# Limit behavior of the $q$-Pólya urn *

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**Abstract**

The $q$-Pólya urn is a $q$-analog of the Pólya urn and is a model of ball extraction from an urn with balls of two colors, A and B. Balls of color B have priority to be picked over those of color A. We prove that, in an infinite sequence of extractions, almost surely, the number of balls of color A that are picked has a finite limit and we identify its distribution. Then we prove functional limit theorems for the number of balls of color A extracted. The limit is either a pure birth process or a diffusion, depending on the initial composition of the urn. Finally, we discuss basic results for the $q$-Pólya urn with more than two colors.

## 1 Introduction and results

**The Pólya urn.** This is the model where in an urn that has initially a finite number of white and black balls we draw, successively and uniformly at random, a ball from it and then we return the ball back together with $k$ balls of the same color as the one drawn. The number $k \in \mathbb{N}^+$ is fixed.

Standard references for the theory and the applications of Pólya urn and related models are [14] and [17].

**The $q$-Pólya urn**. This is a $q$-analog of the Pólya urn (see [10], [15] for more on $q$-analogs) introduced in [16] and studied further in [4] (see also [5]).

A $q$-analog of a mathematical object $A$ is another object $A(q)$ so that when $q \to 1$, $A(q)$ "tends" to $A$. Take $q \in (0, \infty)\backslash\{1\}$. The $q$-analog of any $x \in \mathbb{C}$ is defined as

$$[x]_q := \frac{q^x - 1}{q - 1}. \tag{1.1}$$

Note that $\lim_{q\to 1}[x]_q = x$.

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Now consider an urn that contains a finite number of white and black balls. We perform a sequence of additions of balls in the urn according to the following rule. If at a given time the urn contains $A_1$ white and $A_2$ black balls ($A_1, A_2 \in \mathbb{N}, A_1 + A_2 > 0$), then we add $k$ white balls with probability

$$\mathbf{P}_q(\text{white}) = \frac{[A_1]_q}{[A_1 + A_2]_q}. \tag{1.2}$$

Otherwise, we add $k$ black balls, and this has probability

$$\mathbf{P}_q(\text{black}) = 1 - \mathbf{P}_q(\text{white}) = q^{A_1} \frac{[A_2]_q}{[A_1 + A_2]_q}. \tag{1.3}$$

This stochastic process we call $q$-Pólya urn. To understand how it works, it helps to realize the probabilities $\mathbf{P}_q(\text{white})$, $\mathbf{P}_q(\text{black})$ through the following experiment.

If $q \in (0, 1)$, then we put the balls in a line with the $A_1$ white coming first and the $A_2$ black following. To pick a ball, we go through the line, starting from the beginning and picking each ball with probability $1 - q$ independently of what happened with the previous balls. If we finish the line without picking a ball, we start from the beginning. Once we pick a ball, we return it to its position together with $k$ balls of the same color. Given these rules, the probability of picking a white ball is

$$(1 - q^{A_1}) \sum_{j=0}^{\infty} (q^{A_1 + A_2})^j = \frac{1 - q^{A_1}}{1 - q^{A_1 + A_2}} = \frac{[A_1]_q}{[A_1 + A_2]_q}, \tag{1.4}$$

which is (1.2), because before picking a white ball, we will go through the entire list a random number of times, say $j$, without picking any ball and then, going through the white balls, we pick one (probability $1 - q^{A_1}$).

If $q > 1$, we place in the line first the black balls and we go through the list picking each ball with probability $1 - q^{-1}$. According to the above computation, the probability of picking a black ball is

$$\frac{[A_2]_{q^{-1}}}{[A_1 + A_2]_{q^{-1}}} = q^{A_1} \frac{[A_2]_q}{[A_1 + A_2]_q},$$

which is (1.3).

We extend the notion of drawing a ball from a $q$-Pólya urn to the case where exactly one of $A_1, A_2$ is infinity. Then the probability to pick a white (resp. black) ball is determined again by (1.2) (resp. (1.3)), where this is understood as the limit of the right hand side as $A_1$ or $A_2$ goes to $\infty$. For example, assuming that $A_1 = \infty$ and $A_2 \in \mathbb{N}$, we have $\mathbf{P}_q(\text{white}) = 1$ if $q < 1$ and $\mathbf{P}_q(\text{white}) = q^{-A_2}$ if $q > 1$. Again these probabilities are realized through the experiment described above. Thus, we can run the process even if we start with an infinite number of balls from one color and finite from the other.

Consider now a $q$–Pólya urn having $A_1(0), A_2(0)$ white and black balls respectively and start an infinite sequence of drawings. For $n \in \mathbb{N}^+$, denote by $A_1(n), A_2(n)$ the numbers of white and black balls respectively after $n$ drawings.

We want to study two aspects of the asymptotic behavior of the sequence $\{A_1(n)\}_{n \in \mathbb{N}}$.

1) The first concerns the limit, in any sense, of $A_1(n)$ properly normalized. In the Pólya urn, if we keep the same notation, the following convergence in distribution is a well known

fact: $\frac{A_1(n)}{A_1(n)+A_2(n)} \xrightarrow{d} \text{Beta}\left(A_1(0)/k, A_2(0)/k\right)$ as $n \to \infty$. For the $q$–Pólya urn, things are less exciting. If $q > 1$, after some point, we will be drawing only black balls, and consequently $A_1(n)$ becomes eventualy (a random) constant $A_1(\infty)$. We identify the distribution of $A_1(\infty)$. By the above discussion, this answers the case $q \in (0,1)$ too. Then, it is $A_2(n)$ that becomes eventualy constant.

2) The second concerns the entire path $\{A_1(n)\}_{n\in\mathbb{N}}$. Is it possible, by applying appropriate, natural transformations, to get convergence to a stochastic process? That is, an analogous result to Donsker's theorem for simple symmetric random walk in $\mathbb{Z}$. For the Pólya urn, this question has been investigated in the works [7], [3] .

The results concering these two points are exhibited in the following two subsections.

### 1.1 Basic results for the $q$-Pólya urn

We recall some notation from $q$-calculus (see [5], [15]). For $q \in (0,\infty)\backslash\{1\}, x \in \mathbb{C}, k \in \mathbb{N}^+$, we define

$$[x]_q := \frac{q^x - 1}{q - 1} \qquad \text{the } q\text{-number of } x, \tag{1.5}$$

$$[k]_q! := [k]_q[k-1]_q \cdots [1]_q \qquad \text{the } q\text{-factorial}, \tag{1.6}$$

$$[x]_{k,q} := [x]_q[x-1]_q \cdots [x-k+1]_q \qquad \text{the } q\text{-factorial of order } k, \tag{1.7}$$

$$\begin{bmatrix} x \\ k \end{bmatrix}_q := \frac{[x]_{k,q}}{[k]_q!} \qquad \text{the } q\text{-binomial coefficient} \tag{1.8}$$

$$(x;q)_\infty := \prod_{i=0}^{\infty}(1 - xq^i) \text{ when } q \in [0,1) \qquad \text{the } q\text{-Pochhammer symbol}, \tag{1.9}$$

We extend these definitions in the case $k = 0$ by letting $[0]_q! = 1, [x]_{0,q} = 1$.

Now consider a $q$-Pólya urn that has initially $A_1$ white and $A_2$ black balls, where $A_1 \in \mathbb{N} \cup \{\infty\}$ and $A_2 \in \mathbb{N}$. Call $H_1(n)$ the number of drawings that give white ball in the first $n$ drawings. Its distribution is specified by the following.

**Fact 1:** Let $\hat{A}_1 := A_1/k$ and $\hat{A}_2 := A_2/k$.
(i) If $A_1 \in \mathbb{N}$, then the probability mass function of $H_1(n)$ is

$$\mathbf{P}\left(H_1(n) = x\right) = q^{k(n-x)(\hat{A}_1+x)} \frac{\begin{bmatrix} -\hat{A}_1 \\ x \end{bmatrix}_{q^{-k}} \begin{bmatrix} -\hat{A}_2 \\ n-x \end{bmatrix}_{q^{-k}}}{\begin{bmatrix} -\hat{A}_1-\hat{A}_2 \\ n \end{bmatrix}_{q^{-k}}} \tag{1.10}$$

$$= q^{-A_2 x} \frac{\begin{bmatrix} \hat{A}_1+x-1 \\ x \end{bmatrix}_{q^{-k}} \begin{bmatrix} \hat{A}_2+n-x-1 \\ n-x \end{bmatrix}_{q^{-k}}}{\begin{bmatrix} \hat{A}_1+\hat{A}_2+n-1 \\ n \end{bmatrix}_{q^{-k}}} \tag{1.11}$$

$$= q^{-kx(\hat{A}_2+n-x)} \frac{\begin{bmatrix} -\hat{A}_1 \\ x \end{bmatrix}_{q^k} \begin{bmatrix} -\hat{A}_2 \\ n-x \end{bmatrix}_{q^k}}{\begin{bmatrix} -\hat{A}_1-\hat{A}_2 \\ n \end{bmatrix}_{q^k}} \tag{1.12}$$

for all $x \in \{0, 1, \ldots, n\}$.
(ii) If $A_1 = \infty$ and $q > 1$, then the probability mass function of $H_1(n)$ is

$$\mathbf{P}\left(H_1(n) = x\right) = q^{-A_2 x}(1 - q^{-k})^{n-x} \begin{bmatrix} \hat{A}_2 + n - x - 1 \\ n - x \end{bmatrix}_{q^{-k}} \frac{[n]_{q^{-k}}!}{[x]_{q^{-k}}!} \tag{1.13}$$

for all $x \in \{0, 1, \dots, n\}$.

If $A_1 = \infty$ and $q \in (0,1)$, then $\mathbf{P}(H_1(n) = n) = 1$ obviously. Relation (1.10) is (3.2) in [4] where it is proved through recursion. In Section 2 we give an alternative proof.

According to the experiment described in Section 1, the balls that are placed first in the line have an advantage to be picked (the white if $q \in (0,1)$, the black if $q > 1$). In fact, this leads to the extinction of drawings from the balls of the other color; there is a point after which the number of balls in the urn of that color stays fixed to a random number. In the next theorem, we identify the distribution of this number. We treat the case $q > 1$.

**Theorem 1.1** (Extinction of the second color)**.** *Assume that $q > 1, A_1 \in \mathbb{N} \cup \{\infty\}, A_2 \in \mathbb{N}$. As $n \to \infty$, with probability one, $\{H_1(n)\}_{n \ge 1}$ converges to a random variable $H_1(\infty)$ with values in $\mathbb{N}$ and probability mass function*
*(i)*

$$f(x) = q^{-A_2 x} \begin{bmatrix} \frac{A_1}{k} + x - 1 \\ x \end{bmatrix}_{q^{-k}} \frac{(q^{-A_2}; q^{-k})_\infty}{(q^{-A_1 - A_2}; q^{-k})_\infty} \tag{1.14}$$

*for all $x \in \mathbb{N}$ in the case $A_1 \in \mathbb{N}$ and*
*(ii)*

$$f(x) = \left( \frac{q^{-A_2}}{1 - q^{-k}} \right)^x \frac{1}{[x]_{q^{-k}}!} (q^{-A_2}; q^{-k})_\infty \tag{1.15}$$

*for all $x \in \mathbb{N}$ in the case $A_1 = \infty$.*

When $A_1 \in \mathbb{N}$ and $k | A_1$, $H_1(\infty)$ has the negative $q$-binomial distribution of the second kind with parameters $A_1/k, q^{-A_2}, q^{-k}$ (see §3.1 in [5] for its definition). When $A_1 = \infty$, $H_1(\infty)$ has the Euler distribution with parameters $q^{-A_2}/(1 - q^{-k}), q^{-k}$ (see §3.3 in [5] again).

### 1.2 Functional scaling limits

Consider a $q$–Pólya urn whose initial composition depends on $m \in \mathbb{N}^+$. That is, it has $A_1^{(m)}(0), A_2^{(m)}(0)$ white and black balls respectively. Start an infinite sequence of drawings and for $n \in \mathbb{N}^+$, denote by $A_1^{(m)}(n), A_2^{(m)}(n)$ the numbers of white and black balls respectively after $n$ drawings.
To see a new process arising out of the path of $\{A_1^{(m)}(n)\}_{n \ge 0}$, we start with an initial number of balls that tends to infinity as $m \to \infty$. We assume that $A_2^{(m)}(0)$ grows linearly with $m$. Regarding $A_1^{(m)}(0)$, we study three regimes:

a) $A_1^{(m)}(0)$ stays fixed with $m$.

b) $A_1^{(m)}(0)$ grows to infinity but sublinearly with $m$.

c) $A_1^{(m)}(0)$ grows linearly with $m$.

The regime where $A_1^{(m)}(0)$ grows superlinearly with $m$ follows from regime b) by changing the roles of the two colors. We remark on this after Theorem 1.3.

The other parameter that we have to tune is $q$. If $q$ is kept fixed, then:

(i) if $q > 1$, then nothing interesting happens because the assumption $\lim_{m \to \infty} A_2^{(m)}(0) = \infty$ implies that the process $\{A_1^{(m)}(n)\}_{n \ge 0}$ converges (as $m \to \infty$) to the one that never increases (we always pick a black ball) and

(ii) if $q < 1$, then in the scenario $\lim_{m\to\infty} A_1^{(m)}(0) = \infty$ the situation is analogous to (i) while in the scenario that $A_1^{(m)}(0)$ stays fixed with $m$ the process $\{A_1^{(m)}(n)\}_{n\geq 0}$ converges (as $m \to \infty$) to the $q$-Polya urn with $A_2 = \infty$.

Interesting limits appear once we take $q = q_m$ to depend on $m$ and approach 1 as $m \to \infty$. We study the case that $q_m > 1$ and the distance of $q_m$ from 1 is $\Theta(1/m)$ and remark on the case that the distance is $o(1/m)$.

In the regimes a) and b), the scarcity of white balls has as a result that the time between two consecutive drawings of a white ball is large. We expect then that speeding up time by an appropriate factor we will see a birth process. And indeed this is the case as our first two theorems show.

All processes appearing in this work with index set $[0, \infty)$ and values in some Euclidean space $\mathbb{R}^d$ are elements of $D_{\mathbb{R}^d}[0, \infty)$, the space of functions $f : [0, \infty) \to \mathbb{R}^d$ that are right continuous and have limits from the left at each point of $[0, \infty)$. This space is endowed with the Skorokhod topology (defined in §5 of Chapter 3 of [9]), and convergence in distribution of processes with values on that space is defined through that topology.

We remind the reader that the negative binomial distribution with parameters $\nu \in (0, \infty)$ and $p \in (0, 1)$ is the distribution with support in $\mathbb{N}$ and probability mass function

$$f(x) = \binom{x + \nu - 1}{x} p^\nu (1 - p)^x \tag{1.16}$$

for all $x \in \mathbb{N}$. When $\nu \in \mathbb{N}^+$, this is the distribution of the number of failures until we obtain the $\nu$-th success in a sequence of independent trials, each having probability of success $p$. For a random variable $X$ with this distribution, we write $X \sim NB(\nu, p)$.

In all results of this subsection we assume that the parameter of the urn is $q_m = c^{1/m}$ with $c > 1$.

**Theorem 1.2.** *Fix $w_0 \in \mathbb{N}^+$ and $b > 0$. If $A_1^{(m)}(0) = w_0$ and $\lim_{m\to\infty} A_2^{(m)}(0)/m = b$, then the process $(k^{-1}\{A_1^{(m)}([mt]) - A_1^{(m)}(0)\})_{t\geq 0}$ converges in distribution as $m \to \infty$ to an inhomogeneous in time pure birth process $Z$ with $Z(0) = 0$ and such that for all $0 \leq t_1 < t_2, j \in \mathbb{N}$, the random variable*

$$Z(t_2) - Z(t_1)|Z(t_1) = j \text{ has distribution } NB\Big(\frac{w_0}{k} + j, \frac{1 - c^{-b-kt_1}}{1 - c^{-b-kt_1}}\Big).$$

*In particular, $Z$ has rates*

$$\lambda_{t,j} = \frac{w_0 + jk}{c^{b+kt} - 1} \log c \tag{1.17}$$

*for all $(t, j) \in [0, \infty) \times \mathbb{N}$.*

**Theorem 1.3.** *Assume that $A_1^{(m)}(0) = g_m$ and $\lim_{m\to\infty} A_2^{(m)}(0)/m = b$, where $b \in (0, \infty)$ and $g_m \in \mathbb{N}^+, g_m \to \infty, g_m = o(m)$ as $m \to \infty$. Then the process*

$$(k^{-1}\{A_1^{(m)}([tm/g_m]) - A_1^{(m)}(0)\})_{t\geq 0}$$

*converges in distribution, as $m \to \infty$, to the Poisson process on $[0, \infty)$ with rate*

$$\frac{\log c}{c^b - 1}. \tag{1.18}$$

We return to the discussion at the beginning of the subsection. The regime where $\lim_{m\to\infty} A_2^{(m)}(0)/m = b > 0$ and $A_1^{(m)}(0)/m \to \infty$ is covered by the previous theorem. We need to change the roles of the colors and remark that the role of $m$ as a scaling parameter is played now by $a_m := A_1^{(m)}(0)$. The result that we obtain is that in the $q$-Pólya urn with $q_m := c^{1/a_m}$ and $c > 1$, the process

$$\frac{1}{k}\left(A_2^{(m)}([ta_m/(bm)]) - A_2^{(m)}(0)\right)_{t\geq 0}$$

converges in distribution, as $m \to \infty$, to the Poisson process on $[0,\infty)$ with rate $(\log c)/(c-1)$.

**Theorem 1.4.** *Assume that* $A_1^{(m)}(0), A_2^{(m)}(0)$ *are such that* $\lim_{m\to\infty} A_1^{(m)}(0)/m = a$, $\lim_{m\to\infty} A_2^{(m)}(0)/m = b$, *where* $a, b \in [0,\infty)$ *are not both zero. Then the process* $\left(A_1^{(m)}([mt])/m\right)_{t\geq 0}$ *converges in distribution, as* $m \to +\infty$, *to the unique solution of the differential equation*

$$X_0 = a, \tag{1.19}$$

$$dX_t = k\frac{1 - c^{X_t}}{1 - c^{a+b+kt}}dt, \tag{1.20}$$

*which is*

$$X_t := a - \frac{1}{\log c}\log\left(\frac{c^b - 1 + c^{-kt}(1 - c^{-a})}{c^b - c^{-a}}\right) \tag{1.21}$$

*for all* $t \geq 0$.

Next, we determine the fluctuations of the process $(A_1^{(m)}([mt])/m)_{t\geq 0}$ around its $m \to \infty$ limit, $X$. Let

$$C_t^{(m)} = \sqrt{m}\left(\frac{A_1^{(m)}([mt])}{m} - X_t\right) \tag{1.22}$$

for all $m \in \mathbb{N}^+$ and $t \geq 0$.

**Theorem 1.5.** *Let* $a, b \in [0,\infty)$, *not both zero,* $\theta_1, \theta_2 \in \mathbb{R}$, *and assume that* $A_1^{(m)}(0) := [am + \theta_1\sqrt{m}], A_2^{(m)}(0) = [bm + \theta_2\sqrt{m}]$ *for all large* $m \in \mathbb{N}$. *Then the process* $(C_t^{(m)})_{t\geq 0}$ *converges in distribution, as* $m \to \infty$, *to the unique solution of the stochastic differential equation*

$$\begin{aligned}
Y_0 &= \theta_1, \\
dY_t &= \frac{k\log c}{c^{a+b+kt} - 1}\left\{\frac{(c^{a+b} - 1)Y_t - c^b(c^a - 1)(\theta_1 + \theta_2)}{c^b - 1 + c^{-kt}(1 - c^{-a})}\right\}dt \\
&\quad + k\sqrt{(c^a - 1)(c^b - 1)}\frac{c^{(a+kt)/2}}{c^{a+b+kt} - c^{a+kt} + c^a - 1}dW_t,
\end{aligned} \tag{1.23}$$

*which is*

$$\begin{aligned}
Y_t =& \frac{c^{a+b+kt} - 1}{c^{a+b+kt} - c^{a+kt} + c^a - 1}\Bigg(\theta_1 - (\theta_1 + \theta_2)\frac{c^{a+b}(c^a - 1)}{c^{a+b} - 1}\frac{c^{kt} - 1}{c^{a+b+kt} - 1} \\
&+ k\sqrt{(c^a - 1)(c^b - 1)}\int_0^t \frac{c^{(a+kt)/2}}{c^{a+b+kt} - 1}\,dW_s\Bigg)
\end{aligned} \tag{1.24}$$

*for all* $t \geq 0$. $W$ *is a standard Brownian motion*

**Remark 1.1.** *If we assume that* $q = q(m) := c^{\varepsilon_m/m}$ *where* $c > 1$ *and* $\varepsilon_m \to 0^+$ *as* $m \to \infty$*, then* $q = 1 + o(m^{-1})$*. With computations analogous to those of the results of the previous subsection, it is easy to see that the limits of the processes considered in all theorems of this subsection coincide with those in the case of the plain Pólya urn (i.e., when* $q = 1$*), which are described in the work [7]. Of course, in* (1.22)*, the role of* $X_t$ *will be played by the limit one gets from the analogous to Theorem 1.4.*

## 1.3 $q$-Pólya urn with many colors

In this paragraph, we give a $q$-analog for the Pólya urn with more than two colors. The way to do the generalization is inspired by the experiment we used in order to explain relation (1.2).

Let $l \in \mathbb{N}, l \geq 2$, and $q \in (0, 1)$. Assume that we have an urn containing $A_i$ balls of color $i$ for each $i \in \{1, 2, \dots, l\}$. To draw a ball from the urn, we do the following. We order the balls in a line, first those of color 1, then those of color 2, and so on. Then we visit the balls, one after the other, in the order that they have been placed, and we select each with probability $1 - q$ independently of what happened with the previous balls. If we go through all balls without picking any, we repeat the same procedure starting from the beginning of the line. Once a ball is selected, the drawing is completed. We return the ball to its position together with another $k$ of the same color. For each $i = 0, 1, \dots, l$, let $s_i = \sum_{1 \leq j \leq i} A_j$. Notice that $s_l$ is the total number of balls in the urn. Then, working as for (1.4), we see that

$$\mathbf{P}(\text{color } i \text{ is drawn}) = q^{s_{i-1}} \frac{1 - q^{A_i}}{1 - q^{s_l}} = \frac{q^{s_{i-1}} - q^{s_i}}{1 - q^{s_l}} = q^{s_{i-1}} \frac{[A_i]_q}{[s_l]_q}. \tag{1.25}$$

Call $p_i$ the number in the last display for all $i = 1, 2, \dots, l$. Note that when $q \to 1$, $p_i$ converges to $A_i/s_l$, which is the probability for the usual Pólya urn with $l$ colors. It is clear that for any given $q \in (0, \infty) \backslash \{1\}$, the numbers $p_1, p_2, \dots, p_l$ are non-negative and add to 1 (the second fraction in (1.25) shows this). We define then for this $q$ the $q$-Pólya urn with colors $1, 2, \dots, l$ to be the sequential procedure in which, at each step, we add $k$ balls of a color picked randomly among $\{1, 2, \dots, l\}$ so that the probability that this color is $i$ is $p_i$ .

When $q > 1$, these probabilities come out of the experiment described above but in which we place the balls in reverse order (that is, first those of color $l$, then those of color $l - 1$, and so on) and we go through the list selecting each ball with probability $1 - q^{-1}$. It is then easy to see that the probability to pick a ball of color $i$ is $p_i$.

**Theorem 1.6.** *Assume that* $q \in (0, 1)$ *and that we start with* $A_1, A_2, \dots, A_l$ *balls from colors 1, 2,* $\dots, l$ *respectively, where* $A_1, A_2, \dots, A_l \in \mathbb{N}$ *are not all zero. Call* $H_i(n)$ *the number of times in the first* $n$ *drawings that we picked color* $i$*. The probability mass function for the* *vector* $(H_2(n), H_3(n), \dots, H_l(n))$ *is*

$$\mathbf{P}\left(H_2(n) = x_2, \dots, H_l(n) = x_l\right) = q^{\sum_{i=2}^{l} x_i \sum_{j=1}^{i-1}(A_j + kx_j)} \frac{\prod_{i=1}^{l} \left[{-\frac{A_i}{k} \atop x_i}\right]_{q^{-k}}}{\left[{-\frac{A_1+A_2\dots+A_l}{k} \atop n}\right]_{q^{-k}}} \tag{1.26}$$

$$= \begin{bmatrix} n \\ x_1, x_2, \dots, x_l \end{bmatrix}_{q^{-k}} \frac{q^{\sum_{i=2}^{l} x_i \sum_{j=1}^{i-1}(A_j + kx_j)} \prod_{i=1}^{l} \left[-\frac{A_i}{k}\right]_{x_i, q^{-k}}}{\left[-\frac{A_1+A_2+\dots+A_l}{k}\right]_{n, q^{-k}}} \tag{1.27}$$

*for all* $x_2, \dots, x_l \in \{0, 1, 2, \dots, n\}$ *with* $x_2 + \cdot + x_l \le n$*, where* $x_1 := n - \sum_{i=2}^{l} x_i$ *and* $\left[{n \atop x_1, x_2, \dots, x_l}\right]_{q^{-k}} := \frac{[n]_{q^{-k}}!}{[x_1]_{q^{-k}}! \cdot \dots \cdot [x_l]_{q^{-k}}!}$ *is the q-multinomial coefficient.*

It follows from Theorem 1.1 that when $q \in (0, 1)$, after some random time, we will be picking only balls of color 1. So that the number of times, say $H_i$, that we pick color $i$, where $i = 2, 3, \dots, l$, is finite. The next theorem identifies the joint distribution of $H_2, H_3, \dots, H_l$.

**Theorem 1.7.** *Under the assumptions of Theorem 1.6, as* $n \to +\infty$*, with probability one, the vector* $(H_2(n), H_3(n), \dots, H_l(n))$ *converges to a random vector* $(H_2(\infty), H_3(\infty), \dots, H_l(\infty))$ *with values in* $\mathbb{N}^{l-1}$ *and probability mass function*

$$f(x_2, x_3, \dots, x_l) = q^{\sum_{i=2}^{l} x_i \sum_{j=1}^{i-1} A_j} \prod_{i=2}^{l} \begin{bmatrix} x_i + \frac{A_i}{k} - 1 \\ x_i \end{bmatrix}_{q^k} \frac{(q^{A_1}; q^k)_\infty}{(q^{A_1+\dots+A_l}; q^k)_\infty} \tag{1.28}$$

*for all* $x_2, \dots, x_l \in \mathbb{N}$.

Note that the random variables $H_2(\infty), \dots, H_l(\infty)$ are independent although $(H_2(n), H_3(n), \dots, H_l(n))$ are dependent.

Next, we look for a scaling limit for the path of the process. For each $m \in \mathbb{N}^+$, we consider a $q$-Pólya urn with initial composition $(A_1^{(m)}(0), A_2^{(m)}(0), \dots, A_l^{(m)}(0))$ and $q_m$ that will be specified below. Let $A_i^{(m)}(j)$ be the number of balls of color $i$ in this urn after $j$ drawings.

**Theorem 1.8.** *Assume that* $c \in (0, 1), q_m = c^{1/m}$ *for all* $m \in \mathbb{N}^+$*, and*

$$\frac{1}{m}\left(A_1^{(m)}(0), A_2^{(m)}(0), \dots, A_l^{(m)}(0)\right) \overset{m\to\infty}{\to} (a_1, a_2, \dots, a_l),$$

*where* $a_1, \dots, a_l \in [0, \infty)$ *are not all zero. Set* $\sigma_0 = 0$ *and* $\sigma_i := \sum_{j \le i} a_j$ *for all* $i = 1, 2, \dots, l$. *Then the process* $\frac{1}{m}\left(A_1^{(m)}([mt]), A_2^{(m)}([mt]), \dots, A_l^{(m)}([mt])\right)_{t \ge 0}$ *converges in distribution, as* $m \to +\infty$*, to* $(X_{t,1}, X_{t,2}, \dots, X_{t,l})_{t \ge 0}$ *with*

$$X_{t,i} = a_i + \frac{1}{\log c} \log \frac{(1 - c^{\sigma_l + kt}) - c^{\sigma_{i-1}}(1 - c^{kt})}{(1 - c^{\sigma_l + kt}) - c^{\sigma_i}(1 - c^{kt})} \tag{1.29}$$

*for all* $i = 1, 2, \dots, l$.

**Theorem 1.9.** *Assume that* $c \in (0, 1), q_m = c^{\varepsilon_m/m}$ *for all* $m \in \mathbb{N}^+$ *with* $\lim_{m\to\infty} \varepsilon_m = 0$*, and*

$$\frac{1}{m}\left(A_{0,1}^{(m)}, A_{0,2}^{(m)}, \dots, A_{0,l}^{(m)}\right) \overset{m\to\infty}{\to} (a_1, a_2, \dots, a_l),$$

*where $a_1, \dots, a_l \in [0, \infty)$ are not all zero. Then the process $\frac{1}{m}\big(A^{(m)}_{[mt],1}, A^{(m)}_{[mt],2}, \dots, A^{(m)}_{[mt],l}\big)_{t\geq 0}$ converges in distribution, as $m \to +\infty$, to $(X_t)_{t\geq 0}$ with*

$$X_t = \left(1 + \frac{kt}{a_1 + \cdots + a_l}\right)(a_1, a_2, \dots, a_l) \tag{1.30}$$

*for all $t \geq 0$.*

**Remark**. Discussing this preprint with Prof. Ch. Charalambides, we were informed that he considered this $q$-Pólya urn with many colors in a work that was then in progress and now has appeared ([6]). That work studies other aspects of the urn, and the only common result with the present work is Theorem 1.6.

**Orientation**. In Section 2, we prove Fact 1 and Theorem 1.1, which are basic results for the $q$-Pólya urn. Section 3 (Section 4) contains the proofs of the theorems that give convergence to a jump process (to a continuous process). Finally, Section 5 contains the proofs for the results that refer to the $q$-Pólya urn with arbitrary, finite number of colors.

## 2 Prevalence of a single color

In this section, we prove the claims of Section 1.1. Before doing so, we mention three properties of the $q$-binomial coefficient. For all $q \in (0, \infty) \setminus \{1\}, x \in \mathbb{C}, n, k \in \mathbb{N}$ with $k \leq n$ it holds

$$[-x]_q = -q^{-x}[x]_q, \tag{2.1}$$

$$\begin{bmatrix} -x \\ k \end{bmatrix}_q = (-1)^k q^{-k(k+2x-1)/2} \begin{bmatrix} x+k-1 \\ k \end{bmatrix}_q, \tag{2.2}$$

$$\begin{bmatrix} x \\ k \end{bmatrix}_{q^{-1}} = q^{-k(x-k)} \begin{bmatrix} x \\ k \end{bmatrix}_q, \tag{2.3}$$

$$\sum_{1\leq i_1<i_2<\cdots<i_k\leq n} q^{i_1+i_2+\cdots+i_k} = q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q. \tag{2.4}$$

The first is trivial, the second follows from the first, the third is easily shown, while the last is Theorem 6.1 in [15].

***Proof of Fact 1***. (i) The probability to get black balls exactly at the drawings $i_1 < i_2 < \cdots < i_{n-x}$ is

$$g(i_1, i_2, \dots, i_{n-x}) = \frac{\prod_{j=0}^{x-1}[A_1 + jk]_q \prod_{j=0}^{n-x-1}[A_2 + jk]_q}{\prod_{j=0}^{n-1}[A_1 + A_2 + jk]_q} q^{\sum_{\nu=1}^{n-x} r + (i_\nu - \nu)k}. \tag{2.5}$$

To see this, note that, due to (1.2) and (1.3), the required probability would be equal to the above fraction if in (1.3) the term $q^w$ were absent. This term appears whenever we draw a black ball. Now, when we draw the $\nu$-th black ball, there are $A_1 + (i_\nu - \nu)k$ white balls in the urn, and this explains the exponent of $q$ in (2.5).

Since $[x + jk]_q = \frac{1-q^{x+jk}}{1-q} = [-\frac{x}{k} - j]_{q^{-k}}[-k]_q$ for all $x, j \in \mathbb{R}$, the fraction in (2.5) equals

$$\frac{[-\hat{A}_1]_{x,q^{-k}}[-\hat{A}_2]_{n-x,q^{-k}}}{[-\hat{A}_1 - \hat{A}_2]_{n,q^{-k}}}. \tag{2.6}$$

Then

$$\sum_{1\le i_1<i_2<\cdots<i_{n-x}\le n} q^{\sum_{\nu=1}^{n-x} A_1+(i_\nu-\nu)k} \tag{2.7}$$

$$= q^{(n-x)A_1-k(n-x)(n-x+1)/2} \sum_{1\le i_1<i_2<\cdots<i_{n-x}\le n} (q^k)^{i_1+i_2+\cdots+i_{n-x}} \tag{2.8}$$

$$= q^{(n-x)A_1-k(n-x)(n-x+1)/2} q^{k\binom{n-x+1}{2}} \begin{bmatrix} n \\ x \end{bmatrix}_{q^k} \tag{2.9}$$

$$= q^{(n-x)A_1} q^{kx(n-x)} \begin{bmatrix} n \\ x \end{bmatrix}_{q^{-k}} = q^{k(n-x)(\hat{A}_1+x)} \begin{bmatrix} n \\ x \end{bmatrix}_{q^{-k}}. \tag{2.10}$$

The second equality follows from (2.4) and the equality $\begin{bmatrix} n \\ x \end{bmatrix}_{q^k} = \begin{bmatrix} n \\ n-x \end{bmatrix}_{q^k}$. The third, from (2.3). Thus, employing (1.8) too, we obtain that the sum $\sum_{1\le i_1<i_2<\cdots<i_{n-x}\le n} g(i_1, i_2, \dots, i_{n-x})$ equals the right hand side of (1.10). Then (1.11) and (1.12) follow by using (2.2) and (2.3) respectively.

(ii) In this scenario, we take $A_1 \to \infty$ in (1.11). We will explain shortly why this gives the probability we want. Since $q^{-k} \in (0,1)$, we have $\lim_{t\to\infty}[t]_{q^{-k}} = (1-q^{-k})^{-1}$ and thus, for each $\nu \in \mathbb{N}$, it holds

$$\lim_{t\to\infty} \begin{bmatrix} t+\nu-1 \\ \nu \end{bmatrix}_{q^{-k}} = \frac{1}{[\nu]_{q^{-k}}!} \frac{1}{(1-q^{-k})^\nu}. \tag{2.11}$$

Applying this twice in (1.11) (there $\hat{A}_1 = A_1/k \to \infty$), we get as limit the right hand side of (1.13).

Now, to justify that passage to the limit $A_1 \to \infty$ in (1.11) gives the required result, we argue as follows. For clarity, denote the probability $\mathbf{P}_q(\text{white})$ when there are $w$ white and $b$ black balls in the urn by $\mathbf{P}_q^{w,b}(\text{white})$. And when there are $A_1$ white and $A_2$ black balls in the urn in the beginning of the procedure, denote the probability of the event $H_1(n) = x$ by $\mathbf{P}^{A_1,A_2}(H_1(n) = x)$. It is clear that the probability $\mathbf{P}^{A_1,A_2}(H_1(n) = x)$ is a continuous function (in fact, a polynomial) of the quantities

$$\mathbf{P}_q^{A_1+ki,A_2+kj}(\text{white}) : i = 0, 1, \dots, x-1, j = 0, 1, \dots, n-x-1,$$

for all values of $A_1 \in \mathbb{N} \cup \{\infty\}, A_2 \in \mathbb{N}$. In $\mathbf{P}^{\infty,A_1}(H_1(n) = x)$, each such quantity, $\mathbf{P}_q^{\infty,m}(\text{white})$, equals $\lim_{A_1\to\infty} \mathbf{P}^{A_1,m}(\text{white})$.
Thus $\mathbf{P}^{\infty,A_2}(H_1(n) = x) = \lim_{A_1\to\infty} \mathbf{P}^{A_1,A_2}(H_1(n) = x)$. ■

Before proving Theorem 1.1, we give a simple argument that shows that eventually we will be picking only black balls. That is, the number $H_1(\infty) := \lim_{n\to\infty} H_1(n)$ of white balls drawn in an infinite sequence of drawings is finite. It is enough to show it in the case that $A_1 = \infty$ and $A_2 = 1$ since, by the experiment that realizes the $q$-Pólya urn, we have (using the notation from the proof of Fact 1 (ii))

$$\mathbf{P}^{A_1,A_2}(H_1(\infty) = \infty) \le \mathbf{P}^{\infty,1}(H_1(\infty) = \infty).$$

For each $n \in \mathbb{N}^+$, call $E_n$ the event that at the $n$-th drawing we pick a white ball, $B_n$ the number of black balls present in the urn after that drawing (also, $B_0 := 1$), and write $\hat{q} := 1/q$. Then $\mathbf{P}(E_n) = \mathbf{E}(\mathbf{P}(E_n|B_{n-1})) = \mathbf{E}(\hat{q}^{B_{n-1}})$. We will show that this decays exponentially with

$n$. Indeed, since at every drawing there is probability at least $1-\hat{q}$ to pick a black ball, we can construct in a common probability space the random variables $(B_n)_{n\geq 1}$ and $(Y_i)_{i\geq 1}$ so that the $Y_i$ are i.i.d. with $Y_1 \sim \text{Bernoulli}(1-\hat{q})$ and $B_n \geq 1+k(Y_1+\cdots+Y_n)$ for all $n \in \mathbb{N}^+$. Consequently,

$$\mathbf{P}(E_n) \leq \mathbf{E}(\hat{q}^{1+k(Y_1+\cdots+X_{n-1})}) = \hat{q}\{\mathbf{E}(\hat{q}^{kY_1})\}^{n-1}.$$

This implies that $\sum_{n=1}^{\infty} \mathbf{P}(E_n) < \infty$, and the first Borel-Cantelli lemma gives that $\mathbf{P}^{\infty,1}(H_1(\infty) = \infty) = 0$.

***Proof of Theorem 1.1.*** Since $\{H_1(n)\}_{n\geq 1}$ is increasing, it converges to a random variable $H_1(\infty)$ with values in $\mathbb{N}\cup\{\infty\}$. In particular, it converges to this variable in distribution. Our aim is to take the limit as $n\to\infty$ in (1.11) and in (1.13) in order to determine the distribution of $H_1$. Note that for $a\in\mathbb{R}$ and $\theta\in[0,1)$ it is immediate that (recall (1.9) for the notation)

$$\lim_{n\to\infty} \begin{bmatrix} a+n \\ n \end{bmatrix}_\theta = \frac{(\theta^{a+1};\theta)_\infty}{(\theta;\theta)_\infty}. \tag{2.12}$$

(i) Taking $n\to\infty$ in (1.11) and using (2.12), we get the required expression, (1.14), for $f$. Then relation (2.2) in [4] (or (8.1) in [15]) shows that $\sum_{x\in\mathbb{N}} f(x) = 1$, so that it is a probability mass function of a random variable $H_1$ with values in $\mathbb{N}$.

(ii) This follows after taking limit in (1.13) and using (2.12) and $\lim_{n\to\infty}(1-q^{-k})^n [n]_{q^{-k}}! = (q^{-k};q^{-k})_\infty$. ■

# 3 Jump process limits. Proof of Theorems 1.2, 1.3

In the case of Theorem 1.2, we let $g_m := 1$ for all $m\in\mathbb{N}^+$, and for both theorems we let $v := v_m := m/g_m$. Our interest is in the sequence of the processes $(Z^{(m)})_{m\geq 1}$ with

$$Z^{(m)}(t) = \frac{1}{k}\left\{A_1^{(m)}([vt]) - A_1^{(m)}(0)\right\} \tag{3.1}$$

for all $t\geq 0$.

To show convergence in distribution, according to Theorem 7.8 of Chapter 3 of [9], it is enough to show that the sequence $(Z^{(m)})_{m\geq 1}$ is tight and its finite dimensional distributions converge. The description of the limiting process is obtained on the way.

An easy argument shows that tightness follows from the convergence of the finite dimensional distributions becauce each $Z^{(m)}$ has non decreasing paths. It thus remains to establish the convergence of the finite dimensional distributions.

**Notation:** For sequences $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ with values in $\mathbb{R}$, we will say that they are asymptotically equivalent, and will write $a_n \sim b_n$ as $n\to\infty$, if $\lim_{n\to\infty} a_n/b_n = 1$. We use the same expressions for functions $f, g$ defined in a neighborhood of $\infty$ and satisfy $\lim_{x\to\infty} f(x)/g(x) = 1$.

## 3.1 Convergence of finite dimensional distributions

By definition, $Z^{(m)}(0) = 0 = Z(0)$ for all $m\in\mathbb{N}^+$.

Since for each $m \geq 1$ the process $Z^{(m)}$ is Markov taking values in $\mathbb{N}$ and non decreasing in time, it is enough to show that the conditional probability

$$\mathbf{P}(Z^{(m)}(t_2) = k_2 | Z^{(m)}(t_1) = k_1) \tag{3.2}$$

converges as $m \to \infty$ for each $0 \leq t_1 < t_2$ and nonnegative integers $k_1 \leq k_2$.
Define

$$n := [vt_2] - [vt_1], \tag{3.3}$$
$$x := k_2 - k_1, \tag{3.4}$$
$$\sigma := \frac{A_1^{(m)}(0) + kk_1}{k}, \tag{3.5}$$
$$\tau := \frac{k[vt_1] - kk_1 + A_2^{(m)}(0)}{k}, \tag{3.6}$$
$$r := q_m^{-k} = c^{-k/m}. \tag{3.7}$$

Then, the probability in (3.2), with the help of (1.11), is computed as

$$r^{\tau x} \begin{bmatrix} \sigma + x - 1 \\ x \end{bmatrix}_r \frac{\begin{bmatrix} \tau+n-x-1 \\ n-x \end{bmatrix}_r}{\begin{bmatrix} \sigma+\tau+n-1 \\ n \end{bmatrix}_r}$$
$$= r^{\tau x} \begin{bmatrix} \sigma + x - 1 \\ x \end{bmatrix}_r \Big( \prod_{i=n-x+1}^{n} (1 - r^i) \Big) \frac{1}{\prod_{i=n-x}^{n-1} (1 - r^{\tau+i})} \frac{[\tau + n - 1]_{n,r}}{[\sigma + \tau + n - 1]_{n,r}}. \tag{3.8}$$

The last ratio is

$$\prod_{i=0}^{n-1} \frac{1 - r^{\tau+i}}{1 - r^{\sigma+\tau+i}} = \prod_{i=0}^{n-1} \left( 1 - (1 - r^\sigma) r^\tau \frac{r^i}{1 - r^{\sigma+\tau+i}} \right). \tag{3.9}$$

Denote by $1 - a_{m,i}$ the $i$-th term of the product. The logarithm of the product equals

$$-(1 - r^\sigma) r^\tau \sum_{i=0}^{n-1} \frac{r^i}{1 - r^{\sigma+\tau+i}} + o(1) \tag{3.10}$$

as $m \to \infty$. To justify this, note that $1 - r^\sigma \sim \frac{1}{m}(A_0^{(m)} + kk_1) \log c$ and $r^{\tau+i}/(1 - r^{\sigma+\tau+i}) \leq 1/(1 - c^{-b})$ for all $i \in \mathbb{N}$. Thus, for all large $m$, $|a_{m,i}| < 1/2$ for all $i = 0, 1, \ldots, n-1$, and the error in approximating the logarithm of $1 - a_{m,i}$ by $-a_{m,i}$ is at most $|a_{m,i}|^2$ (by Taylor's expansion, we have $|\log(1 - y) + y| \leq |y|^2$ for all $y$ with $|y| \leq 1/2$). The sum of all errors is at most $n \max_{0 \leq i < n} |a_{m,i}|^2$, which goes to zero as $m \to \infty$ because $1 - r^\sigma \sim C/n$ for some appropriate constant $C > 0$.

We will compute the limit of (3.8) as $m \to \infty$ under the assumptions of Theorems 1.2, 1.3.

***The computation for Theorem 1.2.*** As $m \to \infty$, the first term of the product in (3.8) converges to $c^{-x(b+kt_1)}$. The $q$-binomial coefficient converges to $\binom{k^{-1}w_0+k_2-1}{k_2-k_1}$. The third term converges to $(1 - c^{-k(t_2-t_1)})^x$, while the denominator of the fourth term converges to

$(1-\rho_2)^x$, where we set $\rho_i := c^{-b-kt_i}$ for $i = 1, 2$. The expression preceding $o(1)$ in (3.10) is asymptotically equivalent to

$$-\frac{k}{m}\sigma(\log c)\rho_1 \sum_{i=0}^{n-1} \frac{c^{-ki/m}}{1 - r^{\sigma+\tau} c^{-ki/m}} \tag{3.11}$$

$$= -\rho_1 k\sigma(\log c)\frac{1}{m}\sum_{i=0}^{n-1} \frac{c^{-ki/m}}{1-\rho_1 c^{-ki/m}} + o(1) \tag{3.12}$$

$$= -\rho_1 k\sigma \log c \int_0^{t_2-t_1} \frac{1}{c^{ky}-\rho_1}\, dy + o(1) = \sigma \log \frac{1-\rho_1}{1-\rho_2} + o(1). \tag{3.13}$$

The first equality is true because $\lim_{m\to\infty} r^{\sigma+\tau} = \rho_1$ and the function $x \mapsto c^{-ki/m}/(1 - xc^{-ki/m})$ has derivative bounded uniformly in $i, m$ when $x$ is confined to a compact subset of $[0, 1)$. Thus, the limit of (3.8), as $m \to \infty$, is

$$\binom{\sigma + x - 1}{x} \left(\frac{\rho_1 - \rho_2}{1 - \rho_2}\right)^x \left(\frac{1-\rho_1}{1-\rho_2}\right)^\sigma, \tag{3.14}$$

which means that, as $m \to \infty$, the distribution of $\{Z^{(m)}(t_2) - Z^{(m)}(t_1)\}|Z^{(m)}(t_1) = k_1$ converges to the negative binomial distribution with parameters $\sigma, (1-\rho_1)/(1-\rho_2)$. ■

***The computation for Theorem 1.3.*** Now the term $r^{\tau x}$ converges to $c^{-xb}$, while

$$\begin{bmatrix} \sigma + x - 1 \\ x \end{bmatrix}_r \Big(\prod_{i=n-x+1}^{n} (1 - r^i)\Big) = \frac{\prod_{i=0}^{x-1}(1 - r^{\sigma+i})}{\prod_{i=1}^{x}(1-r^i)} \Big(\prod_{i=n-x+1}^{n} (1 - r^i)\Big) \tag{3.15}$$

$$\sim \frac{\prod_{i=0}^{x-1}(\sigma + i)}{\prod_{i=1}^{x} i} \frac{((t_2-t_1)k\log c)^x}{g_m^x} \sim \frac{1}{x!}((t_2 - t_1)\log c)^x. \tag{3.16}$$

The denominator of the fourth term in (3.8) converges to $(1-c^{-b})^x$. The expression in (3.10) is asymptotically equivalent to

$$-r^\tau(1-r^\sigma)\sum_{i=0}^{n-1}\frac{r^i}{1-r^{\sigma+\tau+i}} \sim -c^{-b}\frac{g_m}{m}\log c\frac{n}{1-c^{-b}} \sim -\frac{\log c}{c^b - 1}(t_2 - t_1). \tag{3.17}$$

In the first $\sim$, we used the fact that the terms of the sum, as $m \to \infty$, converge uniformly in $i$ to $(1-c^{-b})^{-1}$. Thus, the limit of (3.8), as $m \to \infty$, is

$$\frac{1}{x!}\left(\frac{\log c}{c^b - 1}(t_2 - t_1)\right)^x e^{-\frac{\log c}{c^b-1}(t_2 - t_1)}, \tag{3.18}$$

which means that, as $m \to \infty$, the distribution of $\{Z^{(m)}(t_2) - Z^{(m)}(t_1)\}|Z^{(m)}(t_1) = k_1$ converges to the Poisson distribution with parameter $\frac{t_2-t_1}{c^b-1}\log c$. ■

## 3.2 Conclusion

It is clear from the form of the finite dimensional distributions that in both Theorems 1.2, 1.3 the limiting process $Z$ is a pure birth process that does not explode in finite time. Its rate at the point $(t, j) \in [0, \infty) \times \mathbb{N}$ is

$$\lambda_{t,j} = \lim_{h\to 0^+} \frac{1}{h}\mathbf{P}(Z(t + h) = j + 1|Z(t) = j)$$

and is found as stated in the statement of each theorem.

# 4 Deterministic and diffusion limits. Proof of Theorems 1.4, 1.5

These theorems are proved with the use of Theorem 7.1 in Chapter 8 of [8], which is concerned with convergence of time-homogeneous Markov chains to diffusions. The chains whose convergence is of interest to us are time inhomogeneous, but we reduce their study to the time-homogenous setting by considering for each such chain $\{Z_n\}_{n\in\mathbb{N}}$ the time homogeneous chain $\{(Z_n, n)\}_{n\in\mathbb{N}}$. The following consequence of the aforementioned theorem suffices for our purposes.

**Corollary 4.1.** *Assume that for each $m \in \mathbb{N}^+$, $(Z_n^{(m)})_{n\in\mathbb{N}}$ is a Markov chain in $\mathbb{R}$. For each $m \in \mathbb{N}^+$ and $n \in \mathbb{N}$, let $\Delta Z_n^{(m)} := Z_{n+1}^{(m)} - Z_n^{(m)}$ and*

$$\mu^{(m)}(x,n) := m\boldsymbol{E}(\Delta Z_n^{(m)} \boldsymbol{1}_{|\Delta Z_n^{(m)}|\le 1} | Z_n^{(m)} = x), \tag{4.1}$$

$$a^{(m)}(x,n) := m\boldsymbol{E}((\Delta Z_n^{(m)})^2 \boldsymbol{1}_{|\Delta Z_n^{(m)}|\le 1} | Z_n^{(m)} = x) \tag{4.2}$$

*for all $x \in \mathbb{R}$ with $\mathbf{P}(Z_n^{(m)} = x) > 0$. Also, for $R > 0$ and for the same $m, n$ as above, let $A(m,n,R) := \{(x,n) : |x| \le R, n/m \le R, \mathbf{P}(Z_n^{(m)} = x) > 0\}$.*

*Assume that there are continuous functions $\mu, a : \mathbb{R}\times[0,\infty) \to \mathbb{R}$ and $x_0 \in \mathbb{R}$ so that: For every $R, \varepsilon > 0$, it holds*

*(i)* $\sup_{(x,n)\in A(m,n,R)} |\mu^{(m)}(x,n) - \mu(x,n/m)| \to 0$ *as* $m \to \infty$.

*(ii)* $\sup_{(x,n)\in A(m,n,R)} |a^{(m)}(x,n) - a(x,n/m)| \to 0$ *as* $m \to \infty$.

*(iii)* $\sup_{(x,n)\in A(m,n,R)} m\mathbf{P}(|\Delta Z_n^{(m)}| \ge \varepsilon | Z_n^{(m)} = x) \to 0$ *as* $m \to \infty$.

*And also*

*(iv)* $Z_0^{(m)} \to x_0$ *as $m \to \infty$ with probability 1.*

*(v) For each $x \in \mathbb{R}$, the stochastic differential equation*

$$\begin{aligned} dZ_t &= \mu(Z_t,t)\,dt + \sqrt{a(Z_t,t)}\,dB_t, \\ Z_0 &= x, \end{aligned} \tag{4.3}$$

*where $B$ is a one dimensional Brownian motion, has a weak solution which is unique in distribution.*

*Then, the process $(Z_{[mt]}^{(m)})_{t\ge 0}$ converges in distribution to the weak solution of (4.3) with $x = x_0$.*

*Proof.* For each $m \in \mathbb{N}^+$, we consider the process $Y_n^{(m)} := (Z_n^{(m)}, n/m), n \in \mathbb{N}$, which is a time-homogeneous Markov chain with values in $\mathbb{R}^2$, and we apply Theorem 7.1 in Chapter 8 of [8] Conditions $(i), (ii), (iii)$ of that theorem follow from our conditions $(ii), (i), (iii)$ respectively, while condition (A) there translates to the requirement that the martingale problem for the functions $\mu$ and $\sqrt{a}$ is well posed, and this follows from condition $(v)$. ■

The tool we will use in checking that condition $(v)$ of the corollary is satisfied is the well known existence and uniqueness theorem for strong solutions of SDEs which requires that for all $T > 0$, the coefficients $\mu(x,t)$, $\sqrt{a(x,t)}$ are Lipschitz in $x$ uniformly for $t \in [0,T]$ and $\sup_{t\in[0,T]}\{|\mu(0,t)|+a(0,t)\} < \infty$ (e.g., Theorem 2.9 of Chapter 5 or [8]). The same conditions imply uniqueness in distribution.

## 4.1 Proof of Theorem 1.4

We will apply Corollary 4.1. For each $m \in \mathbb{N}^+$, consider the Markov chain $Z_n^{(m)} = \frac{A_1^{(m)}(n)}{m}, n \in \mathbb{N}$. From any given state $x$ of $Z_n^{(m)}$, the chain moves to either of $x+km^{-1}, x$ with corresponding probabilities $p(x,n,m), 1-p(x,n,m)$, where

$$p(x,n,m) := \frac{1-q_m^{mx}}{1-q_m^{A_1^{(m)}(0)+A_2^{(m)}(0)+kn}}.$$

In particular, for any $\varepsilon > 0$, is holds $|\Delta Z_n^{(m)}| < 1 \wedge \varepsilon$ for $m$ large enough. Thus, condition $(iii)$ of the corollary is satisfied trivially. Also, for large $m$, with the notation of the corollary, we have

$$\mu^{(m)}(x,n) = kp(x,n,m), \tag{4.4}$$

$$a^{(m)}(x,n) = \frac{k}{m}p(x,n,m). \tag{4.5}$$

And it is easy to see that conditions $(i)$, $(ii)$ are satisfied by the functions $a, \mu$ with $a(x,t) = 0$ and $\mu(x,t) = kp(x,t)$ where

$$p(x,t) := \frac{1-c^x}{1-c^{a+b+kt}}. \tag{4.6}$$

Now, for each $x \in \mathbb{R}$, the equation

$$\begin{aligned} dZ_t &= kp(Z_t,t)\,dt, \\ Z_0 &= x \end{aligned} \tag{4.7}$$

has a unique solution. Thus, Corollary 4.1 applies. In fact, (4.7) is a separable ordinary differential equation and its unique solution is the one given in the statement of the theorem.

## 4.2 Proof of Theorem 1.5

For each $m \in \mathbb{N}^+$, consider the Markov chain

$$Z_n^{(m)} = \sqrt{m}\Big(\frac{A_1^{(m)}(n)}{m} - X_{n/m}\Big), n \in \mathbb{N}.$$

From any given state $x$ of $Z_n^{(m)}$, the chain moves to either of

$$x + km^{-1/2} + \sqrt{m}(X_{n/m} - X_{(n+1)/m}), \tag{4.8}$$

$$x + \sqrt{m}(X_{n/m} - X_{(n+1)/m}) \tag{4.9}$$

with corresponding probabilities $p(x,n,m), 1-p(x,n,m)$, where

$$p(x,n,m) = \frac{[A_1^{(m)}(n)]_{q_m}}{[A_1^{(m)}(0) + A_2^{(m)}(0) + kn]_{q_m}} \tag{4.10}$$

and

$$A_1^{(m)}(n) = mX_{n/m} + x\sqrt{m}, \tag{4.11}$$

$$A_2^{(m)}(n) = A_1^{(m)}(0) + A_2^{(m)}(0) + kn - A_1^{(m)}(n). \tag{4.12}$$

For convenience, let $\Delta X_{n/m} = X_{(n+1)/m} - X_{n/m}$. We compute

$$\mathbf{E}\left[\Delta Z_n^{(m)} | Z_n^{(m)} = x\right] = \frac{k}{\sqrt{m}} p(x,n,m) - \sqrt{m}\Delta X_{n/m}, \tag{4.13}$$

$$\mathbf{E}\left[(\Delta Z_n^{(m)})^2 | Z_n^{(m)} = x\right] = \left(\frac{k^2}{m} - 2k\Delta X_{n/m}\right) p(x,n,m) + m(\Delta X_{n/m})^2. \tag{4.14}$$

The asymptotics of these expectations are as follows.

Claim: Fix $R > 0$. For $n$ such that $\tau := n/m \le R$ and as $m \to \infty$, we have

$$(a) \quad \mathbf{E}\left[\Delta Z_n^{(m)} | Z_n^{(m)} = x\right]$$
$$= \frac{1}{m}\frac{k\log c}{c^{a+b+k\tau}-1}\left(c^{X_\tau}x - \frac{(c^{X_\tau}-1)c^{a+b+k\tau}}{c^{a+b+k\tau}-1}(\theta_1+\theta_2)\right) + O(\frac{1}{m^{3/2}}) \tag{4.15}$$

$$(b) \quad \mathbf{E}\left[(\Delta Z_n^{(m)})^2 | Z_n^{(m)} = x\right] = \frac{1}{m}k^2 g(\tau)\{1-g(\tau)\} + O(\frac{1}{m^{3/2}}) \tag{4.16}$$

where $g(t) := \frac{c^{X_t}-1}{c^{a+b+kt}-1}$ for all $t \ge 0$.

Proof of the claim. We examine the asymptotics of $p(x,n,m)$ and $\Delta X_{n/m}$. As $\tau \le R$ and $m \to \infty$, we have

$$p(x,n,m) = \frac{c^{X_\tau + \frac{1}{\sqrt{m}x}} - 1}{c^{\frac{A_1^{(m)}(0)+A_2^{(m)}(0)}{m}+k\tau} - 1} = \frac{c^{X_\tau + \frac{1}{\sqrt{m}x}} - 1}{c^{a+b+k\tau+\frac{\theta_1+\theta_2}{\sqrt{m}}+O(\frac{1}{m})} - 1} \tag{4.17}$$

$$= g(\tau) + \frac{\log c}{c^{a+b+k\tau}-1}\left(c^{X_\tau}x - \frac{(c^{X_\tau}-1)c^{a+b+k\tau}}{c^{a+b+k\tau}-1}(\theta_1+\theta_2)\right)\frac{1}{\sqrt{m}} \tag{4.18}$$
$$+ O\Big(\frac{1}{m}\Big).$$

The third equality follows from a Taylor's development. Also

$$\Delta X_{n/m} = X'_{n/m}\frac{1}{m} + O(m^{-2}) = kg(\tau)\frac{1}{m} + O(m^{-2}). \tag{4.19}$$

For $X'$ we used the differential equation, (1.20), that $X$ satisfies instead of the explicit expression for it. Substituting these expressions in (4.13), (4.14), we get the claim.

Relation (1.21) implies that $c^{X_\tau} = (c^{a+b}-1)/\{c^b - 1 + c^{-k\tau}(1-c^{-a})\}$, and this gives that the parenthesis following $\frac{1}{m}$ in equation $(a)$ of the claim above equals

$$\frac{(c^{a+b}-1)x - c^b(c^a-1)(\theta_1+\theta_2)}{c^b - 1 + c^{-k\tau}(1-c^{-a})} \tag{4.20}$$

and also that

$$g(\tau)\{1-g(\tau)\} = \frac{(c^a-1)(c^b-1)c^{a+k\tau}}{(c^{a+b+k\tau} - c^{a+k\tau} + c^a - 1)^2}. \tag{4.21}$$

Thus, the claim implies that conditions $(i)$, $(ii)$ of Corollary 4.1 are satisfied by the functions

$$\mu(x,t) = \frac{k \log c}{c^{a+b+kt}-1} \left\{ \frac{(c^{a+b}-1)x - c^b(c^a-1)(\theta_1+\theta_2)}{c^b - 1 + c^{-kt}(1-c^{-a})} \right\}, \tag{4.22}$$

$$a(x,t) = k^2(c^a-1)(c^b-1)\frac{c^{a+kt}}{(c^{a+b+kt} - c^{a+kt} + c^a - 1)^2}. \tag{4.23}$$

As in the proof of Theorem 1.4, condition $(iii)$ of the corollary holds trivially, while $\lim_{m\to\infty} Z_0^{(m)} = \theta_1$ (condition $(iv)$). Finally, for each $x \in \mathbb{R}$ and for the choice of $\mu, a$ above, equation (4.3) has a strong solution and uniqueness in distribution holds. Thus, the process $(Z_{[mt]}^{(m)})_{t\geq 0}$ converges, as $m \to \infty$, to the unique solution of the stochastic differential equation (1.23).

The same is true for the process $(C_t^{(m)})_{t\geq 0}$ because $\sup_{t\geq 0} |Z_{[mt]}^{(m)} - C_t^{(m)}| \leq k/\sqrt{m}$ for all $m \in \mathbb{N}^+$ (we use the fact that $0 < X_t' \leq k$ for all $t \geq 0$). To solve (1.23), we remark that a solution of an equation of the form

$$dY_t = (\alpha(t)Y_t + \beta(t))dt + \gamma(t)dW_t \tag{4.24}$$

with $\alpha, \beta, \gamma : [0,\infty) \to \mathbb{R}$ continuous functions is given by

$$Y_t = e^{\int_0^t \alpha(s)\,ds}\left(Y_0 + \int_0^t \beta(s) e^{-\int_0^s \alpha(r)\,dr}\,ds + \int_0^t \gamma(s) e^{-\int_0^s \alpha(r)\,dr}\,dW_s\right). \tag{4.25}$$

[To discover the formula, we apply Itó's rule to $Y_t \exp\{-\int_0^t \alpha(s)\,ds\}$ and use (4.24).] Applying this formula for the values of $\alpha, \beta, \gamma$ dictated by (1.23) we arrive at (1.24).

## 5 Proofs for the $q$-Pólya urn with many colors

***Proof of Theorem 1.6***. First, the equality of the expressions in (1.26), (1.27) follows from the definition of the $q$-multinomial coefficient.

We will prove (1.26) by induction on $l$. When $l = 2$, (1.26) holds because of (1.10). In that relation, we have $x_1 = x, x_2 = n - x$. Assuming that (1.26) holds for $l \geq 2$ we will prove the case $l+1$. The probability $\mathbf{P}((H_2(n) = x_2, \ldots, H_{l+1}(n) = x_{l+1})$ equals

$$\begin{aligned}
&\mathbf{P}\left(H_3(n) = x_3, \ldots, H_{l+1}(n) = x_{l+1}\right) \mathbf{P}(H_2(n) = x_2 \mid H_3(n) = x_3, \ldots, H_{l+1}(n) = x_{l+1}) \quad (5.1)\\
&= q^{\sum_{i=3}^{l+1} x_i \sum_{j=1}^{i-1}(w_j + kx_j)} \frac{\left[{-\frac{w_1+w_2}{k} \atop x_1+x_2}\right]_{q^{-k}} \prod_{i=3}^{l+1} \left[{-\frac{w_i}{k} \atop x_i}\right]_{q^{-k}}}{\left[{-\frac{w_1+\ldots w_{l+1}}{k} \atop n}\right]_{q^{-k}}} q^{x_2(w_1+kx_1)} \frac{\left[{-\frac{w_1}{k} \atop x_1}\right]_{q^{-k}} \left[{-\frac{w_2}{k} \atop x_2}\right]_{q^{-k}}}{\left[{-\frac{w_1+w_2}{k} \atop x_1+x_2}\right]_{q^{-k}}}\\
&= q^{\sum_{i=2}^{l+1} x_i \sum_{j=1}^{i-1}(w_j + kx_j)} \frac{\prod_{i=1}^{l+1} \left[{-\frac{w_i}{k} \atop x_i}\right]_{q^{-k}}}{\left[{-\frac{w_1+\ldots w_{l+1}}{k} \atop n}\right]_{q^{-k}}}.
\end{aligned}$$

This finishes the induction provided that we can justify these two equalities. The second is obvious, so we turn to the first. The first probability in (5.1) is specified by the inductive hypothesis. That is, given the description of the experiment, in computing this probability it is as if we merge colors 1 and 2 into one color which is placed in the line before the remaining $l-1$ colors. This color has initially $a_1 + a_2$ balls and we require that in the first $n$ drawings we choose it $x_1 + x_2$ times. The second probability in (5.1) is specified by the $l = 2$ case of (1.26), which we know. More specifically, since the number of drawings from colors $3, 4, \ldots, l+1$ is given, it is as if we have an urn with just two colors $1, 2$ that have initially $w_1$ and $w_2$ balls respectively. We do $x_1 + x_2$ drawings with the usual rules for a $q$-Pólya urn, placing in a line all balls of color 1 before all balls of color 2, and we want to pick $x_1$ times color 1 and $x_2$ times color 2. ■

***Proof of Theorem 1.7.*** The components of $(H_2(n), H_3(n), \ldots, H_l(n))$ are increasing in $n$, and from Theorem 1.1 we have that each of them has finite limit (we treat all colors $2, \ldots, l$ as one color). Thus the convergence of the vector with probability one to a random vector with values is $\mathbb{N}^{l-1}$ follows. In particular, we also have convergence in distribution, and it remains to compute the distribution of the limit. Let $x_1 := n - (x_2 + \cdots + x_l)$. Then the probability in (1.26) equals

$$\mathbf{P}\left(H_2(n) = x_2, \ldots, H_l(n) = x_l\right) = q^{-\sum_{1 \le i < j \le l} w_j x_i} \frac{\prod_{i=1}^{l} \left[\begin{smallmatrix} \frac{w_i}{k} + x_i - 1 \\ x_i \end{smallmatrix}\right]_{q^{-k}}}{\left[\begin{smallmatrix} \frac{\sum_{i=1}^{l} w_i}{k} + n - 1 \\ n \end{smallmatrix}\right]_{q^{-k}}} \tag{5.2}$$

$$= q^{\sum_{1 \le j < i \le l} x_i w_j} \frac{\prod_{i=1}^{l} \left[\begin{smallmatrix} \frac{w_i}{k} + x_i - 1 \\ x_i \end{smallmatrix}\right]_{q^k}}{\left[\begin{smallmatrix} n + \frac{\sum_{i=1}^{l} w_i}{k} - 1 \\ n \end{smallmatrix}\right]_{q^k}} \tag{5.3}$$

$$= q^{\sum_{i=2}^{l} \left(x_i \sum_{j=1}^{i-1} w_j\right)} \left\{ \prod_{i=2}^{l} \left[\begin{matrix} \frac{w_i}{k} + x_i - 1 \\ x_i \end{matrix}\right]_{q^k} \right\} \frac{\left[\begin{smallmatrix} x_1 + \frac{w_1}{k} - 1 \\ x_1 \end{smallmatrix}\right]_{q^k}}{\left[\begin{smallmatrix} n + \frac{\sum_{i=1}^{l} w_i}{k} - 1 \\ n \end{smallmatrix}\right]_{q^k}}. \tag{5.4}$$

In the first equality, we used (2.2) while in the second we used (2.3). When we take $n \to \infty$ in (5.4), the only terms involving $n$ are those of the last fraction, and (2.12) determines their limit. Thus, the limit of (5.4) is found to be the function $f(x_2, \ldots, x_l)$ in the statement of the theorem. ■

***Proof of Theorem 1.8.*** For each $m \in \mathbb{N}^+$, we consider the discrete time-homogeneous Markov chain

$$Z_n^{(m)} := \left( \frac{n}{m}, \frac{A_2^{(m)}(n)}{m}, \frac{A_3^{(m)}(n)}{m}, \ldots, \frac{A_l^{(m)}(n)}{m} \right), n \in \mathbb{N}.$$

From any given state $(t, x) := (t, x_2, x_3, \ldots, x_l)$ that $Z^{(m)}$ finds itself it moves to one of

$$\left( t + \frac{1}{m}, x_2, \ldots, x_i + \frac{1}{m}, \ldots, x_l \right), i = 2, \ldots, l,$$
$$\left( t + \frac{1}{m}, x_2, \ldots, x_i, \ldots, x_l \right)$$

with corresponding probabilities

$$p_i(x_2, \ldots, x_l, t, m) = q^{m s_{i-1}(t)} \frac{[m x_i]_q}{[m s_l(t)]_q}, i = 2, \ldots, l, \tag{5.5}$$

$$p_1(x_2, \ldots, x_l, t, m) = \frac{[m x_1(t)]_q}{[m s_l(t)]_q}, \tag{5.6}$$

where

$$s_i(t) = x_1(t) + \sum_{1 < j \le i} x_j \tag{5.7}$$

for $i \in \{1, 2, \ldots, l\}$ and

$$x_1(t) := m^{-1} \sum_{j=1}^{l} A_j^{(m)}(0) + kt - \sum_{2 \le j \le l} x_i. \tag{5.8}$$

These follow from (1.25) once we count the number of balls of each color present at the state $(t, x)$. To do this, we note that $Z_n^{(m)} = (t, x)$ implies that $n = mt$ drawings have taken place so far, the total number of balls is $A_{0,1}^{(m)} + \cdots + A_{0,l}^{(m)} + kmt$, and the number of balls of color $i$, for $2 \le i \le l$, is $mx_i$. Thus, the number of balls of color 1 is $A_1^{(m)}(0) + \cdots + A_l^{(m)}(0) + kmt - m \sum_{2 \le j \le l} x_i = mx_1(t)$. The required relations follow.

Let $x_1 := \lim_{m\to\infty} x_1(t) = \sigma_l + kt - \sum_{2\le j\le l} x_i$ and $s_i := \lim_{m\to\infty} s_i(t) = \sum_{1\le j\le i} x_i$ for all $i \in \{1, 2, \ldots, l\}$. Then, since $q = c^{1/m}$, for fixed $(t, x_2, \ldots, x_l) \in [0, \infty)^l$ with $(x_2, \ldots, x_l) \neq 0$, we have

$$\lim_{m\to\infty} p_i(x_2, \ldots, x_l, t, m) = c^{s_{i-1}} \frac{[x_i]_c}{[s_l]_c} \tag{5.9}$$

for all $i = 2, \ldots, l$. We also note the following.

$$Z_{n+1,1}^{(m)} - Z_{n,1}^{(m)} = \frac{1}{m}, \tag{5.10}$$

$$\mathbf{E}\left[Z_{n+1,i}^{(m)} - Z_{n,i}^{(m)} | Z_n^{(m)} = (t, x_2, \ldots, x_l)\right] = \frac{k}{m} p_i(x_2, \ldots, x_l, t, m), \tag{5.11}$$

$$\mathbf{E}\left[(Z_{n+1,i}^{(m)} - Z_{n,i}^{(m)})^2 | Z_n^{(m)} = (t, x_2, \ldots, x_l)\right] = \frac{k^2}{m^2} p_i(x_2, \ldots, x_l, t, m), \tag{5.12}$$

$$\mathbf{E}\left[(Z_{n+1,i}^{(m)} - Z_{n,i}^{(m)})(Z_{n+1,j}^{(m)} - Z_{n,j}^{(m)}) | Z_n^{(m)} = (t, x_2, \ldots, x_l)\right] = 0 \tag{5.13}$$

for $i, j = 2, 3, \ldots, l$ with $i \neq j$.

Therefore, with similar arguments as in the proof of Theorem 1.4, as $m \to +\infty$, $(Z_{[mt]}^{(m)})_{t\ge0}$ converges in distribution to $Y$, the solution of the ordinary differential equation

$$\begin{aligned} dY_t &= b(Y_t)dt, \\ Y_0 &= (0, a_2, \ldots, a_l), \end{aligned} \tag{5.14}$$

where $b(t, x_2, \ldots, x_l) = \big(1, b^{(2)}(t, x), b^{(3)}(t, x), \ldots, b^{(l)}(t, x)\big)$ with

$$b^{(i)}(t, x) = kc^{s_{i-1}} \frac{[x_i]_c}{[s_l]_c}$$

for $i = 2, 3, \ldots, l$. Note that $s_l = \sigma_l + kt$ does not depend on $x$.

Since $A_1^{(m)}([mt]) + A_2^{(m)}([mt]) + \cdots + A_l^{(m)}([mt]) = kmt + A_1^{(m)}(0) + A_2^{(m)}(0) + \cdots + A_l^{(m)}(0)$, we get that the process $(A_{[mt],1}^{(m)}/m, A_{[mt],2}^{(m)}/m + \cdots + A_{[mt],l}^{(m)}/m)_{t\ge0}$ converges in distribution to a process $(X_{t,1}, X_{t,2}, \ldots, X_{t,l})_{t\ge0}$ so that $X_{t,1} + \cdots + X_{t,l} = a_1 + a_2 + \cdots + a_l + kt$, while the $X_{t,i}, i = 2, \ldots, l$, satisfy the system

$$X'_{t,i} = kc^{\sigma_l + kt - \sum_{j=i}^{l} X_{t,i}} \frac{1 - c^{X_{t,i}}}{1 - c^{\sigma_l + kt}} \qquad \text{for all } t > 0, \tag{5.15}$$

$$X_{0,i} = a_i, \tag{5.16}$$

with $i = 2, 3, \dots, l$. Letting $Z_{r,i} = c^{X_{\frac{1}{k \log c} \log r, i}}$ for all $r \in (0, 1]$ and $i \in \{1, 2, \dots, l\}$, we have for the $Z_{r,i}, i \in \{2, 3, \dots, l\}$ the system

$$\frac{Z'_{r,i}}{1 - Z_{r,i}} = \frac{\sigma_l}{1 - \sigma_l r} \frac{1}{\prod_{i<j\le l} Z_{r,j}}, \tag{5.17}$$

$$Z_{1,i} = c^{a_i}. \tag{5.18}$$

In the case $i = l$, the empty product equals 1. It is now easy to prove by induction (starting from $i = l$ and going down to $i = 2$) that

$$Z_{r,i} = \frac{c^{\sigma_l - \sigma_{i-1}}(1 - c^{\sigma_l} r) - c^{\sigma_l}(1 - r)}{c^{\sigma_l - \sigma_i}(1 - c^{\sigma_l} r) - c^{\sigma_l}(1 - r)} \tag{5.19}$$

for all $r \in (0, 1]$. Since $Z_{r,1} Z_{r,2} \cdots Z_{r,l} = c^{\sigma_l} r$, we can check that (5.19) holds for $i = 1$ too. The fraction in (5.19) equals

$$c^{a_i} \frac{(1 - c^{\sigma_l} r) - c^{\sigma_{i-1}}(1 - r)}{(1 - c^{\sigma_l} r) - c^{\sigma_i}(1 - r)}. \tag{5.20}$$

Recalling that $X_{t,i} = (\log c)^{-1} \log Z_{c^{kt}}$, we get (1.29) for all $i \in \{1, 2, \dots, l\}$ . ■

***Proof of Theorem 1.9.*** This is proved in the same way as Theorem 1.8. We keep the same notation as there. The only difference now is that $\lim_{m\to\infty} p_i(t, x_2, ..., x_l, m) = x_i/s_l$. As a consequence, the system of ordinary differential equations for the limit process $Y_t := (t, X_{t,2}, \dots, X_{t,l})$ is (5.14) but with

$$b^{(i)}(t, x) = \frac{k x_i}{s_l}.$$

Recall that $s_l = \sigma_l + kt$. Thus, for $i = 2, 3, \dots, l$, the process $X_{t,i}$ satisfies $X'_{t,i} = kX_{t,i}/(\sigma_l + kt), X_{0,i} = a_i$, which give immediately the last $l - 1$ coordinates of (1.30). The formula for the first coordinate follows from $X_{t,1} + X_{t,2} + \cdots + X_{t,l} = kt + \sigma_l$. ■

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