order to prove that $\mathrm{Gcd}_k G = n+m$, it suffices to show (in view of Holm's result mentioned above) that the cohomology functor $H^{n+m}(G, \_)$ does not vanish on a suitable free kG-module. We shall prove that this functor does not vanish on the free kG-module V of rank α. We can compute the cohomology groups of G with coefficients in V by means of the Lyndon-Hochschild-Serre spectral sequence

$$E^2_{pq} = H^p(Q, H^q(N, V)) \Longrightarrow H^{p+q}(G, V).$$

As we have already noted above, assumption (i) implies that the cohomology functors $H^q(N, \_)$ commute with direct sums. Therefore, $H^q(N, V)$ is isomorphic as a kQ-module with the direct sum $H^q(N, kG)^{(\alpha)}$ of α copies of $H^q(N, kG)$ for all $q \geq 0$. Since $\mathrm{Gcd}_k N = n$ and kG is kN-projective, the cohomology groups $H^q(N, V) = H^q(N, kG)^{(\alpha)}$ vanish when $q > n$; hence, $E^2_{pq} = H^p(Q, H^q(N, V)) = H^p(Q, 0) = 0$ if $q > n$. If $q < n$, then assumption (iii) implies that the cohomology groups $H^q(N, V) = H^q(N, kG)^{(\alpha)}$ are projective as kQ-modules (cf. Corollary 4.4(i)). Since $\mathrm{Gcd}_k Q = m$, it follows that $E^2_{pq} = H^p(Q, H^q(N, V)) = 0$ if $p > m$ and $q < n$. The E^2-page of the spectral sequence is thus concentrated on the square $[0, m] \times [0, n]$ and the line $\{(p, n) : p \geq 0\}$. It follows readily that

$$H^{n+m}(G, V) = E^\infty_{mn} = E^2_{mn} = H^m(Q, H^n(N, V)).^1$$

Assumption (iii) also implies that the cohomology group $H^n(N, V) = H^n(N, kG)^{(\alpha)}$ contains a copy of the free kQ-module L of rank α as a kQ-module direct summand (cf. Corollary 4.4(ii)). It follows that the abelian group $H^m(Q, H^n(N, V))$ contains a copy of $H^m(Q, L)$ as a direct summand. As the latter group is non-zero (by the choice of the cardinal number α), we conclude that $H^{n+m}(G, V) = H^m(Q, H^n(N, V)) \neq 0$, as needed. □

Corollary 4.6. *Let k be a commutative ring and consider an extension of groups*

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1,$$

such that:
(i) N is a group of type FP_∞ over k,
(ii) both $Gcd_k N$ and $Gcd_k Q$ are finite and

[1] It also follows that $H^i(G, V) = E^\infty_{i-n\ n} = E^2_{i-n\ n} = H^{i-n}(Q, H^n(N, V))$ for all $i > n+m$. Since we already know that $\mathrm{Gcd}_k G \leq n+m$, we conclude that the cohomology groups $H^j(Q, H^n(N, V))$ are trivial in degrees $j > m$.

(iii) if $Gcd_kN = n$, then the k-modules $H^i(N, kN)$ are free for all $i \leq n$.
Then, we have $Gcd_kG = Gcd_kN + Gcd_kQ$.

Proof. As we have already noted before, our assumption that N is a group of type FP_∞ with $\mathrm{Gcd}_kN = n$ implies that $H^n(N, kN) \neq 0$. Therefore, the result is an immediate consequence of Theorem 4.5. □

In the special case where the coefficient ring is a field, assumption (iii) in the statement of Corollary 4.6 is redundant. The following result is stated in the Introduction as Theorem A.

Corollary 4.7. *Let F be a field and consider an extension of groups*

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1,$$

where N is a group of type FP_∞ over F and $Gcd_FQ < \infty$. Then, $Gcd_FG = Gcd_FN + Gcd_FQ$.

Proof. Since N is a subgroup of G, [10, Proposition 2.4] implies that $\mathrm{Gcd}_FN \leq \mathrm{Gcd}_FG$; hence, the equality to be proved is clear if $\mathrm{Gcd}_FN = \infty$. If $\mathrm{Gcd}_FN < \infty$, the result is an immediate consequence of Corollary 4.6. □

5. An application

Having in mind the geometric interpretation of the various cohomological invariants that are associated with a group, the most interesting case of a commutative coefficient ring is that of the ring $\mathbb{Z}$ of integers. In this final section, we apply the previous results in order to compute the Gorenstein cohomological dimension over $\mathbb{Z}$ of certain group extensions. In this way, we obtain genuine generalizations for some of the results in [9], where the ordinary cohomological dimension of certain products of geometrically finite groups is considered.

If N is a group and m a positive integer, then we define inductively an iterated m-fold extension G of N by itself, as follows: If $m = 1$, then $G = N$. If $m > 1$, then G must fit into an extension of groups

$$0 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1,$$

where Q is an iterated $(m-1)$-fold extension of N by itself. In the special case where $m = 2$, the following result reduces to Theorem B, as stated in the Introduction.

Theorem 5.1. *Let N be a group of type FP_∞ over $\mathbb{Z}$. If m is a positive integer and G is an iterated m-fold extension of N by itself, then $Gcd_{\mathbb{Z}}G = m\,Gcd_{\mathbb{Z}}N$.*

Proof. Since N is a subgroup of G, [10, Proposition 2.4] implies that $\mathrm{Gcd}_{\mathbb{Z}}N \leq \mathrm{Gcd}_{\mathbb{Z}}G$; hence, the equality to be proved is clear if $\mathrm{Gcd}_{\mathbb{Z}}N = \infty$. We may therefore assume below that $\mathrm{Gcd}_{\mathbb{Z}}N < \infty$. In view of Theorem 3.6, there exists a field F, such that $\mathrm{Gcd}_FN = \mathrm{Gcd}_{\mathbb{Z}}N$. We shall prove by induction on m that $\mathrm{Gcd}_FG = m\mathrm{Gcd}_FN$. This is clear if $m = 1$. Assume now that $m > 1$ and the result holds for any iterated $(m-1)$-fold extension of N by itself. The group G fits into an extension

$$0 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1,$$

for a suitable $(m-1)$-fold extension Q of N by itself. Since N is of type FP_∞ over $\mathbb{Z}$, it is also of type FP_∞ over F. The induction hypothesis implies that $\mathrm{Gcd}_FQ = (m-1)\mathrm{Gcd}_FN < \infty$ and hence we can apply Corollary 4.7, in order to conclude that $\mathrm{Gcd}_FG = \mathrm{Gcd}_FN + \mathrm{Gcd}_FQ$, i.e. that $\mathrm{Gcd}_FG = m\mathrm{Gcd}_FN$. We note that [10, Proposition 2.1] implies that $\mathrm{Gcd}_FG \leq \mathrm{Gcd}_{\mathbb{Z}}G$, whereas a repeated application of [10, Proposition 2.9] shows that $\mathrm{Gcd}_{\mathbb{Z}}G \leq m\mathrm{Gcd}_{\mathbb{Z}}N$. Since

$$m\mathrm{Gcd}_{\mathbb{Z}}N = m\mathrm{Gcd}_FN = \mathrm{Gcd}_FG \leq \mathrm{Gcd}_{\mathbb{Z}}G \leq m\mathrm{Gcd}_{\mathbb{Z}}N < \infty,$$

it follows that $\mathrm{Gcd}_{\mathbb{Z}}G = m\mathrm{Gcd}_{\mathbb{Z}}N$, as needed. □

Corollary 5.2. *Let N be a group of type FP_∞ over $\mathbb{Z}$. If m is a positive integer and N^m is the product of m copies of N, then $Gcd_{\mathbb{Z}} N^m = m Gcd_{\mathbb{Z}} N$.* □

As we mentioned in the Introduction, the Gorenstein cohomological dimension of a group coincides with its virtual cohomological dimension, in the case where the latter is defined and finite. Therefore, the following result is an immediate consequence of Theorem 5.1.

Corollary 5.3. *Let N be a group of type FP_∞ over $\mathbb{Z}$ with $vcd_{\mathbb{Z}} N = n < \infty$ and consider a positive integer m. Then:*
(i) If G is an iterated m-fold extension of N by itself, then $Gcd_{\mathbb{Z}} G = mn$.
(ii) If N^m is the product of m copies of N, then $Gcd_{\mathbb{Z}} N^m = vcd_{\mathbb{Z}} N^m = mn$. □

Remarks 5.4. (i) The argument in the proof of Theorem 5.1 breaks down if we consider arbitrary iterated extensions of groups of type FP_∞, even if these groups have finite Gorenstein cohomological dimension. In particular, if

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$

is an extension, where N, Q are two (possibly different) groups of type FP_∞ with finite Gorenstein cohomological dimension over $\mathbb{Z}$, then the inequality $\mathrm{Gcd}_{\mathbb{Z}} G \leq \mathrm{Gcd}_{\mathbb{Z}} N + \mathrm{Gcd}_{\mathbb{Z}} Q$ may be strict. Indeed, since there is no way of knowing that there exists a field F for which both equalities $\mathrm{Gcd}_F N = \mathrm{Gcd}_{\mathbb{Z}} N$ and $\mathrm{Gcd}_F Q = \mathrm{Gcd}_{\mathbb{Z}} Q$ hold, the reduction to the case of a field of coefficients may not be always possible. In fact, the analogue of Corollary 5.3(ii) does not hold for the product of two groups of type FP_∞ with finite virtual cohomological dimension over $\mathbb{Z}$. Dranishnikov has constructed in [8] groups G_1, G_2 of type FP_∞ with finite virtual cohomological dimension over $\mathbb{Z}$, such that $\mathrm{Gcd}_{\mathbb{Z}}(G_1 \times G_2) = \mathrm{vcd}_{\mathbb{Z}}(G_1 \times G_2) < \mathrm{vcd}_{\mathbb{Z}} G_1 + \mathrm{vcd}_{\mathbb{Z}} G_2$. This example also shows that assumption (iii) in the statement of Theorem 4.5 cannot be completely omitted.

(ii) All results in this section are also valid more generally if the ring $\mathbb{Z}$ of integers is replaced by any principal ideal domain.

Appendix A. Characteristic modules for $\mathbf{LH}\mathfrak{F}$-groups

let k be a commutative ring, G a group and consider the kG-module $B(G,k)$ of all functions from G to k, whose image is a finite subset of k (cf. the end of §1). If $B(G,k)$ is a characteristic module for G over k, then Proposition 1.2(ii) implies that $\mathrm{Gcd}_k G < \infty$. Our goal in this Appendix is to show that the converse of the latter assertion is true, in the case where k has finite global dimension and G is a group in Kropholler's class $\mathbf{LH}\mathfrak{F}$.

We note that $B(G,k)$ is generated as a k-module by the characteristic functions χ_A, where A runs through the power set of G. Any $f \in B(G,k)$ admits a unique expression as a sum $\sum_{i=1}^n \lambda_i \chi_{A_i}$, where $A_1, \ldots, A_n \subseteq G$ form a partition of G and $\lambda_1, \ldots, \lambda_n \in k$ are the distinct values that f assumes. If $k \longrightarrow k'$ is a homomorphism of commutative rings, then there is a natural $k'G$-linear map

$$B(G,k) \otimes_k k' \longrightarrow B(G,k'),$$

which maps $f \otimes \lambda'$ onto the function $f' \in B(G,k')$, which is defined by letting $g \mapsto f(g)\lambda' \in k'$, $g \in G$, for all $f \in B(G,k)$ and $\lambda' \in k'$. This map is bijective and its inverse

$$B(G,k') \longrightarrow B(G,k) \otimes_k k'$$

maps an element $\sum_{i=1}^n \lambda_i' \chi_{A_i} \in B(G,k')$, where $A_1, \ldots, A_n$ form a partition of G and $\lambda_1', \ldots, \lambda_n'$ are distinct elements of k' as above, onto $\sum_{i=1}^n \chi_{A_i} \otimes \lambda_i' \in B(G,k) \otimes_k k'$. In particular, for any commutative ring k there is an isomorphism of kG-modules $B(G,k) \simeq B(G,\mathbb{Z}) \otimes_{\mathbb{Z}} k$.

The kG-module $B(G,k)$ is k-free; more generally, $B(G,k)$ is kH-free for any finite subgroup $H \subseteq G$. This is proved in [15] if $k = \mathbb{Z}$; in the general case, the result follows since $B(G,k) \simeq B(G,\mathbb{Z}) \otimes_{\mathbb{Z}} k$. For any element $\lambda \in k$ the constant function $\iota(\lambda) \in B(G,k)$ with value λ is invariant under the action of G. The map $\iota : k \longrightarrow B(G,k)$ which is defined in this way is therefore kG-linear. It is clear that ι is k-split; for any fixed element $g \in G$, we may obtain a k-linear splitting for ι by evaluating functions at g.

In order to examine whether the kG-module $B(G,k)$ has finite projective dimension, we restrict our attention to the case where G is a group contained in the class $\mathbf{LH}\mathfrak{F}$ defined by Kropholler in [14]. The class $\mathbf{H}\mathfrak{F}$ is the smallest class of groups, which contains the class $\mathfrak{F}$ of finite groups and is such that whenever a group G admits a finite dimensional contractible G-CW-complex with stabilizers in $\mathbf{H}\mathfrak{F}$, then we also have $G \in \mathbf{H}\mathfrak{F}$. Then, the class $\mathbf{LH}\mathfrak{F}$ consists of those groups, all of whose finitely generated subgroups are in $\mathbf{H}\mathfrak{F}$. All soluble groups, all groups of finite virtual cohomological dimension and all automorphism groups of Noetherian modules over a commutative ring are $\mathbf{LH}\mathfrak{F}$-groups. The class $\mathbf{LH}\mathfrak{F}$ is closed under extensions, ascending unions, free products with amalgamation and HNN extensions.

Theorem A.1. *Let k be a commutative ring of finite global dimension and consider an $\mathbf{LH}\mathfrak{F}$-group G. Then:*

(i) $B(G,k)$ is a characteristic module for G over k if and only if $Gcd_k G < \infty$ and

(ii) $Gcd_k G = pd_{kG} B(G,k)$.

Proof. As we noted above, the kG-module $B(G,k)$ is k-free and admits a k-split kG-linear monomorphism $\iota : k \longrightarrow B(G,k)$. Therefore, $B(G,k)$ is a characteristic module for G over k if and only if $\text{pd}_{kG} B(G,k) < \infty$.

(i) If $B(G,k)$ is a characteristic module for G over k, then $\text{Gcd}_k G < \infty$, in view of Proposition 1.2(ii). Conversely, assume that $\text{Gcd}_k G < \infty$. Then, [10, Corollary 1.6] implies that any projective kG-module has injective dimension bounded by $\text{Gcd}_k G + \text{gl.dim}\, k$. It follows that for any kG-module M of finite projective dimension we have $\text{pd}_{kG} M \leq \text{Gcd}_k G + \text{gl.dim}\, k$. Since $B(G,k)$ is free as a kH-module for any finite subgroup $H \subseteq G$, the argument in the proof of [20, Theorem 6], which is based on [6, Theorem C], shows that $\text{pd}_{kG} B(G,k) < \infty$.

(ii) It follows from (i) that $\text{Gcd}_k G = \infty$ if and only if $\text{pd}_{kG} B(G,k) = \infty$. In the case where $\text{Gcd}_k G$ is finite, the equality to be proved follows from (i) and Proposition 1.2(i),(iii). □

Theorem A.2. *If G is an $\mathbf{LH}\mathfrak{F}$-group of type FP_∞, then $Gcd_{\mathbb{Z}} G = pd_{\mathbb{Z}G} B(G,k) < \infty$.*

Proof. This is an immediate consequence of Theorem A.1 above and [6, Corollary B.2(2)], which is valid for $\mathbf{LH}\mathfrak{F}$-groups (and not just for $\mathbf{H}\mathfrak{F}$-groups, as stated therein). □

References

[1] Aramayona, J., Martinez-Perez, C.: The proper geometric dimension of the mapping class group. Algebraic and Geometric Topology **14**, 217-227 (2014)

[2] Bahlekeh, A., Dembegioti, F., Talelli, O.: Gorenstein dimension and proper actions. Bull. London Math. Soc. **41**, 859-871 (2009)

[3] Bieri, R.: Homological dimension of discrete groups. Queen Mary College (1977)

[4] Brown, K.S.: Cohomology of groups. Grad. Texts Math. **87**, Berlin Heidelberg New York: Springer 1982

[5] Cornick, J., Kropholler, P.H.: On complete resolutions. Topology Appl. **78**, 235-250 (1997)

[6] Cornick, J., Kropholler, P.H.: Homological finiteness conditions for modules over group algebras. J. London Math. Soc. **58**, 49-62 (1998).

[7] Degrijse, D., Martinez-Perez, C.: Dimension invariants for groups admitting a cocompact model for proper actions. J. Reine Angew. Math. **721**, 233-249 (2016)

[8] Dranishnikov, A.: On the virtual cohomological dimension of Coxeter groups. Proc. Amer. Math. Soc. **125**, 1885-1891 (1997)

[9] Dranishnikov, A.: On dimension of product of groups. Alg. Discr. Math. **28**, 203-212 (2019)

[10] Emmanouil, I., Talelli, O.: Gorenstein dimension and group cohomology with group ring coefficients. J. London Math. Soc. **97**, 306-324 (2018)

[11] Fel'dman, G.I.: On the homological dimension of group algebras of soluble groups. Izv. Akad. Nauk SSR Ser. Mat. Tom **35** (1971), AMS Transl. **2**, 1231-1244 (1972)

[12] Gandini, G.: Cohomological invariants and the classifying space for proper actions. Groups Geom. Dyn. **6**, 659-675 (2012)

[13] Holm, H.: Gorenstein homological dimensions. J. Pure Appl. Algebra **189**, 167-193 (2004)

[14] Kropholler, P.H.: On groups of type FP_∞. J. Pure Appl. Algebra **90**, 55-67 (1993)

[15] Kropholler, P.H., Talelli, O.: On a property of fundamental groups of graphs of finite groups. J. Pure Appl. Algebra **73**, 57-59 (1991)

[16] Leary, I.J., Nucinkis, B.E.A.: Some groups of type VF. Invent. Math. **151**, 135-165 (2003)

[17] Lück, W,: Transformation groups and algebraic K-theory. Lecture Notes in Mathematics **1408**. Springer-Verlag, Berlin, 1989

[18] Nucinkis, B.E.A.: Cohomology relative to a G-set and finiteness conditions. Topology Appl. **92**, 153-171 (1999)

[19] St. John-Green, S.: On the Gorenstein and $\mathfrak{F}$-cohomological dimensions. Bull. London Math. Soc. **46**, 747-760 (2014)

[20] Talelli, O.: A characterization of cohomological dimension for a big class of groups. J. Algebra **326**, 238-244 (2011)

[21] Talelli, O.: On characteristic modules of groups. Geometric and Cohomological Group Theory, P.H. Kropholler et al. (Eds.), London Math. Soc. Lecture Note Ser. **444**, Cambridge Univ. Press (2017), 172-181

Department of Mathematics, University of Athens, Athens 15784, Greece
E-mail addresses: emmanoui@math.uoa.gr and otalelli@math.uoa.gr