Neutrino Physics: A Brief Introduction

D. Katsinis

University of Athens

November , 2009

Lecture 1

Introduction & Oscillations

Outline of Lecture 1

The Early Years

- 1930: Pauli proposed the existence of a neutral fermion that is emitted in the beta decay!
- 1933: Fermi concluded that neutrino may be massless!
- 1934: Fermi formulated a theory for beta decay!
- 1936: Gamow & Teller added axial vectors into the theory!
- 1937: Anderson discovered muon!
- 1947: Pontecorvo proposed the universality of Fermi interaction!
- 1953: Lepton number conservation! Later such a number was assigned on each generation!
- 1956: Reines & Cowan finally discovered neutrino!
- 1956: Lee & Yang solved $\theta - \tau$ puzzle! Parity Violation!

Maturity

- 1958: The $V - A$ theory was established! Indeed neutrino helicity was found negative!
- 1962: Lederman, Schwartz & Steinberg discovered ν_μ!
- 1964: CP violation was discovered on K^0!
- 1967: Glashow, Weinberg & Salam formulated Electroweak Unification!
- 1973: Kobayashi & Maskawa introduced mixing of three generations!
- 1974: GIM Mechanism!
- 1975: Perl discovered tau! 3rd generation of particles includes b quark (1977) & t quark (1995)!
- 1989: Light generations were fixed at 3!
- 2000: Finally ν_τ was observed!

History

- Pontecorvo proposed in 1957 neutrino oscillations!
- Nakagawa, Maki & Sakata formulated in 1967 neutrino mixing!
- Solar Neutrino Problem was discovered by Davies in 1968!
- Atmospheric neutrino oscillations were observed in 80's!
- SNP was solved in 2002 by SNO!

Where we stand? Part 1

What we know:

- We have a well established theory concerning neutrino interaction, oscillations and mixing!
- Neutrinos are massive for sure!
- Solar & atmospheric mixing angles have been measured with very good accuracy! There is bound for reactor mixing angle!
- Square mass splittings are known with great accuracy too!

Where we stand? Part 2

What we want to learn:

- We want to learn the sign of the atmospheric square mass splitting!
- We definitely want to measure the reactor mixing angle!
- We have absolutely no idea about the value(s) of the mixing phase(s)!
- The absolute scale of neutrino mass must be determined!
- The nature of neutrino should be determined too!
- We want to find out what is going on with LSND and MiniBooNe!

Neutrinos Probe New Physics

- Sterile neutrinos are candidates for Hot Dark Matter!
- Relic neutrinos will give information for early universe!
- Neutrino mass may be understood via See-Saw mechanism!
- Extra dimensions may explain the tiny neutrino mass too!
- Some GUTs are connected with neutrino physics!
- Majorana neutrinos break the L!

How does the story begin?

As a result of mixing we have the decay:

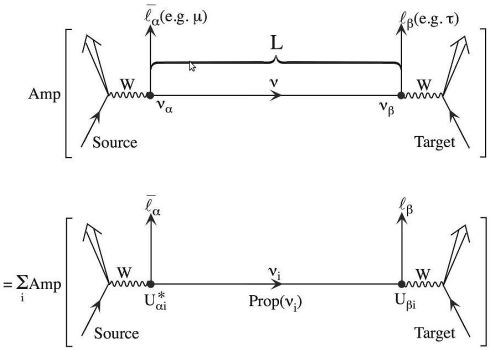
$$W^+ \to \bar{\ell}_\alpha + \nu_i : \tag{1}$$

- ℓ_α is a charged lepton of flavor α.
- ν_i is a neutrino which is mass eigenstate.

We denote the amplitude as $U^*_{\alpha i}$.
Physical particles are mass eigenstates!
Flavor eigenstates are in general superposition of mass eigenstates:

$$|\nu_\alpha >= \sum_i U^*_{\alpha i} |\nu_i > . \tag{2}$$

What does neutrino oscillation mean?



Putting everything together

The total amplitude is:

$$Amp(\nu_\alpha \to \nu_\beta) = \sum_i U^*_{\alpha i} Prop(\nu_i) U_{\beta i}. \tag{3}$$

We must calculate the propagator.
In the rest frame, the Schrödinger equation is:

$$\imath \frac{\partial}{\partial \tau_i} |\nu_i(\tau_i) >= m_i |\nu_i(\tau_i) > . \tag{4}$$

The solution is trivial:

$$|\nu_i(\tau_i) >= e^{-\imath m_i \tau_i} |\nu_i(0) > . \tag{5}$$

Calculating the Phase

The product $m_i\tau_i$ is Lorentz invariant, thus:

$$m_i\tau_i = E_i t - p_i L. \tag{6}$$

We assume that neutrinos are ultrarelativistic and have same energy E, then:

$$p_i = \sqrt{E^2 - m_i^2} \approx E - \frac{m_i^2}{2E}. \tag{7}$$

This leads to:

$$m_i\tau_i \approx E(t - L) + \frac{m_i^2}{2E}L. \tag{8}$$

Obtaining the Probability

The phase $E(t-L)$ will be canceled when we calculate the probability, therefore we omit that term. As a result:

$$Prop(\nu_i) = e^{-\imath m_i^2 \frac{L}{2E}}. \tag{9}$$

The amplitude becomes:

$$Amp(\nu_\alpha \to \nu_\beta) = \sum_i U_{\alpha i}^* e^{-\imath m_i^2 \frac{L}{2E}} U_{\beta i}. \tag{10}$$

Finally, the probability is:

$$P(\nu_\alpha \to \nu_\beta) = \sum_{i,j} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* exp\left(-\imath \Delta m_{ij}^2 \frac{L}{2E}\right). \tag{11}$$

$$\Delta m_{ij}^2 = m_i^2 - m_j^2. \tag{12}$$

Oscillation Probability

After some math we get:

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - 4 \sum_{i>j} \Re(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}) \sin^2\left(\Delta m^2_{ij} \frac{L}{4E}\right)$$
$$+2 \sum_{i>j} \Im(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}) \sin\left(\Delta m^2_{ij} \frac{L}{2E}\right). \quad (13)$$

- The case $\alpha = \beta$ corresponds to Survival Probability.
- The case $\alpha \neq \beta$ corresponds to Transition Probability.

Antineutrino Case

The process:

$$\bar{\nu}_\alpha \longrightarrow \bar{\nu}_\beta \tag{14}$$

is the CPT image of the process:

$$\nu_\beta \longrightarrow \nu_\alpha. \tag{15}$$

We observe that:

$$P(\nu_\beta \rightarrow \nu_\alpha, U) = P(\nu_\alpha \rightarrow \nu_\beta, U^*). \tag{16}$$

Therefore in the antineutrino case we have:

$$P(\nu_\alpha \rightarrow \nu_\beta) = \delta_{\alpha\beta} - 4\sum_{i>j} \Re(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}) \sin^2\left(\Delta m^2_{ij} \frac{L}{4E}\right) \\ -2\sum_{i>j} \Im(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}) \sin\left(\Delta m^2_{ij} \frac{L}{2E}\right). \tag{17}$$

What about CP Violation?

In general:

$$\Im(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}) \neq 0, \qquad (18)$$

which means:

$$P(\nu_\alpha \to \nu_\beta) \neq P(\bar{\nu}_\alpha \to \bar{\nu}_\beta). \qquad (19)$$

Possible CP violation!!!

Does this approach have results? Part 1

Even with this elementary approach we get valuable conclusions!

- Massless neutrinos leads to $P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta}$.
- Does flavor change have any relation to matter effect? NO!
- The probability depends on the ratio L/E, which is nothing but the eigentime elapsed in the rest frame of the neutrino!
- Assume no mixing. Then $U^*_{\alpha j} = 0$ for $i \neq j$, as a result $P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta}$.
- Oscillations depends on square mass splitting.
- Oscillations depend on the quadratic product of U's. This term is phase invariant, thus oscillations are insensitive to the nature of neutrinos and CP violation depends only on Dirac phase.

Does this approach have results? Part 2

- The total flux is invariant!

$$\sum_{\beta} P(\nu_\alpha \to \nu_\beta) = 1. \tag{20}$$

- Only sterile neutrinos change the flux!
- Putting the constants back we get:

$$\Delta m_{ij}^2 \frac{L}{4E} = 1.27 \Delta m_{ij}^2 (eV^2) \frac{L[km]}{E[GeV]}. \tag{21}$$

- The oscillation length is $L^{osc} = 2.47 \frac{E\ [GeV]}{\Delta m^2\ [eV^2]}\ km$.
  If $L \gg L^{osc}$ the probability oscillates rapidly and we get:

$$\langle P_{\nu_\alpha \to \nu_\beta} \rangle = \sum_i |U_{\alpha i}|^2 |U_{\beta i}|^2. \tag{22}$$

- In the case of incoherent sum we get the same result.

Assumptions

We have made two assumptions:

- We expressed flavor eigenstates as superposition of mass eigenstates.
- We assumed that neutrinos have common energy and time.

Symmetries

- We cant apply parity transformations on neutrino!
- In general we can apply CP transformation, that transforms the neutrino to antineutrino, no matter what antineutrino does mean:

$$\nu_\alpha \to \nu_\beta \overset{CP}{\Longleftrightarrow} \overline{\nu}_\alpha \to \overline{\nu}_\beta . \tag{23}$$

- Time reversal simply exchanges initial and final states:

$$\nu_\alpha \to \nu_\beta \overset{T}{\Leftrightarrow} \nu_\beta \to \nu_\alpha . \tag{24}$$

- CPT combines both transformations:

$$\nu_\alpha \to \nu_\beta \overset{CPT}{\Longleftrightarrow} \overline{\nu}_\beta \to \overline{\nu}_\alpha . \tag{25}$$

2 Neutrino Case Part 1

It is interesting to study the 2 neutrino case for 2 reasons:

1. The relations are much more simpler!
2. Its a very good approach!

The mixing matrix is:

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{26}$$

The Transition Probability is:

$$P_{\nu_\alpha \to \nu_\beta}(L, E) = \sin^2 2\theta \sin^2 \left(\frac{\Delta m^2 L}{4E} \right), \quad (\alpha \neq \beta). \tag{27}$$

The Survival Probability is:

$$P_{\nu_\alpha \to \nu_\alpha}(L, E) = 1 - \sin^2 2\theta \sin^2 \left(\frac{\Delta m^2 L}{4E} \right). \tag{28}$$

The square mass splitting $\Delta m^2 = m_2^2 - m_1^2$ is defined positive.

2 Neutrino Case Part 2

The average Transition Probability is:

$$\langle P(\nu_\alpha \to \nu_\beta) \rangle = \frac{1}{2}\sin^2 2\theta. \tag{29}$$

There is a symmetry:

$$\theta \Longleftrightarrow \frac{\pi}{2} - \theta. \tag{30}$$

As the mixing matrix is real we have:

$$P(\nu_\alpha \to \nu_\beta) = P(\nu_\beta \to \nu_\alpha) = P(\bar{\nu}_\alpha \to \bar{\nu}_\beta) = P(\bar{\nu}_\beta \to \bar{\nu}_\alpha) \tag{31}$$

What about Matter Effect?

The neutrino interacts is two ways with matter:

- If the neutrino is ν_e it exchanges W bosons with electrons.
- All neutrinos exchange Z bosons with protons, electrons & neutrons.

The interactions to first order give raise to the potentials:

$$V_W = +\sqrt{2} G_F N_e \qquad V_Z = -\frac{\sqrt{2}}{2} G_F N_{(n,p,e)} \tag{32}$$

If we change neutrinos to antineutrinos the potentials change sign.
At zero momentum transfer contributions of electrons and protons to V_Z cancel.

Matter Effect Calculation Part 1

The Schödinger equation is:

$$\imath \frac{\partial}{\partial t} |\nu(t)\rangle = \mathcal{H} |\nu(t)\rangle \quad |\nu(t)\rangle = \begin{pmatrix} f_e(t) \\ f_\mu(t) \end{pmatrix}. \tag{33}$$

The Hamiltonian is a 2×2 matrix:

$$\langle \nu_\alpha | \mathcal{H}_{vac} | \nu_\beta \rangle = \sum_i U_{\alpha i} U^*_{\beta i} \sqrt{p^2 + m_i^2}. \tag{34}$$

We use the ultrarelativistic approximation:

$$\sqrt{p^2 + m_i^2} \approx p + \frac{m_i^2}{2p} \approx E + \frac{m_i^2}{2E}. \tag{35}$$

Matter Effect Calculation Part 2

The nontrivial part of the Hamiltonian is:

$$\mathcal{H}_{vac} = \frac{\Delta m^2}{4E} \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}. \tag{36}$$

In the presence of matter the Hamiltonian becomes:

$$\mathcal{H}_M = \frac{\Delta m^2}{4E} \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} + V_W \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + V_Z \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{37}$$

Matter Effect Calculation Part 3

Terms proportional to the identity matrix are irrelevant, thus we get:

$$\mathcal{H}_M = \frac{\Delta m^2}{4E} \begin{pmatrix} -(\cos 2\theta - x) & \sin 2\theta \\ \sin 2\theta & (\cos 2\theta - x) \end{pmatrix}, \tag{38}$$

where

$$x = \frac{V_W/2}{\Delta m^2/4E} = \frac{2\sqrt{2} G_F N_e E}{\Delta m^2}. \tag{39}$$

We define:

$$\Delta m_M^2 = \Delta m^2 \sqrt{\sin^2 2\theta + (\cos 2\theta - x)^2} \tag{40}$$

$$\sin^2 2\theta_M = \frac{\sin^2 2\theta}{\sin^2 2\theta + (\cos 2\theta - x)^2} \tag{41}$$

Matter Effect Calculation Part 4

Then the Hamiltonian is nothing else but the vacuum Hamiltonian with matter parameters:

$$\mathcal{H}_M = \frac{\Delta m_M^2}{4E} \begin{pmatrix} -\cos 2\theta_M & \sin 2\theta_M \\ \sin 2\theta_M & \cos 2\theta_M \end{pmatrix} . \tag{42}$$

If electron density is constant then

$$P_M(\nu_e \to \nu_\mu) = \sin^2 2\theta_M \sin^2 \left(\Delta m_M^2 \frac{L}{4E} \right) \tag{43}$$

How strong is matter effect?

$$\Delta m^2 \approx 2.4 \times 10^{-3}\ eV^2 \quad \Rightarrow \quad |x| \simeq \frac{E}{12\ GeV} . \tag{44}$$

Neutrino Antineutrino Asymmetry

The sign of x is

	ν	$\overline{\nu}$
$m_{\nu_2} > m_{\nu_1}$	+	-
$m_{\nu_2} < m_{\nu_1}$	-	+

The asymmetry has nothing to do with genuine CP violation!

MSW Effect

Matter effect can cause a very interesting phenomenon in the case of resonance! We have seen that:

$$\sin^2 2\theta_M = \frac{\sin^2 2\theta}{\sin^2 2\theta + (\cos 2\theta - x)^2} \tag{45}$$

If

$$x \approx \cos 2\theta \tag{46}$$

then even tiny $\sin^2 2\theta$ correspond to large $\sin^2 2\theta_M$.
If electron density variates smoothly and neutrino propagation can be considered adiabatic and we have the MSW Effect.

Solution to SNP

The propagation is adiabatic therefore we can solve Schrödinger equation at every distance r and then combine the solutions.

$$\mathcal{H}_M^{(r)} = \frac{\Delta m^2}{4E} \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} + \sqrt{2} G_F N_e \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \quad (47)$$

At $r = 0$ the vacuum part must be neglected. The neutrino is born being the high energy eigenstate of the Hamiltonian and propagates being at that state. As the neutrino emerges the sun it is just the high energy state of the vacuum Hamiltonian! In other words we have the transition

$$\nu_e \longrightarrow \nu_2. \quad (48)$$

SNP simply means:

$$|U_{e2}|^2 = 1/3 \quad (49)$$

What about Wave Packets?

It a QFT framework we must use wave packets. In fact we don't really need them!
The wave packet treatment gives:

$$P_{\nu_\alpha \to \nu_\beta}(\vec{L}) = \sum_{k,j} U^*_{\alpha k} U_{\alpha j} U_{\beta k} U^*_{\beta j} exp\left[-2\pi\imath \frac{L}{L^{osc}_{kj}} - \left(\frac{L}{L^{coh}_{kj}} \right)^2 \right. \\ \left. -2\pi^2 \left(1 - \frac{\vec{L} \cdot \vec{\xi}}{L} \right)^2 \left(\frac{\sigma_x}{L^{osc}_{kj}} \right)^2 \right]. \tag{50}$$

Where

$$\vec{\tilde{p}}_k \simeq \vec{p} - \vec{\xi} \frac{m_k^2}{2E} \quad \sigma_x^2 \sim \left(\sigma_x^P \right)^2 + \left(\sigma_x^D \right)^2 \quad L^{osc}_{kj} = \frac{4\sqrt{2}E^2}{|\Delta m^2_{kj}|} \sigma_x \tag{51}$$

Identifying the Terms

The exponent has 3 terms:

- The first term is the standard oscillation phase.

$$-2\pi\imath\frac{L}{L_{kj}^{osc}} \tag{52}$$

- The second term is the coherence term.

$$-\left(\frac{L}{L_{kj}^{coh}}\right)^2 \tag{53}$$

- The third term is the localization term.

$$-2\pi^2\left(1-\frac{\vec{L}\cdot\vec{\xi}}{L}\right)^2\left(\frac{\sigma_x}{L_{kj}^{osc}}\right)^2 \tag{54}$$

The Localization Term

This terms reflects the decoherance suppressing the oscillations if $\sigma_x \gg L^{osc}_{kj}$.
This way we separate neutrino oscillation experiments and neutrino mass measurement experiments! The mass measurement accuracy is:

$$\delta m_k^2 = \sqrt{\left(2\tilde{E}_k \delta \tilde{E}_k\right)^2 + \left(2|\vec{\tilde{p}}_k|\delta|\vec{\tilde{p}}_k|\right)^2} \simeq 2\sqrt{2}E\sigma_p, \tag{55}$$

If $\delta m_k^2 < \Delta m_{kj}^2$ there is not enough energy to produce ν_j, therefore oscillations are suppressed:

$$-2\pi^2 \left(\frac{\sigma_x}{L^{osc}_{kj}}\right)^2 \simeq -\frac{1}{4}\left(\frac{\Delta m_{kj}^2}{\delta m_k^2}\right)^2 . \tag{56}$$

In oscillation experiments we can neglect the localization term.

Comparison to our simple Approach

Neglecting the localization term the probability is given by:

$$P_{\nu_\alpha \to \nu_\beta}(\vec{L}) = \sum_{k,j} U^*_{\alpha k} U_{\alpha j} U_{\beta k} U^*_{\beta j} exp\left[-2\pi\imath \frac{L}{L^{osc}_{kj}} - \left(\frac{L}{L^{coh}_{kj}}\right)^2\right] \quad (57)$$

If we overage the probability found in the simple approach over a Gaussian distribution E/L we get:

$$\langle P(\nu_\alpha \to \nu_\beta)\rangle = \sum_{ij} U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} exp\left[-\imath \frac{\Delta m^2_{ij}}{2}\left\langle \frac{L}{E}\right\rangle - \frac{1}{2}\left(\frac{\Delta m^2_{ij}}{2}\sigma_{L/E}\right)^2\right] \quad (58)$$

If we add the quantum space and momentum uncertainties to the classical L/E uncertainty everything is fine!

Summary

Lecture 2

Mixing & Mass

Outline of Lecture 2

$$
\begin{aligned}
\mathcal{L}_{EW} = & \, \imath \sum_{\alpha=e,\mu,\tau} \overline{L}'_{\alpha L} \not{D} L'_{\alpha L} + \imath \sum_{\alpha=1,2,3} \overline{Q}'_{\alpha L} \not{D} Q'_{\alpha L} \\
& + \imath \sum_{\alpha=e,\mu,\tau} \overline{\ell}'_{\alpha R} \not{D} \ell'_{\alpha R} + \imath \sum_{\alpha=d,s,b} \overline{q}'^{D}_{\alpha R} \not{D} q'^{D}_{\alpha R} + \imath \sum_{\alpha=u,c,t} \overline{q}'^{U}_{\alpha R} \not{D} q'^{U}_{\alpha R} \\
& - \frac{1}{4} A_{\mu\nu} A^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \\
& + (D_\mu \Phi)^\dagger (D^\mu \Phi) - \mu^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^2 \\
& - \sum_{\alpha,\beta=e,\mu,\tau} \left(Y'^{\ell}_{\alpha\beta} \overline{L}'_{\alpha L} \Phi \ell'_{\beta R} + \left(Y'^{\ell}_{\alpha\beta} \right)^* \overline{\ell}'_{\beta R} \Phi^\dagger L'_{\alpha L} \right) \\
& - \sum_{\alpha=1,2,3} \sum_{\beta=d,s,b} \left(Y'^{D}_{\alpha\beta} \overline{Q}'_{\alpha L} \Phi q'^{D}_{\beta R} + \left(Y'^{D}_{\alpha\beta} \right)^* \overline{q}'^{D}_{\beta R} \Phi^\dagger Q'_{\alpha L} \right) \\
& - \sum_{\alpha=1,2,3} \sum_{\beta=u,c,t} \left(Y'^{U}_{\alpha\beta} \overline{Q}'_{\alpha L} \tilde{\Phi} q'^{U}_{\beta R} + \left(Y'^{U}_{\alpha\beta} \right)^* \overline{q}'^{U}_{\beta R} \tilde{\Phi}^\dagger Q'_{\alpha L} \right) .
\end{aligned}
\tag{59}
$$

Leptonic Mass Terms

The coupling of leptons to the Higgs field is:

$$\mathcal{L}_{H,L} = -\frac{v+H}{\sqrt{2}} \sum_{\alpha,\beta=e,\mu,\tau} Y'^{\ell}_{\alpha\beta} \overline{\ell}'_{\alpha L} \ell'_{\beta R} + HC. \tag{60}$$

The matrix containing the Yukawa cuplings is not diagonal. The primed fields don't have definite mass! We diagonalize the matrix with a biunitary transformation:

$$\left(V^{\ell}_{L}\right)^{\dagger} Y'^{\ell} V^{\ell}_{R} = Y^{\ell}, \quad Y^{\ell}_{\alpha\beta} = y^{\ell}_{\alpha} \delta_{\alpha\beta} \quad y^{\ell}_{\alpha} > 0, \tag{61}$$

$$\mathcal{L}_{H,L} = -\frac{v+H}{\sqrt{2}} \sum_{\alpha=e,\mu,\tau} y^{\ell}_{\alpha} \overline{\ell}_{\alpha L} \ell_{\alpha R} + HC. \tag{62}$$

Charged Weak Current

Now the leptons have definite mass:

$$\ell_L = \left(V_L^\ell\right)^\dagger \ell'_L = \begin{pmatrix} e_L \\ \mu_L \\ \tau_L \end{pmatrix}. \tag{63}$$

The charged weak current becomes:

$$j^k_{W,L} = 2\overline{\nu}'_L\gamma^k\ell'_L = 2\overline{\nu}'_L\gamma^k V_L^\ell \ell_L = 2\overline{\nu}_L\gamma^k\ell_L, \tag{64}$$

$$\nu_L = \left(V_L^\ell\right)^\dagger \nu'_L = \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix}. \tag{65}$$

These neutrinos have definite flavor & mass.

Symmetries

The Lagrangian is invariant under global phase transformations:

$$\nu_{\alpha L} \to e^{\imath\phi_\alpha}\nu_{\alpha L}, \quad \ell_{\alpha L} \to e^{\imath\phi_\alpha}\ell_{\alpha L}, \quad \ell_{\alpha R} \to e^{\imath\phi_\alpha}\ell_{\alpha R}, \tag{66}$$

$$\nu_{\alpha L} \to e^{\imath\phi}\nu_{\alpha L}, \quad \ell_{\alpha L} \to e^{\imath\phi}\ell_{\alpha L}, \quad \ell_{\alpha R} \to e^{\imath\phi}\ell_{\alpha R}. \tag{67}$$

The Noether currents are:

$$j_\alpha^k = \overline{\nu}_{\alpha L}\gamma^k\nu_{\alpha L} + \overline{\ell}_{\alpha L}\gamma^k\ell_{\alpha L} \tag{68}$$

and the corresponding conserved charges are:

$$L_e, \quad L_\mu, \quad L_\tau, \quad L. \tag{69}$$

Neutral Weak & EM Currents

The weak neutral current is invariant:

$$\begin{aligned} j^k_{Z,L} &= 2g^\nu_L \overline{\nu}'_L \gamma^k \nu'_L + 2g^\ell_L \overline{\ell}'_L \gamma^k \ell'_L + 2g^\ell_R \overline{\ell}'_R \gamma^k \ell'_R \\ &= 2g^\nu_L \overline{\nu}_L \gamma^k \nu_L + 2g^\ell_L \overline{\ell}_L \gamma^k \ell_L + 2g^\ell_R \overline{\ell}_R \gamma^k \ell_R \end{aligned} \tag{70}$$

EM current is invariant too:

$$j^k_{\gamma,L} = -\overline{\ell}'\gamma^k \ell' = -\overline{\ell}\gamma^k \ell. \tag{71}$$

Quark Mass Terms

The coupling of quarks to the Higgs field is:

$$\mathcal{L}_{H,Q} = -\frac{v+H}{\sqrt{2}}\left(\sum_{\alpha,\beta=d,s,b} Y'^{D}_{\alpha\beta}\overline{q}'^{D}_{\alpha L}q'^{D}_{\beta R} + \sum_{\alpha,\beta=u,c,t} Y'^{U}_{\alpha\beta}\overline{q}'^{U}_{\alpha L}q'^{U}_{\beta R}\right)+HC. \tag{72}$$

Yet again we diagonalize the matrices

$$\left(V^{D}_{L}\right)^{\dagger} Y'^{D} V^{D}_{R} = Y^{\ell}, \quad Y^{D}_{\alpha\beta} = y^{D}_{\alpha}\delta_{\alpha\beta},\ y^{D}_{\alpha} > 0, \tag{73}$$

$$\left(V^{U}_{L}\right)^{\dagger} Y'^{U} V^{U}_{R} = Y^{\ell}, \quad Y^{U}_{\alpha\beta} = y^{U}_{\alpha}\delta_{\alpha\beta},\ y^{U}_{\alpha} > 0, \tag{74}$$

Quarks

The quarks having definite mass are defined as:

$$q_L^D = \left(V_L^D\right)^\dagger q_L'^D = \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix}, \quad q_R^D = \left(V_R^D\right)^\dagger q_R'^D = \begin{pmatrix} d_R \\ s_R \\ b_R \end{pmatrix}, \tag{75}$$

$$q_L^U = \left(V_L^U\right)^\dagger q_L'^U = \begin{pmatrix} u_L \\ c_L \\ t_L \end{pmatrix}, \quad q_R^U = \left(V_R^U\right)^\dagger q_R'^U = \begin{pmatrix} u_R \\ c_R \\ t_R \end{pmatrix} \tag{76}$$

Charged & Neutral Weak Current

The charged weak current is not invariant:

$$j^k_{W,Q} = 2\overline{q}'^U_L \gamma^k q'^D_L = 2\overline{q}^U_L \gamma^k \left(V^U_L\right)^\dagger V^D_L q^D_L = 2\overline{q}^U_L \gamma^k V_{CKM} q^D_L \tag{77}$$

$$V_{CKM} = \left(V^U_L\right)^\dagger V^D_L. \tag{78}$$

Flavors are not conserved, only baryonic number is!
The weak current is invariant due to GIM mechanism:

$$\begin{aligned} j^k_{Z,Q} &= 2g^D_L \overline{q}'^D_L \gamma^k q'^D_L + 2g^D_R \overline{q}'^D_R \gamma^k q'^D_R + 2g^U_L \overline{q}'^U_L \gamma^k q'^U_L + 2g^U_R \overline{q}'^U_R \gamma^k q'^U_R \\ &= 2g^D_L \overline{q}^D_L \gamma^k q^D_L + 2g^D_R \overline{q}^D_R \gamma^k q^D_R + 2g^U_L \overline{q}^U_L \gamma^k q^U_L + 2g^U_R \overline{q}^U_R \gamma^k q^U_R \end{aligned} \tag{79}$$

EM current is invariant too:

$$j^k_{\gamma,Q} = \frac{2}{3}\overline{q}^U \gamma^k q^U - \frac{1}{3}\overline{q}^D \gamma^k q^D. \tag{80}$$

CKM Mixing Matrix

The components of CKM matrix are:

$$V = \left(V_L^U\right)^\dagger V_L^D = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \quad (81)$$

At 90 % C.L. the norms of the elements of the CKM matrix are:

$$|V| = \begin{pmatrix} 0.9739 - 0.9751 & 0.221 - 0.227 & 0.0029 - 0.0045 \\ 0.221 - 0.227 & 0.9730 - 0.9744 & 0.039 - 0.044 \\ 0.0048 - 0.014 & 0.037 - 0.043 & 0.9990 - 0.9992 \end{pmatrix} \quad (82)$$

Mixing Parameters

Every $N \times N$ complex matrix has N^2 mixing parameters:

$$\frac{N(N-1)}{2}, \qquad \text{mixing angles}, \tag{83}$$

$$\frac{N(N+1)}{2}, \qquad \text{phases}. \tag{84}$$

Some of the phases can be eliminated.
Physical phases are only the phases that affect the weak currents!

Rephasing

Apart from the coupling to W & Z bosons $\mathcal{L}_{EW}$ is invariant to rephasing the quarks:

$$q_\alpha^U \to e^{\imath\psi_\alpha^U} q_\alpha^U, \quad q_k^D \to e^{\imath\psi_k^D} q_k^D. \tag{85}$$

$$\begin{aligned} j_{W,Q}^k &= 2 \sum_{\alpha=u,c,t} \sum_{k=d,s,b} \overline{q}_{\alpha L}^U e^{-\imath\psi_\alpha^U} \gamma^k V_{\alpha k} e^{\imath\psi_k^D} q_{kL}^D \\ &= 2e^{-\imath\left(\psi_c^U-\psi_s^D\right)} \sum_{\alpha=u,c,t} \sum_{k=d,s,b} \overline{q}_{\alpha L}^U e^{-\imath\left(\psi_\alpha^U-\psi_c^U\right)} \gamma^k V_{\alpha k} e^{\imath\left(\psi_k^D-\psi_s^D\right)} q_{kL}^D \end{aligned} \tag{86}$$

This rephasing can eliminate in total the following phases:

$$\begin{aligned} e^{-\imath\left(\psi_c^U-\psi_s^D\right)} &\to 1 \\ e^{-\imath\left(\psi_\alpha^U-\psi_c^U\right)} &\to N-1 \\ e^{\imath\left(\psi_k^D-\psi_s^D\right)} &\to N-1. \end{aligned} \tag{87}$$

Physical Parameters

The remaining phases are the physical ones:

$$\frac{N(N+1)}{2} - (2N-1) = \frac{(N-1)(N-2)}{2}. \tag{88}$$

The total physical parameters are:

$$\frac{N(N-1)}{2} + \frac{(N-1)(N-2)}{2} = (N-1)^2 \tag{89}$$

Parametrization of the Mixing Matrix Part 1

We define the matrices:

$$\left[W^{\alpha\beta}(\theta_{\alpha\beta}, n_{\alpha\beta})\right]_{rs} = \delta_{rs} + (\cos\theta_{\alpha b} - 1)(\delta_{r\alpha}\delta_{s\alpha} + \delta_{r\beta}\delta_{s\beta}) + \sin\theta_{\alpha\beta}\left(e^{\imath n_{\alpha\beta}}\delta_{r\alpha}\delta_{s\beta} - e^{-\imath n_{\alpha\beta}}\delta_{r\beta}\delta_{s\alpha}\right). \quad (90)$$

This matrices are complex rotations in the $\alpha - \beta$ plane. For example for $N = 3$ we have:

$$W^{12}(\theta_{12}, n_{12}) = \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} e^{\imath n_{12}} & 0 \\ -\sin\theta_{12} e^{-\imath n_{12}} & \cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (91)$$

Parametrization of the Mixing Matrix Part 2

We define the diagonal unitary matrices:

$$D(\omega) = diag(e^{\imath\omega_1}, \cdots, e^{\imath\omega_N}) \tag{92}$$

Any mixing matrix can be parametrized as:

$$V = D(\omega) \prod_{\alpha<\beta} W^{\alpha\beta}(\theta_{\alpha\beta}, n_{\alpha\beta}). \tag{93}$$

The parameters allowed values are:

$$0 \leq \theta_{\alpha\beta} \leq \pi \quad 0 \leq \omega_k < 2\pi \quad 0 \leq n_{\alpha\beta} < 2\pi. \tag{94}$$

Elimination of Unphysical Phases

One can see that:

$$D(\phi)W^{\alpha\beta}(\theta_{\alpha\beta}, n_{\alpha\beta})D^{\dagger}(\phi) = W^{\alpha\beta}(\theta_{\alpha\beta}, n_{\alpha\beta} + \phi_{\alpha} - \phi_{\beta}) \qquad (95)$$

Expressing the mixing matrix as:

$$V = D(\omega - \phi)\left[\prod_{\alpha<\beta} D(\phi)W^{\alpha\beta}(\theta_{\alpha\beta}, n_{\alpha\beta})D^{\dagger}(\phi)\right] D(\phi) \qquad (96)$$

We can eliminate $N - 1$ of the $n_{\alpha\beta}$ phases:

$$V = D(\omega - \phi)\left[\prod_{\alpha<\beta} W^{\alpha\beta}(\theta_{\alpha\beta}, n_{\alpha\beta} + \phi_{\alpha} - \phi_{\beta})\right] D(\phi). \qquad (97)$$

The Parametrization of PDG

In the case N=3 we use:

$$\phi = (\phi_2 - n_{12}, \phi_2, \phi_2 + n_{23}) \tag{98}$$

The mixing matrix becomes:

$$V = R^{23} W^{13} R^{12}. \tag{99}$$

We define $\delta = -n_{13}$ and we get the mixing matrix:

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-\imath\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{\imath\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{\imath\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{\imath\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{\imath\delta} & c_{23}c_{13} \end{pmatrix}. \tag{100}$$

Degenerate Masses

In the case of degenerate masses the mixing matrix can be simplified further. Consider two down-type, for example d & s quarks having degenerate masses. Then rotations in d-s plane have no physical effects:

$$q^D \to U^{12} q^D \quad U^{12} = \begin{pmatrix} U^{12}_{11} & U^{12}_{12} & 0 \\ U^{12}_{21} & U^{12}_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (101)$$

The mixing matrix can be expressed as

$$V = D^L R^{23} R^{13} W^{12} D^R. \qquad (102)$$

The diagonal matrices can be eliminated an once, but by choosing $U^{12} = \left(W^{12}\right)^{\dagger}$ we can eliminate W^{12} too. The mixing matrix becomes rotations on the s-b & d-b planes:

$$V = R^{23} R^{13} \qquad (103)$$

Maximal & Minimal Parameters

If an angle has its minimal or its maximal value then at least one of the elements of the mixing matrix vanishes. Using the unitarity relations:

$$V^{\dagger}V = 1 \qquad VV^{\dagger} = 1. \tag{104}$$

Starting from the PDG parametrization we can rearrange the mixing matrix columns and lines and move $sin\theta_{13}$ to the vanishing element. This way the maxing matrix becomes real. Obviously in the cases

$$\delta = 0, \pi \tag{105}$$

the mixing matrix is real.

The Jarlskog Invariant

In order to measure the CP Violation we use the Jarlskog invariant:

$$J = \Im\left[V_{us} V_{cb} V_{ub} V_{cs}\right] \tag{106}$$

In the PDG parametrization:

$$\begin{aligned} J &= c_{12} s_{12} c_{23} s_{23} c_{13}^2 s_{13} \sin\delta \\ &= \frac{1}{8} \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \cos\theta_{13} \sin\delta. \end{aligned} \tag{107}$$

Maximal CP Violation

Maximal CP Violation corresponds to

$$|J_{max}| = \frac{1}{6\sqrt{3}}, \tag{108}$$

which means $\theta_{12} = \theta_{23} = \pi/4$, $s_{13} = 1/\sqrt{3}$ & $\sin\delta = \pm 1$. The norm of each element of the mixing matrix is $1/\sqrt{3}$.

$$V = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & \mp\imath \\ -e^{\pm\imath\pi/6} & e^{\mp\imath\pi/6} & 1 \\ e^{\mp\imath\pi/6} & e^{\pm\imath\pi/6} & 1 \end{pmatrix} \tag{109}$$

Expressing the Mixing Matrix in terms of the Jarlskog Invariant

The mixing matrix can be determined almost uniquely by

$$|V_{us}|, \quad |V_{ub}|, \quad |V_{cb}|, \quad J. \tag{110}$$

using the unitarity relations and the followings:

$$\tan\theta_{12} = \frac{|V_{us}|}{|V_{ud}|}, \quad \tan\theta_{23} = \frac{|V_{cb}|}{|V_{tb}|}, \quad \sin\theta_{13} = |V_{ub}|, \tag{111}$$

$$\sin\delta_{13} = \frac{8J}{\sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \cos\theta_{13}}. \tag{112}$$

The mixing phase is not unique as $\sin\delta = \sin(\pi - \delta)$.

Conditions for CP Violation

If the mixing matrix is real then CP must be conserved!
This means that we have CP Violation if:

- No two up-type or down-type quarks have degenerate masses. (6)
- No mixing angle is maximal or minimal. (6)
- The phase is not maximal or minimal. (2)

In total we have 14 conditions! This can be summarized as

$$detC \neq 0. \tag{113}$$

C is the commutator:

$$C = \imath \left[\left(M^U \right)^2, V \left(M^D \right)^2 V^\dagger \right] \tag{114}$$

Thus the conditions are:

$$detC = 2J(m_c^2 - m_u^2)(m_t^2 - m_u^2)(m_t^2 - m_c^2)(m_s^2 - m_d^2)(m_b^2 - m_d^2)(m_b^2 - m_s^2). \tag{115}$$

Righthanded Neutrinos

A Dirac mass term for neutrinos arises through the Higgs mechanism just like all other fermions. We can introduce a righthanded neutrino for each generation:

$$\nu_{\alpha R}. \tag{116}$$

This is a gauge singlet therefore we call righthanded neutrinos sterile.
The coupling of leptons to Higgs field is:

$$\mathcal{L}_{H,L} = -\left(\frac{v+H}{\sqrt{2}}\right)\left[\overline{\ell}'_L Y'^{\ell} \ell'_R + \overline{\nu}'_L Y'^{\nu} \nu'_R\right] + H.C. \tag{117}$$

Coupling to Higgs Field

We diagonalize the matrices containing the Yukawa couplings:

$$\begin{aligned} \left(V_L^\ell\right)^\dagger Y'^\ell V_R^\ell = Y^\ell, \quad Y_{ij}^\ell = y_i^\ell \delta_{ij}, \; y_i^\ell > 0 \\ (V_L^\nu)^\dagger Y'^\nu V_R^\nu = Y^\nu, \quad Y_{ij}^\nu = y_i^\nu \delta_{ij}, \; y_i^\nu > 0. \end{aligned} \tag{118}$$

The Lagrangian becomes:

$$\mathcal{L}_{H,L} = -\left(\frac{v+H}{\sqrt{2}}\right)\left[\overline{\ell}_L Y^\ell \ell_R + \overline{\nu}_L Y^\nu \nu_R\right] + H.C. \tag{119}$$

The neutrinos having definite mass are

$$\nu_L = (V_L^\nu)^\dagger \nu'_L = \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \\ \nu_{3L} \end{pmatrix} \qquad \nu_R = (V_R^\nu)^\dagger \nu'_R = \begin{pmatrix} \nu_{1R} \\ \nu_{2R} \\ \nu_{3R} \end{pmatrix}. \tag{120}$$

Having such a tiny neutrino mass is unphysical as it requires extremely small Yukawa couplings.

Charged Weak Current

The charged weak current becomes

$$j^{k}_{W,L} = 2\overline{\nu}'_L \gamma^k \ell'_L = 2\overline{\nu}_L \left(V^{\nu}_L\right)^{\dagger} \gamma^k V^{\ell}_L \ell_L = 2\overline{\nu}_L U^{\dagger}_{PMNS} \gamma^k \ell_L. \qquad (121)$$

We define the leptonic mixing matrix:

$$U_{PMNS} = \left(V^{\ell}_L\right)^{\dagger} V^{\nu}_L \qquad (122)$$

Comparison to SM Neutrinos

We can define the neutrinos having definite flavor as:

$$\nu_L^F = U\nu_L = \left(V_L^\ell\right)^\dagger \nu_L', \quad \nu_L^F = \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix}. \tag{123}$$

Then the current takes the form

$$j_{W,L}^k = 2\overline{\nu}_L^F \gamma^k \ell_L. \tag{124}$$

but this neutrinos are not independent:

$$\mathcal{L}_{H,L} = -\frac{v+H}{\sqrt{2}} \sum_{\alpha=e,\mu,\tau} \left[y_\alpha^\ell \overline{\ell}^{\alpha L} \ell_{\alpha R} + \overline{\nu}_{\alpha L}^F \sum_{k=1,2,3} U_{\alpha k} y_k^\nu \nu_{kR}^F \right] + H.C. \tag{125}$$

Symmetries

The Lagrangian is no more invariant to phase transformations per generation:

$$\nu_{\alpha L} \to e^{\imath\phi_\alpha}\nu_{\alpha L},, \quad \nu_{\alpha R} \to e^{\imath\phi_\alpha}\nu_{\alpha R}, \quad \ell_{\alpha L} \to e^{\imath\phi_\alpha}\ell_{\alpha L}, \quad \ell_{\alpha R} \to e^{\imath\phi_\alpha}\ell_{\alpha R} \tag{126}$$

Only common phase transformations for all leptons leave the Lagrangian invariant:

$$\nu_{kL} \to e^{\imath\phi}\nu_{kL},\ \nu_{kR} \to e^{\imath\phi}\nu_{kR},\ \ell_{\alpha L} \to e^{\imath\phi}\ell_{\alpha L},\ \ell_{\alpha R} \to e^{\imath\phi}\ell_{\alpha R}. \tag{127}$$

This correspond to the conservation of leptonic number.

Are Lefthanded Majorana Mass Terms for Neutrino allowed in SM?

In general Majorana lefthanded particles have mass terms in the form:

$$\bar{f}_L^C f_L. \qquad (128)$$

Its not so easy to include lefthanded Majorana mass term for neutrinos in the frame work of SM. The term:

$$\bar{\nu}_L^C \nu_L \qquad (129)$$

has $I_3 = 1$ & $Y = -2$, therefore the Higgs doublet cant couple to this term. There are two ways to deal with this problem:

- We can add a Higgs triplet.
- We can add a non-renormalizable term.

The non-renormalizable term is not as bad as it sound, as it can be understand in the framework of the low energy limit of new physics.

5-D Operator

The simplest case is a 5-D operator:

$$\mathcal{L}_5 = \frac{g}{M}\left(L_L^T \sigma_2 \Phi\right) C^\dagger \left(\Phi^T \sigma_2 L_L\right) + H.C., \tag{130}$$

After symmetry breaking the neutrinos acquire mass:

$$\mathcal{L}^M_{Mass} = \frac{1}{2}\frac{gv^2}{M}\overline{\nu}^C_L \nu_L + H.C. \tag{131}$$

As Dirac mass terms are proportional to Higgs Field VEV:

$$m \sim \frac{m_D^2}{M}. \tag{132}$$

The similarity with the See-Saw relation is not accidental.

Mixing Part 1

The mass term is:

$$\mathcal{L}^{M}_{Mass} = \frac{1}{2}\nu_L^{T\prime} C^{\dagger} M^{L} \nu_L^{\prime} + H.C. = \frac{1}{2}\sum_{\alpha\beta} \nu_{\alpha L}^{T\prime} C^{\dagger} M^{L}_{\alpha\beta} \nu^{\prime}_{\beta L} + H.C. \tag{133}$$

As the mass matrix is symmetric it can be diagonalized with the transformation:

$$\left(V_L^{\nu}\right)^T M^L V_L^{\nu} = M \quad M_{ij} = m_i \delta_{ij} \; m_i > 0. \tag{134}$$

The neutrinos of definite mass are defined as

$$\nu' = V_L^{\nu} \nu_L \quad \nu_L = \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \\ \nu_{2L} \end{pmatrix}. \tag{135}$$

Mixing Part 2

These neutrinos satisfy the Majorana condition:

$$\nu_k^C = \nu_k. \tag{136}$$

We can express the mass term as

$$\mathcal{L}^M = \frac{1}{2}\overline{\nu}\left(\imath\partial\!\!\!/ - M\right)\nu. \tag{137}$$

One can see that the mass term is no more invariant under transformations:

$$\nu_{kL} \rightarrow e^{\imath\phi}\nu_{kL}. \tag{138}$$

Thus arises lepton number violation.

Charged Weak Current

The charged weak current is transformed as in the Dirac case, but now we cant eliminate all that phases. The mixing matrix can be expressed as:

$$U = U^D U^M. \quad (139)$$

U^D is the mixing matrix of the Dirac case & U^M is a diagonal matrix containing Majorana phases:

$$D^M = diag(e^{\imath\alpha_1}, e^{\imath\alpha_2}, e^{\imath\alpha_3}). \quad (140)$$

Only 2 Majorana phases are physical. Usually we set $\alpha_1 = 0$. Neutrino oscillations cant give information for the Majorana phases.

Mixing in the Framework of the 5-D Operator

The 5-D operator is:

$$\mathcal{L}^{M}_{Mass} = \frac{1}{2}\frac{v^2}{M}\sum_{\alpha\beta} g_{\alpha\beta}\nu'^{T}_{\alpha L} C^{\dagger}\nu'_{\beta L} + H.C. \tag{141}$$

The mass matrix is

$$M^{L} = \frac{v^2}{M} g_{\alpha\beta} \tag{142}$$

thus the matrix $g_{\alpha\beta}$ must be symmetric.

Dirac Majorana Mass Term

If we accept the existence of righthanded neutrinos the most general mass term is a Dirac - Majorana:

$$\begin{aligned}\mathcal{L}^{D+M}_{mass} &= \mathcal{L}^{L}_{mass} + \mathcal{L}^{D}_{mass} + \mathcal{L}^{R}_{mass} \\ &= \frac{1}{2} m_L \overline{\nu}^T_L C^\dagger \nu_L - m_D \overline{\nu}_R \nu_L + \frac{1}{2} m_R \overline{\nu}^T_R C^\dagger \nu_R + H.C.\end{aligned} \tag{143}$$

This Lagrangian is only possible for neutrinos.
As we can redefine only two fields and we have three masses, one may be complex. We assume that this mass is m_L.

A more Convenient Form

It is enlightening to define

$$N_L = \begin{pmatrix} \nu_L \\ \nu_R^C \end{pmatrix} = \begin{pmatrix} \nu_L \\ C\nu_R^T \end{pmatrix}. \tag{144}$$

Then the Lagrangian can now be expressed as:

$$\mathcal{L}_{mass}^{D+M} = \frac{1}{2} N_L^T C^\dagger M N_L + H.C. \tag{145}$$

The mass matrix is has the symmetric form:

$$M = \begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix} \tag{146}$$

As the mass matrix is not diagonal neutrinos don't have definite mass.

Mixing

We diagonalize the mass matrix:

$$N_L = U n_L, \quad n_L = \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \end{pmatrix}. \tag{147}$$

Then the mass matrix becomes:

$$U^T M U = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \quad m_i \geq 0 \tag{148}$$

Diagonalization Part 1

The masses and the elements of the mixing matrix are defined through the eigenvalues and the eigenvectors of the matrix:

$$\mathcal{M} = \begin{pmatrix} \Re[m_L] & m_D & -\Im[m_L] & 0 \\ m_D & m_R & 0 & 0 \\ -\Im[m_L] & 0 & -\Re[m_L] & -m_D \\ 0 & 0 & -m_D & -m_R \end{pmatrix} . \tag{149}$$

The eigenvalues are:

$$\begin{aligned} m_{2,1}^2 = \frac{1}{2}\Bigg[& |m_L|^2 + m_R^2 + 2m_D^2 \\ & \pm \Big((\Re[m_L] + m_R)^2 \left[(\Re[m_L] - m_R)^2 + 4m_D^2 \right] \\ & + (\Im[m_L])^4 + 2\,(\Im[m_L])^2 \left((\Re[m_L])^2 - m_R^2 + 2m_D^2 \right) \Big)^{1/2} \Bigg] \end{aligned} \tag{150}$$

Diagonalization Part 2

For real m_L the relation is simplified:

$$m'_{2,1} = \frac{1}{2}\left[m_L^2 + m_R^2 \pm \sqrt{(m_L - m_R)^2 + 4m_D^2}\right] \qquad (151)$$

The prime symbolizes that the mass is not always positive. If it's negative the minus sign can be absorbed in the mass matrix.Thus the mass is the absolute value of m'.
The mixing matrix can be written as:

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} e^{\imath\lambda} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta e^{\imath\lambda} & \sin\theta \\ -\sin\theta e^{\imath\lambda} & \cos\theta \end{pmatrix}. \qquad (152)$$

Diagonalization Part 3

The first eigenvector is:

$$\mathcal{M}\begin{pmatrix}\sin\theta\\ \cos\theta\\ 0\\ 0\end{pmatrix} = m_2\begin{pmatrix}\sin\theta\\ \cos\theta\\ 0\\ 0\end{pmatrix} \tag{153}$$

This result corresponds to:

$$\tan 2\theta = \frac{2m_D}{m_R - \Re[m_L]} \tag{154}$$

Diagonalization Part 4

The second eigenvector is:

$$\mathcal{M}\begin{pmatrix}\cos\theta\cos\lambda\\ -\sin\theta\cos\lambda\\ \cos\theta\sin\lambda\\ -\sin\theta\sin\lambda\end{pmatrix} = m_1\begin{pmatrix}\cos\theta\cos\lambda\\ -\sin\theta\cos\lambda\\ \cos\theta\sin\lambda\\ -\sin\theta\sin\lambda\end{pmatrix}. \tag{155}$$

This result corresponds to:

$$\tan 2\lambda = \frac{2\Im[m_L]}{\Re[m_L] + m_R - \sqrt{(\Re[m_L] - m_R)^2 + 4m_D^2}} \tag{156}$$

Since $0 \leq 2\lambda \leq 4\pi$ there are 4 allowed values λ We choose the value that makes both masses positive.

Diagonalization Summary

For 1 generation of lefthanded and righthanded neutrinos after the diagonalization occur 2 Majorana neutrinos ν_1 & ν_2 of definite mass. The neutrinos ν_L & ν_R^C are lefthanded on the flavor base. The neutrino ν_L is active, while the neutrino ν_R is sterile, thus oscillations between active and sterile neutrinos is possible. This oscillations have:

$$\Delta m^2 = \Big[(\Re[m_L] + m_R)^2 \left[(\Re[m_L] - m_R)^2 + 4 m_D \right] + (\Im[m_L])^4 + 2 (\Im[m_L])^2 \left((\Re[m_L])^2 - m_R^2 + 2 m_D^2 \right) \Big]^{1/2} \tag{157}$$

Weak Interactions

The neutrinos of definite flavor in terms of the neutrinos of definite mass are:

$$\begin{aligned} \nu_L &= U_{11}\nu_{1L} + U_{12}\nu_{2L} \\ \nu_R^C &= U_{21}\nu_{1L} + U_{22}\nu_{2L} \end{aligned} \tag{158}$$

According to this mixing the Lagrangian of charged current weak interactions is:

$$L^{CC} = -\frac{g}{\sqrt{2}} \sum_{i=1,2} U_{1i}^* \overline{\nu}_{iL} \gamma^\mu \ell_L W_\mu + H.C. \tag{159}$$

The Lagrangian of neutral current weak interactions is

$$L^{NC} = -\frac{g}{2\cos\theta_W} \sum_{ij=1,2} U_{1i}^* U_{1j} \overline{\nu}_{iL} \gamma^\mu \nu_{jL} Z_\mu \tag{160}$$

A very strange phenomenon is that GIM mechanism doesn't work.

Maximal Mixing Part 1

Maximal mixing occurs when

$$m_L = m_R. \tag{161}$$

In this case the masses are:

$$m'_{2,1} = m_L \pm m_D. \tag{162}$$

If $m_L > m_D$ then both masses are real and the mixing matrix is:

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \tag{163}$$

If $m_L < m_D$ then $m_1 < 0$, thus the mixing matrix is:

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} \imath & 1 \\ -\imath & 1 \end{pmatrix}. \tag{164}$$

Maximal Mixing Part 2

In the case $m_L < m_D$ since $\theta = \pi/4$ we get:

$$\nu_{1L} = -\frac{\imath}{\sqrt{2}}\left(\nu_L - \nu_R^C\right) \tag{165}$$

$$\nu_{2L} = \frac{1}{\sqrt{2}}\left(\nu_L + \nu_R^C\right). \tag{166}$$

The neutrinos of definite mass are:

$$\nu_1 = \nu_{1L} + \nu_{1L}^C = -\frac{\imath}{\sqrt{2}}\left[(\nu_L + \nu_R) - (\nu_L^C + \nu_R^C)\right] \tag{167}$$

$$\nu_2 = \nu_{2L} + \nu_{2L}^C = \frac{1}{\sqrt{2}}\left[(\nu_L + \nu_R) + (\nu_L^C + \nu_R^C)\right]. \tag{168}$$

The square mass splitting is:

$$\Delta m^2 = m_2^2 - m_1^2 = 4 m_L m_D \tag{169}$$

Dirac Limit

The Dirac limit corresponds to the case:

$$m_L = m_R = 0 \tag{170}$$

$$m'_{2,1} = \pm m_D. \tag{171}$$

The mixing matrix in this case is:

$$U = \begin{pmatrix} \imath & 1 \\ -\imath & 1 \end{pmatrix}. \tag{172}$$

This way we get two Majorana neutrinos of the same mass with opposite CP parities. This two neutrinos can be combined into a Dirac neutrino as:

$$\nu = \frac{1}{\sqrt{2}}(\imath\nu_1 + \nu_2) = \nu_L + \nu_R. \tag{173}$$

On the same reasoning a Dirac neutrino can split into two Majorana neutrinos of the same mass with opposite parities.

Pseudodirac Neutrinos Part 1

Pseudodirac neutrinos correspond to the case:

$$|m_L|, m_R \ll m_D \tag{174}$$

The neutrinos have masses:

$$m'_{2,1} = \frac{m_L + m_R}{2} \pm m_D \tag{175}$$

Thus the neutrinos have opposite CP parities and the acquired masses are:

$$m_{2,1} = m_D \pm \frac{m_L + m_R}{2} \tag{176}$$

The mass splitting is

$$(m_L + m_R) \ll m_{2,1}, \tag{177}$$

Pseudodirac Neutrinos Part 2

The Majorana neutrinos are almost degenerate. We call them pseudodirac because its very difficult distinguish them from the Dirac neutrinos. The best way to distinguish the two cases is the oscillations between active and sterile neutrinos with square mass splitting:

$$\Delta m^2 \simeq m_D(m_L + m_R) \tag{178}$$

and a practically maximal mixing angle:

$$\tan 2\theta = \frac{2m_D}{m_R - m_L} \gg 1 \to \theta \simeq \frac{\pi}{4} \tag{179}$$

See-Saw Mechanism Part 1

The most realistic and interesting case is when:

$$m_L = 0 \quad m_R \gg m_D \tag{180}$$

The mass m_L is protected by the symmetries of SM and renormalizability.
In this case the mass eigenstates are:

$$m'_1 \simeq -\frac{m_D^2}{m_R} \tag{181}$$

$$m'_2 \simeq m_R \tag{182}$$

Mixing is minimal

$$\tan 2\theta = 2\frac{m_D}{m_R} \ll 1 \tag{183}$$

See-Saw Mechanism Part 2

- The huge difference between neutrinos and other fermions are explained in a very natural manner.
- The mass m_D is generated through Higgs mechanism can't be greater than the energy scale of EW symmetry breaking. Its natural to set $m_D \sim 100 GeV$.
- m_R may be associated with symmetry breaking of a greater symmetry. If m_R is associated with GUT then $m_R \sim 10^{14} - 10^{16}$.
- The similarity of See-Saw relations with the relations obtained for the 5-D operator is not accidental. See-Saw mechanism is a special case of 5-D operators.

See-Saw Mechanism as Special Case of 5-D Operators Part 1

For $m_L = 0$ the Lagrangian is:

$$\mathcal{L}^{D+M}_{mass} = -m_D\left(\overline{\nu}_R\nu_L + \overline{\nu}_L\nu_R\right) + \frac{1}{2}m_R\left(\overline{\nu}^T_R C^\dagger \nu_R + \nu^\dagger_R C \nu^*_R\right) \quad (184)$$

Above symmetry breaking:

$$\mathcal{L}^{D+M}_{mass} = -y^\nu\left(\overline{\nu}_R\tilde{\Phi}^\dagger L_L + \overline{L}_L\tilde{\Phi}\nu_R\right) + \frac{1}{2}m_R\left(\overline{\nu}^T_R C^\dagger \nu_R + \nu^\dagger_R C \nu^*_R\right) \quad (185)$$

Considering ν_R static E-L equation becomes:

$$0 \simeq \frac{\partial \mathcal{L}^{D+M}_{mass}}{\partial \nu_R} = m_R \nu^T_R C^\dagger - y^\nu \overline{L}_L \tilde{\Phi}. \quad (186)$$

Solving for ν_R we have:

$$\nu_R \simeq -\frac{y^\nu}{m_R}\tilde{\Phi}^T C \overline{L}^T_L. \quad (187)$$

Substitution of ν_R to the Lagrangian gives a 5-D operator for ν_L.

Non Vanishing m_L

Consider the case:

$$m_L \ll m_D \ll m_R \quad m_L = g\frac{m_D^2}{M} \tag{188}$$

M is at the energy scale of physics Beyond SM, maybe at the scale of L or B-L symmetry breaking. We obtain:

$$m_1 \simeq \left| g\frac{m_D^2}{M} - \frac{m_D^2}{m_R} \right| \qquad m_2 \simeq m_R \tag{189}$$

- Type I See-Saw mechanism corresponds to the case $m_L \ll m_D^2/m_R$.
- Type II See-Saw mechanism corresponds to the case $m_L \gg m_D^2/m_R$.

Mass Terms Part 1

We consider 3 active lefthanded neutrinos and N_S sterile righthanded neutrinos. A general Lagrangian has the form:

$$\mathcal{L}^{D+M}_{mass} = \mathcal{L}^{L}_{mass} + \mathcal{L}^{D}_{mass} + \mathcal{L}^{R}_{mass}. \tag{190}$$

The Majorana mass terms are:

$$\mathcal{L}^{L}_{mass} \frac{1}{2} \sum_{\alpha,\beta=e,\mu,\tau} \left(\nu'\right)^{T}_{\alpha L} C^{\dagger} M^{L}_{\alpha\beta} \nu'_{\beta L} + H.C. \tag{191}$$

$$\mathcal{L}^{R}_{mass} \frac{1}{2} \sum_{s,s'=s_1,\cdots,s_{N_S}} \left(\nu'\right)^{T}_{sR} C^{\dagger} M^{R}_{ss'} \nu'_{s'R} + H.C. \tag{192}$$

while the Dirac mass term is:

$$\mathcal{L}^{D}_{mass} = - \sum_{s=1,\cdots,N_S} \sum_{\alpha=e,\mu,\tau} \overline{\nu}_{sR} M^{D}_{s\alpha} \nu'_{\alpha L} + H.C. \tag{193}$$

Mass Terms Part 2

All mass matrices are complex, especially the Majorana mass matrices are symmetric. The matrices dimensions are as follows: M^L is 3×3, M^R is $N_S \times N_S$ & M^D is $N_S \times 3$. We express the neutrinos as a $N = 3 + N_S$ column:

$$N'_L = \begin{pmatrix} \nu'_L \\ \nu_R^{C\prime} \end{pmatrix} \qquad \nu_R^{C\prime} = \begin{pmatrix} \nu_{s_1 R}^{C\prime} \\ \vdots \\ \nu_{s_{N_S} R}^{C\prime} \end{pmatrix} \tag{194}$$

Then the Lagrangian becomes:

$$\mathcal{L}_{mass}^{D+M} = \frac{1}{2} \left(N'_L\right)^T C^\dagger M^{D+M} N'_L + H.C. \tag{195}$$

Diagonalization

The mass matrix M^{D+M} is symmetric $N \times N$:

$$M^{D+M} = \begin{pmatrix} M^L & \left(M^D\right)^T \\ M^D & M^R \end{pmatrix} . \tag{196}$$

If we define the neutrinos as

$$N'_L = V_L^{\nu} n_L \qquad n_L = \begin{pmatrix} \nu_{1L} \\ \vdots \\ \nu_{NL} \end{pmatrix} , \tag{197}$$

then the mass matrix is diagonalized:

$$\left(V_L^{\nu}\right)^T M^{D+M} V_L^{\nu} = M \quad M_{ij} = m_i \delta_{ij} \; m_i > 0. \tag{198}$$

The Lagrangian can be written as:

$$\mathcal{L}^{D+M}_{mass} = \frac{1}{2} \left(n_L\right)^T C^\dagger M n_L + H.C. \tag{199}$$

If we define the column:

$$n = \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_N \end{pmatrix} \qquad \nu_i = \nu_{iL} + \nu^C_{iL} \tag{200}$$

with neutrinos satisfy the Majorana condition:

$$\nu^C_i = \nu_i \tag{201}$$

the Lagrangian becomes:

$$\mathcal{L}^{D+M}_{mass} = \overline{n} \left(\imath \partial\!\!\!/ - M\right) n. \tag{202}$$

Weak Interactions Part 1

The leptonic weak charged current is:

$$j^{\mu}_{W,L} = 2\overline{\nu}'_L \gamma^{\mu} \ell'_L \tag{203}$$

In term of neutrinos with definite mass the current is:

$$j^{\mu}_{W,L} = 2\overline{\nu}_L U^{\dagger} \gamma^{\mu} \ell_L . \tag{204}$$

The mixing matrix has elements:

$$U_{\alpha k} = \sum_{\beta=e,\mu,\tau} \left[\left(V^{\ell}_L \right)^{\dagger} \right]_{\alpha\beta} [V^{\nu}_L]_{\beta k} , \tag{205}$$

The mixing matrix is 3×3 and diagonalizes the charged leptons mass matrix. U is $N \times N$ but its not unitary as:

$$UU^{\dagger} = 1 \qquad U^{\dagger} U \neq 1 \tag{206}$$

Weak Interactions Part 2

The neutrinos of definite flavor are defined as:

$$\nu_L = U n_L = \left(V_L^\ell\right)^\dagger \nu_L' \qquad \nu_L = \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix} \tag{207}$$

This way we get the SM expression for the weak charged current:

$$j^\mu_{W,L} = 2\overline{\nu}_L \gamma^\mu \ell_L \tag{208}$$

As in the case of one generation GIM mechanism doesn't work:

$$j^\mu_{Z,\nu} = \overline{n}_L \gamma^\mu U^\dagger U n_L, \tag{209}$$

See-Saw Mechanism Part 1

If we set $M_L = 0$ the mass matrix becomes:

$$M^{D+M} = \begin{pmatrix} 0 & \left(M^D\right)^T \\ M^D & M^R \end{pmatrix}, \tag{210}$$

the elements of M^R are much greater than the elements of M^D. We diagonalize to first order of $\left(M^R\right)^{-1} M^D$:

$$W^T M^{D+M} W \simeq \begin{pmatrix} M_{light} & o \\ 0 & M_{heavy} \end{pmatrix}, \tag{211}$$

where the matrix W is:

$$W \simeq \begin{pmatrix} 1 - \frac{1}{2} M^{D\dagger} \left(M^R M^{R\dagger}\right)^{-1} M^D & \left(M^{R^{-1}} M^D\right)^\dagger \\ -M^{R^{-1}} M^D & 1 - \frac{1}{2} M^{R^{-1}} M^D M^{D\dagger} \left(M^{R\dagger}\right)^{-1} \end{pmatrix} \tag{212}$$

See-Saw Mechanism Part 2

The elements blocks of the mass matrix are:

$$M_{light} \simeq -M^{D^T} M^{R^{-1}} M^D \qquad M_{heavy} \simeq M^R. \tag{213}$$

The heavy masses are the eigenvalues of the heavy mass matrix, while the light masses are suppressed by the factor $M^{D^T} M^{R^{-1}}$. The masses can have a wide range depending on the elements of the matrices.

Quadratic See-Saw

Quadratic See-Saw Mechanism corresponds to the case:

$$M^R = \mathcal{M} I \tag{214}$$

then

$$M_{light} \simeq -\frac{M^{D^T} M^D}{\mathcal{M}}, \tag{215}$$

which means that the masses are:

$$m_i \simeq -\frac{(m_i^D)^2}{\mathcal{M}}. \tag{216}$$

We expect the mass ratios to be:

$$m_1 : m_2 : m_3 = \left(m_1^D\right)^2 : \left(m_2^D\right)^2 : \left(m_3^D\right)^2 \tag{217}$$

Linear See-Saw

The case of linear See-Saw $N_S = 3$ and

$$M^R = \frac{\mathcal{M}}{\mathcal{M}^D} M^D, \tag{218}$$

$\mathcal{M}^D$ is the energy scale of the elements of M^D. $\mathcal{M} \gg \mathcal{M}^D$ is the GUT scale.
The matrix M_{light} becomes:

$$M_{light} \simeq -\frac{\mathcal{M}^D}{\mathcal{M}} M^D \tag{219}$$

Thus the light masses are:

$$m_i = \frac{\mathcal{M}^D}{\mathcal{M}} m_i^D. \tag{220}$$

The ratios are linear:

$$m_1 : m_2 : m_3 = m_1^D : m_2^D : m_3^D \tag{221}$$

Both ratios hold at GUT scale, we need R.G.E.

GUTs Part 1

We are interested in 3+1 models. The special case is the symmetric expansion of SM, which only need righthanded neutrinos and U(1) symmetry corresponds to B-L:

$$SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L} \tag{222}$$

The GUTs are:

$$SU(5) \quad SU(4) \times SU(2)_L \times SU(2)_R \quad SO(10) \tag{223}$$

SU(5) is the simplest case. It doesn't provide space for righthanded neutrinos which means that in order to have massive neutrinos we must include either Higgs triplets or non-renormalizable mass terms.

GUTs Part 2

Pati - Salam $SU(4) \times SU(2)_L \times SU(2)_R$ includes righthanded currents but they are suppressed by $m^2_{W_R}$.
$SU(4) \times SU(2)_L \times SU(2)_R$ is a subgroup of SO(10).
SO(10) does not only include righthanded neutrino but the neutrino is SU(5) singlet too. Nevertheless the righthanded neutrino couples to other gauge bosons. As SO(10) not necessarily conserve B-L Majorana neutrinos are allowed.

Summary

Lecture 3

Phenomenology & Experimental Aspects

Outline of Lecture 3

Introduction

- Appearance Transition Probability
- Disappearance Survival Probability

Oscillations are suppressed for:

$$\frac{\Delta m^2 L}{2E} \ll 1 \tag{224}$$

Oscillations are averaged for:

$$\frac{\Delta m^2 L}{2E} \gg 1 \tag{225}$$

Thus we need:

$$\frac{\Delta m^2 L}{2E} \sim 1. \tag{226}$$

Sources

- Sun: ν_e
- Atmosphere: $\overline{\nu}_e$, ν_e, $\overline{\nu}_\mu$, ν_μ
- Reactor: $\overline{\nu}_e$
- Accelerator experiments are divided as follows:
  - Pion Decay In Flight:

$$\pi^+ \rightarrow \mu^+ + \nu_\mu \tag{227}$$

  - Muon Decays at Rest:

$$\mu^+ \rightarrow e^+ + \nu_e + \bar{\nu}_\mu \tag{228}$$

  - Beam Dump: $\overline{\nu}_e$, ν_e, $\overline{\nu}_\mu$, ν_μ
  - Beta Beams: pure ν_e or $\overline{\nu}_e$
  - Nu Factories: pure ν_μ or $\overline{\nu}_\mu$

Classification of Experiments

Type	L	E	Δm^2
Reactor SBL	$\sim 10\ m$	$\sim 1\ MeV$	$\sim 0.1\ eV^2$
Ac. SBL			
Pion DIF	$\sim 1\ km$	$\geq 1\ GeV$	$\sim 1\ eV^2$
Muon DAR	$\sim 10\ m$	$\sim 10\ MeV$	$\sim 1\ eV^2$
Beam Dump	$\sim 10\ km$	$\sim 10^2\ GeV$	$\sim 10^2\ eV^2$
Reactor LBL	$\sim 10\ km$	$\sim 1\ MeV$	$\sim 10^{-3}\ eV^2$
Ac. LBL	$\sim 10^3\ km$	$\geq 1\ GeV$	$\geq 10^{-3}\ eV^2$
Atmospheric	$20 - 10^4\ km$	$0.5 - 10^2\ GeV$	$\sim 10^{-4}\ eV^2$
Reactor VLBL	$\sim 10^2\ km$	$\sim 10^2\ MeV$	$\sim 10^{-5}\ eV^2$
Ac. VLBL	$\sim 10^4\ km$	$\geq 10^2\ GeV$	$\geq 10^{-4}\ eV^2$
Solar	$\sim 10^{11}\ km$	$0.2 - 15\ MeV$	$\sim 10^{-12}\ eV^2$

Taking into Account Uncertainties Part 1

We will include in our analysis that the ratio L/E follows a distribution. It reasonable to consider:

$$\phi\left(\frac{L}{E}\right) = \frac{1}{\sqrt{2\pi\sigma_{L/E}}}\, exp\left(\frac{L/E - \langle L/E\rangle}{2\sigma^2_{L/E}}\right), \tag{229}$$

As a result

$$\langle P_{\nu_\alpha \to \nu_\beta}(L,E)\rangle = \frac{1}{2}\sin^2 2\theta \left[1 - \left\langle \cos\left(\frac{\Delta m^2 L}{2E}\right)\right\rangle\right] \quad (\alpha \neq \beta), \tag{230}$$

where

$$\left\langle \cos\left(\frac{\Delta m^2 L}{2E}\right)\right\rangle = \int \cos\left(\frac{\Delta m^2 L}{2E}\right) \phi\left(\frac{L}{E}\right) d\frac{L}{E} \tag{231}$$

Taking into Account Uncertainties Part 2

With the Gaussian distribution we can calculate:

$$\left\langle \cos\left(\frac{\Delta m^2 L}{2E}\right)\right\rangle = \cos\left(\frac{\Delta m^2}{2}\left\langle\frac{L}{E}\right\rangle\right)\ exp\left[-\frac{1}{2}\left(\frac{\Delta m^2}{2}\sigma_{L/E}\right)^2\right]. \tag{232}$$

We can consider that:

$$\sigma_{L/E} \sim \left\langle \frac{L}{E}\right\rangle. \tag{233}$$

Assume that:

$$\langle P_{\nu_\alpha\to\nu_\beta}(L,E)\rangle \leq P^{max}_{\nu_\alpha\to\nu_\beta}. \tag{234}$$

This bound can be used to set the following bound:

$$\sin^2 2\theta \leq \frac{2P^{max}_{\nu_\alpha\to\nu_\beta}}{1-\left\langle\cos\left(\frac{\Delta m^2 L}{2E}\right)\right\rangle}. \tag{235}$$

Special Cases

In the case:

$$\Delta m^2 \left\langle \frac{L}{E} \right\rangle \gg 1 \Rightarrow sin^2 2\theta \rightarrow 2P^{max}_{\nu_\alpha \rightarrow \nu_\beta} \tag{236}$$

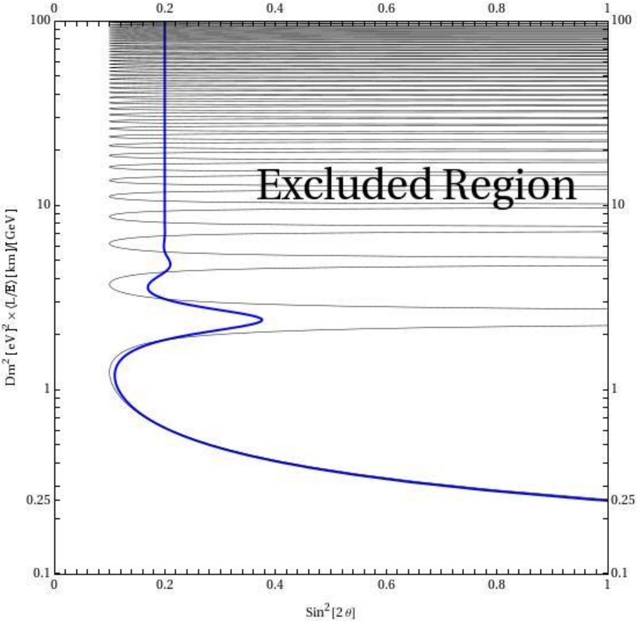
In the opposite case:

$$\Delta m^2 \left\langle \frac{L}{E} \right\rangle \ll 1 \Rightarrow \sin^2 2\theta \leq \frac{0.62 P^{max}_{\nu_\alpha \rightarrow \nu_\beta}}{\left(\Delta m^2 [eV^2] \left\langle \frac{L}{E} \right\rangle \frac{[km]}{[GeV]}\right)^2}. \tag{237}$$

This means that we can't bound Δm^2 if it is lower than:

$$\Delta m^2 [eV^2] \left\langle \frac{L}{E} \right\rangle \frac{[km]}{[GeV]} = 0.79 \sqrt{P^{max}_{\nu_\alpha \rightarrow \nu_\beta}}. \tag{238}$$

Exclusion Curves



A Different Case

If we have 2 detectors we can measure the flux ratio.

$$\frac{\langle P_{\nu_\alpha\to\nu_\alpha}(L,E)\rangle_{far}}{\langle P_{\nu_\alpha\to\nu_\alpha}(L,E)\rangle_{near}} \geq R, \quad 0 \leq R < 1 \tag{239}$$

We "average" the survival probability:

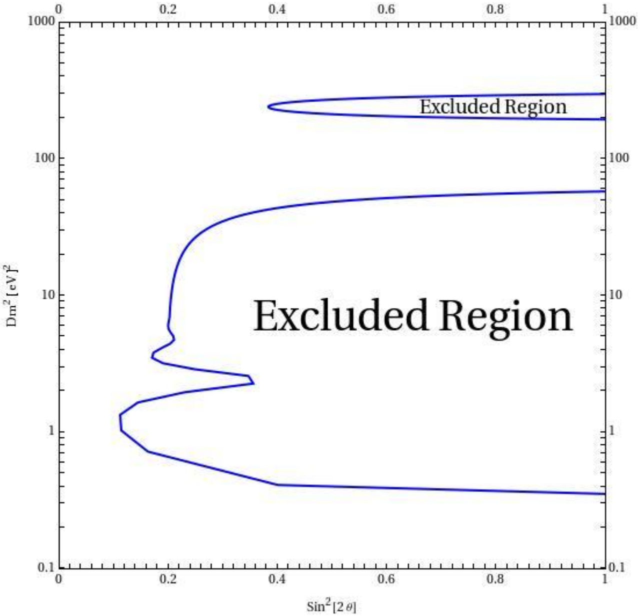
$$P_{\nu_\alpha\to\nu_\alpha}(L,E) = 1 - \frac{1}{2}\sin^2 2\theta\left[1-\cos^2\left(\frac{\Delta m^2 L}{2E}\right)\right], \tag{240}$$

Thus we get:

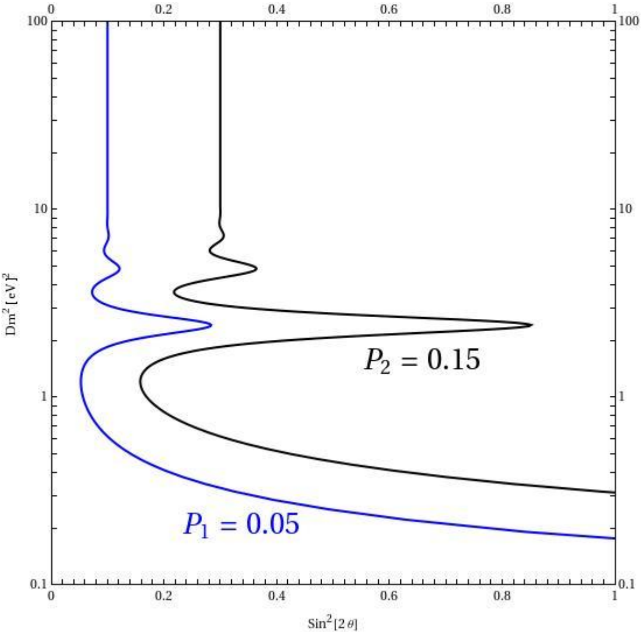
$$2(1-R) \leq \sin^2 2\theta\left[1 - R - \left\langle\cos\frac{\Delta m^2 L}{2E}\right\rangle_{far} + R\left\langle\cos\frac{\Delta m^2 L}{2E}\right\rangle_{near}\right] \tag{241}$$

$$\sin^2 2\theta \leq \frac{2(1-R)}{1 - R - \left\langle\cos\frac{\Delta m^2 L}{2E}\right\rangle_{far} + R\left\langle\cos\frac{\Delta m^2 L}{2E}\right\rangle_{near}}, \tag{242}$$

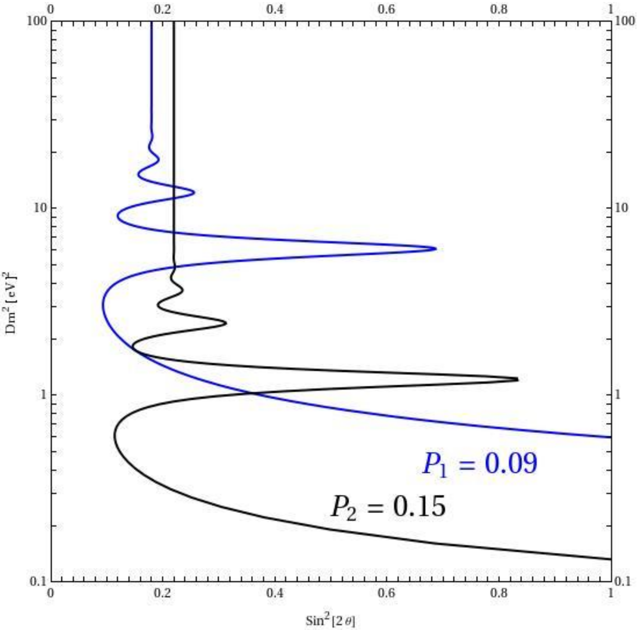
Exclusion Curves



Exclusion Curves



Exclusion Curves



3 Neutrino Case

In the three neutrino case:

$$\langle P(\nu_\alpha \to \nu_\beta)\rangle = \delta_{\alpha\beta} - 2\sum_{i>j}\Re(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j})\left(1 - \left\langle \cos\left(\Delta m^2_{ij}\frac{L}{2E}\right)\right\rangle\right) + 2\sum_{i>j}\Im(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j})\left\langle \sin\left(\Delta m^2_{ij}\frac{L}{2E}\right)\right\rangle , \tag{243}$$

The sine's overage is given by:

$$\left\langle \sin\left(\frac{\Delta m^2 L}{2E}\right)\right\rangle = \sin\left(\frac{\Delta m^2}{2}\left\langle\frac{L}{E}\right\rangle\right)\ exp\left[-\frac{1}{2}\left(\frac{\Delta m^2}{2}\sigma_{L/E}\right)^2\right]. \tag{244}$$

It's obvious that this analysis is much more complicated.

Solar Neutrino Production

In the Sun energy is produced by the reaction:

$$4p + 2e^- \rightarrow {}^4He + 2\nu_e + Q, \tag{245}$$

The Q-Value is $Q = 25.731\ MeV$.
Each branch contributes to solar constant:

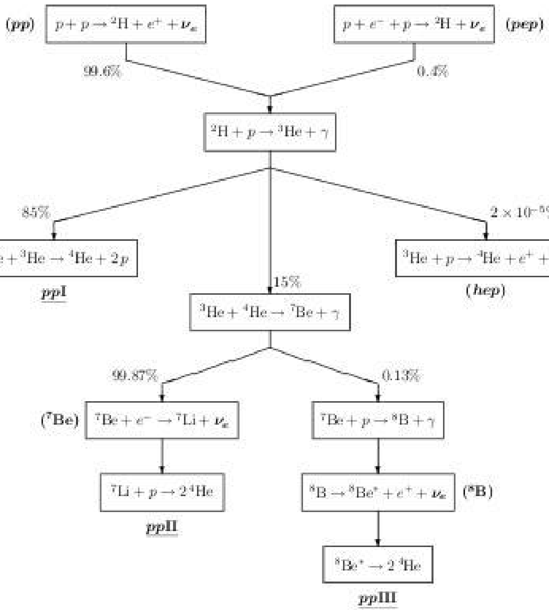
$$\sum_r \alpha_r \Phi_r = K_o, , \quad r = pp, pep, hep, {}^7Be, {}^8B, {}^{13}N, {}^{15}O, {}^{17}F \tag{246}$$

Therefore the total flux is:

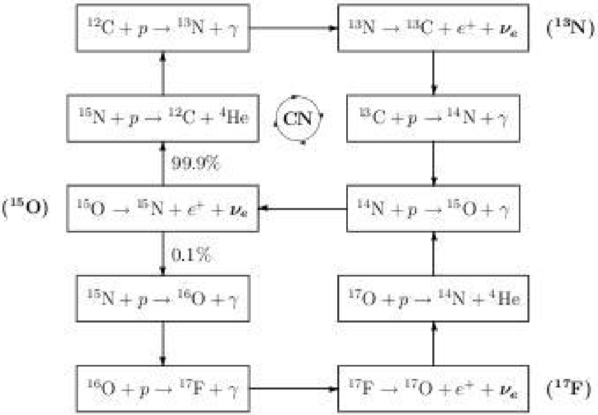
$$\Phi \simeq \frac{2K_o}{Q}. \tag{247}$$

The flux is measured in SNU, where $1\ SNU = 10^{-36}$ events per second.

Solar Neutrinos



Solar Neutrinos



Solar Results

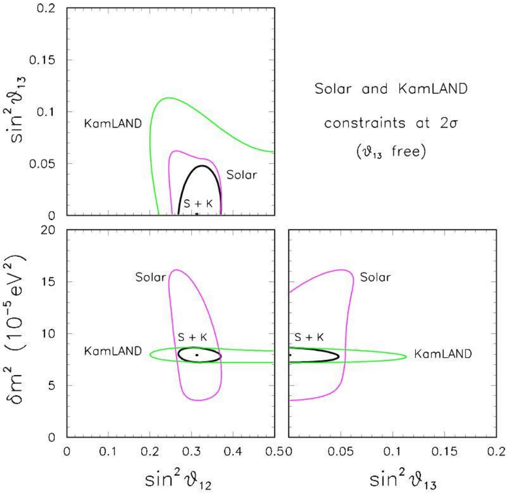
The solar data leads to:

$$\Delta m^2 = 6.5^{+4.4}_{-2.3} \times 10^{-5} eV^2, \qquad \tan^2\theta = 0.45^{+0.09}_{-0.08} \tag{248}$$

If we take KamLAND's data into account we get:

$$\Delta m^2 = 8.0^{+0.6}_{-0.4} \times 10^{-5} eV^2, \qquad \tan^2\theta = 0.45^{+0.09}_{-0.07}, \tag{249}$$

Solar Neutrinos



Atmospheric Overview

Pions produced by cosmic rays produce neutrinos and muons.

$$\pi^+ \to \mu^+ + \nu_\mu \qquad \pi^- \to \mu^- + \bar{\nu}_\mu. \tag{250}$$

Muons may or may not decay before they hit the ground:

$$\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu \qquad \mu^- \to e^- + \bar{\nu}_e + \nu_\mu \tag{251}$$

We estimate the following flux ratios:

$$\frac{\phi_{\nu_\mu} + \phi_{\bar{\nu}_\mu}}{\phi_{\nu_e} + \phi_{\bar{\nu}_e}} \simeq 2, \quad \frac{\phi_{\nu_\mu}}{\phi_{\bar{\nu}_\mu}} \simeq 1, \quad \frac{\phi_{\nu_e}}{\phi_{\bar{\nu}_e}} \simeq \frac{\phi_{\mu^+}}{\phi_{\mu^-}} \tag{252}$$

We compare experimental values with Monte Carlo results.
The best fit values for atmospheric experiments are:

$$\sin^2 2\theta = 1.00 \qquad \Delta m^2 = 2.1 \times 10^{-3}\ eV^2 \tag{253}$$

Reactor Overview

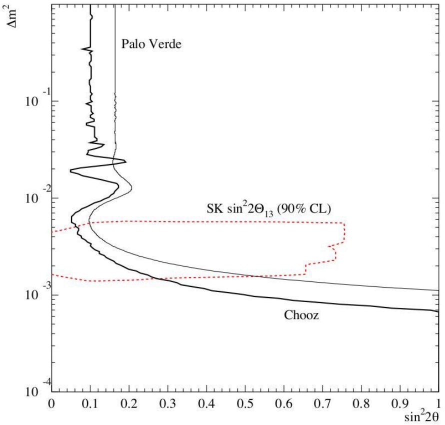
Neutrinos $\overline{\nu}_e$ are produced in nuclear reactors. Each *GW* of energy corresponds to 2×10^{20} antineutrinos. Due to low neutrino's energy we can only measure the disappearance of $\overline{\nu}_e$.
We use the inverse beta decay :

$$\overline{\nu}_e + p \rightarrow n + e^+ \quad E_{th} = 1.806\ MeV. \tag{254}$$

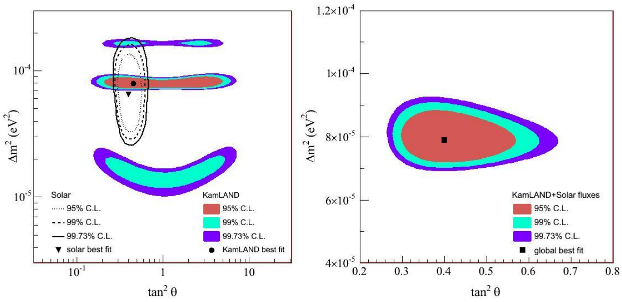
Only 25% of the flux has energy above this threshold.
KamLAND result taking into account solar data is:

$$\Delta m^2 = 7.9^{+0.6}_{-0.5} \times 10^{-5}\ eV^2 \quad \tan^2\theta = 0.40^{+0.10}_{-0.07} \tag{255}$$

Reactor Experiments



Reactor Experiments



Accelerator Overview

There are 3 types of beams:

- Wide Band: Neutrino energies may differ 2 orders of magnitude, but the beam has great intensity.
- Narrow Band: These beams have low intensity.
- Off Axis: These beams are almost monochromatic.

Only K2K detected oscillation. Its data implied that:

$$\sin^2 2\theta = 1.0 \quad \Delta m^2 = 2.8 \times 10^{-3}\ eV \tag{256}$$

Atmospheric results were confirmed. No transformations $\nu_\mu \rightarrow \nu_e$ were detected, leading to the bound:

$$\sin^2 2\theta_{\mu e} < 0.13\ (90\% C.L.) \quad \Delta m^2 = 2.8 \times 10^{-3}\ eV \tag{257}$$

New Point of View

So far experiments were analyzed using 2 neutrino oscillations.
Now we are going one step further.
Square mass splitting don't depend on the analysis, as they are fixed by nature. Only 2 out of 3 are independent as:

$$\Delta m_{32} + \Delta m_{21} - \Delta m_{31} = 0 \tag{258}$$

The experiments imply that:

$$\Delta m^2_{SOL} \ll \Delta m^2_{ATM}, \tag{259}$$

We symbolize:

$$\Delta m^2_{SOL} = \Delta m^2_{21} \quad \Delta m^2_{ATM} = |\Delta m^2_{31}| \simeq |\Delta m^2_{32}| \tag{260}$$

Mixing in 3 Neutrinos Analysis

Mixing may change dramatically, but as we will see it won't. Solar experiments measure the disappearance of ν_e, as a result they depend only on U_{ei}. Atmospheric oscillations depend on the mixing angles θ_{23} & θ_{13}. This angles are determined as:

$$\sin\theta_{23} = \frac{|U_{\mu 3}|}{\sqrt{1-|U_{e3}|^2}} \qquad \sin\theta_{13} = |U_{e3}| \tag{261}$$

The only common elements is $|U_{e3}|$ which is very small, if not zero.

Two types of oscillations

As a result of

$$\Delta m^2_{SOL} \ll \Delta m^2_{ATM}, \qquad (262)$$

we have two different types of oscillations depending on the active square mass splitting.

Large Square Mass Splitting Part 1

In atmospheric or LBL experiments we have:

$$\frac{\Delta m^2_{31}}{2}\left\langle\frac{L}{E}\right\rangle \sim \pi \tag{263}$$

Oscillations due to Δm^2_{21} are averaged. As a result the oscillation can be interpreted as effective two neutrinos oscillations with the following probabilities and mixing angles:

$$P^{eff}_{\nu_\alpha\to\nu_\beta}(L,E) = \sin^2 2\theta^{eff}_{\alpha\beta}\sin^2\left(\frac{\Delta m^2_{31}L}{4E}\right), \quad \alpha\neq\beta \tag{264}$$

$$P^{eff}_{\nu_\alpha\to\nu_\alpha}(L,E) = 1-\sin^2 2\theta^{eff}_{\alpha\alpha}\sin^2\left(\frac{\Delta m^2_{31}L}{4E}\right) \tag{265}$$

$$\sin^2 2\theta^{eff}_{\alpha\beta} = 4|U_{\alpha 3}|^2|U_{\beta 3}|^2, \quad \alpha\neq\beta \tag{266}$$

$$\sin^2 2\theta^{eff}_{\alpha\alpha} = 4|U_{\alpha 3}|^2\left(1-|U_{\alpha 3}|^2\right) \tag{267}$$

Large Square Mass Splitting Part 2

Oscillations between any type of neutrinos are allowed and they have the same oscillation length:

$$L^{osc} = \frac{4\pi}{\Delta m^2_{31}}. \tag{268}$$

As the probabilities don't depend on the phase, no information for CP violation can be obtained. Oscillations depend only on Δm^2_{31}, $|U_{e3}|$ & $|U_{\mu 3}|$. In terms of mixing parameters we have $|U_{e3}| = \sin^2\theta_{13}$ & $|U_{\mu 3}| = \cos^2\theta_{13}\sin^2\theta_{23}$. This analysis implies that 2 neutrino analysis uses the same square mass splitting and the effective mixing angle.

Small Square Mass Splitting Part 1

In solar of VLBL experiments we have:

$$\frac{\Delta m^2_{21}}{2}\left\langle\frac{L}{E}\right\rangle \sim \pi, \tag{269}$$

Oscillation due to the small square mass splitting are washed out. Either we measure the disappearance of ν_e or as we can't distinguish ν_μ & ν_τ we measure their total appearance. The second process is equivalent to the first as:

$$P_{\nu_e\to\nu_\mu} + P_{\nu_e\to\nu_\tau} = 1 - P_{\nu_e\to\nu_e}. \tag{270}$$

Small Square Mass Splitting Part 2

The probability for this oscillations is given by:

$$P^{eff}_{\nu_\alpha \to \nu_\alpha}(L,E) = \left(1 - |U_{\alpha 3}|^2\right)^2 P^{(1,2)}_{\nu_\alpha \to \nu_\alpha}(L,E) + |U_{\alpha 3}|^2, \quad (271)$$

where the effective two neutrino probability is given by:

$$P^{(1,2)}_{\nu_\alpha \to \nu_\alpha}(L,E) = 1 - \sin^2 2\theta^{eff}_{\alpha\alpha} \sin^2\left(\frac{\Delta m^2_{21} L}{4E}\right), \quad (272)$$

and the effective mixing angle is given by:

$$\sin^2 2\theta^{eff}_{\alpha\alpha} = 4 \frac{|U_{\alpha 1}|^2 |U_{\alpha 2}|^2}{(|U_{\alpha 1}|^2 + |U_{\alpha 2}|^2)^2} \quad (273)$$

Bound on $|U_{e3}|$

As we mentioned in oscillations duo to Δm^2_{31} the effective mixing angle depends only on $|U_{\alpha 3}|$. If we use ν_e or $\overline{\nu}_e$ then:

$$\sin^2 2\theta^{eff}_{ee} = 4|U_{\alpha 3}|^2 \left(1 - |U_{\alpha 3}|^2\right) = \sin^2 2\theta_{13}. \tag{274}$$

If experimentally we have an upper bound $\left(\sin^2 2\theta_{13}\right)_{max}$ then:

$$|U_{e3}|^2 \leq \frac{1}{2}\left(1 - \sqrt{1 - \left(\sin^2 2\theta_{13}\right)_{max}}\right). \tag{275}$$

Analysis based on data of CHOOZ and Palo Verde in combined with data of Super-Kamiokande gives:

$$|U_{e3}|^2 < 5 \times 10^{-5} \quad 99.73\%\ CL \tag{276}$$

As LBL experiments didn't detect any $\nu_\mu \to \nu_e$ oscillation we have the bound:

$$|U_{e3}|^2 < 7 \times 10^{-5} \quad 90\%\ CL \quad @\ \Delta m^2 = 2.8 \times 10^{-3}\ eV^2. \tag{277}$$

Tribimaximal Analysis Part 1

We can assume $|U_{e3}| = 0$, then atmospheric & solar oscillations are decoupled. We denote:

$$\theta_{SOL} = \theta_{12} \qquad \theta_{ATM} = \theta_{23} \tag{278}$$

Then the mixing matrix becomes:

$$U = \begin{pmatrix} \cos\theta_{SOL} & \sin\theta_{SOL} & 0 \\ -\sin\theta_{SOL}\cos\theta_{ATM} & \cos\theta_{SOL}\cos\theta_{ATM} & \sin\theta_{ATM} \\ \sin\theta_{SOL}\sin\theta_{ATM} & -\cos\theta_{SOL}\sin\theta_{ATM} & \cos\theta_{ATM} \end{pmatrix} \tag{279}$$

Neutrinos ν_e are the superposition:

$$\nu_e = \cos\theta_{SOL}\nu_1 + \sin\theta_{SOL}\nu_2. \tag{280}$$

As they oscillate they transit to the orthogonal state:

$$\begin{aligned} \nu_{SOL} &= -\sin\theta_{SOL}\nu_1 + \cos\theta_{SOL}\nu_2 \\ &= \cos\theta_{ATM}\nu_\mu - \sin\theta_{ATM}\nu_\tau \end{aligned} \tag{281}$$

Tribimaximal Analysis Part 2

As the atmospheric mixing angle is maximal, thus:

$$\nu_{SOL} = \frac{1}{\sqrt{2}} \left(\nu_\mu - \nu_\tau \right). \tag{282}$$

From SNO we know that the ration charged current to neutral current events is 1/3 thus ν_e, ν_μ & ν_τ have same flux. We can use the approximation $\theta_{SOL} = \pi/6$, then:

$$U = \begin{pmatrix} \sqrt{3}/2 & 1/2 & 0 \\ -1/2\sqrt{2} & \sqrt{3}/2\sqrt{2} & 1/\sqrt{2} \\ 1/2\sqrt{2} & -\sqrt{3}/2\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \tag{283}$$

The Framework

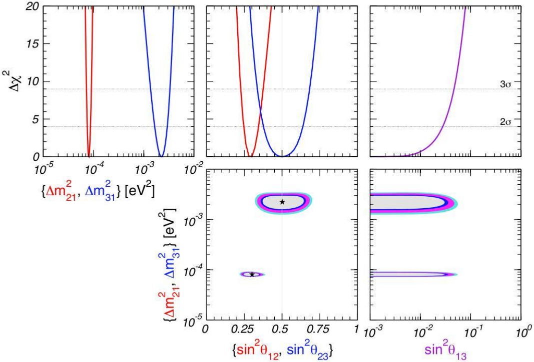
Now we are going to review experimental results without any assumption for $|U_{e3}|$. We use 5 free parameters:

$$\Delta m_{21} \quad \Delta m_{31} \quad \theta_{12} \quad \theta_{23} \quad \theta_{13} \tag{284}$$

So far experiments are not sensitive to the phase δ therefore we will use the mixing matrix:

$$U = R^{23}R^{13}R^{12}$$
$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13} & c_{12}c_{23} - s_{12}s_{23}s_{13} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13} & -c_{12}s_{23} - s_{12}c_{23}s_{13} & c_{23}c_{13} \end{pmatrix} \tag{285}$$

Global Results



Results Part 1

An analysis gives:

$$\Delta m^2_{21} = 7.92(1 \pm 0.09) \times 10^{-5}\ eV^2 \qquad \sin^2 2\theta_{12} = 0.314(1^{+0.18}_{-0.15}),$$
$$\Delta m^2_{23} = 2.4(1^{+0.21}_{-0.26}) \times 10^{-3}\ eV^2 \qquad \sin^2 2\theta_{23} = 0.44(1^{+0.41}_{-0.22}),$$
$$\sin^2 2\theta_{13} = 0.9(1^{+2.3}_{-0.9} \times 10^{-2}), \tag{286}$$

the range corresponds to 2σ. Other analysis give similar results. The mixing matrix for these values is:

$$|U|_{bf} = \begin{pmatrix} 0.82 & 0.56 & 0.09 \\ 0.31 - 0.43 & 0.51 - 0.59 & 0.75 \\ 0.37 - 0.47 & 0.59 - 0.66 & 0.66 \end{pmatrix} \tag{287}$$

Results Part 2

The ranges are a consequence of the lack of information about δ. In 2σ we have:

$$|U|_{2\sigma} = \begin{pmatrix} 0.78 - 0.86 & 0.51 - 0.61 & 0.00 - 0.18 \\ 0.19 - 0.57 & 0.39 - 0.73 & 0.61 - 0.80 \\ 0.20 - 0.47 & 0.40 - 0.74 & 0.59 - 0.79 \end{pmatrix} \quad (288)$$

Summary

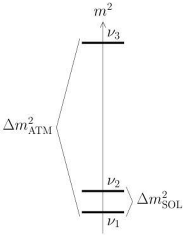
Lecture 4

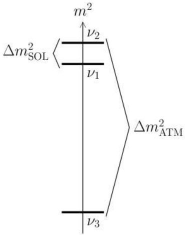
Direct Mass Measurement

Outline of Lecture 4

Neutrinos Hierarchy

NORMAL





INVERTED

The 2 Hierarchies

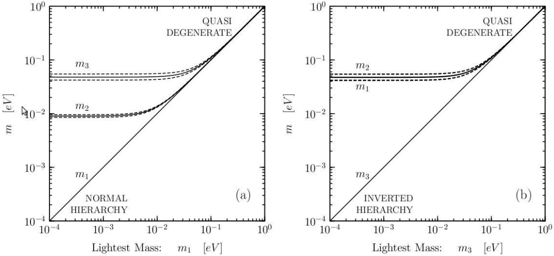
We can express neutrino masses in terms of the lightest neutrino mass. In the normal hierarchy:

$$\begin{aligned} m_2^2 &= m_1^2 + \Delta m_{21}^2 = m_1^2 + \Delta m_{SOL}^2 \\ m_3^2 &= m_1^2 + \Delta m_{31}^2 = m_1^2 + \Delta m_{ATM}^2. \end{aligned} \tag{289}$$

In the inverted hierarchy:

$$\begin{aligned} m_1^2 &= m_3^2 - \Delta m_{31}^2 = m_3^2 + \Delta m_{ATM}^2 \\ m_2^2 &= m_1^2 + \Delta m_{21}^2 = m_2^2 + \Delta m_{ATM}^2 + \Delta m_{SOL}^2. \end{aligned} \tag{290}$$

Neutrinos Hierarchy



Pattern Analysis

In both cases there is a degenerate region, where:

$$m_1 \simeq m_2 \simeq m_3 \simeq m_\nu, \quad m_\nu \gg \sqrt{\Delta m^2_{ATM}} \simeq 5 \times 10^{-2}\ eV \quad (291)$$

If the lightest mass is smaller than $\sqrt{\Delta m^2_{ATM}}$ the 2 patterns can be distinguished. In the normal hierarchy:

$$m_1 \ll m_2 \ll m_3, \quad (292)$$

In the inverted hierarchy:

$$m_3 \ll m_1 \simeq m_2 \quad (293)$$

In any case at least two neutrino has mass greater than $\sqrt{\Delta m^2_{SOL}} > 8 \times 10^{-3}\ eV$ and one of them has mass greater than $\Delta m^2_{ATM} > 4 \times 10^{-2}\ eV$.

Introduction

In beta decay the energy released becomes kinetic energy of the electron and neutrino energy.

$$Q_\beta = E_e + E_\nu . \tag{294}$$

When the neutrino is produced at rest the electron has its maximum energy:

$$E_{e-max} = Q_\beta - m_{\nu_e} \tag{295}$$

In allowed beta decays we have:

$$\begin{aligned} \frac{d\Gamma}{dE_e} = & \frac{G_F^2 m_e^5}{2\pi^3} \cos^2\theta_C |\mathcal{M}|^2 F(Z, E_e) E_e p_e \\ & \times (Q_\beta - E_e) \sqrt{(Q_\beta - E_e)^2 - m_{\nu_e}^2} . \end{aligned} \tag{296}$$

Curie Function

Unfortunately events near maximum are very rare. Tritium beta decay gives the best bound:

$$^{3}H \rightarrow \, ^{3}He + e^{-} + \overline{\nu}_e \quad Q_\beta = 18.754\, keV. \tag{297}$$

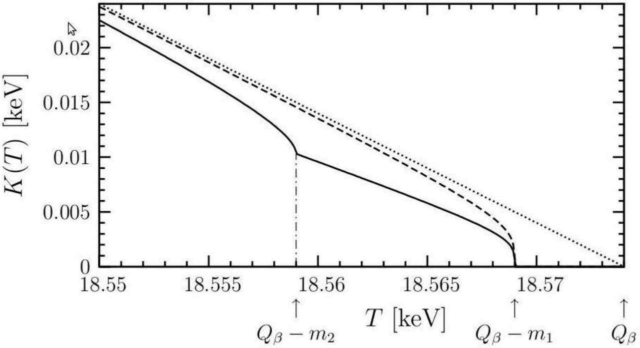
Curie function is defined as:

$$K(E_e) = \left[(Q_\beta - E_e) \sqrt{(Q_\beta - E_e)^2 - m_{\nu_e}^2} \right]^{1/2} \tag{298}$$

For massless neutrinos:

$$K(E_e) = Q_\beta - E_e, \tag{299}$$

Without Mixing



Results

This way Mainz & Troitzk collaborations got the following limits:

$$m_{\nu_e} < 2.3\ eV\ (95\% C.L.) \tag{300}$$

$$m_{\nu_e} < 2.5\ eV\ (95\% C.L.) \tag{301}$$

These collaboration are now joined and they work on the KATRIN experiments which has sensitivity down to 0.2 eV

Taking Mixing Into Account Part 1

We see the decay as:

$$^3H \rightarrow\ ^3He + e^- + \overline{\nu}_k \tag{302}$$

Now Curie function is defined as:

$$K(E_e) = \left[(Q_\beta - E_e) \sum_k |U_{ek}|^2 \sqrt{(Q_\beta - E_e) - m_k^2} \right]^{1/2} \tag{303}$$

The shift of the end point gives the mass of the lightest mass eigenstate:

$$m_{light} = Q_\beta - E_{e-max}. \tag{304}$$

There are kinks at the points:

$$E_{e-k} = Q_\beta - m_k \quad m_k \neq m_{light} \tag{305}$$

Taking Mixing Into Account Part 2

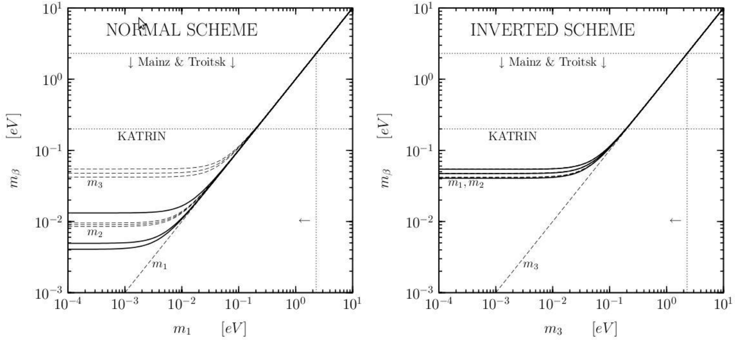
For $m_k \ll Q_\beta - E_e$ we have:

$$K^2 \simeq (Q_\beta - E_e)\sqrt{(Q - E_e)^2 - m_\beta^2} \quad m_\beta^2 = \sum_k |U_{ek}| m_k^2. \quad (306)$$

Thus m_β is the effective mass of the neutrino in beta decay:

$$m_\beta^2 = c_{12}^2 c_{13}^2 m_1^2 + s_{12}^2 c_{13}^2 m_2^2 + s_{13}^2 m_3^2. \quad (307)$$

With Mixing


NORMAL SCHEME
↓ Mainz & Troitsk ↓
KATRIN
m_3
m_2
m_1
m_β [eV]
m_1 [eV]
INVERTED SCHEME
↓ Mainz & Troitsk ↓
KATRIN
m_1, m_2
m_3
m_β [eV]
m_3 [eV]

Introduction

Double Beta Decay was proposed by M. Goeppert - Meyer back to 1935. It's the process:

$$\mathcal{N}(A,Z) \to \mathcal{N}(A,Z\pm 2) + e^{\mp} + 2^{\overline{\nu}_e}_{\nu_e}. \qquad (308)$$

Neutrinoless Double Beta Decay on the other hand is the process:

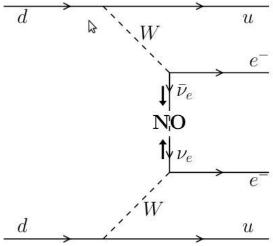
$$\mathcal{N}(A,Z) \to \mathcal{N}(A,Z\pm 2) + 2e^{\mp}. \qquad (309)$$

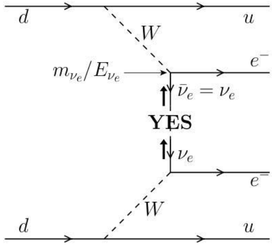
As a second order weak interaction it's extremely suppressed. Currently from Heidelberg - Moscow we have the bound:

$$T^{0\nu}_{1/2}(^{76}Ge) > 1.9 \times 10^{25}\ y \quad (90\% C.L.) \qquad (310)$$

Neutrinoless Double Beta Decay requires massive Majorana neutrinos. Positive helicity is proportional to m_{ν_e}/E_{ν_e}.

Basics


d
u
W
e⁻
$\bar{\nu}_e$
NO
ν_e
e⁻
W
d
u


d
u
W
e⁻
m_{ν_e}/E_{ν_e}
$\bar{\nu}_e = \nu_e$
YES
ν_e
e⁻
W
d
u

Effective Mass

The effective mass involved into the process is:

$$m_{2\beta} = \sum_k U_{ek}^2 m_k . \tag{311}$$

$$\begin{aligned} m_{2\beta} &= c_{12}^2 c_{13}^2 m_1 + e^{2\imath\lambda_2} s_{12}^2 c_{13}^2 m_2 + e^{2\imath(\lambda_3-\delta)} m_3 \\ &= |U_{e1}|^2 m_1 + e^{\imath\alpha_2}|U_{e2}|^2 m_2 + e^{\imath\alpha_3}|U_{e3}|^2 m_3 \\ \alpha_2 &= 2\lambda_2, \alpha_3 = 2(\lambda_3 - \delta) \end{aligned} \tag{312}$$

CP Conserving Cases

CP is conserved when $\delta = 0, \pi$ & $\lambda_k = 0, \pi/2, \pi, 3\pi/2$. As a result:

$$\alpha_k = 0, \pi \quad e^{\imath\alpha_k} = \pm 1. \tag{313}$$

There are 4 cases:

$$(++) \quad m_{2\beta} = |U_{e1}|^2 m_1 + |U_{e2}|^2 m_2 + |U_{e3}|^2 m_3 \tag{314}$$

$$(+-) \quad m_{2\beta} = |U_{e1}|^2 m_1 + |U_{e2}|^2 m_2 - |U_{e3}|^2 m_3 \tag{315}$$

$$(-+) \quad m_{2\beta} = |U_{e1}|^2 m_1 - |U_{e2}|^2 m_2 + |U_{e3}|^2 m_3 \tag{316}$$

$$(--) \quad m_{2\beta} = |U_{e1}|^2 m_1 - |U_{e2}|^2 m_2 - |U_{e3}|^2 m_3 \tag{317}$$

Maximal effective mass is achieved in ++ case:

$$m_{2\beta}^{max} = \sum_k |U_{ek}|^2 m_k \tag{318}$$

Halflife and Matrix Elements

Neutrinoless Double Beta Decay halflife is:

$$\left[T^{0\nu}_{1/2}(\mathcal{N})\right]^{-1} = G^{\mathcal{N}}_{0\nu}|\mathcal{M}^{\mathcal{N}}_{0\nu}|^2\frac{|m_{2\beta}|^2}{m_e^2}, \tag{319}$$

where $G^{\mathcal{N}}_{0\nu}$ & $\mathcal{M}^{\mathcal{N}}_{0\nu}$ are the phase space factor and nuclear matrix element respectivly. The phase space factor can be calculation with great accuracy, but there are uncertainties into the calculation. For ^{76}Ge we have:

$$G^{^{76}Ge}_{0\nu} = 6.31\times 10^{-15}y^{-1} \tag{320}$$

$$1.5 \leq |\mathcal{M}^{^{76}Ge}_{0\nu}| \leq 4.6 \tag{321}$$

As a result :

$$|m_{2\beta}| \leq 0.3 - 1.0\ eV. \tag{322}$$

Normal Hierarchy Part 1

The lightest mass is m_1, then:

$$m_{2\beta} = |U_{e1}|^2 m_1 + e^{\imath\alpha_2}|U_{e2}|^2\sqrt{m_1^2 + \Delta m^2_{SOL}} + e^{\imath\alpha_3}|U_{e3}|^2\sqrt{m_1^2 + \Delta m^2_{ATM}} \tag{323}$$

The last term can be neglected as $|U_{e3}| \ll |U_{e1}|, |U_{e2}|$. In the degenerate region $m_1 \gg \Delta m^2_{ATM}$ thus:

$$m_{2\beta} \simeq m_1 \left(|U_{e1}|^2 + e^{\imath\alpha_2}|U_{e2}|\right) \tag{324}$$

In the 4 CP concerving cases:

$$(++), (+-) \quad m_{2\beta} \simeq m_1 \tag{325}$$

$$(-+), (--) \quad m_{2\beta} \simeq m_1 \left(|U_{e1}|^2 - |U_{e2}|^2\right) \simeq m_1 \cos 2\theta_{12}. \tag{326}$$

Normal Hierarchy Part 2

In the hierarchical region:

$$m_2 \simeq \sqrt{\Delta m^2_{SOL}} \quad m_3 \simeq \sqrt{\Delta m^2_{ATM}} \tag{327}$$

Using the experimanetal values we observe that the effective mass may vanish in the cases $(-+)$ & $(--)$, if

$$m_1 = \tan^2\theta_{12}\sqrt{\Delta m^2_{SOL}} \simeq 4\times 10^{-3} \rightarrow m_{2\beta} = 0 \tag{328}$$

For even smaller m_1 we have:

$$m_{2\beta} \simeq |U_{e2}|^2\sqrt{\Delta m^2_{SOL}} \simeq 2.7\times 10^{-3} \tag{329}$$

In the normal hierarchy in the hierarchical region there is no lower bound, but there is an upper bound:

$$|m_{2\beta}| \leq 6\times 10^{-3}\ eV \quad m_1 \leq 10^{-3}\ eV \tag{330}$$

Inverted Hierarchy

In the inverted hierarchy $m_1 \simeq m_2 \gg m_3$, then:

$$m_{2\beta} = \left(|U_{e1}|^2 + e^{\imath\alpha_2}|U_{e2}|^2\right)\sqrt{m_3^2 + \Delta m^2_{ATM}} + e^{\imath\alpha_3}|U_{e3}|^2 m_3 \tag{331}$$

Again the last term can be neglected, but now the effective mass cann't vanish. In the degenerate region everything is as in normal hierarchy if we substitute m_3 to m_1. In the hierarchical region in the 4 CP conserving cases:

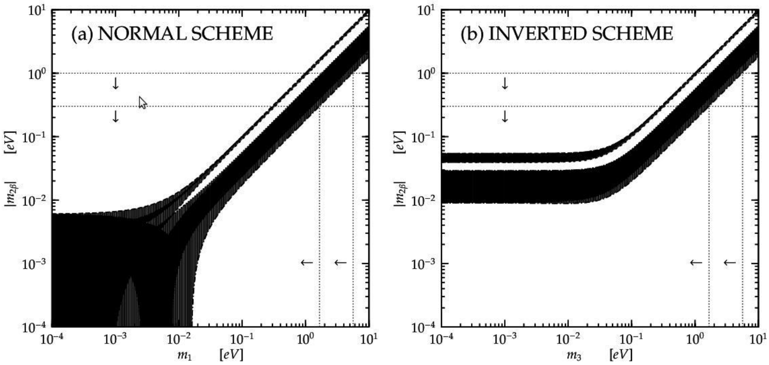
$$(++),(+-) \quad m_{2\beta} \simeq \sqrt{\Delta m^2_{ATM}} \tag{332}$$

$$(-+),(--) \quad m_{2\beta} \simeq \sqrt{\Delta m^2_{ATM}}\left(|U_{e1}|^2 - |U_{e2}|^2\right) \simeq \sqrt{\Delta m^2_{ATM}}\cos 2\theta_{12} \tag{333}$$

The effective mass is bounded:

$$9\times 10^{-3}\ eV \leq |m_{2\beta}| \leq 5\times 10^{-2}\ eV \quad m_3 \leq 10^{-2}\ eV. \tag{334}$$

Inverted Hierarchy



Pion & Tau Decays

There are bounds on neutrino mass from pion and tau decays. This bound are not so strickt but their importance was that they excluded the existance of neutrino heavier than the bound. From pion decays we got the bound:

$$m_k < 0.17\ MeV \quad (90\% C.L.), \tag{335}$$

while from tau decays we got:

$$m_k < 18.2\ MeV \quad (90\% C.L.) \tag{336}$$

Supernova & Cosmology

Analysis based on data from SN1987A implied the model independent bound:

$$m_k \leq 30\ eV, \tag{337}$$

If the analysis is performed based on assumption the bound varies between 5 to 30 eV.
Global analysis of cosmological data set a bound for the sum of neutrino masses:

$$\sum_j m_j \leq 0.5 \sim 1.0\ eV \tag{338}$$

Summary

That's All Folks!