PROBLEMS AND SOLUTIONS


Proposed problems and solutions should be sent in duplicate to the MONTHLY problems address on the back of the title page. Submitted solutions should arrive at that address before June 30, 2011. Additional information, such as generalizations and references, is welcome. The problem number and the solver’s name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

11551. Proposed by Gregory Galperin, Eastern Illinois University, Charleston, IL, and Yury Ionin, Central Michigan University, Mount Pleasant, MI. Given a finite set $S$ of closed bounded convex sets in $\mathbb{R}^n$ having positive volume, prove that there exists a finite set $X$ of points in $\mathbb{R}^n$ such that each $A \in S$ contains at least one element of $X$ and any $A, B \in S$ with the same volume contain the same number of elements of $X$.

11552. Proposed by Weidong Jiang, Weihai Vocational College, Weihai, China. In triangle $ABC$, let $A_1, B_1, C_1$ be the points opposite $A, B, C$ at which the angle bisectors of the triangle meet the opposite sides. Let $R$ and $r$ be the circumradius and inradius of $ABC$. Let $a, b, c$ be the lengths of the sides opposite $A, B, C$, and let $a_1, b_1, c_1$ be the lengths of the line segments $B_1C_1, C_1A_1, A_1B_1$. Prove that

$$\frac{a_1}{a} + \frac{b_1}{b} + \frac{c_1}{c} \geq 1 + \frac{r}{R}.$$

11553. Proposed by Mihály Bencze, Brasov, Romania. For a positive integer $k$, let $\alpha(k)$ be the largest odd divisor of $k$. Prove that for each positive integer $n$,

$$\frac{n(n + 1)}{3} \leq \sum_{k=1}^{n} \frac{n - k + 1}{k} \alpha(k) \leq \frac{n(n + 3)}{3}.$$

11554. Proposed by Zhang Yun, Xi’an Jiao Tong University Sunshine High School, Xi’an, China. In triangle $ABC$, let $I$ be the incenter, and let $A’, B’, C’$ be the reflections of $I$ through sides $BC, CA, AB$, respectively. Prove that the lines $AA’, BB’,$ and $CC’$ are concurrent.

11555. Proposed by Duong Viet Thong, National Economics University, Hanoi, Vietnam. Let $f$ be a continuous real-valued function on $[0, 1]$ such that $\int_{0}^{1} f(x) \, dx = 0$. Prove that there exists $c$ in the interval $(0, 1)$ such that $c^2 \int_{0}^{c} (x + x^2) f(x) \, dx = 0$.

doi:10.4169/amer.math.monthly.118.02.178

© THE MATHEMATICAL ASSOCIATION OF AMERICA [Monthly 118]
**11556.** Proposed by Pál Péter Dályay, Deák Ferenc High School, Szeged, Hungary.

For positive real numbers $a, b, c, d$, show that

$$
\frac{9}{a(b + c + d)} + \frac{9}{b(c + d + a)} + \frac{9}{c(d + a + b)} + \frac{9}{d(a + b + c)} \geq \frac{16}{(a + b)(c + d)} + \frac{16}{(a + c)(b + d)} + \frac{16}{(a + d)(b + c)}.
$$

**11557.** Proposed by Marius Cavachi, “Ovidius” University of Constanța, Constanța, Romania.

Let $S$ be a finite set of circles in the Cartesian plane having the property that any two circles in $S$ intersect in exactly two points, each circle encloses the origin, but no three circles share a common point. Construct a graph $G$ by taking as the vertices the set of all intersection points of circles in $S$, with edges corresponding to arcs of a circle in $S$ connecting vertices without passing through any intermediate vertex. (Thus, with four circles, there are 12 vertices and 24 edges.) Show that the resulting graph contains a Hamiltonian path.

---

**SOLUTIONS**

---

**An Arctan Series**

**11438 [2009, 464].** Proposed by David H. Bailey, Lawrence Berkeley National Laboratory, Berkeley, CA, Jonathan M. Borwein, University of Newcastle, Newcastle, Australia and Dalhousie University, Halifax, Canada, and Jörg Waldvogel, Swiss Federal Institute of Technology ETH, Zurich, Switzerland.

Let

$$
P(x) = \sum_{k=1}^{\infty} \arctan \left( \frac{x - 1}{(k + x + 1)\sqrt{k + 1} + (k + 2)\sqrt{k + x}} \right).
$$

(a) Find a closed-form expression for $P(n)$ when $n$ is a nonnegative integer.

(b) Show that $\lim_{x \to -1^+} P(x)$ exists, and find a closed-form expression for it.

*Solution by Hongwei Chen, Christopher Newport University, Newport News, VA.*

(a) Notice that

$$
\frac{1}{\sqrt{k+1}} - \frac{1}{\sqrt{k+n}} = \frac{\sqrt{k+n} - \sqrt{k+1}}{1 + \sqrt{k+1} \cdot \sqrt{k+n}}.
$$

Rationalizing the numerator gives

$$
\frac{1}{\sqrt{k+1}} - \frac{1}{\sqrt{k+n}} = \frac{n - 1}{(k + n + 1)\sqrt{k + 1} + (k + 2)\sqrt{k + n}}.
$$

From the identity

$$
\arctan \alpha - \arctan \beta = \arctan \left( \frac{\alpha - \beta}{1 + \alpha \beta} \right),
$$
we obtain

\[ P(n) = \sum_{k=1}^{\infty} \arctan \left( \frac{n - 1}{(k + n + 1)\sqrt{k + 1} + (k + 2)\sqrt{k + n}} \right) \]

\[ = \sum_{k=1}^{\infty} \left( \arctan \frac{1}{\sqrt{k + 1}} - \arctan \frac{1}{\sqrt{k + n}} \right). \]

Clearly, \( P(1) = 0 \). The series for \( P(0) \) telescopes to give

\[ P(0) = -\arctan 1 + \lim_{k \to \infty} \arctan \frac{1}{\sqrt{k + 1}} = -\frac{\pi}{4}. \]

In general, for \( n \geq 2 \), the series telescopes into the form

\[ P(n) = \sum_{k=2}^{n} \arctan \frac{1}{\sqrt{k}}. \]

(b) Now use the inequality \( \arctan t < t \) for \( t > 0 \). If \( k \geq 2 \) and \( x \geq -1 \), then

\[ \arctan \left( \frac{|x - 1|}{(k + x + 1)\sqrt{k + 1} + (k + 2)\sqrt{k + x}} \right) \leq \frac{|x| + 1}{k\sqrt{k + 1} + (k + 2)\sqrt{k - 1}}. \]

By the Weierstrass M-test, the series \( P(x) \) converges uniformly, and therefore it is continuous for \( x > -1 \). As in (a), we have

\[ P(x) = \sum_{k=1}^{\infty} \left( \arctan \frac{1}{\sqrt{k + 1}} - \arctan \frac{1}{\sqrt{k + x}} \right), \]

so

\[ P(x + 1) = P(x) + \arctan \frac{1}{\sqrt{1 + x}}. \]

Thus,

\[ \lim_{x \to -1^+} P(x) = \lim_{x \to -1^+} \left( P(x + 1) - \arctan \frac{1}{\sqrt{1 + x}} \right) = P(0) - \frac{\pi}{2} = -\frac{3\pi}{4}. \]

Editorial comment. The proposers report that they discovered the value \(-3\pi/4\) experimentally. They ask whether there are more general closed forms for \( P \), say at half-integers.

Also solved by R. Bagby, N. Bagis (Greece) & M. L. Glasser, D. Beckwith, M. Benito, Ó. Ciaurri, E. Fernández & L. Roncal (Spain), M. Chamberland, R. Chapman (U.K.), Y. Dumont (France), M. Goldenberg & M. Kaplan, O. Kouba (Syria), G. Lamb, O. P. Lossers (Netherlands), R. Nandan, M. Omarjee (France), A. H. Sabuwala, R. Stong, M. Tetcă (Romania), M. Vowe (Switzerland), GCHQ Problem Solving Group (U.K.), Microsoft Research Problems Group, and the proposers.

A Vector Differential Equation

11440 [2009, 547]. Proposed by Stefano Siboni, University of Trento, Trento, Italy. Consider the vector differential equation

\[ x''(t) = p(t, x(t), x'(t))x'(t) \times \left( \frac{x(t)}{\|x(t)\|} \right) \] (1)
where \( x(t) = (x_1(t), x_2(t), x_3(t)) \), \( \|u\| \) denotes the usual Euclidean norm of a vector \( u \), \( \times \) is the standard cross-product, and \( p \) and its first partial derivatives are real-valued and continuous.

(a) Show that all solutions to (1) are defined on all of \( \mathbb{R} \).
(b) Show that any nonconstant solution tends to infinity as \( t \to +\infty \).
(c) Show that for any nonzero solution \( x(t) \), \( \lim_{t \to +\infty} \frac{x(t)}{||x(t)||} \) exists.

Solution by Robin Chapman, University of Exeter, Exeter, U.K.

(a) Consider a nonconstant solution of (1) on an open interval \( I \). From (1),

\[
x(t) \cdot x''(t) = x'(t) \cdot x''(t) = 0
\]

on \( I \). Therefore,

\[
\frac{d}{dt} (x'(t) \cdot x'(t)) = 2x'(t) \cdot x''(t) = 0,
\]

which implies \( x'(t) \cdot x'(t) = A \), where \( A \) is a constant. Certainly \( A \geq 0 \). If \( A = 0 \), then \( x'(t) = 0 \) on \( I \), so \( x(t) \) would have to be constant. Hence \( A > 0 \). Next,

\[
\frac{d}{dt} (x(t) \cdot x'(t)) = x'(t) \cdot x'(t) + x(t) \cdot x''(t) = A.
\]

This implies that \( x(t) \cdot x'(t) = At + B \) for some constant \( B \). Also, \( \frac{d}{dt} (x(t) \cdot x(t)) = 2x(t) \cdot x'(t) = 2At + 2B \). This in turn implies that

\[
x(t) \cdot x(t) = At^2 + 2Bt + C,
\]

where \( C \) is a constant.

By the Cauchy–Schwarz inequality,

\[
(x(t) \cdot x'(t))^2 \leq (x(t) \cdot x(t))(x'(t) \cdot x'(t)).
\]

Upon substituting the results above, this becomes

\[
(At + B)^2 \leq A(At^2 + 2Bt + C).
\]

Thus \( B^2 \leq AC \). If \( B^2 = AC \), then \( x(t) \) and \( x'(t) \) are linearly dependent, so \( x'(t) \propto (x(t)/||x(t)||) = 0 \). Thus by (1), \( x''(t) = 0 \), so \( x'(t) \) is constant. In this case the solution has the form \( x(t) = (t + k)u \) for a fixed \( k \in \mathbb{R} \) and vector \( u \); this extends to all of \( \mathbb{R} \). Moreover, \( x(t)/||x(t)|| = u/||u|| \) for \( t \neq k \).

Suppose now that \( B^2 < AC \). We then have

\[
x(t) \cdot x(t) = \frac{(At + B)^2 + (AC - B^2)}{A} \geq \frac{AC - B^2}{A} > 0.
\]

Therefore, a problem with the differential equation in (1) being ill-defined when \( x(t) = 0 \) does not arise. From the theory of ordinary differential equations, a solution on an open interval \( I \) with an endpoint \( a \) extends to a larger open interval \( J \) containing \( a \) provided neither \( x(t) \) nor \( x'(t) \) tends to infinity as \( t \to a \). Our formulas for \( x(t) \cdot x(t) \) and \( x'(t) \cdot x'(t) \) prevent this. Hence \( x(t) \) extends to a solution on all of \( \mathbb{R} \).

(b), (c) Since \( x(t) \cdot x(t) \to \infty \), it follows that \( x(t) \to \infty \) as \( t \to \infty \). Let \( y(t) = x(t)/||x(t)|| \). Now \( \frac{d}{dt} ||x(t)|| = \frac{x(t) \cdot x'(t)}{||x(t)||} \). Hence,

\[
y'(t) = \frac{x(t)}{||x(t)||} - \frac{(x(t) \cdot x'(t)) x(t)}{||x(t)||^3} = \frac{(x(t) \cdot x(t)) x'(t) - (x(t) \cdot x'(t)) x(t)}{||x(t)||^3}.
\]

February 2011] PROBLEMS AND SOLUTIONS 181
Consequently,
\[ ||\mathbf{y}'(t)||^2 = \frac{||\mathbf{x}(t)||^2||\mathbf{x}'(t)||^2 - (\mathbf{x}(t) \cdot \mathbf{x}'(t))^2}{||\mathbf{x}(t)||^4} \]
\[ = \frac{A(At^2 + 2Bt + C) - (At + B)^2}{(At^2 + 2Bt + C)^2} = \frac{AC - B^2}{(At^2 + 2Bt + C)^2}. \]

Hence \( ||\mathbf{y}'(t)|| = O(1/t^2) \) as \( t \to \infty \). The integral \( \int_0^\infty y'(s) \, ds \) then converges absolutely, and so
\[ y(t) = y(0) + \int_0^t y'(s) \, ds \to y(0) + \int_0^\infty y'(s) \, ds, \]
a finite limit as \( t \to \infty \).


**Triangles in a Subdivided Polygon**

11441 [2009, 547]. Proposed by Y. N. Aliyev, Qafqaz University, Khyrdalan, Azerbaijan. Let \( n \geq 4 \), let \( A_0, \ldots, A_{n-1} \) be the vertices of a convex polygon, and for each \( i \) let \( B_i \) be a point in the interior of the segment \( A_iA_{i+1} \). (Here, and throughout, indices of points are taken modulo \( n \).) Let \( C_i \) denote the intersection of diagonals \( B_{i-2}B_i \) and \( B_{i-1}B_{i+1} \). Let \( a(p, q, r) \) denote the area of the triangle with vertices \( p, q, r \).

Show that
\[ \sum_{i=0}^{n-1} \frac{1}{a(A_i, B_i, B_{i-1})} \geq \sum_{i=0}^{n-1} \frac{1}{a(C_i, B_i, B_{i-1})}. \]

Solution by Jim Simons, Cheltenham, U.K. Fix all the \( B_i \) (and therefore all the \( C_i \)), and all the \( A_i \) except \( A_i \) and \( A_{i+1} \) for some particular \( i \), and consider varying the line \( A_iA_{i+1} \) through \( B_i \). Let \( |B_{i-1}B_{i+1}| = l, \alpha = \angle B_iB_{i-1}B_{i+1}, \beta = \angle B_iB_{1-1}A_i, \gamma = \angle B_iB_{i+1}B_{i-1}, \delta = \angle B_iB_{i+1}A_{i+1} \) and \( \theta = \angle A_iB_iB_{i-1} \), so that \( \alpha + \gamma - \theta = \angle A_iB_iB_{i+1} \).

Now
\[ |B_{i-1}B_i| = \frac{l \sin \gamma}{\sin(\alpha + \gamma)} \quad \text{and} \quad |B_iB_{i+1}| = \frac{l \sin \alpha}{\sin(\alpha + \gamma)}. \]

If \( h_i \) is the distance of \( A_i \) from the line \( B_{i-1}B_i \), then
\[ h_i = \frac{|B_{i-1}B_i|}{\cot \beta + \cot \theta} \quad \text{and} \quad h_{i+1} = \frac{|B_iB_{i+1}|}{\cot \delta + \cot(\alpha + \gamma - \theta)}. \]

Writing \( \Delta \) for \( 1/a(A_iB_iB_{i-1}) + 1/a(A_{i+1}B_{i+1}B_i) \), we conclude that
\[ \Delta = \frac{2 \sin^2(\alpha + \gamma)}{l^2 \sin^2 \gamma} \left( \cot \beta + \cot \theta + \frac{2 \sin^2(\alpha + \gamma)}{l^2 \sin^2 \alpha} \left( \cot \delta + \cot(\alpha + \gamma - \theta) \right) \right). \]

Differentiating with respect to \( \theta \) here gives
\[ \frac{d \Delta}{d \theta} = \frac{2 \sin^2(\alpha + \gamma)}{l^2} \left( \frac{-1}{\sin^2 \gamma \sin^2 \theta} + \frac{1}{\sin^2 \alpha \sin^2(\alpha + \gamma - \theta)} \right). \]

Thus \( d\Delta/d\theta = 0 \) when \( \sin \gamma \sin \theta = \pm \sin \alpha \sin(\alpha + \gamma - \theta) \). Since the sines are all positive, the only valid case is when \( \sin \gamma \sin \theta = \sin \alpha \sin(\alpha + \gamma - \theta) \), and this gives a
minimum of $\Delta$ since $\Delta \to \infty$ as $\theta \to 0$ and as $\theta \to \alpha + \beta$. Therefore $\Delta$ is minimized when $\sin \gamma \sin \theta = \sin \alpha \sin(\alpha + \gamma) \cos \theta - \sin \theta \cos(\alpha + \gamma)$, or equivalently, when

$$\tan \theta = \frac{\sin \alpha \sin(\alpha + \gamma)}{\sin \gamma + \sin \alpha \sin(\alpha + \gamma)} = \frac{\sin \alpha \sin(\alpha + \gamma)}{\sin \gamma + \sin \alpha \cos \alpha \cos \gamma - \sin^2 \alpha \sin \gamma} = \frac{\sin \alpha \sin(\alpha + \gamma)}{\sin \gamma \cos \alpha + \sin \alpha \cos \alpha \cos \gamma} = \tan \alpha.$$ 

This occurs when $A_i A_{i+1}$ is parallel to $B_{i-1} B_{i+1}$. Thus in a configuration that minimizes $\sum_{i=0}^{n-1} \frac{1}{a(A_i, B_i, B_{i+1})}$ for a given value of $\sum_{i=0}^{n-1} \frac{1}{a(C_i, B_i, B_{i-1})}$, every $A_i A_{i+1}$ is parallel to the corresponding $B_{i-1} B_{i+1}$. In that case every $A_i B_{i-1} C_i B_i$ is a parallelogram, so that every $a(A_i, B_i, B_{i-1}) = a(C_i, B_i, B_{i-1})$, and therefore

$$\sum_{i=0}^{n-1} \frac{1}{a(A_i, B_i, B_{i-1})} = \sum_{i=0}^{n-1} \frac{1}{a(C_i, B_i, B_{i-1})}.$$ 

Also solved by R. Chapman (U.K.), P. P. Dályay (Hungary), J. H. Lindsey II, Á. Plaza & S. Falcón (Spain), R. Stong, GCHQ Problem Solving Group (U.K.), and the proposer.

**That’s Sum Inequality**

11442 [2009, 547]. Proposed by José Luis Díaz-Barrero and José Gibergans-Báguena, Universidad Politécnica de Cataluña, Barcelona, Spain. Let $\langle a_k \rangle$ be a sequence of positive numbers defined by $a_n = \frac{1}{2}(a_{n-1}^2 + 1)$ for $n > 1$, with $a_1 = 3$. Show that

$$\left[ \sum_{k=1}^{n} \frac{a_k}{1 + a_k} \left( \sum_{k=1}^{n} \frac{1}{a_k(1 + a_k)} \right) \right]^{1/2} \leq \frac{1}{4} \left( \frac{a_1 + a_n}{\sqrt{a_1 a_n}} \right).$$

Solution by Jim Simons. Cheltenham, U.K. This is an extraordinarily weak inequality! The left side exceeds 1 for the first time when $n = 9$, at which point the right side exceeds $10^{23}$. To see that it is true, we first note that $a_k > 2^k$. To prove this by induction, note $a_1 = 3 > 2$ and $a_2 = 5 > 4$; beyond that, $a_k > 4$ and $a_{k+1} > 2a_k$. (A stronger bound of $a_k \geq 2(3/2)^{2^k-1}$ is also easy.) Since $a_k/(1 + a_k) < 1$, we have $\sum_{k=1}^{n} a_k/(1 + a_k) < n$. Since $1 + a_k \geq 4$ and $a_k > 2^k$, we have $1/(a_k(1 + a_k)) < 2^{-k-2}$ and $\sum_{k=1}^{n} 1/a_k(1 + a_k) < 1/4$. Combining these, we see that the left side is less than $\sqrt{n}/2$. For $n \geq 8$, the right side satisfies

$$\frac{1}{4} \left( \frac{a_1 + a_n}{\sqrt{a_1 a_n}} \right) > \frac{\sqrt{a_n}}{4\sqrt{3}} > \frac{2^{n/2}}{4\sqrt{3}} > \sqrt{\frac{n}{2}}.$$ 

Direct calculation shows that the inequality holds for smaller $n$, the closest call being at $n = 3$.


**An Integral with Fractional Parts**

11447 [2009, 647]. Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania. Let $a$ be a positive number, and let $g$ be a continuous, positive, increasing function on $[0, 1]$.
Prove that
\[
\lim_{n \to \infty} \int_{0}^{1} \left\{ \frac{n}{x} \right\}^{a} g(x) \, dx = \frac{1}{a + 1} \int_{0}^{1} g(x) \, dx,
\]
where \(a > 0\) and \(\{x\}\) denotes the fractional part of \(x\).

**Solution by Ralph Howard, University of South Carolina, Columbia, SC.** The result holds in greater generality; we claim that:

*If \(\beta : \mathbb{R} \to \mathbb{R}\) be a bounded measurable function that is periodic with period 1, so that \(\beta\) satisfies \(\beta(z + 1) = \beta(z)\), and if \(g \in L^{1}([0, 1])\), then*

\[
\lim_{n \to \infty} \int_{0}^{1} \beta \left( \frac{n}{x} \right) g(x) \, dx = \int_{0}^{1} \beta(z) \, dz \int_{0}^{1} g(x) \, dx.
\]

Assuming (1) and taking \(\beta(z) = \{z\}^{a}\), which has period one and is bounded for \(a > 0\), we have

\[
\lim_{n \to \infty} \int_{x=0}^{1} \left\{ \frac{n}{x} \right\}^{a} g(x) \, dx = \int_{0}^{1} z^{a} \, dz \int_{x=0}^{1} g(x) \, dx = \frac{1}{a + 1} \int_{x=0}^{1} g(x) \, dx.
\]

For the proof of (1), we extend the Riemann–Lebesgue lemma:

**Lemma.** If \(f \in L^{1}(\mathbb{R})\) and \(\beta : \mathbb{R} \to \mathbb{R}\) is a bounded measurable function such that \(\beta(z + 1) = \beta(z)\), then

\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} \beta(ny) f(y) \, dy = \int_{0}^{1} \beta(z) \, dz \int_{-\infty}^{\infty} f(y) \, dy.
\]

The proof of this lemma proceeds just as in one of the standard proofs of the Riemann–Lebesgue lemma: It is easy to check that it holds for \(f = \chi_{[a, b]}\), the characteristic function of an interval. By linearity, it then holds for finite linear combinations of characteristic functions of intervals, that is, for step functions. However, step functions are dense in \(L^{1}(\mathbb{R})\), so the result holds for all \(f \in L^{1}(\mathbb{R})\) by approximation.

To obtain (1) from the lemma, let \(g \in L^{1}([0, 1])\), and define \(f : \mathbb{R} \to \mathbb{R}\) by \(f(y) = y^{-2} g(1/y)\) for \(y \geq 1\) and \(f(y) = 0\) otherwise. Letting \(y = 1/x\) in the change of variable formula yields

\[
\int_{-\infty}^{\infty} f(y) \, dy = \int_{0}^{1} y^{-2} g(1/y) \, dy = \int_{0}^{1} g(x) \, dx.
\]

This equation holds as well with absolute value bars on the integrands, and therefore \(f \in L^{1}(\mathbb{R})\). The same change of variable yields

\[
\int_{-\infty}^{\infty} \beta(ny) f(y) \, dy = \int_{0}^{1} \beta(ny) \frac{g(1/y)}{y^{2}} \, dy = \int_{0}^{1} \beta \left( \frac{n}{x} \right) g(x) \, dx.
\]

The required result is now an application of the lemma.

Asymptotics of a Product

11456 [2009, 747]. Proposed by Raymond Mortini, Université Paul Verlaine, Metz, France. Find

$$\lim_{n \to \infty} n \prod_{m=1}^{n} \left( 1 - \frac{1}{m} + \frac{5}{4m^2} \right).$$

Solution by Oliver Geupel, Brühl, Germany. We use Stirling’s formula, which says that

$$n! \approx \sqrt{2\pi} \, n^{n+1/2} e^{-n},$$

together with the infinite product

$$\cosh z = \prod_{m=1}^{\infty} \left[ 1 + \frac{4z^2}{(2m-1)^2 \pi^2} \right].$$

(Abramowitz & Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Dover 1972, Formula 4.5.69, p. 85).

We have

$$\lim_{n \to \infty} n \prod_{m=1}^{n} \left( 1 - \frac{1}{m} + \frac{5}{4m^2} \right) = \lim_{n \to \infty} n \prod_{m=1}^{n} \frac{(2m-1)^2 + 4}{(2m)^2} = \lim_{n \to \infty} n \prod_{m=1}^{n} \frac{(2m-1)^2}{(2m)^2} \cdot \frac{(2m-1)^2 + 4}{(2m-1)^2} = \lim_{n \to \infty} \frac{n \cdot (2n)!^2}{16^n \cdot n!^4} \cdot \prod_{m=1}^{n} \left[ 1 + \frac{4m^2}{(2m-1)^2 \pi^2} \right] = \lim_{n \to \infty} \frac{n \cdot 2\pi \cdot (2n)^4 e^{-4n}}{16^n \cdot (2\pi)^2 \cdot n^{4n+2} \cdot e^{-4n} \cdot \cosh(\pi) = \frac{\cosh(\pi)}{\pi}.}

Editorial comment. A generalization was provided by Jerry Minkus (San Francisco, CA): Let \( b \) be a positive integer, and \( c \) a nonzero constant such that \( \prod_{m=1}^{b} (m^2 - bm + c) \neq 0 \). Letting \( \Delta = b^2/4 - c \), we have

$$\lim_{n \to \infty} n(n-1)(n-2) \cdots (n-b+1) \prod_{m=1}^{n} \left( 1 - \frac{b}{m} + \frac{c}{m^2} \right) = \frac{1}{c} \prod_{m=1}^{b} (m^2 - bm + c) \cdot \frac{1}{\Gamma(b/2 - \sqrt{\Delta}) \Gamma(b/2 + \sqrt{\Delta})}.$$

Several other solvers provided the generalization with \( b = 1 \) but general \( c \).