

Metastable states in Brownian energy landscape

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Abstract

Random walks and diffusions in symmetric random environment are known to exhibit metastable behavior: they tend to stay for long times in wells of the environment. For the case that the environment is a one-dimensional two-sided standard Brownian motion, we study the process of depths of the consecutive wells of increasing depth that the motion visits. When these depths are looked in logarithmic scale, they form a stationary renewal cluster process. We give a description of the structure of this process and derive from it the almost sure limit behavior and the fluctuations of the empirical density of the process.

1 Introduction and statement of the results

Consider $(X_t)_{t \geq 0}$ Brownian motion with drift in \mathbb{R} , starting from 0, with the drift at each point $x \in \mathbb{R}$ being $-f'(x)/2$ for a certain differentiable function f . That is, $(X_t)_{t \geq 0}$ satisfies the SDE

$$dX_t = d\beta_t - \frac{1}{2}f'(X_t)dt,$$

with β a standard Brownian motion. This is called diffusion in the environment f , and it has $e^{-f(x)} dx$ as an invariant measure. In statistical mechanics terms, f gives the energy profile, and the above SDE defines the Langevin dynamics for the corresponding measure $e^{-f(x)} dx$. The diffusion likes to go downhill on the environment f , decreasing the energy, and thus it tends to stay around local minima of f . If the set M_f of local minima of f is non-empty, the diffusion exhibits metastable behavior, with metastable states being the points of M_f (see Bovier (2006), Section 8).

Now, for each point x_0 of local minimum, there are intervals $[a, c]$ containing x_0 with the property that $f(x_0)$ is the minimum value of f in $[a, c]$ and $f(a), f(c)$ are the maximum values of f in the intervals $[a, x_0], [x_0, c]$ respectively. Let $J(x_0) := [a_{x_0}, c_{x_0}]$ be the maximal such interval. This is the “interval of influence” for x_0 . We call $f|_{J(x_0)}$ the **well** of x_0 , the number $\min\{f(a_{x_0}) - f(x_0), f(c_{x_0}) - f(x_0)\}$ the **depth** of the well, and x_0 the **bottom** of the well. If the diffusion starts inside $J(x_0)$, typically it is trapped in that interval for a time that depends predominantly on the depth of the well.

Also, for $h > 0$, we say that the local minimum x_0 is a point of **h -minimum** for f if the depth of its well is at least h , while a point x_0 is called a point of **h -maximum** for f if it is a point of h -minimum for $-f$.

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A case of particular interest is the one where the function f above is a “typical” two sided Wiener path with $f(0) = 0$. Of course, an f picked from the Wiener measure is not differentiable, but there is a way to make sense of the above SDE defining X through a time and space transformation. See Shi(2001) for the construction.

From now on, we will denote the two sided Wiener path with B . Due to the nature of a typical Wiener path, once the diffusion exits an interval $J(x_0)$, it is trapped in another well. We will define a process x_B that records some local minima of the path of B in the order that are visited by a typical diffusion path, but not all of them. Roughly, assuming that the value of the process at some point is x_0 , its next value is going to be the unique local minimum x_1 whose interval of influence is the smallest one satisfying $J(x_1) \supsetneq J(x_0)$. The well $B|J(x_1)$ is the minimal one containing strictly $B|J(x_0)$, it is the first well right after $J(x_0)$ that can trap the diffusion for considerably more time, and this because it has greater depth.

The formal definition of the process x_B goes as follows. With probability one, for all $h > 0$, there are $z_{-1}(h) < 0 < z_1(h)$ points of h -extremum (h -minimum or h -maximum) for B closest to zero from the left and right respectively. Exactly one of them is a point of h -minimum for B . This we denote by $x_B(h)$.

The process $(x_B(h))_{h>0}$ has piecewise constant paths, it is left continuous, and there are several results showing its impact on the behavior of the diffusion. For example, $X_t - x_B(\log t)$ converges in distribution as $t \rightarrow +\infty$ (Tanaka(1988)), i.e., x_B gives a good prediction for the location X_t of the diffusion at large times. Note also that, by Brownian scaling, for $a > 0$ the process x_B satisfies

$$(x_B(ah))_{h>0} \stackrel{d}{=} (a^2 x_B(h))_{h>0}. \quad (1)$$

We would like to study the set of points where x_B jumps, because this shows how frequently the diffusion discovers the bottom of a well that is deeper than any well encountered by then. It turns out that it is more convenient to consider this set in logarithmic scale, that is, the point process

$$\xi := \{t \in \mathbb{R} : x_B \text{ has a jump at } e^t\}.$$

The purpose of this work is to describe the structure of ξ . A crucial observation is that the law of ξ is translation invariant because of the scaling relation (1) for x_B . Since B is continuous, the set ξ has no finite accumulation point.

For any set A define $N(A) := |\xi \cap A|$, the cardinality of $\xi \cap A$, i.e., N is the counting measure induced by ξ . When A is an interval, we will write NA instead of $N(A)$.

The following result (Theorem 2.4.13 in Zeitouni(2004)) gives the probability that ξ does not hit an interval.

Theorem 1 (Dembo, Guionnet, Zeitouni). *For $t > 0$,*

$$\mathbf{P}(N[0, t] = 0) = \frac{1}{t^2} \left(\frac{5}{3} - \frac{2}{3} e^{1-t} \right). \quad (2)$$

This allows us to compute the mean density, $\mathbf{E}N(0, 1]$, of the process, because for a simple stationary point process, its mean density equals also its intensity $\lim_{t \rightarrow 0^+} t^{-1} \mathbf{P}(N(0, t] > 0)$ (Proposition 3.3 IV in Daley and Vere-Jones (2003)). Thus we get the following result, which has been predicted by physicists (relation (84) in Le Dousal et al. (1999)) via renormalization arguments.

Corollary 1 (Mean density). *For every Borel set $A \subset \mathbb{R}$, $\mathbf{E}N(A) = \frac{4}{3} \lambda(A)$, where λ is Lebesgue measure. Moreover,*

$$\lim_{t \rightarrow \infty} \frac{N[0, t]}{t} = \frac{4}{3} \quad a.s. \quad (3)$$

In Section 5, we give an easy proof of this corollary which avoids the use of Theorem 1.

Combining this with well known localization results for the diffusion, we infer that the diffusion jumps to a deeper well extremely rarely, at times that progress roughly as $\exp(\exp(3n/4))$. We also remark that for the process $\hat{\xi} := \{t : x_B \text{ changes sign at } e^t\}$, which is a subset of ξ , it was shown in Cheliotis (2005) that it has mean density $1/3$. On average, one in every four consecutive jumps is a sign change.

The description of ξ given in the coming subsection has the following implication.

Theorem 2 (Fluctuations). *As $t \rightarrow \infty$, the following convergence in distribution holds:*

$$\frac{1}{\sqrt{t}} \left(N[0, t] - \frac{4}{3}t \right) \Rightarrow \mathcal{N}(0, \sigma^2), \quad (4)$$

with $\sigma^2 = \frac{64}{27} - \frac{4}{9} \int_0^\infty e^{-t}(1+t)^{-1} dt \approx 2.105327$

1.1 The structure of the process ξ

ξ is a renewal cluster process in \mathbb{R} . That is, it consists of:

- (i) a skeleton of points that serve as “centers” of clusters, together with
- (ii) the cluster points.

The centers form a stationary renewal process in \mathbb{R} . Then each cluster is distributed in a certain way relative to its center (to be exact, relative to the skeleton).

More specifically, let ψ be a stationary renewal process in \mathbb{R} with interarrival distribution that of the sum $W_1 + W_2$ of two independent random variables with $W_1 \sim \text{Exponential}(1)$, $W_2 \sim \text{Exponential}(2)$. ψ is the “centers” process.

Next, we describe the law of a cluster with center at 0.

Count the points of a Poisson point process in $[0, \infty)$ with rate 1 as $(t_k)_{k \geq 2}$ in increasing order, and let $t_1 = 0$. Out of the points

$$t_1, t_2, \dots$$

we will keep only the first \mathcal{N} , where \mathcal{N} is defined as follows. Take a sequence $(Y_i)_{i \geq 1}$ of i.i.d. random variables, independent of $(t_k)_{k \geq 2}$, each with distribution $\text{Exponential}(1)$. Define recursively a sequence $(z_k)_{k \geq 1}$ as follows:

$$\begin{aligned} z_1 &:= 1 \\ z_{k+1} &:= z_k + Y_k e^{t_k} \text{ for } k \geq 1, \end{aligned}$$

and let

$$\begin{aligned} \mathcal{N} &:= \max\{i : t_i \leq \log z_i\}, \\ \mathcal{T} &:= \{t_1, t_2, \dots, t_{\mathcal{N}}\}. \end{aligned}$$

\mathcal{N} is finite with probability 1 as we will see in Theorem 4.

A cluster with center at 0 has the law of \mathcal{T} .

Let also

$$F := \log z_{\mathcal{N}+1}.$$

Independent of $(t_k)_{k \geq 1}, (Y_i)_{i \geq 1}$ take another random variable $Z \sim \text{Exponential}(2)$, and let

$$R := F + Z.$$

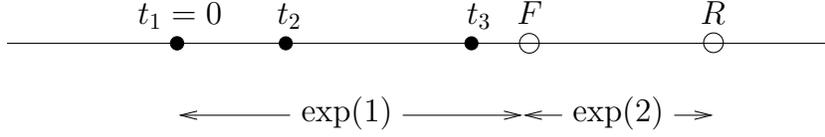


Figure 1: A typical cluster with center at t_1 . Points appear at an interval with length distribution $\exp(1)$. This cluster has 3 points, marked with black dots. The next cluster right to it will have its center at R . The interval between F and R has length distribution $\exp(2)$, and it is not allowed to have points.

Note that $\mathcal{T} \subset [0, F)$. We will see in Section 2.2 that $F \sim \text{Exponential}(1)$, while, by construction, $R - F \sim \text{Exponential}(2)$.

The role of F and R is the following. Given that x is a point in the process of the centers, the cluster at x has law $x + \mathcal{T}$, while the next cluster to the right of it has center at $x + R$, and thus distributed as $x + R + \mathcal{T}'$, with \mathcal{T}' an independent copy of \mathcal{T} .

And we are now ready to give the formal description of ξ . For each $x \in \psi$ let $x^+ := \inf\{y \in \psi : y > x\}$, the nearest right neighbor of x in ψ .

Theorem 3. ξ has the same law as

$$\bigcup_{x \in \psi} \{x + \mathcal{T}_x(x^+ - x)\},$$

where $\{\mathcal{T}_x(x^+ - x) : x \in \psi\}$ are independent, and $\mathcal{T}_x(x^+ - x)$ is distributed as \mathcal{T} given that $R = x^+ - x$.

Finally, we look closer into the law of a cluster. The random variables \mathcal{N}, F are positively correlated, and the following result captures their joint distribution. For its statement, we will use the confluent hypergeometric function of the second kind, which is usually denoted by Ψ . This has three arguments, and its value at a point (x, y, z) is denoted by $\Psi(x, y; z)$.

Theorem 4. The moment generating function of (\mathcal{N}, F) equals

$$\mathbf{E}(e^{\lambda \mathcal{N} + \mu F}) = e^\lambda \frac{\Psi(1 - e^\lambda, 1 + \mu; 1)}{\Psi(-e^\lambda, \mu; 1)} \quad (5)$$

for all $(\lambda, \mu) \in \mathbb{R}^2$ where the generating function is finite. This set of (λ, μ) is open, convex, and contains $(0, 0)$. In particular, $\mathbf{E}(\mathcal{N}) = 2$.

The main ingredient in the proof of the above results is a new way to follow the evolution of x_B , using excursion theory. This point of view has also been useful in the study of large deviations for the family of paths $\{\varepsilon x_B(\cdot) : \varepsilon > 0\}$ as $\varepsilon \rightarrow 0$ (see Cheliotis and Virag (2013)). Two other ways of studying x_B have been exhibited in Zeitouni(2004) and Le Dousal et al. (1999).

Theorems 2, 3 and 4 are proven in Sections 4, 2 and 3 respectively. An alternative proof of Corollary 1 is given in Section 5, while Section 6 gives an elementary computation of a certain expectation which makes the proof of Corollary 1 independent of Theorem 4.

2 Description of the process ξ . Proof of Theorem 3

In this section, we study how the process x_B evolves, and justify the description of the structure of the process ξ given in Section 1.1, thus proving Theorem 3.

We will use elements of excursion theory, for which we refer the reader to Bertoin (1996), Chapter IV. For ease in exposition, when working with the excursions of a real valued process $(Y_t)_{t \geq 0}$ away from 0, by the term “actual domain” of an excursion ε we will mean the interval $[c, d]$ in the domain of Y where the excursion happens and not $[0, d - c]$ or $[0, \infty)$, which are the two common conventions for the domain of ε in the literature (Bertoin (1996) adopts the first). Also we will abuse notation (notice the conflict with (7) below) and denote by $\bar{\varepsilon}$, the **height** of ε , that is, the supremum of ε in its domain.

For any process $(Y_t)_{t \in I}$ defined in an interval I containing 0, we define the processes \underline{Y}, \bar{Y} of the running infimum and supremum of Y respectively as

$$\underline{Y}_t := \inf\{Y_s : s \text{ between } 0 \text{ and } t\}, \quad (6)$$

$$\bar{Y}_t := \sup\{Y_s : s \text{ between } 0 \text{ and } t\} \quad (7)$$

for all $t \in I$. This notation will be used throughout the paper.

Now let $(B_s)_{s \in \mathbb{R}}$ be a two sided standard Brownian motion. For $\ell > 0$, we define

$$\begin{aligned} H_\ell^- &:= \sup\{s < 0 : B_s = \ell\}, \\ H_\ell^+ &:= \inf\{s > 0 : B_s = \ell\}, \\ \Theta_\ell &:= -\min\{B_s : s \in [H_\ell^-, H_\ell^+]\}. \end{aligned} \quad (8)$$

Following the path $B[[H_\ell^-, H_\ell^+]$ as ℓ increases reveals the consecutive values of $(x_B(h))_{h > 0}$ in the same order that the diffusion typically discovers them. Adopting this view, leads us to consider the processes $\{e_t^+ : t \geq 0\}$ and $\{e_t^- : t \geq 0\}$ of excursions away from 0 of $(\bar{B}_s - B_s)_{s \geq 0}$ and $(\bar{B}_s - B_s)_{s \leq 0}$ respectively. Both processes are parametrized by the inverse of the local time processes $(\bar{B}_s)_{s \geq 0}$ and $(\bar{B}_s)_{s \leq 0}$ respectively, and of course they are independent and identically distributed.

The continuity of B implies that Θ is piecewise constant, left continuous, and the set of points where it jumps, call it \mathcal{L} , has 0 as only accumulation point.

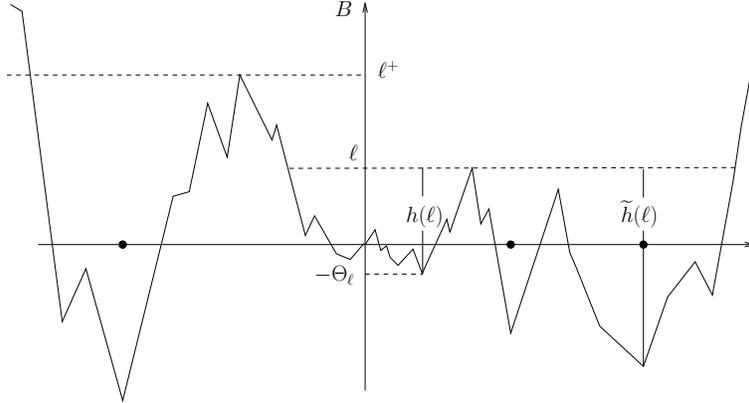


Figure 2: Following the evolution of x_B . The dots mark three consecutive values of x_B . Θ jumps at the values ℓ, ℓ^+ .

Pick $\ell \in \mathcal{L}$. With probability 1, exactly one of $(\bar{B}_s - B_s)_{s \geq 0}, (\bar{B}_s - B_s)_{s \leq 0}$ has at the value ℓ of its local time a nontrivial excursion, call it ε , and moreover that excursion makes the graph of B go deeper than $-\Theta_\ell$. In Figure 2, the excursion comes from $(\bar{B}_s - B_s)_{s \geq 0}$. Let

$$h(\ell) := \ell + \Theta_\ell,$$

call $\tilde{h}(\ell)$ the height of ε , and $\ell^+ := \min\{x \in \mathcal{L} : x > \ell\}$. x_B jumps at the “time” $h(\ell)$, and its value just after $h(\ell)$ is contained in the “actual domain” of the excursion ε . The excursion may contain more than one value of x_B (e.g., in Figure 2 it contains two, marked with a dot). After we take into account the jumps that happen in moving through these values, we wait until Θ jumps again at ℓ^+ because of a new excursion that goes deeper.

2.1 The underlying renewal

We will now examine the distribution of the points $\{h(\ell) : \ell \in \mathcal{L}\}$. Fix $\ell \in \mathcal{L}$. For simplicity, we will denote $h(\ell), \tilde{h}(\ell), h(\ell^+)$ by h, \tilde{h}, h^+ respectively.

Lemma 1. (i) *The random variables $\tilde{h}/h, h^+/\tilde{h}$ are independent of each other and of $B[[H_\ell^-, H_\ell^+]$, and have density $x^{-2}1_{x \geq 1}$ and $2x^{-3}1_{x \geq 1}$ respectively.*

(ii)

$$\log h^+ - \log \tilde{h}, \log \tilde{h} - \log h$$

have exponential distribution with means $1/2$ and 1 respectively.

Proof. (i) Pick $\delta > 0$ arbitrary. First, we prove the claim for ℓ being the smallest element of $\mathcal{L} \cap [\delta, \infty)$. Let (see Figure 3)

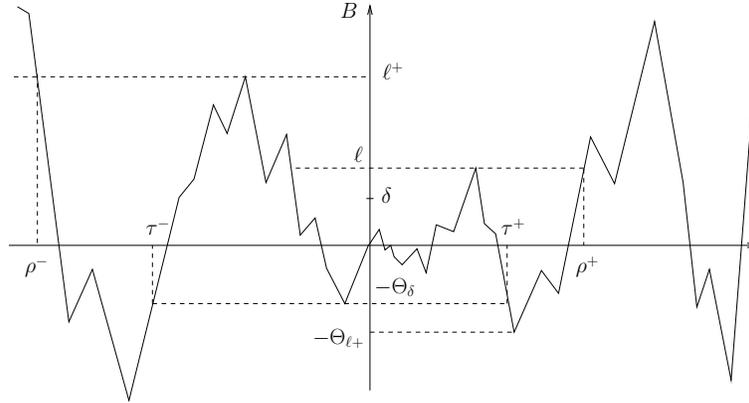


Figure 3:

$$\begin{aligned} \tau^+ &:= \inf\{s \geq H_\delta^+ : B_s = -\Theta_\delta\}, \\ M^+ &:= \overline{B}_{\tau^+}, \\ \rho^+ &:= \inf\{s > \tau^+ : B_s = M^+\}, \\ J^+ &:= M^+ + \Theta_\delta, \end{aligned}$$

and similarly on the negative semiaxis,

$$\begin{aligned} \tau^- &:= \sup\{s \leq H_\delta^- : B_s = -\Theta_\delta\}, \\ M^- &:= \overline{B}_{\tau^-}, \\ \rho^- &:= \sup\{s < \tau^- : B_s = M^-\}, \\ J^- &:= M^- + \Theta_\delta. \end{aligned}$$

Then $\ell = M^- \wedge M^+$ and

$$\frac{\tilde{h}}{h} = \begin{cases} (M^+ - \underline{B}_{\rho^+})/J^+ & \text{if } M^- \geq M^+, \\ (M^- - \underline{B}_{\rho^-})/J^- & \text{if } M^- < M^+. \end{cases} \quad (9)$$

For $x \geq 1$, we compute

$$\begin{aligned} \mathbf{P}\left(\frac{M^+ - \underline{B}_{\rho^+}}{J^+} \geq x \mid J^+\right) &= \mathbf{P}(\text{Brownian Motion starting from } -\Theta_\delta \text{ hits } M^+ - xJ^+ \text{ before } M^+ \mid J^+) \\ &= \frac{M^+ + \Theta_\delta}{xJ^+} = \frac{1}{x}. \end{aligned}$$

Since τ^+ is a stopping time, $\{B_{\tau^++s} - B_{\tau^+} : s \geq 0\}$ is independent of $B|[\tau^-, \tau^+]$. Thus, given J^+ , $(M^+ - \underline{B}_{\rho^+})/J^+$ is independent of $B|[\tau^-, \tau^+]$, and the previous computation shows that it is independent of J^+ as well and has density $x^{-2}1_{x \geq 1}$. Thus, $(M^+ - \underline{B}_{\rho^+})/J^+$ is independent of $B|[\tau^-, \tau^+]$. Similarly $(M^- - \underline{B}_{\rho^-})/J^-$ has the same density, $x^{-2}1_{x \geq 1}$, and is independent of $B|[\tau^-, \tau^+]$. Since the event $M^- \geq M^+$ is in the σ -algebra generated by $B|[\tau^-, \tau^+]$, these observations combined with (9) imply the claim of the lemma for \tilde{h}/h .

We turn now to h^+/\tilde{h} . Let

$$\begin{aligned} \hat{\tau}^+ &:= \inf\{s \geq H_{\ell^+}^+ : B_s = -\Theta_{\ell^+}\}, \\ \hat{M}^+ &:= \overline{B}_{\hat{\tau}^+}, \\ \hat{\tau}^- &:= \sup\{s \leq H_{\ell^+}^- : B_s = -\Theta_{\ell^+}\}, \\ \hat{M}^- &:= \overline{B}_{\hat{\tau}^-}. \end{aligned}$$

Here Θ_{ℓ^+} denotes the limit of Θ at ℓ from the right, and the same remark applies to $H_{\ell^+}^-, H_{\ell^+}^+$. Then $\ell^+ = \hat{M}^- \wedge \hat{M}^+$, $\tilde{h} = \ell + \Theta_{\ell^+}$, $h^+ = \ell^+ + \Theta_{\ell^+}$. So that for $x \geq 1$,

$$\mathbf{P}(h^+ > x\tilde{h} \mid \tilde{h}) = \mathbf{P}(\text{Brownian starting from } \ell \text{ hits } x\tilde{h} - \Theta_{\ell^+} \text{ before } -\Theta_{\ell^+} \mid \tilde{h})^2 = \left(\frac{\ell + \Theta_{\ell^+}}{x\tilde{h}}\right)^2 = \frac{1}{x^2}.$$

The strong Markov property implies that, given \tilde{h} , h^+ is independent of $B|[\underline{H}_{\ell^+}^-, H_{\ell^+}^+]$, and the above computation shows that h^+/\tilde{h} is independent of $B|[\underline{H}_{\ell^+}^-, H_{\ell^+}^+]$. Note that \tilde{h}/h is determined by $B|[\underline{H}_{\ell^+}^-, H_{\ell^+}^+]$. Thus the claim about h^+/\tilde{h} is proved.

Having proved the result for $\ell := \min\{\mathcal{L} \cap [\delta, \infty)\}$, we can prove it similarly for ℓ^+ by repeating the above procedure with the role of δ played now by ℓ^+ . Doing the appropriate induction, we get the result for all elements of $\mathcal{L} \cap [\delta, \infty)$. But δ was arbitrary, so the claim is true for all $\ell \in \mathcal{L}$.

(ii) It is an immediate consequence of part (i). ■

Lemma 1 shows that $\{\tilde{h}(\ell)/h(\ell), h^+(\ell)/\tilde{h}(\ell) : \ell \in \mathcal{L}\}$ are all independent because for given $\ell \in \mathcal{L}$, the ones with index strictly less than ℓ are functions of $B|[\underline{H}_\ell^-, H_\ell^+]$, while $\tilde{h}(\ell)/h(\ell), h^+(\ell)/\tilde{h}(\ell)$ are independent of that path and of each other. Also their distribution is known. Thus

$$\{\log h(\ell^+) - \log h(\ell) : \ell \in \mathcal{L}\}$$

are i.i.d. each with law the same as $W_1 + W_2$, with $W_1 \sim \text{Exponential}(1)$, $W_2 \sim \text{Exponential}(2)$ independent. Let

$$\psi := \{\log h(\ell) : \ell \in \mathcal{L}\}.$$

For a given $a > 0$, scaling invariance of Brownian motion implies that $\{h(\ell) : \ell \in \mathcal{L}\} \stackrel{d}{=} \{ah(\ell) : \ell \in \mathcal{L}\}$. Combining these observations, we have that ψ is a stationary renewal process with interarrival times distributed as $W_1 + W_2$ mentioned above.

x_B jumps at each point of $h(\mathcal{L})$, thus $\psi \subset \xi$. In fact, the inclusion is strict, and the points in $\xi \setminus \psi$ are the subject of the next subsection.

2.2 Jumps inside an excursion. Distribution of the clusters

Now we examine the behavior of x_B in each interval $[h(\ell), h(\ell^+)]$, where $\ell \in \mathcal{L}$. Again, we abbreviate $h(\ell), h(\ell^+)$ to h, h^+ . Assume that the jump at ℓ is caused by an excursion, ε , of $(\overline{B}_s - B_s)_{s \geq 0}$. This excursion is simply $(B_{H_\ell^+} - B_{H_\ell^+ + s} : 0 \leq s \leq H_{\ell^+}^+ - H_\ell^+)$ and contains all the information on the jumps of x_B in $[h, h^+)$.

CLAIM: Given h , the excursion ε has law $n(\cdot \mid \overline{\varepsilon} \geq h)$.

Recall the excursion processes $\{e_t^+ : t \geq 0\}$ and $\{e_t^- : t \geq 0\}$ introduced just after relation (8). They are independent and identically distributed, and we call n their characteristic measure. We prove the claim for $\ell := \inf(\mathcal{L} \cap [\delta, \infty))$, where $\delta > 0$ is arbitrary. An argument similar with the one used in the proof of Lemma 1 gives the result for any $\ell \in \mathcal{L}$.

With probability 1, \mathcal{L} does not contain δ . Let

$$\begin{aligned}\tau^- &= \inf\{t \geq \delta : \overline{e_t^-} > t + \Theta_\delta\}, \\ \tau^+ &= \inf\{t \geq \delta : \overline{e_t^+} > t + \Theta_\delta\}.\end{aligned}$$

Then $\ell = \tau^- \wedge \tau^+$, and

$$\varepsilon = \begin{cases} e_{\tau^+}^+ & \text{if } \tau^- \geq \tau^+, \\ e_{\tau^-}^- & \text{if } \tau^- < \tau^+. \end{cases} \quad (10)$$

The process $t \mapsto (t, e_t^+)$ is a Poisson point process with characteristic measure $\lambda \times n$ (λ is Lebesgue measure), and τ^+ is the first entrance time of this process in the set $A := \{(s, \varepsilon) : s \geq \delta, \overline{\varepsilon} > s + \Theta_\delta\}$. The law of the pair $(\tau^+, e_{\tau^+}^+)$ is that of $\lambda \times n(\cdot \mid (s, \varepsilon) \in A)$, and given that $\tau^+ + \Theta_\delta = h$, the law of $e_{\tau^+}^+$ is independent of τ^+ and equals $n(\cdot \mid \overline{\varepsilon} > h)$, which the same as $n(\cdot \mid \overline{\varepsilon} \geq h)$. The analogous assertion holds for the pair $(\tau^-, e_{\tau^-}^-)$, which is independent of $(\tau^+, e_{\tau^+}^+)$. These observations together with (10) imply the claim.

We pause for a moment to define for any excursion ε_0 of $\overline{B} - B$ and $a > 0$, a positive integer $\mathcal{N}(\varepsilon_0, a)$.

Assume that ε_0 has domain $[0, \zeta]$ and height $\tilde{h} := \overline{\varepsilon_0} > 0$. We consider the path $\gamma = -\varepsilon_0$, see Figure 4. To the process $(\gamma_s - \underline{\gamma}_s)_{s \in [0, \zeta]}$ corresponds the process $(\varepsilon_r)_{r \in [0, \tilde{h}]}$ of its excursions away from zero. This is parametrized by the inverse of the local time process defined by the absolute value of the running minimum (i.e., $-\underline{\gamma}_s$). Since ε_0 is continuous defined on a compact interval, the subset of excursions with height $\geq a$ constitutes a finite, possibly empty, set $(\varepsilon_{r_i})_{1 \leq i \leq K}$, with (r_i) increasing. We define recursively a finite sequence j as follows (see Figure 4).

$$\begin{aligned}j_0 &:= 0 \\ j_1 &:= \min\{i \leq K : \overline{\varepsilon_{r_i}} > a\} \quad \text{if the set is nonempty,} \\ j_{k+1} &:= \min\{i \leq K : \overline{\varepsilon_{r_i}} > \overline{\varepsilon_{r_{j_k}}}\} \quad \text{if the set is nonempty and } k \geq 1.\end{aligned}$$

If any of the sets involved in the definition is empty, the corresponding j_k is not defined, and the recursive definition stops. Let

$$\mathcal{N}(\varepsilon_0, a) := \begin{cases} \max\{k : j_k \text{ is defined}\} + 1 & \text{if } \overline{\varepsilon} \geq a, \\ 0 & \text{if } \overline{\varepsilon} < a. \end{cases}$$

Recalling the definition of x_B , we can say informally that $\mathcal{N}(\varepsilon_0, a)$ counts the number of jumps caused in x_B by ε_0 with starting benchmark a .

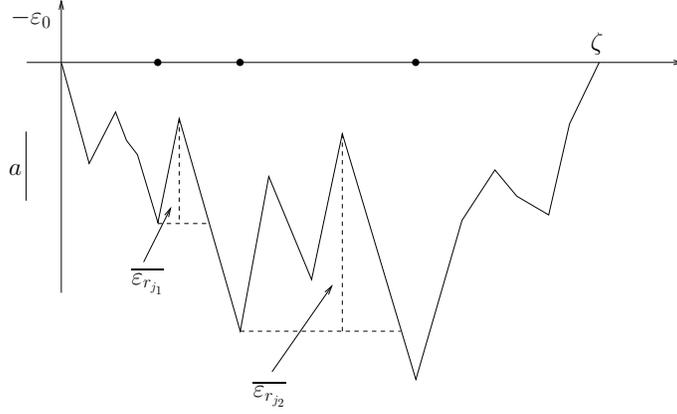


Figure 4: The graph of $-\varepsilon_0$. For this path, only j_0, j_1, j_2 are defined, thus $\mathcal{N}(\varepsilon_0, a) = 3$. The three points on the x -axis mark the values of x_B after $x_B(a)$ that are contained in the “actual domain” of the excursion.

Thus $\mathcal{N}(\varepsilon, h)$ counts the jumps of x_B in $[h, h^+)$, and note that $\mathcal{N}(\varepsilon, h) \geq 1$ because of the jump at h , while in the interval $[\tilde{h}, h^+)$ there are no jumps. The excursions $\varepsilon_{r_{j_k}}$ in the definition of $\mathcal{N}(\varepsilon, h)$ give rise to the jumps in (h, \tilde{h}) . And in fact, if we let $\nu := \mathcal{N}(\varepsilon, h)$, the jumps happen exactly at the points

$$\overline{\varepsilon_{r_{j_1}}} < \dots < \overline{\varepsilon_{r_{j_{\nu-1}}}},$$

assuming that $\nu > 1$. Otherwise, there are no jumps in (h, \tilde{h}) .

We will determine the law of these points given the value of h .

Let $a = h$. The law of $-\varepsilon$ is described as follows (see Revuz and Yor (1999), Chapter XII, Theorem 4.1). It starts from zero as the negative of a three dimensional Bessel process until it hits $-a$. After that, it continues as Brownian motion until hitting 0. Thus, let $\eta_0 := a, y_0 = -a$, and take W a Brownian motion starting from y_0 . Then let

$$\begin{aligned} \tau_0 &:= \min\{s > 0 : W_s - \underline{W}_s = \eta_0\}, \\ y_1 &:= \underline{W}_{\tau_0}, \\ \sigma_1 &:= \min\{s > \tau_0 : W_s = y_1\}, \\ \eta_1 &:= \overline{W}_{\sigma_1} - y_1. \end{aligned}$$

By well known property of Brownian motion, it holds $\eta_1 > \eta_0$. Repeat the above, with the role of η_0, y_0 played by η_1, y_1 , and define $\tau_1, y_2, \sigma_2, \eta_2$. Continue recursively. Then ν is the largest integer i for which $\eta_{i-1} \leq |y_{i-1}|$, while

$$\overline{\varepsilon_{r_{j_1}}} = \eta_1, \overline{\varepsilon_{r_{j_2}}} = \eta_2, \dots, \overline{\varepsilon_{r_{j_{\nu-1}}}} = \eta_{\nu-1},$$

and $-y_\nu = \tilde{h}$, which is the height of the excursion.

We remark that given y_k and η_k , the random variables y_{k+1}, η_{k+1} are independent of $W|[0, \sigma_k]$ because by the strong Markov property, $W_s^{(k)} := W_{\sigma_k+s} - y_k$ is a standard Brownian motion independent of $W|[0, \sigma_k]$, and y_{k+1}, η_{k+1} are functions of the path $W^{(k)}$ and of y_k, η_k . The dependence on y_k, η_k is removed if we consider

$$\frac{y_k - y_{k+1}}{\eta_k} =: \alpha_k, \quad \frac{\eta_{k+1} - \eta_k}{\eta_k} =: \beta_k.$$

CLAIM: The random variables α_k, β_k are independent of $W|[0, \sigma_k]$, independent of each other, and have densities $e^{-x} 1_{x>0}, (1+x)^{-2} 1_{x>0}$ respectively.

Indeed, consider the excursion process, for the excursions away from zero, of the reflected from the past minimum process $W^{(k)} - \underline{W}^{(k)}$ parametrized by the inverse of the local time process $L_s := |\underline{W}_s^{(k)}|$. $y_k - y_{k+1}$ is the value of the local time when the first excursion with height at least η_k appears, while η_{k+1} is the height of the excursion. Now Proposition 2 from Chapter 0 of Bertoin (1996) gives that, conditional on η_k , $y_k - y_{k+1}$ is an exponential random variable with parameter $n(\varepsilon \geq \eta_k) = 1/\eta_k$, and the excursion is independent of $y_k - y_{k+1}$ and has law $n(\cdot | \bar{\varepsilon} \geq \eta_k)$. The equality $n(\varepsilon \geq \eta_k) = 1/\eta_k$ is true by Exercise 2.10 (1), Chapter XII of Revuz and Yor (1999), which also implies that η_{k+1} has density $\eta_k x^{-2} 1_{x \geq \eta_k}$. So that the conditional law of (α_k, β_k) given η_k does not depend on η_k or y_k and it is a product measure. Thus α_k, β_k do not depend on η_k or y_k , are independent of each other, and have the required density. The proof of the claim is concluded by also taking into account the discussion preceding it.

The above imply that $\{(\alpha_k, \beta_k) : k \geq 0\}$ are i.i.d. Then the random variables

$$\nu, \left(\frac{|y_{k-1}|}{a}, \log \frac{\eta_{k-1}}{a} \right)_{1 \leq k \leq \nu}, \log \frac{|y_\nu|}{a} \quad (11)$$

are related in exactly the same way as

$$\mathcal{N}, (z_k, t_k)_{1 \leq k \leq \mathcal{N}}, F, \quad (12)$$

defined in Section 1.1. In particular, they don't depend on a . For example

$$\log \frac{\eta_{k+1}}{a} - \log \frac{\eta_k}{a} = \log \frac{\eta_{k+1}}{\eta_k} = \log(1 + \beta_k)$$

has exponential distribution with mean 1. The correspondence between (11), (12) proves that F has exponential distribution with mean 1 because $\log(|y_\nu|/a) = \log(\tilde{h}/h)$, whose distribution was determined in Lemma 1. Thus

$$\xi \cap [\log h(\ell), \log h(\ell^+)) \mid \ell \in \mathcal{L} \stackrel{d}{=} \log h(\ell) + \mathcal{T} \mid \ell \in \mathcal{L}.$$

Taking into account the structure of the process $\psi := \{\log h(\ell) : \ell \in \mathcal{L}\}$ given at the end of Section 2.1, we get Theorem 3.

3 Jumps inside an excursion. Proof of Theorem 4

For $\lambda, \mu \in \mathbb{R}$, let

$$K(\lambda, \mu) := \mathbf{E}(e^{\lambda \mathcal{N} + \mu F}) \quad (13)$$

the moment generating function of (\mathcal{N}, F) . The aim of this section is to compute K explicitly for all λ, μ for which it is finite. First we compute it for negative λ, μ , and then we use analytic extension.

3.1 The Laplace transform

Recall that we denote by Ψ the confluent hypergeometric function of the second kind.

Proposition 1. *It holds*

$$\mathbf{E}(e^{-\lambda \mathcal{N} - \mu F}) = e^{-\lambda} \frac{\Psi(1 - e^{-\lambda}, 1 - \mu; 1)}{\Psi(-e^{-\lambda}, -\mu; 1)} \quad (14)$$

for all $\lambda, \mu \geq 0$.

Proof. Because of the correspondence between (11) and (12), the pair (\mathcal{N}, F) has the same distribution as $(\mathcal{N}(\varepsilon, 1), \log \bar{\varepsilon})$ where ε is an excursion with law $n(\cdot | \bar{\varepsilon} \geq 1)$. In the following, we use the notation set in Section 2.2, with $h = 1$, and in particular the random variables $\{y_k, \eta_k, \alpha_k, \beta_k : k \geq 0\}$. Let

$$(s_k, x_k) := \left(\frac{|y_k|}{\eta_k}, |y_k| \right) \text{ and } \phi_k = (1 + \beta_k)^{-1} \quad (15)$$

for $k \geq 0$. Then, $(s_0, x_0) = (1, 1)$,

$$s_{k+1} = \frac{x_k + \alpha_k \eta_k}{\eta_k (1 + \beta_k)} = (s_k + \alpha_k) \phi_k, \quad (16)$$

$$x_{k+1} = x_k \left(1 + \frac{\alpha_k}{s_k} \right), \quad (17)$$

for all $k \geq 0$, while by the claim in Section 2.2, $\{\alpha_k, \phi_k : k \geq 0\}$ are independent, with α_k exponential with mean 1 and ϕ_k uniform in $[0, 1]$. Also

$$\mathcal{N}(\varepsilon, 1) = \min\{i > 1 : s_i < 1\}, \quad (18)$$

$$\bar{\varepsilon} = x_{\mathcal{N}(\varepsilon, 1)}. \quad (19)$$

Fix $\lambda, \mu \geq 0$. For given $(s, x) \in [1, \infty) \times (0, \infty)$, consider the Markov process $(s_k, x_k)_{k \geq 0}$ that has $(s_0, x_0) = (s, x)$ and evolves as in (16), (17), and define

$$\begin{aligned} M = M(s, x) &:= \min\{i > 1 : s_i < 1\}, \\ f(s, x) &:= \mathbf{E}_{s_0=s, x_0=x} (e^{-\lambda M - \mu \log x_M} \mathbf{1}_{M < \infty}) = \mathbf{E}_{s_0=s, x_0=x} (e^{-\lambda M} x_M^{-\mu} \mathbf{1}_{M < \infty}). \end{aligned} \quad (20)$$

We will show that $M < \infty$ with probability 1, so that $K(-\lambda, -\mu) = f(1, 1)$. Thus the plan is to show that f is regular enough, derive a differential equation involving it, and solve the equation to get in particular the value $f(1, 1)$.

Using standard arguments, we can see that f is measurable. Also, it is nonnegative and bounded by $\delta^{-\mu}$ in each set of the form $[1, \infty) \times [\delta, \infty)$, with $\delta > 0$, because $\lambda, \mu, M \geq 0$, and by (17), $(x_k)_{k \geq 0}$ is increasing.

Brownian scaling gives that

$$f(s, x) = x^{-\mu} f(s, 1). \quad (21)$$

For $(s, x) \in [1, \infty) \times (0, \infty)$, define

$$H(s, x) := \int_1^s f(t, x) dt + x^{-\mu}. \quad (22)$$

CLAIM: It holds

$$s \partial_{s,s} H(s, x) + x \partial_{s,x} H(s, x) - s \partial_s H(s, x) + e^{-\lambda} H(s, x) = 0 \quad (23)$$

in the interior of

$$\{(s, x) : s \geq 1, x > 0\},$$

and $H(1, x) = x^{-\mu}$ for $x > 0$.

PROOF OF THE CLAIM: The equation is derived through first step analysis. Call $k(dt, dy|s, x)$ the transition law of the chain $(s_n, x_n)_{n \geq 1}$. Then using (16), (17) we have that

$$f(s, x) = e^{-\lambda} \left(\mathbf{E}(x_1^{-\mu} \mathbf{1}_{s_1 < 1}) + \int_{A_{s,x}} f(t, y) k(dt, dy|s, x) \right), \quad (24)$$

with

$$A_{s,x} := \left\{ (t, y) : 1 \leq t \leq \frac{s}{x}y, y \geq x \right\}.$$

For fixed s, x , the measure $k(dt, dy|s, x)$ is supported on

$$B_{s,x} := \left\{ (t, y) : 0 < t \leq \frac{s}{x}y, y \geq x \right\}$$

and is derived from a density, which we now determine. The distribution function of the measure at a $(t, y) \in B_{s,x}$ is

$$\begin{aligned} F(t, y) &:= \mathbf{P} \left((s+r)\phi \leq t, x \left(1 + \frac{r}{s}\right) \leq y \right) = \mathbf{P} \left(r \leq \left(\frac{y}{x} - 1\right) s, \phi \leq \frac{t}{s+r} \right) \\ &= \int_0^{(\frac{y}{x}-1)s} e^{-z} \left(\frac{t}{s+z} \wedge 1 \right) dz = \begin{cases} t \int_0^{(\frac{y}{x}-1)s} \frac{e^{-z}}{z+s} dz & 0 < t \leq s, \\ \int_0^{t-s} e^{-z} dz + t \int_{t-s}^{(\frac{y}{x}-1)s} \frac{e^{-z}}{z+s} dz & s < t \leq \frac{y}{x}s. \end{cases} \end{aligned} \quad (25)$$

In the interior of $B_{s,x}$, $\partial_t F(t, y)$ exists and is continuous in t , and $\partial_{y,t} F(t, y)$ exists and is continuous in y . Also, the integral of $\partial_{y,t} F(t, y)$ in $B_{s,x}$ is 1. Thus, the measure $k(dt, dy|s, x)$ has density

$$\frac{\partial^2 F}{\partial y \partial t}(t, y) \mathbf{1}_{(t,y) \in B_{s,x}} = \frac{1}{y} e^{-(\frac{y}{x}-1)s} \mathbf{1}_{(t,y) \in B_{s,x}}.$$

Let $g(y) = y^{-\mu}$. Then (24) becomes

$$f(s, x) = e^{-\lambda} \left(\int_x^\infty \frac{g(y)}{y} e^{-(\frac{y}{x}-1)s} dy + \int_x^\infty \int_1^{\frac{s}{x}y} \frac{1}{y} e^{-(\frac{y}{x}-1)s} f(t, y) dt dy \right). \quad (26)$$

This, combined with the measurability and boundedness of f in sets of the form $[1, \infty) \times [\delta, \infty)$, with $\delta > 0$, shows that f is continuous in $[1, \infty) \times (0, \infty)$ and differentiable in the interior of the same set. We write the last equation as

$$\begin{aligned} e^{\lambda-s} f(s, x) &= \int_x^\infty \frac{g(y)}{y} e^{-\frac{y}{x}s} dy + \int_x^\infty \int_1^{\frac{s}{x}y} \frac{1}{y} e^{-\frac{y}{x}s} f(t, y) dt dy \\ &= \int_s^\infty \frac{e^{-w}}{w} g\left(\frac{x}{s}w\right) dw + \int_s^\infty \frac{e^{-w}}{w} \int_1^w f\left(t, \frac{x}{s}w\right) dt dw. \end{aligned}$$

Putting $x = hs$ we get

$$e^{\lambda-s} f(s, hs) = \int_s^\infty \frac{e^{-w}}{w} g(hw) dw + \int_s^\infty \frac{e^{-w}}{w} \int_1^w f(t, hw) dt dw,$$

and differentiating with respect to s ,

$$e^{\lambda-s} \left(-f(s, hs) + \partial_s f(s, hs) + h \partial_x f(s, hs) \right) = -\frac{e^{-s}}{s} g(hs) - \frac{1}{s} e^{-s} \int_1^s f(t, hs) dt.$$

Here ∂_s, ∂_x denote differentiation with respect to the first and second argument respectively. Putting back $h = x/s$, this gives

$$e^\lambda \left(-f(s, x) + \partial_s f(s, x) + \frac{x}{s} \partial_x f(s, x) \right) + \frac{1}{s} g(x) + \frac{1}{s} \int_1^s f(t, x) dt = 0,$$

which in terms of $H(s, x)$ is written as

$$s \left(-\partial_s H(s, x) + \partial_{ss} H(s, x) \right) + x \partial_{sx} H(s, x) + e^{-\lambda} H(s, x) = 0.$$

This is (23).

Determination of f .

For $s \geq 1$ define $G(s) := H(s, 1)$. Relation (21) gives $H(s, x) = x^{-\mu}G(s)$, so that (23) is equivalent to

$$sG''(s) + (-\mu - s)G'(s) + e^{-\lambda}G(s) = 0, \quad (27)$$

while the condition $H(1, x) = x^{-\mu}$ translates to $G(1) = 1$.

Let $a := -e^{-\lambda}$. For $\mu \notin \mathbb{N} = \{0, 1, \dots\}$, the general solution of (27) is (see (9.10.11) of Lebedev(1972))

$$C_1\Phi(a, -\mu; s) + C_2\Psi(a, -\mu; s)$$

with Φ, Ψ the confluent hypergeometric functions of the first and second kind respectively.

Restrict first to the case $\mu > 0, \mu \notin \mathbb{N}, \lambda > 0$. Then as $s \rightarrow \infty$, $|\Phi(a, -\mu; s)|$ goes to infinity faster than any polynomial (see relation (9.12.8) of Lebedev (1972)), while $\Psi(a, -\mu; s)/s \rightarrow 0$ because of (32), (39), and noting that $a \in (-1, 0)$. Since $|G(s)| \leq s$, we get $C_1 = 0$. Then $G(1) = 1$ gives that

$$G(s) = \frac{\Psi(a, -\mu; s)}{\Psi(a, -\mu; 1)}. \quad (28)$$

Note that the denominator is not zero because by (32) it equals $\Psi(a + \mu + 1, 2 + \mu; 1)$, which, because of (36), is positive.

Then

$$f(s, x) = x^{-\mu}f(s, 1) = \partial_s H(s, x) = x^{-\mu}G'(s) = x^{-\mu} \frac{(-a)\Psi(a + 1, 1 - \mu; s)}{\Psi(a, -\mu; 1)} \quad (29)$$

because of (35), that is

$$\mathbf{E}_{s_0=s, x_0=x}(e^{-\lambda M} x_M^{-\mu} \mathbf{1}_{M < \infty}) = e^{-\lambda} \frac{\Psi(1 - e^{-\lambda}, 1 - \mu; s)}{\Psi(-e^{-\lambda}, -\mu; 1)}. \quad (30)$$

The quantity in the expectation, for $\mu \in [0, 1], \lambda \geq 0$, is bounded by $\max\{1, x^{-1}\}$ because by (17), $(x_k)_{k \geq 0}$ is increasing, and thus when sending $\lambda, \mu \rightarrow 0^+$ in the last equality, we can invoke the bounded convergence theorem to get $\mathbf{P}_{s_0=s, x_0=x}(M < \infty) = 1$. We used (37), (38) for the evaluation of the right hand side of the equality.

Now using the continuity of both sides of (30) in μ , we infer its validity for $\mu \in \mathbb{N}$ too. And similarly for $\mu \geq 0$ and $\lambda = 0$. In particular,

$$\mathbf{E}(e^{-\lambda \mathcal{N} - \mu F}) = f(1, 1) = e^{-\lambda} \frac{\Psi(1 - e^{-\lambda}, 1 - \mu; 1)}{\Psi(-e^{-\lambda}, -\mu; 1)} \quad (31)$$

for all $\lambda, \mu \geq 0$. ■

3.2 Analytic extension

Our objective in this subsection is to extend equality (14) to all values of λ, μ for which the left hand side is finite. Before proceeding, we collect some facts concerning the function Ψ which we will use in the rest of the paper. For their proof, we refer the reader to Lebedev(1972).

$\Psi(\cdot, \cdot; \cdot)$ is defined in $\mathbb{C} \times \mathbb{C} \times (\mathbb{C} \setminus (-\infty, 0])$ and is analytic in all its arguments (§9.10 of Lebedev (1972)). Differentiation with respect to the first, second, and third argument will be denoted by $\partial_x, \partial_y, \partial_z$ respectively. In its domain, Ψ satisfies

$$\Psi(a, b; z) = z^{1-b}\Psi(a - b + 1, 2 - b; z), \quad (32)$$

$$\Psi(a - 1, b; z) + (b - 2a - z)\Psi(a, b; z) + a(a - b + 1)\Psi(a + 1, b; z) = 0, \quad (33)$$

$$\Psi(a - 1, b; z) - z\Psi(a, b + 1; z) = (a - b)\Psi(a, b; z), \quad (34)$$

$$\partial_z \Psi(a, b; z) = -a\Psi(a + 1, b + 1; z), \quad (35)$$

while for a, z with positive real part, it holds

$$\Psi(a, b; z) = \Gamma(a)^{-1} \int_0^{+\infty} e^{-zt} t^{a-1} (1+t)^{-a+b-1} dt. \quad (36)$$

Relations (32), (33), (34), (35), (36) are respectively (9.10.8), (9.10.17), (9.10.14), (9.10.12), (9.11.6) of Lebedev (1972).

We will also need some special values of Ψ

Lemma 2. *For $a \in (0, \infty), b \in \mathbb{C}, z \in \mathbb{C} \setminus (-\infty, 0]$, it holds*

$$\Psi(0, b; z) = 1, \quad (37)$$

$$\Psi(-1, b; z) = z - b, \quad (38)$$

$$\lim_{z \rightarrow +\infty} z^a \Psi(a, b; z) = 1, \quad (39)$$

while

$$\partial_x \Psi(0, 1; 1) = 0, \quad (40)$$

$$\partial_x \Psi(-1, 0; 1) = 1, \quad (41)$$

$$\partial_x \Psi(0, 0; 1) = - \int_0^\infty e^{-t} (1+t)^{-1} dt. \quad (42)$$

Proof. For (37), note that by (35), $\Psi(0, b; z)$ is a function of b alone, while it is easy to see that for $z > 0$, $\lim_{a \rightarrow 0^+} \Psi(a, b; z) = 1$ (use (36) and $\lim_{a \rightarrow 0^+} a \Gamma(a) = 1$). Then (38) follows from (37) and (33) by setting $a = 0$.

Relation (39) follows from (36) by doing the change of variables $y = zt$ in the integral and applying the dominated convergence theorem.

Regarding (40), note that $\Psi(0, 1; 1) = 1$, and for $x > 0$,

$$\frac{\Psi(x, 1; 1) - 1}{x} = \frac{1}{x \Gamma(x)} \int_0^\infty e^{-t} t^{x-1} \{(1+t)^{-x} - 1\} dt.$$

For $x \rightarrow 0^+$, the denominator goes to 1, while the numerator goes to zero by the dominated convergence theorem.

Finally, (41) follows from (34), (37) and (40), while (42) is proven in the same way as (40) taking into account that $\Psi(0, 0; 1) = 1$. ■

Define

$$\Xi(\lambda, \mu) := e^\lambda \frac{\Psi(1 - e^\lambda, 1 + \mu; 1)}{\Psi(-e^\lambda, \mu; 1)} \quad (43)$$

for all $\lambda, \mu \in \mathbb{C}$ that this makes sense, that is, everywhere except possibly at values where the denominator is 0. Proposition 1 shows that $\Xi(\lambda, \mu) = \mathbf{E}(e^{\lambda \mathcal{N} + \mu F}) =: K(\lambda, \mu)$ for $\lambda, \mu \leq 0$. We show below that this holds throughout

$$D_K := \{(\lambda, \mu) \in \mathbb{R}^2 : K(\lambda, \mu) < \infty\}.$$

The following two lemmas show, among other things, that D_K contains a neighborhood of $(0, 0)$.

Lemma 3. *The number*

$$z_0 := \sup\{z > 0 : \mathbf{E}(z^{\mathcal{N}}) < \infty\} \in (1, 2),$$

and $\mathbf{E}(z_0^{\mathcal{N}}) = \infty$.

Mathematica gives the approximate value $z_0 \approx 1.57391$

Proof. By Proposition 1, we have

$$\mathbf{E}(z^{\mathcal{N}}) = z \frac{\Psi(1-z, 1; 1)}{\Psi(-z, 0; 1)} \quad (44)$$

for all z with $z \in [0, 1]$. Call $W(z)$ the right hand side of (44). The left hand side is a power series in z with positive coefficients $a_k := \mathbf{P}(\mathcal{N} = k)$ for all $k \geq 0$. The right hand side is a meromorphic function on the plane. It is finite at 0, so that it has a power series development centered at zero. Since the coefficients are positive, the radius of convergence coincides with the smallest pole of W on $(0, \infty)$. We will show that this occurs at a point $z_0 \in (1, 2)$.

The denominator in (44) is a continuous function of z and equals $\Psi(1-z, 2, 1)$, which is positive in $[0, 1]$ and has value -1 at $z = 2$ (use (32), (36), (38) correspondingly for the last three claims). Thus, it has a smallest root in $(1, 2)$, call it z_0 . On the other hand, the numerator is positive in $[0, 2)$. To see that, let $y = 2 - z$, and note that, since $y > 0$, (33) and the integral representation (36) give

$$\Psi(y-1, 1, 1) = y(2\Psi(y, 1, 1) - y\Psi(y+1, 1, 1)) = y\Gamma(y)^{-1} \int_0^{+\infty} e^{-t} t^{y-1} (1+t)^{-y-1} (t+2) dt > 0.$$

Thus, the power series for $W(z)$ centered at 0 has radius of convergence z_0 . As we already noted, the expectation on the left hand side of (44) is a power series of z . It follows that it too has radius of convergence z_0 , thus the two sides of (44) are finite and equal for all $z \in \mathbb{C}$ with $|z| < z_0$. The fact that z_0 is a pole of W gives that $\mathbf{E}(z_0^{\mathcal{N}}) = \infty$ and concludes the proof of the lemma. \blacksquare

Next, we list some properties of the set D_K .

Lemma 4. 1. D_K is convex.

2. $(x, y) \in D_K$ implies that $(-\infty, x] \times (-\infty, y] \in D_K$.

3. $(\lambda, 0) \in D_K$ exactly when $\lambda < \lambda_0 := \log z_0 > 0$.

4. $(0, \mu) \in D_K$ exactly when $\mu < 1$.

5. The intersection of D_K with the second and fourth quadrant is under the line that passes through $(\lambda_0, 0), (0, 1)$.

6. The interior of the triangle with vertices $(0, 0), (0, 1), (\lambda_0, 0)$ is inside D_K .

Proof. 1 follows from Hölder's inequality, 2 is true because \mathcal{N} and F take positive values, 3 is shown in Lemma 3, 4 follows from the fact that $F \sim \text{Exponential}(1)$, and finally 5 and 6 follow from 1, 2, 3, 4. \blacksquare

And now we are ready to state the main result of this subsection, which completes the proof of Theorem 4.

Lemma 5. 1. $K(\lambda, \mu) = \Xi(\lambda, \mu)$ for every $(\lambda, \mu) \in D_K$.

2. $K(\lambda, \mu) = \infty$ for every $(\lambda, \mu) \in \partial D_K$. In particular, D_K is open.

3. $\mathbf{E}(\mathcal{N}) = 2$.

Proof. 1 and 2. Fix $\mu \leq 0$. Since $\mathbf{E}(e^{\lambda\mathcal{N} + \mu F})$ is finite for $\lambda \in [0, \lambda_0)$, it follows that the power series in λ

$$\mathbf{E}(e^{\lambda\mathcal{N} + \mu F}) = \sum_{k=0}^{\infty} \frac{1}{k!} \mathbf{E}(e^{\mu F} \mathcal{N}^k) \lambda^k$$

has radius of convergence at least λ_0 . Also

$$\lambda \mapsto e^\lambda \frac{\Psi(1 - e^\lambda, 1 + \mu; 1)}{\Psi(-e^\lambda, \mu; 1)},$$

is analytic near zero because the value of the denominator at 0 is $1 - \mu \neq 0$ (recall (38)), and Ψ is entire in its first argument. Since it agrees with the previous power series in a line segment, they agree on the ball of convergence of the series. In particular, its development around zero has positive coefficients and consequently its radius of convergence, $\hat{\lambda}(\mu)$, coincides with its smallest singularity in $[0, \infty)$ if such exists, otherwise it is infinite. Since the numerator is entire in λ , the only possibility for a singularity is at a zero of the denominator. Thus $K(\lambda, \mu) < \infty$ exactly for $\lambda < \hat{\lambda}(\mu)$ and for all such λ it holds $K(\lambda, \mu) = \Xi(\lambda, \mu)$. Because of Property 5 of the previous lemma, it follows that $\lambda(\mu) < \infty$. Property 1 gives that $\mu \mapsto \hat{\lambda}(\mu)$ is concave in $(-\infty, 0]$, thus continuous in $(-\infty, 0)$, and Properties 1, 2, 3 give that it is also left continuous at zero with value $\hat{\lambda}(0) = \lambda_0$.

Now fix $\lambda < \lambda_0$. $\mathbf{E}(e^{\lambda N + \mu F})$ is finite for small enough positive μ due to Property 6. With similar reasoning as above, we show that there is a concave function $\lambda \mapsto \hat{\mu}(\lambda)$ continuous on $(-\infty, \lambda_0]$, $\hat{\mu}(\lambda_0) = 0$, so that for $\lambda \in (-\infty, \lambda_0]$ it holds $K(\lambda, \mu) < \infty$ iff $\mu < \hat{\mu}(\lambda)$ and moreover $K(\lambda, \mu) = \Xi(\lambda, \mu)$.

Thus

$$\partial D_K = \{(\hat{\lambda}(\mu), \mu) : \mu \leq 0\} \cup \{(\lambda, \hat{\mu}(\lambda)) : \lambda \leq \lambda_0\},$$

and on this set K takes the value ∞ . This finishes the proof of the first two statements.

3. It follows from the first claim of the Lemma, the formula for Ξ , and differentiation. ■

4 Proof of Theorem 2

Let $(S_k)_{k \in \mathbb{Z}}$ be the points of the renewal ψ in increasing order such that $S_{-1} < 0 \leq S_0$, and for $k \in \mathbb{Z}$,

$$\begin{aligned} X_k &:= S_k - S_{k-1}, \\ \mathcal{N}_k &:= N[S_{k-1}, S_k]. \end{aligned}$$

The random variables $\{(X_k, \mathcal{N}_k) : k \geq 1\}$ are i.i.d., each with distribution the same as $(F + Z, \mathcal{N})$, defined in Section 1.1. Then $\mathbf{E}(X_1) = 1 + (1/2) = 3/2$, and $\mathbf{E}(\mathcal{N}) = 2$ by Lemma 5. Let also for $k \geq 1$,

$$Y_k := \mathcal{N}_k - aX_k,$$

where $a := \mathbf{E}(\mathcal{N}_1)/\mathbf{E}(X_1) = 4/3$. Then $\{Y_k : k \geq 1\}$ are i.i.d. with mean value 0, and we will see below that they have finite variance. By the central limit theorem,

$$\begin{aligned} \frac{N[0, S_k] - aS_k}{\sqrt{k}} &= \frac{N[0, S_0] + \mathcal{N}_1 + \cdots + \mathcal{N}_k - a(S_0 + X_1 + \cdots + X_k)}{\sqrt{k}} \\ &= \frac{Y_1 + \cdots + Y_k + N[0, S_1] - aS_0}{\sqrt{k}} \Rightarrow N(0, \text{Var}(Y_1)) \end{aligned}$$

for $k \rightarrow \infty$.

For $t > 0$, let $n_t := \max\{k : S_k \leq t\}$. Then, by the renewal theorem, we have $\lim_{t \rightarrow \infty} n_t/t = 1/\mu$ with $\mu := \mathbf{E}(X_1) = 3/2$, thus in the same way as in Exercise 3.4.6 in Durrett (2010), we get

$$\frac{N[0, S_{n_t}] - aS_{n_t}}{\sqrt{t}} \Rightarrow N(0, \text{Var}(Y_1)/\mu). \quad (45)$$

Now note that the families $\{S_{n_t} - t : t > 0\}, \{N[0, t] - N[0, S_{n_t}) : t > 0\}$ are tight, because by stationarity, for every $t > 0$,

$$\begin{aligned} 0 \leq t - S_{n_t} &\leq S_{n_{t+1}} - S_{n_t} \stackrel{d}{=} S_0 - S_{-1}, \\ 0 \leq N[0, t] - N[0, S_{n_t}) &\leq N[S_{n_t}, S_{n_{t+1}}) \stackrel{d}{=} N[S_{-1}, S_0]. \end{aligned}$$

Thus (45) and Slutsky's theorem give

$$\frac{N[0, t] - at}{\sqrt{t}} \Rightarrow \mathcal{N}(0, \text{Var}(Y_1)/\mu). \quad (46)$$

It remains to compute $\text{Var}(Y_1)$. We have $Y_1 \stackrel{d}{=} \mathcal{N} - a(F + Z)$, and recall that $F \sim \text{Exponential}(1)$, $Z \sim \text{Exponential}(2)$ is independent of (\mathcal{N}, F) , and the moment generating function of (\mathcal{N}, F) is given in Theorem 4. Thus

$$\begin{aligned} \text{Var}(Y_1) &= \text{Var}(\mathcal{N}) + a^2\text{Var}(F + Z) - 2a\text{Cov}(\mathcal{N}, F) \\ &= -\mathbf{E}(\mathcal{N})^2 + a^2(\text{Var}(F) + \text{Var}(Z)) + 2a\mathbf{E}(\mathcal{N})\mathbf{E}(F) + \mathbf{E}(\mathcal{N}^2) - 2a\mathbf{E}(\mathcal{N}F) \\ &= \frac{32}{9} - \partial_{xx}\Psi(-1, 0; 1) + \partial_{xx}\Psi(0, 1; 1) - \frac{8}{3}\{\partial_{xy}\Psi(-1, 0; 1) - \partial_{xy}\Psi(0, 1; 1)\} \\ &= \frac{32}{9} + \frac{2}{3}\partial_x\Psi(0, 0; 1) = \frac{32}{9} - \frac{2}{3}\int_0^\infty e^{-t}(1+t)^{-1} dt. \end{aligned} \quad (47)$$

For the third equality, we use the formula for the moment generating function of (\mathcal{N}, F) , given in Theorem 4, and (40), (41).

The fourth equality is true because by (34),

$$\Psi(x - 1, y; 1) - \Psi(x, y + 1; 1) = (x - y)\Psi(x, y; 1),$$

so that

$$\begin{aligned} \partial_{xx}\Psi(-1, 0; 1) - \partial_{xx}\Psi(0, 1; 1) &= \partial_{xx}\{(x - y)\Psi(x, y; 1)\}|_{x=y=0} = 2\partial_x\Psi(0, 0; 1), \\ \partial_{xy}\Psi(-1, 0; 1) - \partial_{xy}\Psi(0, 1; 1) &= \partial_{xy}\{(x - y)\Psi(x, y; 1)\}|_{x=y=0} \\ &= -\partial_x\Psi(0, 0; 1) + \partial_y\Psi(0, 0; 1) = -\partial_x\Psi(0, 0; 1). \end{aligned}$$

We used (37) in the last equality. The last equality in (47) follows from (42).

5 Proof of Corrolary 1

First we prove (3). We use the notation of Section 4. For $n \geq 1$,

$$N[0, S_n] = N[0, S_0] + 1 + \sum_{k=1}^n \mathcal{N}_k.$$

Thus,

$$\lim_{n \rightarrow \infty} \frac{N[0, S_n]}{S_n} = \lim_{n \rightarrow \infty} \frac{N[0, S_n]/n}{S_n/n} = \frac{\mathbf{E}(\mathcal{N})}{\mathbf{E}(X_1)} = \frac{2}{3}\mathbf{E}(\mathcal{N}).$$

Since the process $(N[0, S_n])_{n \geq 1}$ is increasing in n and $\lim_{n \rightarrow \infty} S_{n+1}/S_n = 1$, with interpolation we get that

$$\lim_{n \rightarrow \infty} \frac{N[0, t]}{t} = \frac{2}{3}\mathbf{E}(\mathcal{N}). \quad (48)$$

The proof of (3) is completed by noting that $\mathbf{E}(\mathcal{N}) = 2$ because of Theorem 4. However, since the proof of that theorem is quite involved, we give in the following section an easy proof of $\mathbf{E}(\mathcal{N}) = 2$.

For $a > 0$, the stationarity of ξ and the ergodic theorem give that for $n \rightarrow \infty$,

$$\frac{N[0, na]}{n} = \frac{1}{n} \sum_{k=1}^n N[(k-1)a, ka] \rightarrow G \quad (49)$$

a.s. and in L^1 , where G is a random variable. By (48), $G = (4/3)a$, and since $\mathbf{E}N[0, na] = n\mathbf{E}N[0, a]$, the L^1 convergence gives that $\mathbf{E}N[0, a] = (4/3)a$. Now the stationarity of ξ together with standard arguments show that $\mathbf{E}N(A) = (3/4)\lambda(A)$ for each Borel $A \subset \mathbb{R}$.

6 The expected value of \mathcal{N}

Although the expectation of \mathcal{N} was computed in Theorem 4, here we give an alternative, elementary derivation based on a double counting argument.

As noted in the proof of Proposition 1 (Subsection 3.1), \mathcal{N} has the same law as $\mathcal{N}(\varepsilon, 1)$ where ε is an excursion with law $n(\cdot | \bar{\varepsilon} \geq 1)$. Expectation with respect to this law will be denoted by $\mathbf{E}_n(\cdot | \bar{\varepsilon} \geq 1)$.

Lemma 6. $\mathbf{E}_n(\mathcal{N}(\varepsilon, 1) | \bar{\varepsilon} \geq 1) = 2$.

Proof. For the path B with $B|(-\infty, 0) = +\infty$ and $B|[0, +\infty)$ a standard Brownian motion, we define the process x_B exactly as in the introduction. Now x_B is an increasing function, it moves always forward to deeper and deeper valleys of B . We will count in two ways the number $\nu^+(x)$ of jumps of x_B in $[1, x]$.

FIRST WAY:

Let $T_0 = 0, h_0 := 1$, and define (see Figure 2)

$$\begin{aligned} \sigma_1 &:= \min\{s > 0 : B_s - \underline{B}_s = h_0\}, \\ R_1 &:= -\underline{B}_{\sigma_1}, \\ \tau_1 &:= \min\{s : B_s = -R_1\}, \\ h_1 &:= \overline{B}_{\tau_1} - \underline{B}_{\tau_1}, \\ T_1 &:= T_0 + \tau_1. \end{aligned}$$

We repeat the same procedure for the process $(B_{s+T_1} - B_{T_1})_{s \geq 0}$ with the roles of h_0, T_0 played now by h_1, T_1 . Thus we define $\sigma_2, \tau_2, h_2, R_2, T_2$, and we continue recursively.

Using the strong Markov property and an argument analogous to the one in the proof of Lemma 1, we see that the random variables $w_n := h_n/h_{n-1}, n \geq 1$ are i.i.d. and each has density $x^{-2}1_{x \geq 1}$. In particular, $\log w_1$ has the exponential distribution with mean 1. Note that $h_n = \prod_{i=1}^n w_i, \nu^+(h_n) = n$, so that

$$\frac{\nu^+(h_n)}{\log h_n} = \frac{n}{\sum_{i=1}^{n-1} \log w_i}.$$

By the law of large numbers, this converges to 1 since $\mathbf{E}(\log w_1) = 1$. With interpolation we show that

$$\lim_{x \rightarrow +\infty} \frac{\nu^+(x)}{\log x} = 1. \quad (50)$$

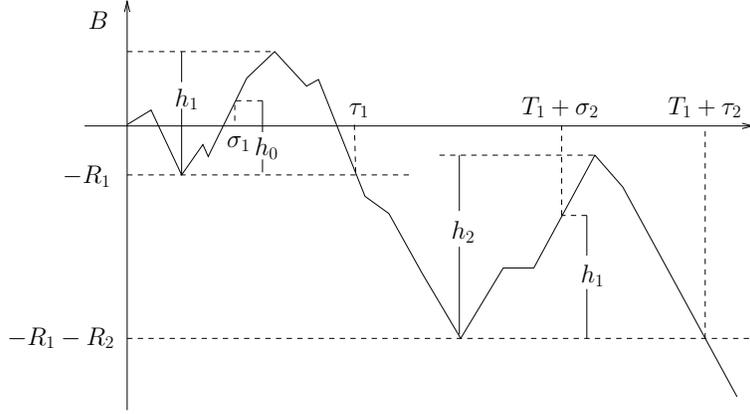


Figure 5: First decomposition of a one sided Brownian motion path.

SECOND WAY: Now we split the path of B using a different strategy. By analogy with Section 2, we define

$$H_\ell^+ := \inf\{s > 0 : B_s = \ell\},$$

$$\Theta_\ell^+ := -\min\{B_s : s \in [0, H_\ell^+]\}.$$

Again we can see that Θ^+ is piecewise constant, left continuous, and the set of points where it jumps, call it \mathcal{L}^+ , has 0 as only accumulation point.

Pick $\ell \in \mathcal{L}^+$. At ℓ , Θ^+ jumps because at height ℓ , the first excursion of $\bar{B} - B$ appeared which goes deeper than $-\Theta_\ell^+$. Let (see Figure 6)

$$h(\ell) := \ell + \Theta_\ell^+.$$

The aforementioned excursion has law $n(\cdot | \bar{\varepsilon} \geq h)$. Call $\tilde{h}(\ell)$ its height. Number the elements of the

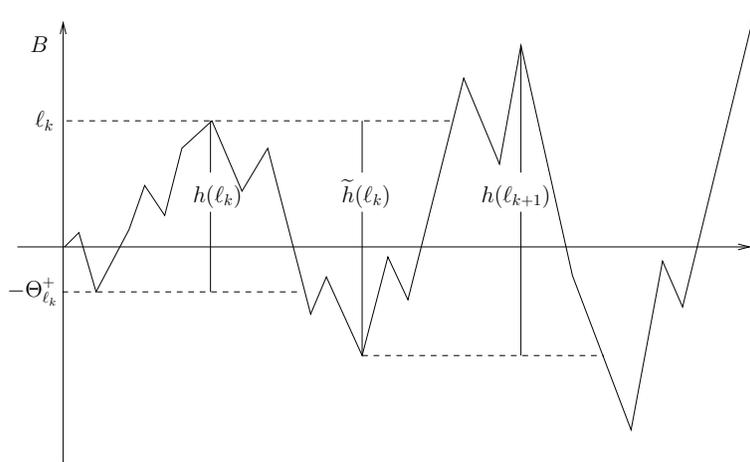


Figure 6: Second decomposition of a one sided Brownian motion path.

set $\mathcal{L}^+ \cap [1, \infty)$ in increasing order as $(\ell_n)_{n \geq 1}$, and for each n , call ε_n the excursion that gives rise to the jump at ℓ_n . Then, with the same arguments as in Lemma 1, we can prove the following.

CLAIM: The random variables

$$\left\{ \frac{\tilde{h}(\ell_k)}{h(\ell_k)}, \frac{h(\ell_{k+1})}{\tilde{h}(\ell_k)} : k \geq 1 \right\}$$

are i.i.d., and each has density $x^{-2}1_{x \geq 1}$.

Now note that

$$\begin{aligned} \nu^+(h(\ell_n)) &= \nu^+(h(\ell_1)) - 1 + \mathcal{N}(\varepsilon_1, h(\ell_1)) + \mathcal{N}(\varepsilon_2, h(\ell_2)) + \cdots + \mathcal{N}(\varepsilon_{n-1}, h(\ell_{n-1})), \\ h(\ell_n) &= h(\ell_1) \prod_{k=1}^{n-1} \frac{h(\ell_{k+1}) \tilde{h}(\ell_k)}{\tilde{h}(\ell_k) h(\ell_k)}. \end{aligned}$$

The above claim gives that for each $k \geq 1$, the random variables $\log(h(\ell_{k+1})/\tilde{h}(\ell_k)), \log(\tilde{h}(\ell_k)/h(\ell_k))$ are exponential with mean 1, so that

$$\lim_{n \rightarrow \infty} \frac{\nu^+(h(\ell_n))}{\log h(\ell_n)} = \frac{\mathbf{E}_n(\mathcal{N}(\varepsilon, 1) \mid \bar{\varepsilon} \geq 1)}{2} \quad (51)$$

The result follows by comparing (50), (51). ■

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