Localization of favorite points for diffusion in random environment

Dimitris Cheliotis *
Department of Mathematics
40 St. George St., 6th floor
Toronto, ON M5S 3G3
Canada
dimitris@math.toronto.edu

April 28, 2007

Abstract

For a diffusion X_t in a one-dimensional Wiener medium W, it is known that there is a certain process $(b_r(W))_{r\geq 0}$ that depends only on the environment, so that $X_t - b_{\log t}(W)$ converges in distribution as $t \to \infty$. The paths of b are step functions. Denote by $F_X(t)$ the point with the most local time for the diffusion at time t. We prove that, modulo a relatively small time change, the paths of the process $(b_r(W))_{r\geq 0}$, $(F_X(e^r))_{r\geq 0}$ are close after some large r.

1 Introduction

One of the simplest and most studied models of random motion in random environment is random walk in random environment in $\mathbb{Z}$. To see a realization of such a walk, first, independently for every integer k, pick a random p_k in $[0,1]$ according to a fixed probability measure (the same for every k). Then, do a walk in $\mathbb{Z}$, with $p_k, 1-p_k$ giving the probabilities of going from k to $k+1, k-1$ respectively. The sequence $(p_k)_{k\in\mathbb{Z}}$ is called the environment, and its random nature gives rise to several new and surprising phenomena.

The continuous time and space analog of this walk is the diffusion satisfying the formal stochastic differential equation

$$\begin{aligned} dX(t) &= d\omega(t) - \tfrac{1}{2}W'(X(t))\,dt, \\ X(0) &= 0, \end{aligned} \tag{1}$$

where ω is a one-dimensional standard Brownian motion, and W is picked from a measure on $\mathbb{R}^{\mathbb{R}}$ before the diffusion starts running. Here, W is called the environment.

In this work, we focus our attention to the case where W has the law of a two sided Brownian motion path. With probability one, the derivative W' does not exist, but we will explain soon what we mean exactly by the above diffusion. Assume for the moment that W is differentiable. In intervals where W' is positive, the diffusion is pushed by the environment to the left, while in intervals where W' is negative, the diffusion is pushed to the right. Thus, in a neighborhood of a local minimum x_0, the diffusion oscillates for some time around x_0 until randomness (i.e., $d\omega$) manages to overcome the effect of the environment, and the diffusion moves away from the "interval of influence" of x_0. The explicit definition of this interval we give now.

*Research partially supported by NSF grant DMS-0072331

Assume that W is continuous and x_0 is point of local minimum for W. There are intervals $[l, r]$ around x_0 so that $W(x_0)$ is the minimum value of W in $[l, r]$, and $W(l), W(r)$ are the maximum values of W in $[l, x_0], [x_0, r]$ respectively. Call $J(x_0) := [l_{x_0}, r_{x_0}]$ the maximal such interval. We will call it the **interval of influence** of x_0, and the number $\min\{W(l_{x_0}) - W(x_0), W(r_{x_0}) - W(x_0)\}$ the **strength** of the interval. Assume now that W is a "typical" two sided Brownian path. A given interval $J(x_0)$ is strictly contained in other intervals $J(x')$ with greater strength. Let $J(x_1) \supsetneqq J(x_0)$ be the minimal such interval. x_1 is uniquely determined this way because W is a Brownian path. We call x_1 the **parent** of x_0. As each $J(x)$ contains many other $J(x)$'s, the process of temporary entrapment in a $J(x)$ and the escape, described in the previous paragraph, happens at all scales.

It is known that there is a real valued process $(b_s(W))_{s>0}$ depending only on the environment and whose values are points of local minimum for W, so that $X(t) - b_{\log t}(W)$ converges in distribution as $t \to \infty$. For stronger results, see, Golosov (1984), Tanaka (1988), Hu (2000). $b_s(W)$ moves from a local minimum x_0 to its parent x_1, and then to the parent of x_1, and so on. Consequently, as time passes, b discerns only rougher details of the path W.

This characteristic of b has been the basis for a renormalization picture in Le Doussal et al. (1999). The authors of that paper call the process b "effective dynamics" of the motion, and give many properties of the path $(b_r(W))_{r>0}$. For example,

$$\lim_{t\to\infty} \frac{\text{number of sign changes of } b \text{ in the interval } [1,t]}{\log t} = \frac{1}{3} \quad \text{a.s.,}$$

and for $t > 1$,

$$\mathbb{P}(b \text{ is constant in } [1,t]) = \frac{1}{3t^2}(5 - 2e^{1-t}).$$

These two results are proved rigorously in Cheliotis (2005) and Zeitouni (2004) (Theorem 2.5.13) respectively.

And a natural question is: what do such properties of b say about the diffusion itself? Or, put differently, what object defined in terms of the diffusion is related to the path of the process b? The diffusion itself is a much different process than b. e.g., it is continuous and recurrent, while b is discontinuous and transient (this will be clear after the rigorous definition of b). The next best thing is to find a process whose value at time t is determined by the knowledge of $(X_s)_{s\le t}$ only which follows b closely (If we have the entire path of X, then we can completely recover the process b with probability 1). The one we put forth is the process of the favorite point of X at time t. And the result justifying this is our theorem, stated below, already announced in Cheliotis (2005).

To make the above precise, we define explicitly the three processes of interest.

The diffusion: On the space $\mathcal{W} := C(\mathbb{R})$, consider the topology of uniform convergence on compact sets, the corresponding σ-field of the Borel sets, and $\mathbb{P}$ the measure on $\mathcal{W}$ under which the coordinate processes $\{W(t) : t \ge 0\}, \{W(-t) : t \ge 0\}$ are independent standard Brownian motions. Also let $\Omega := C([0,\infty))$, and equip it with the σ-field of Borel sets derived from the topology of uniform convergence on compact sets. For $W \in \mathcal{W}$, we denote by P_W the probability measure on Ω such that $\{X(t) : t \ge 0\}$, the coordinate process, is a diffusion with $X(0) = 0$ and generator

$$\frac{1}{2} e^{W(x)} \frac{d}{dx}\left(e^{-W(x)} \frac{d}{dx}\right).$$

Such a diffusion is defined by the formula

$$X_t := A^{-1}(B(T^{-1}(t))), \tag{2}$$

where

$$A(x) := \int_0^x e^{W(s)} ds,$$
$$T(u) := \int_0^u e^{-2W(A^{-1}(B(s)))} ds$$

for all $x \in \mathbb{R}, u \geq 0$. A is the scale function for the diffusion, B is a standard Brownian motion, and T is a time change. Then consider the space $\mathcal{W} \times \Omega$, equip it with the product σ-field, and take the probability measure defined by

$$d\mathcal{P}(W, X) := dP_W(X)\, d\mathbb{P}(W).$$

Finally, take the completion of the above σ-field with respect to $\mathcal{P}$. The marginal of $\mathcal{P}$ in Ω gives a process that is known as diffusion in a random environment; the environment being the function W.

The favorite point process: To the above diffusion corresponds the local time process $\{L_X(t, x) : t \geq 0, x \in \mathbb{R}\}$, which is jointly continuous and its defining property is

$$\int_0^t f(X_s) ds = \int_{\mathbb{R}} f(x) L_X(t, x) dx$$

for all $t \geq 0$ and any bounded Borel function $f \in \mathbb{R}^{\mathbb{R}}$. Using the definition of X in (2), we can see that

$$L_X(t, x) = e^{-W(x)} L_B(T^{-1}(t), A(x)), \tag{3}$$

where L_B is the local time process of the Brownian motion B.

For a fixed $t > 0$, the set $\mathfrak{F}_X(t) := \{x \in \mathbb{R} : L_X(t, x) = \sup_{y \in \mathbb{R}} L_X(t, y)\}$ of the points with the most local time at time t is nonempty and compact. Any point there is called a **favorite point** of the diffusion at time t. One can prove that for fixed $t > 0$, $\mathfrak{F}_X(t)$ has at most two elements, and with probability 1, $\mathfrak{F}_X(t)$ has exactly one element. Also, $Leb(\{t : \mathfrak{F}_X(t) \text{ has two elements }\}) = 0$.

Define $F_X : (0, \infty) \to \mathbb{R}$ with $F_X(t) := \inf \mathfrak{F}_X(t)$, the smallest favorite point at time t. This is a left continuous function. (Note that what we prove does not change if we define F_X as the maximum of $\mathfrak{F}_X(t)$, or as any other choice function. The only difference in the latter case is that some non-measurable sets may appear, e.g. in relation (6) below, and we will have a bound on their outer measure. Our basic tool, the first Borel Cantelli Lemma, works in that case too, and our result is unaffected)

The process b: For a function $f : \mathbb{R} \to \mathbb{R}$, $t > 0$, and $y_0 \in \mathbb{R}$, we say that f **admits a t-minimum at** y_0 if there are $\alpha, \beta \in \mathbb{R}$ with $\alpha < y_0 < \beta$, $f(y_0) = \inf\{f(y) : y \in [\alpha, \beta]\}$ and $f(\alpha) \geq f(y_0) + t$, $f(\beta) \geq f(y_0) + t$. We say that f **admits a t-maximum at** y_0 if $-f$ admits a t-minimum at y_0. We denote by $R_t(f)$ the set of t-extrema of f.

It is easy to see that for a two sided Brownian path W, with probability one, for all $t > 0$, the set $R_t(W)$ has no accumulation point in $\mathbb{R}$, it is unbounded above and below, and the points of t-maxima and t-minima alternate. Thus we can write $R_t(W) = \{x_k(W, t) : k \in \mathbb{Z}\}$, with $(x_k(W, t))_{k \in \mathbb{Z}}$ strictly increasing, and $x_0(W, t) \leq 0 < x_1(W, t)$. One of $x_0(W, t), x_1(W, t)$ is a point of local minimum. This we call $b_t(W)$.

For a fixed t, the part $W|[x_k, x_{k+2}]$ of the path of W between two consecutive t-maxima we call it a **t-valley**, or simply a valley when the value of t is understood. The **depth** of the valley is defined as $\min\{W(x_k) - W(x_{k+1}), W(x_{k+2}) - W(x_{k+1})\}$, and the point x_{k+1} is called the **bottom** of the valley.

Pick any $c > 6$, and for any x with $|x| > 1$, define the interval

$$I(x) := (x - (\log|x|)^c, x + (\log|x|)^c).$$

Our result says that the processes $F_X(\exp(\cdot))$ and b are very close. The precise statement is as follows.

Theorem. *With $\mathcal{P}$-probability 1, there is a $\tau = \tau(W, X) > 0$ so that if we label by $(s_n(W,\tau))_{n\geq 1}$ the strictly increasing sequence of the points in $[\tau, \infty)$ where $b.$ jumps and $x_n(W,\tau)$ its value in (s_n, s_{n+1}), then there is a strictly increasing sequence $(t_n(X,W))_{n\geq 1}$ converging to infinity so that*

(i) $F_X(e^t) \in I(x_n)$ for $n \geq 1, t \in (t_n, t_{n+1})$

(ii) $t_n/s_n \to 1$ $\mathcal{P}$-a.s. as $n \to \infty$.

We abbreviated $s_n(W,\tau), x_n(W,\tau), t_n(X,W)$ to s_n, x_n, t_n.

The times $\{t_i : i \geq 1\}$ will be defined explicitly in the proof of the theorem. Observe also that for big x, the interval $I(x)$ is a relatively small neighborhood of x. Thus, the theorem says that after some point, the function $F_X(\exp(\cdot))$ "almost tracks" the values of the process b with the same order and at about the same time.

A consequence of the theorem and its proof is the following corollary. Note that the typical size of b_s is of the order s^2, so that the result says something nontrivial.

Corollary. *Let $c > 6$ be fixed. With $\mathcal{P}$-probability 1, there is a strictly increasing map λ from $[0, \infty)$ to itself with $\lim_{s\to\infty} \lambda(s)/s = 1$ and*

$$|F_X(e^{\lambda(s)}) - b_s| < (\log s)^c$$

for all large s.

One can show that $\overline{\lim}_{s\to\infty} b_s/(s^2 \log\log s) = 8/\pi^2$ (it follows from the proofs in Hu and Shi (1998). The process b is much easier to handle than X or F_X). Using the corollary, we get

$$\overline{\lim_{s\to\infty}} \frac{F_X(e^s)}{s^2 \log\log s} = \frac{8}{\pi^2}.$$

Remark 1. As we mentioned in the beginning of the paper, a diffusion satisfying (1) is related to a random walk in random environment in $\mathbb{Z}$ for an appropriate law on the transition probabilities. The relative of the case we consider (i.e., W Brownian path) in the discrete world is the so-called Sinai's walk; the two models behave in most respects in the same way. See the survey article Shi (2001) for details.

For Sinai's walk, limiting properties of the process $(\xi(n))_{n\geq 1}$, with $\xi(n)$ being the number of visits paid to the most visited point by time n, have been studied in Révész (1990) and Dembo et al. (2007). More related to our work is Hu and Shi (2000), where the authors study the process $F^+(n)$ of the location of the biggest positive favorite point at time n as well as the analog for the diffusion, i.e., $F_X^+(t) := \max \mathfrak{F}_X^+(t)$, where $\mathfrak{F}_X^+(t) := \{x \in [0,\infty) : L_X(t,x) = \sup_{y\geq 0} L_X(t,y)\}$. The results for the diffusion are that

$$\overline{\lim_{t\to\infty}} \frac{F_X^+(t)}{(\log t)^2 \log\log\log t} = \frac{8}{\pi^2},$$

and that for any non-decreasing function $f > 1$,

$$\underline{\lim}_{t\to\infty} \frac{f(t)}{\log^2 t} F_X^+(t) = \begin{cases} 0 \\ \infty \end{cases} \quad \text{a.s} \Leftrightarrow \int^{\infty} \frac{\log f(t)}{t\sqrt{f(t)}\log t} dt \begin{cases} = \infty \\ < \infty \end{cases}$$

The crucial element in the proofs of the above two results is the fact that F_X and b are closely related. In connection with this paper of Hu and Shi, we should mention that in our work, we use techniques we learned from it.

Remark 2. Our interest in this paper is to connect the paths of the two processes

$$\big(F_X(e^s)\big)_{s>0}, \big(b_s(W)\big)_{s>0}.$$

Regarding only their fixed time values, one can prove, using similar techniques as in the proof of our theorem, that

$$F_X(t) - b_{\log t}(W) \to 0 \text{ in probability as } t \to \infty.$$

Remark 3. Related to the present work is the paper by Andreoletti (2006). The author works with Sinai's walk, and shows that the local time process of the walk at a given time gives an estimate for the environment at many sites. In particular, for the location of $b_s(W)$. He bounds the probability that the estimate the local time gives differs significantly from the true values of the environment (see Theorem 1.8 and Proposition 1.9 in Andreoletti (2006)).

Orientation. The paper is organized as follows. In the remaining of this section, we give an outline of the proof. The main ideas are contained in Section 2, where the theorem is proved assuming that the environment behaves in the way we expect it to. In Section 3 we show that, with high probability, the environment indeed behaves the way we assumed.

1.1 Informal description of the proof

It will be useful for our discussion to imagine that the diffusion actually takes place on the graph of W. If the diffusion at time t is at the point x (i.e., $X_t = x$), we will imagine instead that it is at the point $(x, W(x))$. So that the motion will involve going up or down on the hills of W, with gravity favoring downward movement.

It is known that if the diffusion starts from the bottom of a valley and wants to reach a point inside the valley so that while travelling towards it reaches maximum height h above the bottom, then it takes typically time e^h (see, e.g., Lemma 3.1 in Brox (1986)). Also, if one puts reflecting barriers at the endpoints of an interval and starts the diffusion inside the interval, then the diffusion has stationary probability measure $C\,e^{-W(x)}dx$, with C a normalizing constant. Thus, it spends more time on points that are deeper. Assume that we take an r-valley $[x_k, x_{k+2}]$, with r large. For fixed ε small, define

$$\begin{aligned} y_1 &:= \sup\{s < x_{k+1} : W(s) - W(x_{k+1}) = (1-\varepsilon)r\}, \\ y_2 &:= \inf\{s > x_{k+1} : W(s) - W(x_{k+1}) = (1-\varepsilon)r\}. \end{aligned}$$

So that $I := [y_1, y_2] \subset [x_k, x_{k+2}]$. If one starts a diffusion in I and observes it only in I until it exits the valley $[x_k, x_{k+2}]$, then it looks the same way as if we put reflecting barriers at the endpoints of I. And since until the exit from the valley a large amount of time passes, the points of I will have occupation time very close to the one dictated by the stationary measure. Thus, points in $[x_k, x_{k+2}] \setminus I$ are out of the competition for the favorite point (they are in an area visited rarely), while for the points in I, the occupation time can be read off from the stationary measure $Ce^{-W(x)}dx$.

Now lets look at Figure 1. We drew the valleys around three consecutive values of b. z_1, z_2, z_3. Typically, the diffusion visits first z_1, then moves on to z_2, and then to z_3. The favorite points will be close to z_2 at a certain time interval which we will describe.

Call h_1, h_2 the height of the first and second valley respectively. ζ is the smallest point with the property that $W(\zeta) - W(z_2)$ is a bit greater than h_1 (in fact, we ask that it is $h_1 + k \log h_1$

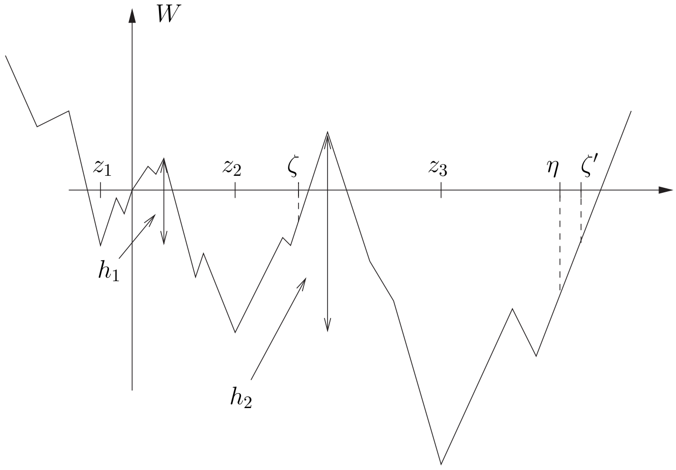

Figure 1:

for some constant $k \geq 18$). η is the smallest point with the property that $W(\eta) - W(z_3)$ is a bit less than h_2 (in fact, we ask that it is $h_2 - l \log h_2$ for a constant $l \geq 10$). The main claim is the following.

Claim: From the time that the diffusion hits ζ up to the time that it hits η, with high probability, the favorite point is near z_2.

This is intuitively expected. When the diffusion leaves the valley of z_1, it has spent time of the order e^{h_1} there. A considerable fraction of it was spent as local time on points near z_1. So that while it travels to z_2, the favorite point is somewhere near z_1. Now we wait until it spends time a bit more than e^{h_1} inside the valley of z_2. This happens when it hits ζ. Then the favorite point will move near z_2. It will stay there until the diffusion moves to the next deeper valley, and climbs up to heights a bit less than h_2. That is, around the time that it hits η. After that, it reaches the ζ corresponding to z_3, call it ζ', where $W(\zeta') - W(z_3)$ is a little over h_2 (the depth of the valley of z_2). Then the favorite point will be near z_3.

In the paper, we prove the above claims. Here we will provide a sample computation, to illustrate how the above intuition comes up in the rigorous argument.
First, some notation that we will use throughout the paper. For $x \in \mathbb{R}$, we define

$$\begin{aligned}\tau(x) &:= \inf\{t \geq 0 : X_t = x\},\\ \rho(x) &:= \inf\{t \geq 0 : B_t = x\},\end{aligned}$$

the hitting times of X and B respectively.
The local time of X has been expressed in (3) in terms of the local time process L_B for Brownian motion. A useful property of L_B is

$$\big(L_B(\rho(a), x)\big)_{x\in\mathbb{R}} \overset{law}{=} \big(|a| L_B(\rho(1), x/a)\big)_{x\in\mathbb{R}}, \tag{4}$$

for $a \neq 0$.

And now the computation we promised. At time $\tau(\eta)$, we will compare the local time of any other point s with the local time of z_2. We have

$$\begin{aligned}\frac{L_X(\tau(\eta), s)}{L_X(\tau(\eta), z_2)} &= e^{W(z_2)-W(s)} \frac{L_B(T^{-1}(\tau(\eta)), A(s))}{L_B(T^{-1}(\tau(\eta)), A(z_2))}\\ &= e^{W(z_2)-W(s)} \frac{L_B(\rho(A(\eta)), A(s))}{L_B(\rho(A(\eta)), A(z_2))} = e^{W(z_2)-W(s)} \frac{Z_{A(\eta)-A(s)}}{Z_{A(\eta)-A(z_2)}}\end{aligned}$$

because $T^{-1}(\tau(\eta)) = \rho(A(\eta))$. Z is a two dimensional squared Bessel process (Ray-Knight theorem). We know that for large t, we have $Z_t \approx t$. So that the above ratio is about

$$e^{W(z_2)-W(s)}\frac{A(\eta)-A(s)}{A(\eta)-A(z_2)} = \frac{\int_s^\eta e^{W(x)-W(s)}dx}{\int_{z_2}^\eta e^{W(x)-W(z_2)}dx}. \tag{5}$$

The dominant contribution to the integrals comes from the points x where $W(x)-W(s), W(x)-W(z_2)$ are maximum. The exponent in the integrand of the denominator has maximum h_2. Regarding the numerator, if the point s is in the valley of z_2 and away from z_r, the values of $W(x)-W(s)$ will be a bit less than h_2 ($W(s)$ will be larger that $W(z_2)$), while if s is in the valley of z_3, then by the definition of η, $W(x)-W(s)$ will always be a bit less than h_2. Consequently the ratio in (5) is less than one for points s that are reachable by time $\tau(\eta)$ and away from z_2.

The proof we sketched for the time $\tau(\eta)$ is done for all times in $[\tau(\zeta), \tau(\eta)]$. As we said before, when the diffusion visits ζ', the favorite point will be near z_3. In this way, we know from about what time the bottom of each valley starts being the favorite point and when it stops. For each valley, the scenario we described happens on the complement of a set with a small probability (which we bound in Lemma 2).

The next step is to glue together all the time intervals (one corresponding to each valley). We prove that the probabilities of all exceptional sets where our scenario fails have finite sum, and we use the first Borel-Cantelli lemma. The main tool for this step is Lemma 12 and the bound given in Lemma 2. Thus we get part (i).

For part (ii), note that the process $F(t)$ jumps from the vicinity of z_2 to the vicinity of z_3 around time e^{h_2}. While the process b jumps exactly at h_2.

2 Proof of the Theorem

The strategy of the proof is the following. Let $(y_i)_{i\geq 1}$ be the consecutive values of b in $[1,\infty)$. First we show that outside a set K_i of very small probability, the favorite point $F_X(t)$ is very close to y_i in a time interval I_i. This is the content of Lemmas 2 and 3. Then we show that the probabilities of the K_i's are summable, and we use the first Borel-Cantelli lemma. This is accomplished through Lemma 12 and the bounds given in Lemmas 2 and 3. Finally, the intervals I_i are such that the right endpoint of I_i coincides with the left endpoint of I_{i+1}. So that we have the required information for $F_X(t)$ on an interval of the form (a,∞).

2.1 A quenched probability estimate

Here we give precisely the argument outlined in Section 1.1. Imagine that (α,γ) is a small neighborhood of $\beta := z_2$, ζ,η are as before, and y_0 is a point on the left of z_1 higher than the tip of the mountain between z_2 and z_3. If this piece of W has some nice properties ((i)-(vii) below), then the favorite point will stay near β for a certain time interval.

Lemma 1. *Assume $W \in C(\mathbb{R})$, and $y_0 < 0 < \zeta \leq \eta$, $\alpha < \beta < \gamma$, $\zeta > \beta$, $H > 1$, $k_4 \geq 10$ are such that*

(i) $P_W(X \text{ hits } y_0 \text{ before } \eta) < H^{-2}$,

(ii) $\max_{s\in[y_0,\eta]} W(s) \leq H^{3/2}$,

(iii) $|\eta - y_0| \leq H^4$,

(iv) $\log \frac{A(\eta)-A(\beta)}{A(\zeta)-A(\beta)} < H^2$,

(v) $A(\beta)/A(\zeta) > -H^{-5}$,

(vi) $\sup_{y\in[\zeta,\eta]}\sup_{y_0<s<y,\ s\notin(\alpha,\gamma)}\int_s^y e^{W(t)-W(s)}dt/\int_\beta^y e^{W(t)-W(\beta)}dt < H^{-k_4+5}$,

(vii) $\sup_{y\in[\zeta,\eta]}\sup_{y_0<s<y}(e^{W(s)}\int_\beta^y e^{W(t)-W(\beta)}dt)^{-1} < \exp(2H^{7/4})$.

Then we have the following quenched probability estimate.

$$P_W(\exists\, t\in[\tau(\zeta),\tau(\eta)] \text{ with } F_X(t)\notin(\alpha,\gamma)) < c\,H^{-2}, \tag{6}$$

where c is a universal constant.

Proof. It is enough to prove that the quantity

$$\sup_{\tau(\zeta)\le t\le\tau(\eta)}\sup_{s\notin(\alpha,\gamma)}\frac{L_X(t,s)}{L_X(t,\beta)}$$

is greater than or equal to one with probability at most cH^{-2}. Using (3), we get

$$\frac{L_X(t,s)}{L_X(t,\beta)} = e^{W(\beta)-W(s)}\frac{L_B(T^{-1}(t),A(s))}{L_B(T^{-1}(t),A(\beta))}.$$

Note that $T^{-1}(\tau(\zeta)) = \rho(A(\zeta))$ and $T^{-1}(\tau(\eta)) = \rho(A(\eta))$. Thus, we are interested in the quantity

$$\sup_{\rho(A(\zeta))<t<\rho(A(\eta))}\sup_{s\notin(\alpha,\gamma)} e^{W(\beta)-W(s)}\frac{L_B(t,A(s))}{L_B(t,A(\beta))}.$$

Let

$$N := \min\{k\in\mathbb{Z} : k\ge(\log 2)^{-1}\log(\frac{A(\eta)-A(\beta)}{A(\zeta)-A(\beta)})\},$$

$u_k = 2^k(A(\zeta)-A(\beta))+A(\beta)$ for $k=0,1,...,N-1$, and $u_N = A(\eta)$.
There are unique points $\zeta =: p_0 < p_1 < \ldots < p_N := \eta$ such that $u_k = A(p_k)$.

Let $A_1 := [\ X$ hits η before $y_0\ ]$. Then

$$\begin{aligned}&P_W\Big(\sup_{\rho(A(\zeta))<t<\rho(A(\eta))}\sup_{s\notin(\alpha,\gamma)} e^{W(\beta)-W(s)}\frac{L_B(t,A(s))}{L_B(t,A(\beta))}\ge 1\Big)\\ &\le P_W(A_1^c) + H^2(\log 2)^{-1}\sup_{k<N}P_W\Big(\{\sup_{\rho(u_k)<t<\rho(u_{k+1})}\sup_{s\notin(\alpha,\gamma)} e^{W(\beta)-W(s)}\frac{L_B(t,A(s))}{L_B(t,A(\beta))}\ge 1\}\cap A_1\Big),\end{aligned}$$

using assumption (iv). Now

$$P_W\Big(\{\sup_{\rho(u_k)<t<\rho(u_{k+1})}\sup_{s\notin(\alpha,\gamma)} e^{W(\beta)-W(s)}\frac{L_B(t,A(s))}{L_B(t,A(\beta))}\ge 1\}\cap A_1\Big)$$

$$\le P_W\Big(\{\sup_{s\notin(\alpha,\gamma)}\frac{e^{W(\beta)-W(s)}}{u_{k+1}-A(\beta)}L_B(\rho(u_{k+1}),A(s)) > \frac{1}{H^4}\}\cap A_1\Big) \tag{7}$$

$$+ P_W\Big(\frac{1}{u_{k+1}-A(\beta)}L_B(\rho(u_k),A(\beta)) \le \frac{1}{H^4}\Big). \tag{8}$$

Bound on the term of (7): We observe that the local time appearing in the expression is zero for $s\ge p_{k+1}$, and we use the Ray-Knight theorem to get

$$\begin{aligned}\{\frac{e^{W(\beta)-W(s)}}{u_{k+1}-A(\beta)}L_B(\rho(u_{k+1}),A(s)) : s<p_{k+1}\} &\overset{law}{=} \{\frac{e^{W(\beta)-W(s)}}{u_{k+1}-A(\beta)}\tilde{Z}_{u_{k+1}-A(s)} : s<p_{k+1}\}\\ &= \{\Big(\int_\beta^{p_{k+1}} e^{W(y)-W(\beta)}dy\Big)^{-1} e^{-W(s)}\tilde{Z}_{\int_s^{p_{k+1}} e^{W(y)}dy} : s<p_{k+1}\},\end{aligned}$$

where $(\tilde{Z}_s)_{s\geq 0}$ is a two dimensional squared Bessel process up to time u_{k+1} and then zero dimensional squared Bessel process. Let also Z be the two dimensional squared Bessel process which is run with the same Brownian motion as $\tilde{Z}$. Then with probability one,

$$Z_t \geq \tilde{Z}_t \text{ for all } t \geq 0$$

by Theorem 3.7 of Chapter IX in Revuz and Yor (1999). The function ρ required by that theorem is in our case $\rho(x) = x$ for all $x \geq 0$. So that

$$\{\frac{e^{W(\beta)-W(s)}}{u_{k+1}-A(\beta)} L_B(\rho(u_{k+1}), \frac{A(s)}{u_k}) : s \leq p_{k+1}\} \overset{law}{\leq} \{\Big(\int_\beta^{p_{k+1}} e^{W(y)}dy\Big)^{-1} e^{W(\beta)-W(s)} Z_{\int_s^{p_{k+1}} e^{W(y)}dy} : s \leq p_{k+1}\}.$$

Let

$$x_1(s) := e^{W(\beta)-W(s)},\ x_2(s) := \int_s^{p_{k+1}} e^{W(y)}dy = A(p_{k+1}) - A(s).$$

Assumptions (vi) and (vii) give respectively

$$\sup_{y_0<s<p_{k+1}, s\notin(\alpha,\gamma)} \frac{x_1(s)x_2(s)}{x_2(\beta)} \leq H^{-k_4+5} \text{ and } \sup_{y_0<s<p_{k+1}, s\notin(\alpha,\gamma)} \frac{x_1(s)}{x_2(\beta)} \leq 2\exp(H^{7/4}). \tag{9}$$

Thus the term of (7) is bounded by

$$\begin{aligned}
&P_W\Big(\{\sup_{s\notin(\alpha,\gamma),\ s<p_{k+1}} \frac{x_1(s)}{x_2(\beta)} Z_{x_2(s)} > \frac{1}{H^4}\} \cap A_1\Big) \\
&\leq P_W\Big(\{\sup_{\substack{s\notin(\alpha,\gamma)\\ x_2(s)\leq e^{-H^2},\ s<p_{k+1}}} \frac{x_1(s)}{x_2(\beta)} Z_{x_2(s)} > \frac{1}{H^4}\}\cap A_1\Big) + P_W\Big(\{\sup_{\substack{s\notin(\alpha,\gamma)\\ x_2(s)\geq e^{-H^2}}} \frac{x_1(s)x_2(s)}{x_2(\beta)} \frac{Z_{x_2(s)}}{x_2(s)} > \frac{1}{H^4}\}\cap A_1\Big) \\
&\qquad \leq P_W\Big(\sup_{t\leq e^{-H^2}} e^{2H^{7/4}} Z_t > \frac{1}{H^4}\Big) + P_W\Big(\sup_{e^{-H^2}\leq t\leq H^4 e^{H^2}} \frac{Z_t}{t} > H^{k_4-9}\Big) \\
&\qquad\qquad = P_W\Big(\sup_{t\leq 1} Z_t > \frac{e^{H^2-2H^{7/4}}}{H^4}\Big) + P_W\Big(\sup_{1\leq t\leq H^4 e^{2H^2}} \frac{Z_t}{t} > H^{k_4-9}\Big)
\end{aligned} \tag{10}$$

To justify the last inequality, we use the bound $x_2(s) \leq H^4 e^{H^2}$ (coming from (ii),(iii)) and (9). By well known property of Brownian motion, $\mathbb{P}(\sup_{t\leq 1} Z_t > a) \leq 4\mathbb{P}(B_1 > \sqrt{\frac{a}{2}}) < 4e^{-a/4}$. So the first term in (10) is bounded by $4\exp(-\frac{e^{H^2-2H^{7/4}}}{4H^4})$. To bound the last term in (10), we use Lemma 4 with the choices $a = \sqrt{\frac{H^{k_4-9}}{2}}$, $M = H^4 e^{2H^2}$, and $\sigma = 2$, and we find that it is bounded by

$$8\Big(\frac{4\log H + 2H^2}{\log 2} + 1\Big)\frac{1}{\sqrt{H^{k_4-9}}} \exp\Big(-\frac{H^{k_4-9}}{8}\Big) < 60\, H^2 \exp\Big(-\frac{H^{k_4-9}}{8}\Big)$$

using the fact that $H > 1$.

Bound on the term of (8): If $\beta \geq 0$, then we use the Ray-Knight theorem to obtain

$$P_W\Big(\frac{1}{u_{k+1}-A(\beta)} Z_{u_k-A(\beta)} \leq \frac{1}{H^4}\Big) = P_W\Big(\frac{u_k - A(\beta)}{u_{k+1}-A(\beta)} Z_1 \leq \frac{1}{H^4}\Big) \leq P_W\Big(Z_1 \leq \frac{2}{H^4}\Big) \leq \frac{1}{H^4}.$$

Here Z is a two dimensional squared Bessel process. We used also the scaling property of Z, the fact that Z_1 has a density bounded by 1/2 (it is exponential with mean 2), and that $(u_{k+1} - A(\beta))/(u_k - A(\beta)) \leq 2$ (see the definition of u_k. We have equality unless $k = N-1$). For the case $\beta < 0$, we will need the inequality $(u_{k+1} - A(\beta))/u_k < 4$. This translates to

$2^{k+1}(A(\zeta)-A(\beta))+4A(\beta)>0$. The last quantity is enough to be positive for $k=0$. Then the inequality becomes $A(\zeta)+A(\beta)>0$ which holds because of (v). Thus, using (4) and $(u_{k+1}-A(\beta))/u_k<4$, we get

$$\frac{1}{u_{k+1}-A(\beta)}L_B(\rho(u_k),A(\beta)) \overset{law}{=} \frac{u_k}{u_{k+1}-A(\beta)}L_B(\rho(1),\frac{A(\beta)}{u_k}) > \frac{1}{4}L_B(\rho(1),\frac{A(\beta)}{u_k}).$$

So that the term is bounded by $P_W(L_B(\rho(1),A(\beta)/u_k)<4/H^4)$. The process $\tilde{Z}_s := L_B(\rho(1),1-s), s\ge 0$, is up to time 1 a two dimensional squared Bessel process, and after that a zero dimensional squared Bessel process. Let $(\hat{Z})_{s\ge 0}$ be a two dimensional squared Bessel process. Then the comparison Theorem IX.3.7 in Revuz and Yor (1999), the fact that $\tilde{Z}_1,\hat{Z}_1$ are exponential with mean 2, and the assumption $0>A(\beta)/A(\zeta)>-H^{-5}$ give

$$\begin{aligned}
P_W\Big(\tilde{Z}_{1-A(\beta)/u_k}\le\frac{4}{H^4}\Big) &\le P_W\Big(\tilde{Z}_1\le\frac{8}{H^4}\Big)+P_W\Big(|\tilde{Z}_1-\tilde{Z}_{1-A(\beta)/u_k}|>\frac{4}{H^4}\Big)\\
&\le \frac{4}{H^4}+P_W\Big(\hat{Z}_{-A(\beta)/u_k}>\frac{4}{H^4}\Big)<\frac{4}{H^4}+P_W\Big(\hat{Z}_1>-\frac{4}{H^4}\frac{u_k}{A(\beta)}\Big)=\frac{4}{H^4}+\exp\Big(\frac{2}{H^4}\frac{u_k}{A(\beta)}\Big)\\
&< \frac{4}{H^4}+\exp(-2H)<5H^{-4}.
\end{aligned}$$

Putting all estimates together, we get for the probability in (6) the bound

$$P_W(A_1^c)+\frac{H^2}{\log 2}\Big(4\exp\big(-\frac{e^{H^2-2H^{7/4}}}{4H^4}\big)+60\,H^2\exp\big(-\frac{H^{k_4-9}}{8}\big)+5H^{-4}\Big)<c\,H^{-2}$$

for some universal constant c and for all $H>1$. The bound on $P_W(A_1^c)$ is assumption (i), and we used the fact that $k_4\ge 10$. ∎

2.2 The typical behavior of the favorite point

In this section, we prove an annealed version of estimate (6) for a certain random valley. We check that this valley satisfies with high probability the assumptions for (6), and then apply the quenched estimate.

Let $r>0$ be fixed, and $x_0(W,r),x_1(W,r)$ be the r-extrema around zero (their definition given together with that of b in the introduction). Assume that $b_r=x_1(W,r)$. Otherwise, all the definitions following should be applied to the path $(s\mapsto W(-s))$. Let $r^-=\sup\{x<r: b_x\ne b_r\}, r^+=\inf\{x>r: b_x\ne b_r\}$ the points where b jumps just before and after r respectively. Since the probability that $r^-=r$ or $r^+=r$ is zero, in the following we assume that $r^-<r<r^+$.

A comment on notation: Note that the function $(r\mapsto b_r)$ (and similarly all functions $(r\mapsto x_i(W,r)), i\in\mathbb{Z}$) is step and left continuous. So for the next value of it after r we will use the notation b_{r^++}, i.e., the right limit of b at the point r^+. We will do the same for the other functions too.

For $x,y\in\mathbb{R}$, we define

$$W^{\#}(x,y):=\sup\{W(s)-W(t):(t-s)(x-y)\ge 0\}.$$

That is, the highest slope the diffusion should climb in order to go from x to y.

In the following, we will use three constants k_1,k_2,k_3. Our assumption for them is that $k_1,k_3\ge 10, k_2\ge 18$. We prefer not to choose values for them so that their role in the proof is clearer.

Let

$$j_r := \sup\{s < b_r : W(s) - W(b_r) = r\}, \tag{11}$$

$$l_r := \inf\{s > b_r : W(s) - W(b_r) = r\}, \tag{12}$$

and

$$\alpha_r := \inf\{t > j_r : W(t) - W(b_r) < k_1 \log r\}, \tag{13}$$

$$\gamma_r := \sup\{t < l_r : W(t) - W(b_r) < k_1 \log r\}. \tag{14}$$

So that, for $t \in [j_r, l_r] \setminus (\alpha_r, \gamma_r)$, we have

$$W(t) \geq W(b_r) + k_1 \log r.$$

Also let $h_1 := W^{\#}(b_{r^-}, b_r), h_2 := W^{\#}(b_r, b_{r^{++}})$, and

$$\zeta_r := \inf\{t > b_r : W(t) - W(b_r) \geq h_1 + k_2 \log h_1\},$$

$$\eta_r := \begin{cases} \inf\{t > b_{r^{++}} : W(t) - W(b_{r^{++}}) \geq h_2 - k_3 \log h_2\} & \text{if } b_{r^{++}} > 0, \\ \sup\{t < b_{r^{++}} : W(t) - W(b_{r^{++}}) \geq h_2 - k_3 \log h_2\} & \text{if } b_{r^{++}} < 0, \end{cases}$$

The two cases $b_{r^{++}} > 0, b_{r^{++}} < 0$ are shown in Figures 2, 3 respectively along with other points which are introduced in the proof of Lemma 2.

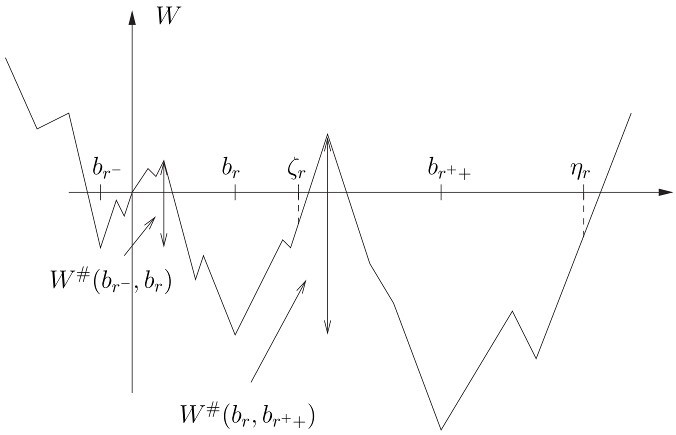

Figure 2: The scenario $b_r > 0$, $b_{r^{++}} > 0$

It will be shown that, with high probability, the diffusion hits first ζ_r and then η_r. And the main claim is that from time $\tau(\zeta_r)$ to $\tau(\eta_r)$, the favorite point is around b_r (recall the definition of $\tau(x)$ from Section 1.1). More precisely, for the event

$$\Delta_r := [\tau(\zeta_r) > \tau(\eta_r) \text{ or there is a } t \in [\tau(\zeta_r), \tau(\eta_r)] \text{ such that } F_X(t) \notin (\alpha_r, \gamma_r)],$$

we have the following lemma.

Lemma 2. *There is a $C > 0$ so that for all $r > 0$,*

$$\mathcal{P}(\Delta_r) < C\, r^{-1/4}. \tag{15}$$

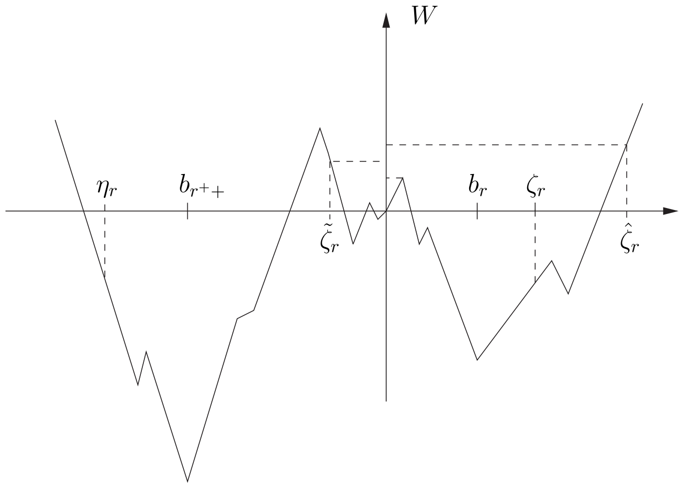

Figure 3: The scenario $b_r > 0$, $b_{r^+ +} < 0$

Proof. We are always under the assumption $b_r > 0$. Consider two cases.

Case 1: $b_{r^+ +} > 0$.

We will apply Lemma 1 for the path W and the choice

$$(y_0, \alpha, \beta, \gamma, \zeta, \eta, H) := (x_0(W, r^+ +), \alpha_r, b_r, \gamma_r, \zeta_r, \eta_r, r).$$

Call E_1 the event that one of (i)-(vii) of Lemma 1 fails. The probability of Δ_r is bounded by

$$\mathbb{P}(E_1) + \mathcal{P}(\Delta_r \cap E_1^c) \leq c r^{-1/4}$$

for some constant c that depends on k_1, k_2, k_3 but not on r. The first term is bounded with the use of Lemma 7, while the bound for the second follows from Lemma 1.

Case 2: $b_{r^+ +} < 0$.

Let

$$\tilde{\zeta}_r := \sup\{s < 0 : W(s) - W(b_r) \geq W^{\#}(b_r, 0) + 2k_2 \log W^{\#}(b_r, 0)\}, \tag{16}$$

$$\hat{\zeta}_r := \inf\{s > \zeta_r : W(s) - W(b_r) \geq W^{\#}(b_r, 0) + 3k_2 \log W^{\#}(b_r, 0)\}, \tag{17}$$

and

$$A_3 := [\tau(\zeta_r) < \tau(\tilde{\zeta}_r) < \tau(\hat{\zeta}_r) < \tau(\eta_r)].$$

See Figure 3. Then

$$\mathcal{P}(\tau(\zeta_r) > \tau(\eta_r) \text{ or } F([\tau(\zeta_r), \tau(\eta_r)]) \nsubseteq (\alpha_r, \gamma_r)) \leq \mathcal{P}(\{F([\tau(\zeta_r), \tau(\eta_r)]) \nsubseteq (\alpha_r, \gamma_r)\} \cap A_3) + \mathcal{P}(A_3^c).$$

As Lemma 11 shows, $\mathcal{P}(A_3^c) < C r^{-1/4}$. The first quantity is bounded by

$$\mathcal{P}(F([\tau(\zeta_r), \tau(\hat{\zeta}_r)]) \nsubseteq (\alpha_r, \gamma_r)) + \mathcal{P}(F([\tau(\tilde{\zeta}_r), \tau(\eta_r)]) \nsubseteq (\alpha_r, \gamma_r)).$$

Both of these two probabilities are bounded with the use of Lemma 1.
For the first, we apply Lemma 1 for the path W and the choice

$$(y_0, \alpha, \beta, \gamma, \zeta, \eta, H) := (x_0(W, r), \alpha_r, b_r, \gamma_r, \zeta_r, \hat{\zeta}_r, r).$$

Working as in Case 1, we obtain the required bound.
For the second, we apply Lemma 1 for the path $W^* := W(-\cdot)$ and the choice

$$(y_0, \alpha, \beta, \gamma, \zeta, \eta, H) := (-x_1(W, r^+ +), -\gamma_r, -b_r, -\alpha_r, -\tilde{\zeta}_r, -\eta_r, r).$$

We will use the notation X^W for the diffusion run in the fixed environment W. Let E_2 be the event that, with these choices, one of (i)-(vii) fails. As in Case 1, we use the bound on $\mathbb{P}(E_2)$ given in Lemma 7 to get that for W outside E_2, we have

$$P_W(F_{X^{W^*}}([\tau(-\tilde{\zeta}_r), \tau(-\eta_r)]) \nsubseteq (-\gamma_r, -\alpha_r)) < cr^{-1/4}$$

But $X^{W^*} \overset{law}{=} -X^W$. So that

$$P_W(F_{X^W}([\tau(\tilde{\zeta}_r), \tau(\eta_r)]) \nsubseteq (\alpha_r, \gamma_r)) < cr^{-1/4}$$

as required. ∎

In the time interval $[\tau(\eta_r), \tau(\zeta_{r^+ +})]$, we will show that F jumps from a neighborhood of b_r to a neighborhood of $b_{r^+ +}$. That is, if we let

$$K_r := \left\{ \begin{array}{c} \text{There is a time } z_r \in [\tau(\eta_r), \tau(\zeta_{r^+ +})] \text{ so that} \\ F(t) \in (\alpha_r, \gamma_r) \text{ for } t \in [\tau(\eta_r), z_r), \\ \text{and } F(t) \in (\alpha_{r^+ +}, \gamma_{r^+ +}) \text{ for } t \in (z_r, \tau(\zeta_{r^+ +})]. \end{array} \right\},$$

then the following holds.

Lemma 3. *There is a constant c so that*

$$\mathcal{P}(K_r^c) < cr^{-1/4}$$

for all $r > 0$.

Proof. Assume that $b_{r^+ +} > 0$. Then $b_r < b_{r^+ +} < \eta_r < \zeta_{r^+ +}$. Let

$$\begin{aligned} m_r &:= \sup\{s < b_{r^+ +} : W(s) - W(b_r) = W^{\#}(b_r, b_{r^+ +})\}, \\ z &:= \sup\{s < b_{r^+ +} : W(s) - W(b_{r^+}) \geq W^{\#}(b_r, b_{r^+ +}) + k_2 \log W^{\#}(b_r, b_{r^+ +}) + k_2 \log r\}, \end{aligned}$$

(see Figure 4), and recall that $W(\zeta_{r^+}) - W(b_{r^+}) = W^{\#}(b_r, b_{r^+ +}) + k_2 \log W^{\#}(b_r, b_{r^+ +})$.
Also call $Y_t := X_{\tau(z)+t}$ the diffusion after time $\tau(z)$. All objects defined for X (e.g., the local time, the process of the favorite point) are defined analogously for Y.
Define the events

$$\begin{aligned} \Sigma_0 &:= [z > m_r], \\ \Sigma_1 &:= [F_X(\tau(\eta_r)) \in (\alpha_r, \gamma_r)], \\ \Sigma_2 &:= [F_X(\tau(\zeta_{r^+ +})) \in (\alpha_{r^+ +}, \gamma_{r^+ +})], \\ \Sigma_3 &:= [(X_{\tau(\eta_r)+s})_{s\geq 0} \text{ hits } \zeta_{r^+ +} \text{ before } z], \\ \Sigma_4 &:= [F_Y([\tau_Y(\eta_r), \tau_Y(\zeta_{r^+ +})]) \subset (\alpha_{r^+ +}, \gamma_{r^+ +})]. \end{aligned}$$

On $\Sigma_1 \cap \Sigma_3 \cap \Sigma_4$ we claim that

$$t \in [\tau(\eta_r), \tau(\zeta_{r^+ +})] \Rightarrow F_X(t) \in (\alpha_r, \gamma_r) \cup (\alpha_{r^+ +}, \gamma_{r^+ +}). \tag{18}$$

Let $t \in [\tau(\eta_r), \tau(\zeta_{r^+ +})]$. Points in $(-\infty, z]$ collect local time only from the part $(X_s)_{s\leq\tau(\eta_r)}$ of the path $(X_s)_{s\leq\tau(\zeta_{r^+ +})}$ by the definition of Σ_3. And by the definition of Σ_1, the ones with the

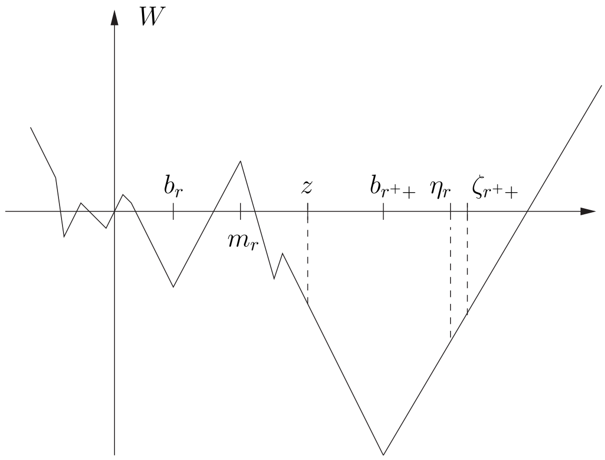

Figure 4:

most local time are in (α_r, γ_r). Points in $[z, \infty)$ collect local time only from the part $(X_s)_{\tau(z)\le s\le t}$ of the path. And by the definition of Σ_4, we know that out of them, the ones with the most local time at time t are in $(\alpha_{r^++}, \gamma_{r^++})$. This proves our claim.
On $\Sigma_0 \cap \Sigma_1 \cap \Sigma_2 \cap \Sigma_3 \cap \Sigma_4$ we know that $F(\tau(\eta_r)) \in (\alpha_r, \gamma_r)$, $F(\tau(\zeta_{r^++})) \in (\alpha_{r^++}, \gamma_{r^++})$, and from time $\tau(\eta_r)$ to $\tau(\zeta_{r^++}), X$ does not visit (α_r, γ_r). These combined with (18) show that $\Sigma_0 \cap \Sigma_1 \cap \Sigma_2 \cap \Sigma_3 \cap \Sigma_4 \subset K_r$.

The proof will be completed after we bound the probability of $(\Sigma_0 \cap \Sigma_1 \cap \Sigma_2 \cap \Sigma_3 \cap \Sigma_4)^c$. Lemmas 10(iii) and 2 give the bound for $\mathcal{P}(\Sigma_0^c)$ and $\mathcal{P}(\Sigma_1^c)$ respectively. To bound $\mathcal{P}(\Sigma_2^c)$, we apply Lemma 1 with the choice

$$(y_0, \alpha, \beta, \gamma, \zeta, \eta, H) := (x_0(W, r^++), \alpha_{r^++}, b_{r^++}, \gamma_{r^++}, \zeta_{r^++}, \zeta_{r^++}, r).$$

Lemma 11 shows that $\mathcal{P}(\Sigma_3^c) < cr^{-1/4}$ for some constant c (independent of r).
For Σ_4^c, we write $\Sigma_4^c \subset (\Sigma_4^c \cap \Sigma_0) \cup \Sigma_0^c$ The probability of $\Sigma_4^c \cap \Sigma_0$ is bounded with the use of Lemma 1 for the environment $W^z := W(z + \cdot)$ and with the choice

$$(y_0, \alpha, \beta, \gamma, \zeta, \eta, H) := (m_r - z, \alpha_{r^++} - z, b_{r^++} - z, \gamma_{r^++} - z, \eta_r - z, \zeta_{r^++} - z, r).$$

Let E_4 be the event that, with this choice, one of (i)-(vii) fails. Lemma 7 shows that $\mathbb{P}(E_4) < cr^{-1/4}$, and as in Lemma 2, we show that $\mathcal{P}(\Sigma_4^c \cap \Sigma_0) < cr^{-1/4}$. This finishes the proof. ■

2.3 Proof of the main results

Proof of the Theorem: The proof is a consequence of Lemmas 2, 3, 12.

Pick any $a \in (0, 1/2)$, and let $r_k = \exp(k^a)$ for $k \ge 1$. Then $\sum_{k=0}^\infty r_k^{-1/4} < \infty$, combined with Lemmas 2, 3, 12, 13, implies that there is a $k_0 > 0$ so that

(1) K_{r_k}, Δ_{r_k} happen for $k \ge k_0$,

if we call $(s_n(W))_{n\ge1}$ the increasing sequence of the point where $b|[r_{k_0}, \infty)$ jumps, then

(2) between any two terms from $(s_n(W))_{n\ge1}$ there is a term from $(r_k)_{k\ge k_0}$, and

(3) $(\alpha_{r_k}, \gamma_{r_k}) \subset I(b_{r_k})$ for $k \ge k_0$.

Now we claim that if $r_k^- > r_{k_0}$, then

$$F_X(t) \in (\alpha_{r_k}, \gamma_{r_k}) \text{ for all } t \in (z_{r_k^-}, z_{r_k}). \tag{19}$$

Indeed, this holds for $t \in [\tau(\zeta_{r_k}), z_{r_k})$ because K_{r_k} happens. Let $j := \max\{n : r_n < r_k^-\}$. The assumption $r_k^- > r_{k_0}$ gives that $j \geq k_0$, and this together with (2) implies that in $[r_j, r_k]$ the path of b jumps only at r_k^-. Since K_{r_j} happens, $F_X(t) \in (\alpha_{r_j^+}, \gamma_{r_j^+})$ for all $t \in (z_{r_j}, \tau(\zeta_{r_j^+})]$. But $r_j^+ = r_k^-$, so $\zeta_{r_j^+} = \zeta_{r_k}$, $z_{r_j} = z_{r_k^-}$, and $(\alpha_{r_j^+}, \gamma_{r_j^+}) \subset (\alpha_{r_k}, \gamma_{r_k})$ (recall the definitions of α_r, γ_r in (13), (14), and note that $r_j^+ < r_k$).

Let $x_n := b_{s_{n+1}}$, the value of b in $(s_n(W), s_{n+1}(W)]$, and $t_n := \log z_{s_n}$. Because of observation (2) above, for $n \geq 1$ there is a k with $s_n < r_k \leq s_{n+1}$. Then $z_{r_k^-} = z_{s_n} = e^{t_n}$, $z_{r_k} = z_{s_{n+1}} = e^{t_{n+1}}$, and $F_X((e^{t_n}, e^{t_{n+1}})) \subset (\alpha_{r_k}, \gamma_{r_k})$ because of (19) (note that $r_k^- = s_n > r_{k_0}$). This together with fact (3) above proves (i) of the theorem.

For the second claim of the theorem, observe that $t_n/s_n = \log z_{s_n}/s_n$, and $z_{s_n} \in [\tau(\eta_{s_n}), \tau(\zeta_{s_n+})]$. One can see that

$$\overline{\lim_{n\to\infty}} \frac{\log \tau(\zeta_{s_n+})}{s_n} \leq 1 \text{ and } \underline{\lim_{n\to\infty}} \frac{\log \tau(\eta_{s_n})}{s_n} \geq 1.$$

The proof of these two is done by modifying the proof of (4.7), (4.8), (4.11) in Hu and Shi (1998). Since no new idea is involved, we omit it. ■

Proof of the Corollary: Let $\lambda(s) = (t_1/s_1)s$ for $s \in [0, s_1]$. Then in any interval $[s_n, s_{n+1}]$ with $n \geq 1$, $\lambda(s)$ is defined as the unique increasing map of the form $\gamma_n s + \delta_n$ mapping that interval to $[t_n, t_{n+1}]$. Since on $[s_n, s_{n+1}]$ the function $\lambda(s)/s = a_n + \delta_n s^{-1}$ is monotone, it maps $[s_n, s_{n+1}]$ to the interval with endpoints $t_n/s_n, t_{n+1}/s_{n+1}$. It follows that $\lim_{s\to\infty} \lambda(s)/s = 1$. Then for all large s, the theorem says that $|F_X(e^{\lambda(s)}) - b_s| < (\log |b_s|)^c$. It is easy to prove that, with probability 1, $\log |b_s| < 3 \log s$ for all large s.

[Recall that the typical size of b_s is s^2. The proof is similar with the proof of (28). We show $\log \beta_s^+ < 3 \log s$ for large s (see the beginning of the next section for notation). The basic ingredient is that for large $A > 0$, it holds $\mathbb{P}(\beta_s^+ > s^2 A) = \mathbb{P}(\beta_1^+ > A) \leq Ce^{-A\pi^2/8}$. The last inequality holds because β_1^+ has Laplace transform $(\sqrt{2\lambda} \coth \sqrt{2\lambda})^{-1}$ (see Neveu and Pitman (1989), Lemma of §1) so that its density is $\sum_{k=0}^{\infty} \exp\big(-\frac{(2k+1)^2\pi^2}{8}x\big) 1_{x>0}$, having tail as we claimed.] This finishes the proof. ■

3 Auxiliary lemmas

3.1 Estimates on the environment

In this section, we prove several facts we needed in the proof of Lemma 2 and of the main theorem. First a useful fact for studying the process b. For any real valued process $(Z_s)_{s\in\mathbb{R}}$ and $t \in \mathbb{R}$, we define

$$\begin{aligned} \underline{Z}_t &: = \inf\{Z_s : s \text{ between 0 and t }\}, \\ \overline{Z}_t &: = \sup\{Z_s : s \text{ between 0 and t }\}. \end{aligned}$$

Call W^+ the process $(W(s) : s \geq 0)$ and W^- the process $(W(-s) : s \geq 0)$. For $r > 0$, let

$$\begin{aligned} \tau_r^+ &:= \min\{s \geq 0 : W^+(s) - \underline{W}^+(s) = r\}, \\ \beta_r^+ &:= \min\{s \geq 0 : W^+(s) = \underline{W}^+(\tau_r^+)\}, \\ \tau_r^- &:= \min\{s \geq 0 : W^-(s) - \underline{W}^-(s) = r\}, \\ \beta_r^- &:= -\min\{s \geq 0 : W^-(s) = \underline{W}^-(\tau_r^-)\}. \end{aligned}$$

One can see that with probability one, it holds $b_r \in \{\beta_r^-, \beta_r^+\}$ (see e.g. Zeitouni (2004)). We will make frequent use of this fact in what follows.

The next statement is from lemma of §1 in Neveu and Pitman (1989).

Fact 1: $-W(\beta_1^+)$ is an exponential random variable with mean 1.

The following uniform continuity result is an immediate consequence of Lemma 1.1.1 in Csörgö and Révész (1981).

Fact 2: There is a constant C so that for any $\rho > 0, h \in [0,\rho], v > 0$, it holds

$$\mathbb{P}(\sup_{y,z\in[0,\rho],|y-z|\le h} |W(y)-W(z)| \ge v\sqrt{h}) \le C\frac{\rho}{h}e^{-v^2/3}.$$

For an interval $I \subset [0,\infty)$, the random variable $\sup\{|B_t|/\sqrt{t} : t \in I\}$ is infinite if one of the endpoints of I is 0 or ∞ (by the law of the iterated logarithm). However if the interval is bounded away from 0 and infinity, the variable is finite, and one can get the following result on its tail.

Lemma 4. *Let $\sigma > 1, M > 1$, and $a > 0$. Then*

$$\mathbb{P}\Big(\sup_{1\le t\le M}\frac{|B_t|}{\sqrt{t}} \ge a\Big) \le 4\Big(\frac{\log M}{\log\sigma}+1\Big)\frac{\sqrt{\sigma}}{a}\exp\Big(-\frac{a^2}{2\sigma}\Big).$$

Proof. Let $N := [\frac{\log M}{\log\sigma}]$, and $t_n := \sigma^n$ for $n = 0,\dots N+1$. Then

$$\begin{aligned}\mathbb{P}\Big(\sup_{1\le t\le M}\frac{|B_t|}{\sqrt{t}} \ge a\Big) &\le \sum_{n=0}^{N}\mathbb{P}\Big(\sup_{t_n\le s\le t_{n+1}}\frac{|B_s|}{\sqrt{s}} \ge a\Big) \le \sum_{n=0}^{N}\mathbb{P}\big(\sup_{0\le s\le t_{n+1}}|B_s| \ge \sqrt{t_n}a\big)\\ &\le 4\sum_{n=0}^{N}\mathbb{P}\Big(B_1 \ge \frac{a}{\sqrt{\sigma}}\Big) \le 4\Big(\frac{\log M}{\log\sigma}+1\Big)\frac{\sqrt{\sigma}}{a}\exp\big(-\frac{a^2}{2\sigma}\big).\end{aligned}$$

In the last line, the first inequality follows by the reflection principle, while the second, from a well known bound on the tail of the standard normal distribution. ■

Lemma 5. *For all $x > 0$, we have*

1. $\mathbb{P}(W^\#(b_{1-}, b_1) < x) \le 4\sqrt{x}$.
2. $\mathbb{P}(W^\#(b_1, b_{1+_+}) > x) < 2x^{-1}$.
3. $\mathbb{P}(W^\#(b_1, 0) < x) \le 2x$.
4. $\mathbb{P}(W^\#(b_1, 0) > x) < 6e^{-x}$.
5. $\mathbb{P}(|b_1| > x) < 2\exp\big(-\frac{\pi^2}{8}x\big)$.
6. $(W(b_1) - W(b_{1+_+}))/W^\#(b_1, b_{1+_+})$ *is an exponential random variable with mean 1.*

Proof. 1. $W^\#(b_{1-}, b_1) \ge \min\{\overline{W}(\beta_1^+), \overline{W}(\beta_1^-)\}$ because $b_1 \in \{\beta_1^-, \beta_1^+\}$. So that $\mathbb{P}(W^\#(b_{1-}, b_1) < x) \le 2\mathbb{P}(\overline{W}(\beta_1^+) < x) < 4\sqrt{x}$ by Lemma 8.

2. From the proof of Lemma 12, it follows that $W^\#(b_1, b_{1+}) \le D_{\ell_1} = \sigma_0\tau_0$. The variables σ_0, τ_0 are independent and have density $2x^{-3}1_{x\ge1}$. It is easy to compute that for $x \ge 1$, it holds $\mathbb{P}(\sigma_0\tau_0 > x) = (2\log x + 1)/x^2$. This proves the claim.

3. $W^\#(b_1, 0) \ge \min\{-W(\beta_1^+), -W(\beta_1^-)\}$. So that $\mathbb{P}(W^\#(b_1, 0) < x) \le 2\mathbb{P}(-W(\beta_1^+) < x) < 2x$, since $-W(\beta_1^+)$ has exponential distribution with mean 1.

4. $W^\#(b_1, 0) = \overline{W}(b_1) - W(b_1) < 1 - W(b_1)$, and $-W(b_1) \le \max\{-W(\beta_1^+), -W(\beta_1^-)\}$. So that $\mathbb{P}(W^\#(b_1, 0) > x) \le \mathbb{P}(-W(b_1) > x - 1) \le 2\mathbb{P}(-W(\beta_1^+) > x - 1) \le 2e^{1-x}$.

5. The density of b_1 is $f_{b_1}(x) := \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\exp\big(-\frac{(2k+1)^2\pi^2}{8}|x|\big)$ (see relation (1.4) in Shi (2001)), which is less than $\frac{2}{\pi}\exp(-\frac{\pi^2}{8}|x|)$. And the required inequality follows after integration.

6. For the proof of this claim, drawing a picture will help the reader. For $\ell \in \mathbb{R}$, let

$$\begin{aligned} H_\ell^- &:= \sup\{s<0 : W_s=\ell\},\\ H_\ell^+ &:= \inf\{s>0 : W_s=\ell\},\\ K_\ell &:= \min\{\max_{H_\ell^-\le s\le 0} W_s, \max_{0\le s\le H_\ell^+} W_s\}-\ell. \end{aligned}$$

Also let $\ell_0 := \sup\{\ell<0 :$ one of H^-, H^+ jumps at ℓ, and $\lim_{x\nearrow\ell} K_x \ge 1\}$. Clearly, $\ell_0 = W(b_1)$. Assume that H^- jumps at ℓ_0. Assume moreover that $\lim_{\ell\nearrow\ell_0}\max_{H_\ell^-\le s\le 0} W_s > \lim_{\ell\nearrow\ell_0}\max_{0\le s\le H_\ell^+} W_s$ (the case that the reverse inequality holds is treated similarly). Then $b_{1++}>0$, and $W^\#(b_1, b_{1++}) = \max_{0\le s\le H_{\ell_0}^+} W_s$. The way to locate $(b_{1++}, W(b_{1++}))$ is as follows. We look at $B=\{W(H_{\ell_0}^+ + s)-W(b_r) : s>0\}$ (which is a standard Brownian motion), and we wait until $B-\underline{B}$ hits $W^\#(b_1,b_{1++})$. When this happens, the value of $-\underline{B}$ is an exponential random variable with mean $W^\#(b_1,b_{1++})$ (see Fact 1 in the beginning of this section). This proves our claim. ∎

For the next lemma, recall the definitions of $x_0(W,1), x_1(W,1)$, given together with that of b in the introduction.

Lemma 6. *$|x_0(W,1)|, x_1(W,1)$ have densities bounded by 1.*

Proof. It is true that $\{x_{k+1}(W,1)-x_k(W,1) : k\in\mathbb{Z}\setminus\{0\}\}$ is a set of i.i.d. random variables with the same distribution as $\ell := \inf\{s\ge 0 : |W_s|=1\}$ (see proposition of §1 in Neveu and Pitman (1989)) . Call $f_\ell(x)$ the density of this random variable. Since for any fixed t, the process $(W_{s-t}-W_{-t} : s\in\mathbb{R})$ is a standard Brownian motion, one can take $t\to\infty$ and use the renewal theorem to show that $x_1(W,1), |x_0(W,1)|$ are respectively the residual waiting time after 0 and the age at time 0 for a renewal process "starting at $-\infty$" and with increments having distribution ℓ. Their densities are computed in Exercise 4.7 of Chapter 3 in Durrett (1996), and they both equal $\int_x^\infty f_\ell(z)dz$, which is less than 1. ∎

An estimate on the hitting time of Brownian motion is as follows.

Fact 3: Let W be standard Brownian motion. For the time $\rho(1) := \inf\{s>0 : W(s)=1\}$, it holds

$$\mathbb{P}(\rho(1)>u) < u^{-1/2} \text{ for all } u>0.$$

This follows from $\mathbb{P}(\rho(1)>u)=\mathbb{P}(\overline{W}(u)<1)=\mathbb{P}(\overline{W}(1)<u^{-1/2})$ and the fact that $\overline{W}(1)$ has density $\sqrt{2/\pi}e^{-x^2/2}1_{x\ge 0}$.

Lemma 7. *There are constants C_1, C_2 (C_2 depends on k_1,k_2,k_3) so that for the three choices of $y_0,\alpha,\beta,\gamma,\zeta,\eta,H$ in Lemma 2 and the two in Lemma 3, we have for $r>0$*

1. $\mathbb{P}(|\eta-y_0|>H^4) < 26\, r^{-1/4}$.
2. $\mathbb{P}(\max_{s\in[y_0,\eta]} W(s) > H^{3/2}) < 4\, r^{-1/4}$.
3. $\mathbb{P}(A(\eta)/(A(\eta)+|A(y_0)|) > H^{-2}) < r^{-2}$.
4. $\mathbb{P}(\frac{A(\eta)-A(\beta)}{A(\zeta)-A(\beta)} > e^{H^2}) < r^{-1/3}$.
5. $\mathbb{P}(A(\beta)/A(\zeta) < -H^{-5}) < C_1 r^{-1/2}$.
6. $\mathbb{P}(\sup_{y\in[\zeta,\eta]}\sup_{y_0<s<y, s\notin(\alpha,\gamma)}\int_s^y e^{W(t)-W(s)}dt/\int_\beta^y e^{W(t)-W(\beta)}dt > H^{-\min\{k_1,k_3\}+5}) < C_2\, r^{-1/4}$.
7. $\mathbb{P}(\sup_{y\in[\zeta,\eta]}\sup_{y_0<s<y, s\notin(\alpha,\gamma)}(e^{W(s)}\int_0^y e^{W(t)-W(\beta)}dt)^{-1} > \exp(2H^{7/4})) < 7\, r^{-1/4}$.

Proof. In all uses of the lemma, it is $H = r$.

1. We prove this claim at once for all the cases that we use it. Let

$$\rho(r) := \inf\{s > 0 : W(s) = r\},$$
$$\tau_1(r) := \inf\{s > \rho(r) : W(s) = \underline{W}(\rho(r))\},$$
$$\tau_2(r) := \inf\{s > \tau_1(r) : W(s) = \overline{W}(\tau_1(r))\}.$$

Through the analogous series of definitions, we define $\tilde{\tau}_2(r)$ for the path $(W(-\cdot))_{s\geq 0}$. In all cases that we use the lemma, it holds $[y_0, \eta] \subset [-\tilde{\tau}_2(r), \tau_2(r)]$. Also let $\tau_0(r) := \inf\{s \in [0, \tau_1(r)] : W(s) = \overline{W}(\tau_1(r))\}$.

Clearly, $\tau_2(r) \overset{law}{=} r^2\tau_2(1)$. We will show that $\mathbb{P}(\tau_2(r) > r^2/2) < r^{-1/4}/2$ for large r. We write

$$\tau_2(1) = \rho(1) + \big(\tau_0(1) - \rho(1)\big) + \big(\tau_2(1) - \tau_0(1)\big), \tag{20}$$

and we will bound separately the probability that each of the three terms in the last expression is large.

For the first term, we note that $\mathbb{P}(\rho(1) > r) < r^{-1/2}$ for all $r > 0$ (Fact 3).

For the second term. The random variable $r_1 := W(\tau_0(1)) - 1/(1 - \underline{W}(\rho(1)))$ has density $(1+x)^{-2}1_{x\geq 0}$ because

$$\begin{aligned}\mathbb{P}(r_1 > x) &= \mathbb{E}\{\mathbb{P}((W(\tau_0(1)) - 1 > x(1 - \underline{W}(\rho(1)))|\underline{W}(\rho(1))))\}\\ &= \mathbb{E}\{\mathbb{P}(W \text{ starting from 1 hits first } 1 + x(1 - \underline{W}(\rho(1)) \text{ and then } \underline{W}(\rho(1))|\underline{W}(\rho(1))))\}\\ &= \mathbb{E}\Big(\frac{1 - \underline{W}(\rho(1))}{1 - \underline{W}(\rho(1)) + x(1 - \underline{W}(\rho(1)))}\Big) = (1+x)^{-1}.\end{aligned}$$

We used the Markov property on the stopping time $\rho(1)$. Now given the values of $\underline{W}(\rho(1)), W(\tau_0(1)) - 1$, the law of $\tau_0(1) - \rho(1)$ is the same as the time it takes for a three dimensional Bessel process starting from $1 + \underline{W}(\rho(1))$ to hit $W(\tau_0(1)) + \underline{W}(\rho(1))$ (Proposition 3.13 (iv), Chapter VI in Revuz and Yor (1999)). So it is bounded stochastically from above by the time it takes for Brownian motion starting from zero to hit $W(\tau_0(1)) - 1$. This last time equals in law to

$$(W(\tau_0(1)) - 1)^2 X = r_1^2(1 - \underline{W}(\rho(1)))^2 X,$$

where r_1 has density $(1+x)^{-2}1_{x\geq 0}$, X has the same law as $\rho(1)$, and $X, r_1, \rho(1)$ are independent. Also it is easy to see that $1 - \underline{W}(\rho(1))$ has density $(1+x)^{-2}1_{x\geq 0}$ (the proof goes as that for r_1 above).

Consequently

$$\mathbb{P}(\tau_0(1) - \rho(1) > r^{3/2}) \leq \mathbb{P}(r_1 > r^{1/4}) + \mathbb{P}(1 - \underline{W}(\rho(1)) > r^{1/4}) + \mathbb{P}(X > r^{1/2}) < 3r^{-1/4}.$$

For the third term in (20). As above, we show that the random variable

$$r_2 := \frac{W(\tau_1(1)) - \underline{W}(\tau_2(1))}{W(\tau_0(1)) - \underline{W}(\rho(1))}$$

has density $(1+x)^{-2}1_{x\geq 0}$. Given $W(\tau_0(1)) - \underline{W}(\tau_2(1))$, the law of $\tau_2(1) - \tau_0(1)$ is the same as the law of $(W(\tau_0(1)) - \underline{W}(\tau_2(1)))^2(Y_1 + Y_2)$, where Y_1, Y_2 (are independent and) have the law of the time it takes for a three dimensional Bessel process starting from zero to hit 1. This follows from Proposition 3.13 (iii), (iv), (v), Chapter VI, in Revuz and Yor (1999), and the scaling property of the Bessel process. Observe that

$$\begin{aligned}W(\tau_0(1)) - \underline{W}(\tau_2(1)) &= W(\tau_0(1)) - W(\tau_1(1)) + W(\tau_1(1)) - \underline{W}(\tau_2(1)) \\ &= W(\tau_0(1)) - W(\tau_1(1)) + r_2(W(\tau_0(1)) - W(\tau_1(1))) \\ &= (1+r_2)(W(\tau_0(1)) - W(\tau_1(1))) = (1+r_2)(1+r_1)(1-\underline{W}(\rho(1))).\end{aligned}$$

So that

$$\tau_2(1) - \tau_0(1) = (1+r_2)^2(1+r_1)^2(1-\underline{W}(\rho(1)))^2(Y_1+Y_2),$$

and

$$\begin{aligned}\mathbb{P}(\tau_2(1) - \tau_0(1) > \frac{r^2}{3}) &\leq \mathbb{P}(1+r_1 > r^{1/4}) + \mathbb{P}(1+r_2 > r^{1/4}) \\ &+ \mathbb{P}(1-\underline{W}(\rho(1)) > r^{1/4}) + \mathbb{P}(Y_1 > \frac{r^{1/2}}{6}) + \mathbb{P}(Y_2 > \frac{r^{1/2}}{6}) < 9r^{-1/4}.\end{aligned}$$

We used $\mathbb{P}(Y_1 > x) \leq \mathbb{P}(\rho(1) > x)$ and Fact 3. Combining all the above estimates, we get that on a set whose complement has probability at most $13r^{-1/4}$, it holds $\tau_2(1) \leq r + r^{3/2} + r^2/3$, which is less than $r^2/2$ for $r > 50$. This finishes the proof of part 1.

2. As we mentioned in the proof of part one, in all uses of the lemma, it holds $[y_0, \eta] \subset [-\tilde{\tau}_2(r), \tau_2(r)]$. So $\max_{s\in[y_0,\eta]} W(s) \leq \max\{W(\tau_2(r)), W(-\tilde{\tau}_2(r))\} = \max\{W(\tau_0(r)), W(-\tilde{\tau}_0(r))\}$. Now $W(\tau_0(r)) \overset{law}{=} rW(\tau_0(1))$, and we saw that $W(\tau_0(1)) = 1+W(\tau_0(1))-1 = 1+r_1(1-\underline{W}(\rho(1)))$. Since

$$\mathbb{P}(1+r_1 > r^{1/4}) = \mathbb{P}(1-\underline{W}(\rho(1)) > r^{1/4}) = r^{-1/4},$$

for $r > 1$, we have outside a set of probability at most $2r^{-1/4}$, that $W(\tau_0(1)) < 1+(r^{1/4}-1)r^{1/4} < r^{1/2}$. Consequently, $\mathbb{P}(\max\{W(\tau_2(r)), W(\tilde{\tau}_2(r))\} > r^{3/2}) \leq 4r^{-1/4}$.

7. On $[\zeta < 1]^c$, for $y \in [\zeta, \eta]$ and $s \in [y_0, \eta]$, we have

$$\frac{e^{-W(s)}}{\int_0^y e^{W(t)-W(\beta)}dt} \leq \exp(-\inf_{y_0<s<\eta_r} W(s))$$

because $W(\beta)$ is the minimum value of W in $[0, \zeta]$, and $\zeta \geq 1$. Recall the definitions made above, in the proof of part 1. The exponent of the last expression is bounded above by $\max\{-\underline{W}(\tau_2(r)), -\underline{W}(\tilde{\tau}_2(r))\}$, which has the same distribution as $r\max\{-\underline{W}(\tau_2(1)), -\underline{W}(\tilde{\tau}_2(1))\}$. Observe that

$$\begin{aligned}-\underline{W}(\tau_2(1)) &= -\underline{W}(\rho(1)) + W(\tau_1(1)) - \underline{W}(\tau_2(1)) = -\underline{W}(\rho(1)) + r_2(W(\tau_0(1)) - \underline{W}(\rho(1))) \\ &= -\underline{W}(\rho(1)) + r_2(W(\tau_0(1)) - 1 + 1 - \underline{W}(\rho(1))) = -\underline{W}(\rho(1)) + r_2(1+r_1)(1-\underline{W}(\rho(1))),\end{aligned}$$

and $\mathbb{P}(\max\{r_2, 1+r_1, 1-\underline{W}(\rho(1))\} > r^{1/4}) < 3r^{-1/4}$. On the complement of $[\max\{r_2, 1+r_1, 1-\underline{W}(\rho(1))\} > r^{1/4}]$, we have $-\underline{W}(\tau_2(1)) \leq r^{1/4} + r^{3/4} < 2r^{3/4}$. Consequently, for $r > 1$,

$$\mathbb{P}(-\inf_{x_0<s<\eta_r} W(s) > 2r^{7/4}) < 6r^{-1/4}.$$

Also $\mathbb{P}(\zeta < 1) < r^{-2}$, in all the uses of the lemma. For example, when $\zeta = \zeta_r$, we have $\mathbb{P}(\zeta_r < 1) = \mathbb{P}(\zeta_1 < r^{-2}) \leq \mathbb{P}(x_1(W,1) < r^{-2}) < r^{-2}$ (the last inequality follows from Lemma 6). So that the probability in the statement is bounded by $6r^{-1/4} + r^{-2} < 7r^{-1/4}$.

Now we move to the proof of the remaining claims. The events

$$A_4 := [x_0(W,r^++) \in (-1,0]] \cup [|x_0(W,r^++)| > r^4] \cup [\zeta_r < 1] \cup [|\zeta_r| > r^4] \cup [|\eta_r| > r^4],$$
$$A_5 := [\sup_{y,z\in[-r^4,r^4],|y-z|\le 1} |W(y)-W(z)| \ge \log r],$$

will be used bellow. Observe that

$$\mathbb{P}(x_0(W,r^++) \in (-1,0]) \le \mathbb{P}(x_0(W,r) \in (-1,0]) = \mathbb{P}(x_0(W,1) \in (-1/r^2,0]) \le r^{-2},$$

and $\mathbb{P}(\zeta_r < 1) < r^{-2}$ as we proved just before. Combining these with part 1 of the lemma, we get $\mathbb{P}(A_4) < cr^{-2}$. Also, applying Fact 2, with $h=1, \rho = r^4, v = \log r/2$, we get

$$\mathbb{P}(A_5) \le Cr^4 e^{-(\log r)^2/12} < cr^{-2}.$$

5. In four of the five cases we use the lemma, it holds $A(b_r)/A(\zeta_r) > 0$, and we have nothing to prove. The only case where something needs a proof is in the claim $P_W(F_{X^{W^*}}([\tau(-\tilde{\zeta}_r), \tau(-\eta_r)]) \not\subseteq (-\gamma_r, -\alpha_r)) < cr^{-1/4}$ contained in the proof of Lemma 2 (Case 2). Let $A_7 := [W^\#(b_r,0) < \sqrt{r}] \cup [|b_r| > r^3] \cup [\tilde{\zeta}_r > -1]$. Then $\mathbb{P}(A_7) \le \mathbb{P}(W^\#(b_1,0) < 1/\sqrt{r}) + \mathbb{P}(|b_1| > r) + \mathbb{P}(\tilde{\zeta}_1 > -r^{-2}) \le 2/\sqrt{r} + 2e^{-r} + cr^{-1/2} < Cr^{-1/2}$. On $(A_4 \cup A_5 \cup A_7)^c$ we have

$$\frac{A(b_r)}{|A(\tilde{\zeta}_r)|} = \frac{\int_0^{b_r} e^{W(y)-W(b_r)}dy}{\int_{\tilde{\zeta}_r}^0 e^{W(y)-W(b_r)}dy} \le \frac{b_r e^{W^\#(b_r,0)}}{e^{W^\#(b_r,0)+k_2\log W^\#(b_r,0)-\log r}} = \frac{rb_r}{(W^\#(b_r,0))^{k_2}} \le r^{4-k_2/2} < r^{-5}$$

since $k_2 \ge 18$.

The remaining parts of the lemma we prove only for the first choice of $y_0, \alpha, \beta, \gamma, \zeta, \eta, H$, i.e., $(y_0, \alpha, \beta, \gamma, \zeta, \eta, H) := (x_0(W, r^++), \alpha_r, b_r, \gamma_r, \zeta_r, \eta_r, r)$. For the other choices, the proof is similar.

3. The quotient inside the probability equals

$$\frac{A(\eta_r)}{A(\eta_r) + |A(x_0(W,r^+))|} = \frac{1}{1 + |A(x_0(W,r^+))|/A(\eta_r)},$$

and we will show that $|A(x_0(W,r^+))|/A(\eta_r)$ is large. We will use x_0 instead of $x_0(W,r^+)$ in the following. On $(A_4 \cup A_5)^c$ we have $\sup_{s\in[x_0,x_0+1]} |W(s)-W(x_0)| < \log r$. So that $W(s) \ge W(x_0) - \log r$ on $[x_0, x_0+1]$, and

$$\frac{|A(x_0(W,r^+))|}{A(\eta_r)} = \frac{\int_{x_0}^0 e^{W(y)}dy}{\int_0^{\eta_r} e^{W(y)}dy} = \frac{\int_{x_0}^0 e^{W(y)-W(b_r)}dy}{\int_0^{\eta_r} e^{W(y)-W(b_r)}dy} \ge \frac{e^{W(x_0)-W(b_r)-\log r}}{\eta_r e^{W^\#(b_r,b_{r^+})-k_3\log W^\#(b_r,b_{r^+})}}$$
$$\ge \frac{e^{k_3 \log W^\#(b_r,b_{r^+})-\log r}}{r^4} = \frac{(W^\#(b_r,b_{r^+}))^{k_3}}{r^5} \ge r^{k_3-5}.$$

We used the fact that $W(x_0) - W(b_r) \ge W^\#(b_r, b_{r^+})$, which holds because we assumed that $b_{r^++} > 0$, and also that $W^\#(b_r, b_{r^+}) \ge r$. Thus, on the complement of $A_4 \cup A_5$, it holds

$$\frac{A(\eta_r)}{A(\eta_r) + |A(x_0(W,r^+))|} \le \frac{1}{1 + r^{k_3-5}} \le r^{-2}$$

since $k_3 \ge 7$.

4. Let $A_6 := [W^\#(b_{r^-}, b_r) < \log r] \cup [W^\#(b_r, b_{r^+}) > r^{3/2}]$. Then

$$\mathbb{P}(A_6) = \mathbb{P}(W^\#(b_{1^-}, b_1) < \log r/r) + \mathbb{P}(W^\#(b_1, b_{1^+}) > \sqrt{r}) < r^{-1/3}/3,$$

for large r, using Lemma 5. Now on $(A_4 \cup A_5 \cup A_6)^c$ we have $\zeta_r - b_r > 1$ (because of the definition of A_5 and the fact that $W(\zeta_r) - W(b_r) > W^{\#}(b_{r-}, b_r) > \log r$ on A_6^c), and

$$\frac{A(\eta_r) - A(b_r)}{A(\zeta_r) - A(b_r)} = \frac{\int_{b_r}^{\eta_r} e^{W(y)-W(b_r)} dy}{\int_{b_r}^{\zeta_r} e^{W(y)-W(b_r)} dy} \leq \frac{\eta_r e^{W^{\#}(b_r, b_{r+}) - k_3 \log W^{\#}(b_r, b_{r+})}}{e^{W^{\#}(b_{r-}, b_r) + k_2 \log W^{\#}(b_{r-}, b_r) - \log r}}$$

$$= \frac{r \eta_r e^{W^{\#}(b_r, b_{r+}) - W^{\#}(b_{r-}, b_r)}}{W^{\#}(b_r, b_{r+})^{k_3} W^{\#}(b_{r-}, b_r)^{k_2}} \leq \frac{r^{5-k_3} e^{W^{\#}(b_r, b_{r+}) - W^{\#}(b_{r-}, b_r)}}{W^{\#}(b_{r-}, b_r)^{k_2}} < r^{5-k_3} e^{r^{3/2}} < e^{r^2}.$$

The last inequality holding for large r, and the same is true for $\mathbb{P}(A_4 \cup A_5 \cup A_6) < r^{-1/3}$.

6. The quantity of interest is

$$\frac{\int_s^y e^{W(t)-W(s)} dt}{\int_{b_r}^y e^{W(t)-W(b_r)} dt}. \tag{21}$$

Let $m_r := \inf\{x > b_r : W(x) - W(b_r) = W^{\#}(b_r, b_{r++})\}$ and

$$A_7 := [\text{ there is an } s \in [x_0(W, r^+ +), m_r] \setminus (\alpha_r, \gamma_r) \text{ with } W(s) - W(b_r) \leq k_1 \log r].$$

For $s \in [j_r, l_r] \setminus (\alpha_r, \gamma_r)$, it holds $W(s) - W(b_r) \geq k_1 \log r$ by the definition of α_r, γ_r. It remains to study the intervals $[x_0(W, r^+ +), j_r]$, $[l_r, m_r]$. We will study only the first; the case of the second is similar. $B := (W(-s + j_r) - W(j_r))_{s \geq 0}$ is a standard Brownian motion. If there is $s \in [x_0(W, r^+ +), j_r]$ with $W(s) - W(b_r) < k_1 \log r$, then B visits $-r + k_1 \log r$ and then returns to 0 before hitting $-r$. This last event has probability $k_1 \log r / r$. So that $\mathbb{P}(A_7) < r^{-1/2}$ for large r. Finally, $\mathbb{P}(m_r > r^4) < r^{-1/4}$ from part 1 of the lemma.

Working as in part 4, we see that on $([m_r > r^4] \cup [\zeta_r < 1] \cup A_5 \cup A_7)^c$ we have the following bound on (21).
If $y < m_r$, then the bound is

$$\frac{(y - s) \exp(W^{\#}(b_r, y) - k_1 \log r)}{\exp(W^{\#}(b_r, y) - \log r)} < (\eta_r - x_0(W, r)) r^{-k_1 + 1} < r^{-k_1 + 5}.$$

If $y > m_r$, then the bound is

$$\frac{(\eta_r - x_0(W, r)) \exp(W^{\#}(b_r, b_{r+}) - k_3 \log W^{\#}(b_r, b_{r+}))}{\exp(W^{\#}(b_r, b_{r_1+}) - \log r)} < (\eta_r - x_0(W, r))(W^{\#}(b_r, b_{r+}))^{-k_3} r < r^{-k_3 + 5}.$$

We used the definition of η_r to bound the numerator. ■

Lemma 8. *The random variable $\overline{W}(\beta_1^+)$ has density $f(x) = -\mathbf{1}_{x \in (0,1]} \log x$. In particular, $\mathbb{P}(\overline{W}(\beta_1^+) < x) < 2\sqrt{x}$ and $\mathbb{P}(|\overline{W}(\beta_1^+) - \overline{W}(\beta_1^-)| < x) < 3\sqrt{x}$ for all $x \in [0, 1]$.*

Proof. The proof uses excursion theory, for which we give the basic setup. The following are standard (see Revuz and Yor (1999), Chapter XII).

Consider the process $Y(t) := W(t) - \underline{W}(t)$. A local time process for Y is $-\underline{W}$. Let $(\varepsilon_t)_{t>0}$ be the corresponding excursion process. For any ε in the space of the excursions, we denote by $\overline{\varepsilon}$ the maximum value of ε. The process $\{(\overline{\varepsilon}_t, t) : t \geq 0\}$ is a Poisson point process in $[0, \infty) \times [0, \infty)$ with characteristic measure $x^{-2} dx dt$. The time $t^* := \inf\{s \geq 0 : \overline{\varepsilon}_s \geq 1\}$ has exponential distribution with mean one, and $\{(\overline{\varepsilon}_t, t) : t \leq t^*\}$ has the same law as the restriction in $[0, 1] \times [0, \tau]$ of a Poisson point process in $[0, 1] \times [0, \infty)$ with characteristic measure $dn := x^{-2} dx dt$, where τ is an exponential random variable independent of the process. Let N be the counting measure of that process. Also for all $t > 0$, let $A_t := \{(y, s) : s \in [0, t] \text{ and } y > x + s\}$. Then $n(A_t) =$

$\int_0^{\min\{t,1-x\}}\int_{s+x}^1 y^{-2}dyds$, which equals $\log(1+t/x)-t$ if $t<1-x$, and $-\log x-1+x$ otherwise. Then

$$\mathbb{P}(\overline{W}(\beta_1^+)<x)=\mathbb{P}(\text{ for all } s<t^*, \text{ it holds } \overline{\varepsilon}_s - s<x)=\int_0^\infty e^{-t}P(N(A_t)=0)dt$$
$$=\int_0^\infty e^{-t}e^{-n(A_t)}dt=\int_0^{1-x}\frac{x}{x+t}dt+\int_{1-x}^\infty e^{-t}xe^{1-x}dt=-x\log x+x.$$

In particular, the density is $f(x)=-\log x$. To bound $\mathbb{P}(\overline{W}(\beta_1^+)<x)$, we observe that for $x\in(0,1]$, it holds $-x\log x+x<2\sqrt{x}$. To bound $\mathbb{P}(|\overline{W}(\beta_1^+)-\overline{W}(\beta_1^-)|<x)$, we use the fact that f is decreasing in $(0,1]$ and the bound we just established to get

$$\mathbb{P}(|\overline{W}(\beta_1^+)-\overline{W}(\beta_1^-)|<x)=\int_0^1 f(y)\mathbb{P}(\overline{W}(\beta_1^+)\in(y-x,y+x))dy\le\int_0^1 f(y)\mathbb{P}(\overline{W}(\beta_1^+)<2x)dy$$
$$<2\sqrt{2x}.$$

■

Lemma 9. *There is a constant c so that for all $x>0$, it holds*

$$\mathbb{P}(\tilde{\zeta}_1\in[-x,0])\le cx^{1/4}.$$

Proof. Recall the definition of $\tilde{\zeta}_1$ (given in the proof of Lemma 2) and the other definitions in the beginning of this section. Also let $\rho_{W^-}(c)=\inf\{t>0: W^-(t)=c\}$. Then

$$\mathbb{P}(\tilde{\zeta}_1\in[-x,0])\le\mathbb{P}(\rho_{W^-}(\overline{W}(\beta_1^+))<x)=-\int_0^1\mathbb{P}(\rho_{W^-}(s)<x)\log sds$$
$$=-\int_0^1\mathbb{P}(\rho_{W^-}(1)<x/s^2)\log sds=\int_1^\infty\frac{\log y}{y^2}\mathbb{P}(\rho_{W^-}(1)<xy^2)dy$$
$$<\int_1^{x^{-1/2}}\frac{\log y}{y^2}(xy^2)^{1/2}dy+\int_{x^{-1/2}}^\infty\frac{\log y}{y^2}dy=\sqrt{x}\int_1^{x^{-1/2}}\frac{\log y}{y}dy+\sqrt{x}(1-\frac{\log x}{2})$$

We used Lemma 8 for the density of $\overline{W}(\beta_1^+)$, and Fact 3. The last quantity is easily shown to have bound of the form $cx^{1/4}$. ■

The next lemma says that, with high probability, the points $\tilde{z}_r,\hat{\zeta}_r,m_r,z$ are as we depict them in Figures 3, 4. Parts 1 and 2 should be used when one proves the versions of Lemma 7 needed in the proof of Lemma 2. Part 3 is used in the proof of Lemma 3.

Lemma 10. *There is a constant c, depending on k_2, so that for all $r>0$ it holds*

(i) $\mathbb{P}(b_{r^{++}}>\tilde{\zeta}_r)<cr^{-1/2}$,
(ii) $\mathbb{P}(\min_{s\in[\zeta_r,\hat{\zeta}_r]}W(s)<W(b_r))<cr^{-1/2}$,
(iii) $\mathbb{P}(z<m_r)<cr^{-1/2}$.

Proof. 1. The probability of interest is bounded by twice the following probability (since b_r will be either β_r^+ or β_r^-).

$$\mathbb{P}(W^- \text{ starting from } \overline{W}(\beta_r^+) \text{ hits } W(\beta_r^+) \text{ before } \overline{W}(\beta_r^+)+k_2\log(\overline{W}(\beta_r^+)-W(\beta_r^+)))$$
$$=\mathbb{E}\Big(\frac{2k_2\log(\overline{W}(\beta_r^+)-W(\beta_r^+))}{\overline{W}(\beta_r^+)-W(\beta_r^+)}\Big)<4k_2\mathbb{E}\Big(\big(\overline{W}(\beta_r^+)-W(\beta_r^+)\big)^{-1/2}\Big)<\frac{4k_2}{\sqrt{r}}\mathbb{E}\Big(\big(\overline{W}(\beta_1^+)\big)^{-1/2}\Big).$$

It follows from Lemma 8 that the last expectation is finite.

2. Let $T_r^+ := \inf\{s > 0 : W(s) - \underline{W}(s) \geq \max\{r, \overline{W}(s) - \underline{W}(s)\}\}$. This is a stopping time. Introduce B a standard Brownian motion independent of W, and denote by P_B its law. The probability in question is bounded by

$$\begin{aligned}
&\mathbb{E}(\mathbb{P}_B(B \text{ hits first } -W^{\#}(b_r,0) \text{ and then } 3k_2 \log W^{\#}(b_r,0) | B(0)=0)) \\
&\quad = \mathbb{E}\Big(\frac{3k_2 \log W^{\#}(b_r,0)}{W^{\#}(b_r,0) + 3k_2 \log W^{\#}(b_r,0)}\Big) < \frac{6k_2}{\sqrt{r}} \mathbb{E}(W^{\#}(b_1,0)^{-1/2}) < \frac{12k_2}{\sqrt{r}} \mathbb{E}(\overline{W}(\beta_1^+)^{-1/2})
\end{aligned}$$

The last expectation is finite as we mentioned above.

3. Let $Y := W(b_1) - W(b_{1^{++}})/W^{\#}(b_1, b_{1^{++}})$. From Lemma 5, Y is an exponential with mean 1. So that

$$\begin{aligned}
\mathbb{P}(z < m_r) &\leq \mathbb{P}(W(b_r) - W(b_{r^{++}}) \leq k_2 \log W^{\#}(b_r, b_{r^{++}}) + k_2 \log r) \\
&\leq \mathbb{P}\Big(W(b_1) - W(b_{1^{++}}) \leq \frac{k_2 \log(r^2 W^{\#}(b_1, b_{1^{++}}))}{r}\Big) \leq \mathbb{P}(W^{\#}(b_1, b_{1^{++}}) > r) \\
&+ \mathbb{P}\Big(W^{\#}(b_1, b_{1^{++}})\, Y \leq \frac{k_2 \log(r^3)}{r}\Big) \leq \frac{2}{r} + \mathbb{P}\Big(Y \leq \frac{3k_2 \log r}{r}\Big) < \frac{2}{r} + \frac{3k_2 \log r}{r}.
\end{aligned}$$

We used part 2 of Lemma 5 and the fact that $W^{\#}(b_1, b_{1^{++}}) \geq 1$. ∎

3.2 Exit from certain intervals

Lemma 11. *With the notation as in Lemmas 2, 3, there is a constant C so that for all $r > 0$ it holds*

$$\mathcal{P}([\tau(\zeta_r) < \tau(\tilde{\zeta}_r) < \tau(\hat{\zeta}_r) < \tau(\eta_r)]^c) < C r^{-1/4},$$
$$\mathcal{P}([(X_{\tau(\eta_r)+s})_{s \geq 0} \text{ hits } \zeta_{r^{++}} \text{ before } z]^c) < C r^{-1/4}.$$

Proof. The first probability is bounded by

$$\begin{aligned}
&\mathcal{P}(\tau(\tilde{\zeta}_r) < \tau(\zeta_r)) + \mathcal{P}(\tau(\hat{\zeta}_r) < \tau(\tilde{\zeta}_r)) + \mathcal{P}(\tau(\eta_r) < \tau(\hat{\zeta}_r)) \\
&\qquad = \mathbb{E}\Big(\frac{A(\zeta_r)}{A(\zeta_r) + |A(\tilde{\zeta}_r)|}\Big) + \mathbb{E}\Big(\frac{|A(\tilde{\zeta}_r)|}{|A(\tilde{\zeta}_r)| + A(\hat{\zeta}_r)}\Big) + \mathbb{E}\Big(\frac{A(\hat{\zeta}_r)}{A(\hat{\zeta}_r) + |A(\eta_r)|}\Big)
\end{aligned}$$

We work as in part 3 of Lemma 7. Let A_5 be defined as there, and

$$A_8 := \left\{ \begin{array}{c} \zeta_r < 1 \text{ or } \tilde{\zeta}_r \in (-1, 0] \text{ or } \hat{\zeta}_r > r^4 \text{ or } |\eta_r| > r^4 \text{ or } |\zeta_r| > r^4 \\ \text{or } W^{\#}(b_r, 0) < \sqrt{r} \text{ or } |\overline{W}(\beta_r^+) - \overline{W}(\beta_r^-)| < \sqrt{r} \end{array} \right\}.$$

Then $\mathbb{P}(\zeta_r < 1 \text{ or } \tilde{\zeta}_r \in (-1,0]) \leq \mathbb{P}(x_1(W,r) < 1) + \mathbb{P}(\tilde{\zeta}_r \in (-1,0]) < r^{-2} + c r^{-1/2}$ (using Lemmas 6, 9), $\mathbb{P}(\hat{\zeta}_r > r^4 \text{ or } |\eta_r| > r^4 \text{ or } |\zeta_r| > r^4) < 26\, r^{-1/4}$ (part 1 of Lemma 7), $\mathbb{P}(W^{\#}(b_r,0) < \sqrt{r}) < 2r^{-1/2}$ (part 3 of Lemma 5), $\mathbb{P}(|\overline{W}(\beta_r^+) - \overline{W}(\beta_r^-)| < \sqrt{r}) < 3\, r^{-1/4}$ (Lemma 8). So that $\mathbb{P}(A_8) < C\, r^{-1/4}$ for a constant C (independent of r).
On $(A_8 \cup A_5)^c$ the quantities $|A(\tilde{\zeta}_r)|/A(\zeta_r)$, $A(\hat{\zeta}_r)/|A(\tilde{\zeta}_r)|$,$|A(\eta_r)|/A(\hat{\zeta}_r)$ are large. Indeed

$$\frac{|A(\tilde{\zeta}_r)|}{A(\zeta_r)} = \frac{\int_{\tilde{\zeta}_r}^0 e^{W(y)-W(b_r)} dy}{\int_0^{\zeta_r} e^{W(y)-W(b_r)} dy} \geq \frac{\exp(W^{\#}(b_r,0) + 2k_2 \log W^{\#}(b_r,0) - \log r)}{\zeta_r \exp(W^{\#}(b_{r^-}, b_r) + k_2 \log W^{\#}(b_{r^-}, b_r))} \geq \frac{(W^{\#}(b_r,0))^{k_2}}{r^5} \geq r^{k_2/2-5},$$

$$\frac{A(\hat{\zeta}_r)}{|A(\tilde{\zeta}_r)|} = \frac{\int_0^{\hat{\zeta}_r} e^{W(y)-W(b_r)} dy}{\int_{\tilde{\zeta}_r}^0 e^{W(y)-W(b_r)} dy} \geq \frac{\exp(W^{\#}(b_r,0) + 3k_2 \log W^{\#}(b_r,0) - \log r)}{|\tilde{\zeta}_r| \exp(W^{\#}(b_r,0) + 2k_2 \log W^{\#}(b_r,0))} \geq \frac{(W^{\#}(b_r,0))^{k_2}}{r^5} \geq r^{k_2/2-5},$$

$$\frac{|A(\eta_r)|}{A(\hat{\zeta}_r)} = \frac{\int_{\eta_r}^0 e^{W(y)-W(b_r)} dy}{\int_0^{\hat{\zeta}_r} e^{W(y)-W(b_r)} dy} \geq \frac{\exp(W^{\#}(b_r, b_{r^+}) - \log r)}{\hat{\zeta}_r \exp(W^{\#}(b_r,0) + 3k_2 \log W^{\#}(b_r,0))} = \frac{\exp(W^{\#}(b_r, b_{r^+}) - W^{\#}(b_r,0))}{\hat{\zeta}_r r (W^{\#}(b_r,0))^{3k_2}}.$$

In the first line, we used the fact that $W^{\#}(b_{r^-}, b_r) \le W^{\#}(b_r, 0)$. Regarding the last quantity of the third line, observe that $W^{\#}(b_r, b_{r^+}) - W^{\#}(b_r, 0) \ge |\overline{W}(\beta_r^+) - \overline{W}(\beta_r^-)| \ge \sqrt{r}$, and $\hat{\zeta}_r r (W^{\#}(b_r, 0))^{3k_2} \le r^{5+3k_2/2}$. So that $|A(\eta_r)|/A(\hat{\zeta}_r) \ge e^{\sqrt{r}} r^{-5-3k_2/2} > r^2$ for large r.
Finally,

$$\mathcal{P}([(X_{\tau(\eta_r)+s})_{s\ge 0} \text{ hits } \zeta_{r^+ +} \text{ before } z]^c) = \mathbb{E}\Big(\frac{A(\zeta_{r^+ +}) - A(\eta_r)}{A(\zeta_{r^+ +}) - A(z)}\Big).$$

The quantity in the expectation is always at most one. And there is a constant c so that for all r, the set $A_5 \cup [\zeta_{r^+ +} < 1 \text{ or } \zeta_{r^+ +} > r^4]$ has probability at most $26\, r^{-1/4} + cr^{-2}$ (because of $0 < x_1(W, r) < \zeta_{r^+ +}$ and Lemma 6, part 1 of Lemma 7, and the bound on $\mathbb{P}(A_5)$ given in the proof of the same lemma). On the complement of that set, it holds

$$\frac{A(\zeta_{r^+ +}) - A(\eta_r)}{A(\zeta_{r^+ +}) - A(z)} = \frac{\int_{\eta_r}^{\zeta_{r^+ +}} e^{W(y) - W(b_{r^+ +})} dy}{\int_z^{\zeta_{r^+ +}} e^{W(y) - W(b_{r^+ +})} dy} \le \frac{\zeta_{r^+} \exp(W(\zeta_{r^+ +}) - W(b_{r^+ +}))}{\exp(W(\zeta_{r^+ +}) - W(b_{r^+ +}) + k_2 \log r - \log r)}$$
$$= \zeta_{r^+ +}/r^{k_2 - 1} < r^{5-k_2} < r^{-2}.$$

This finishes the proof. ■

3.3 The jumps of b, and the intervals around its values

Let $(R_i)_{i\ge 1}$ be the increasing sequence of points where b jumps in $[1, \infty)$. The next lemma gives a measure of how rare these points are as we approach infinity. It is the result that makes possible to move from Lemmas 2, 3 to the theorem by gluing all the time intervals for which the two lemmas give information.

Lemma 12. *Let $a \in (0, 1/2)$. With probability one, ultimately between any two terms from $(R_i)_{i\ge 1}$ there is at least one term from the sequence $(\exp(k^a))_{k\ge 1}$.*

Proof. There are four cases for the signs of the pair $\{b_{R_i+}, b_{R_{i+1}+}\}$. First we show that the sequence $(\exp(k^a))_{k\ge 1}$ enters eventually in the intervals (R_i, R_{i+1}) with $b_{R_i+}, b_{R_{i+1}+} > 0$ (similarly if $b_{R_i+}, b_{R_{i+1}+} < 0$).

The indices i with $b_{R_i+}, b_{R_{i+1}+} > 0$: The process $(\beta_r^+)_{r>0}$ (defined in the beginning of this section) takes only positive values, and it is increasing. The points where b jumps from a positive to a positive value are contained in the points where β^+ jumps. So we will prove our claim for the process β^+. The points where β^+ jumps in $[1, \infty)$ make up an increasing sequence $(h_n)_{n\ge 0}$ with h_0 the first such point, and $h_{n+1} := (1 + r_n)\ h_n$ for $n \ge 0$, where the r_n's are i.i.d. with density $(1 + x)^{-2}\mathbf{1}_{x\ge 0}$ (It is the same idea as in the proof of part 1 of Lemma 7). We note that $\log(1 + r_n)$ is exponential random variable with mean one.
For $n \ge 1$ there is a unique k_n so that $c(k_n) < h_n \le c(k_{n+1})$, i.e.,

$$k_n^a < \log h_n \le (k_n + 1)^a. \tag{22}$$

We want to prove that eventually $\log h_{n+1} > (k_n + 1)^a$. It is enough to prove that $\log h_{n+1} - \log h_n > (k_n + 1)^a - k_n^a$. The last quantity is less than $a k_n^{a-1}$ (we use the fact that $a < 1$ and the mean value theorem). Also $\log h_{n+1} - \log h_n = \log(1 + r_n)$, and by the first Borel-Cantelli, we have eventually $\log(1 + r_n) > (n \log^2 n)^{-1}$. So that a.s. eventually

$$\frac{(k_n + 1)^a - k_n^a}{\log h_{n+1} - \log h_n} < a k_n^{a-1} n \log^2 n < a((\log h_n)^{1/a} - 1)^{a-1} n \log^2 n.$$

In the second inequality, we used (22) and $a - 1 < 0$. Since $\log(1 + r_i)$ is exponential with mean one, we have $\log h_n \approx n$ (for the rigorous argument, we use the strong law of large numbers to say that $\log h_n > n/2$ eventually). So that the above bound is of the order $n^{2-1/a}\log^2 n$ which goes to zero as $n \to \infty$ provided that $a < 1/2$.

The indices i with $b_{R_i+}b_{R_{i+1}+} < 0$: For $\ell \in \mathbb{R}$, recall the definitions of H_ℓ^-, H_ℓ^+ given in the proof of Lemma 5, and moreover define $\Theta_\ell := -\min\{W_s : s \in [H_\ell^-, H_\ell^+]\}$. Let $(\ell_n)_{n\geq 1}$ be the strictly increasing sequence consisting exactly of the points in $[1, \infty)$ where Θ jumps, and $\ell_0 := 1$. At every "time" ℓ, we observe $A_\ell := W|[H_\ell^-, H_\ell^+]$. We call this a well, and $D_\ell := \ell + \Theta_\ell$ its depth. As ℓ increases, in the picture A_ℓ, excursions of $\overline{W^+} - W^+$ and $\overline{W^-} - W^-$ are introduced (on the right and the left respectively). And Θ_l jumps at ℓ if, just after ℓ, an excursion is added that has height strictly greater than D_ℓ. Let $\zeta_0 := 1$, and for $n \geq 0$, let $\zeta_{2n+1} := D_{\ell_n+}, \zeta_{2n+2} := D_{\ell_{n+1}}$, $\sigma_n := \zeta_{2n+1}/\zeta_{2n}, \tau_n := \zeta_{2n+2}/\zeta_{2n+1}$. So that $\zeta_{2n+2} = \prod_{i=0}^n \sigma_i\tau_i$ and $\zeta_{2n+1} = \sigma_n\zeta_{2n}$ for $n \geq 0$. It can be shown that $\{\sigma_n : n \geq 1\}$ are i.i.d. with density $x^{-2}\mathbf{1}_{x\geq 1}$, and $\sigma_0, \{\tau_n : n \geq 0\}$ are i.i.d. with density $2x^{-3}\mathbf{1}_{x\geq 1}$ (similar arguments as in the proof of part 1 of Lemma 7). If i is such that $b_{R_i+}b_{R_{i+1}+} < 0$, then there is a $n \in \mathbb{N}$ with $R_i < \zeta_{2n+1} < \zeta_{2n+2} = R_{i+1}$ (a picture can convince the reader that indeed this is the case). As before, we prove that, a.s. eventually, between $\zeta_{2n+1}, \zeta_{2n+2}$ there is a term from the sequence $(\exp(k^a))_{k\geq 1}$. ■

The interval (α_r, γ_r) has been defined in terms of r. For our theorem, we need to get control on its size in terms of the distance of its center from zero (i.e., $|b_r|$. It makes more pleasing the statement of the theorem). This is accomplished in the next lemma.

Lemma 13. *Let $c > 6$ be fixed. With probability one, there is a random $r_0 > 1$ so that*

$$(\alpha_r, \gamma_r) \subset (b_r - (\log b_r)^c, b_r + (\log b_r)^c)$$

for all $r > r_0$.

Proof. First we will show that with probability one we have

$$(\alpha_r, \gamma_r) \subset (b_r - (\log r)^c, b_r + (\log r)^c) \tag{23}$$

for all big r.
Define

$$\begin{aligned}
J_r^+(W) &:= \inf\{t > 0 : W(t) - \underline{W}(t) = r\}, \\
\beta_r^+(W) &:= \inf\{t > 0 : W(t) = \underline{W}(J_r^+(W))\}, \\
\alpha_r^+(W) &:= \inf\{t > 0 : W(t) - W(\beta_r^+(W)) < 2k_1 \log r\}, \\
\gamma_r^+(W) &:= \sup\{t < J_r^+(W) : W(t) - W(\beta_r^+(W)) < 2k_1 \log r\}.
\end{aligned}$$

The set whose infimum is $\alpha_r^+(W)$ could be empty, in which case $\alpha_r^+(W) = \infty$. We show in (24) below that this is not the case for large r.
Then consider the process $(W^-(s))_{s\in\mathbb{R}}$ defined by $W^-(s) = W(-s)$ for $s \in \mathbb{R}$, and set

$$\begin{aligned}
\beta_r^-(W) &:= -\beta_r^+(W^-), \\
\alpha_r^-(W) &:= -\alpha_r^+(W^-), \\
\gamma_r^-(W) &:= -\gamma_r^+(W^-).
\end{aligned}$$

In the following, we will omit the argument W in the above functionals.
Note that $-W(\beta_r^+), -W(\beta_r^-)$ are i.i.d with density $r^{-1}e^{-x/r}1_{x>0}$ (i.e., exponential with mean r, see Fact 1).

Observation 1: $-W(\beta_r^-), -W(\beta_r^+)$ are not small.

It holds

$$\mathbb{P}(-W(\beta_r^+) \le 4k_1 \log r) < \frac{4k_1 \log r}{r}$$

as $-W(\beta_r^+)$ has density bounded by $1/r$. Now using interpolation and the monotonicity of $(r \mapsto -W(\beta_r^+))$, we can show that a.s.

$$-W(\beta_r^+) > 2k_1 \log r \text{ for all large } r. \tag{24}$$

[We present a similar interpolation argument in detail below in the proof of (26), which is trickier. So we skip the details here.] The same holds for $-W(\beta_r^-)$. And these guarantee that α_r^+, α_r^- are finite for all large r.

Observation 2: The neighborhood (α_r^+, γ_r^+) of β_r^+ is small.

Here we will give a bound on the probability that $\beta_r^+ - \alpha_r^+, \gamma_r^+ - \beta_r^+$ are large. Note first that the process

$$(W(J_r^+) - W(J_r^+ - t))_{0 \le t \le J_r^+ - \beta_r^+}$$

is a three-dimensional Bessel processes starting from zero and killed when hitting r. This follows from the proof of the lemma in §1 of Neveu and Pitman (1989), the structure of Brownian excursions (see Revuz and Yor (1999), Chapter XII, Theorem 4.5), and Proposition 4.8, Chapter VII, of Revuz and Yor (1999). Thus, $\gamma_r^+ - \beta_r^+$ has distribution $\tau_r^{r-2k_1 \log r}(Y)$. Where by $\tau_c^y(Z)$ we denote the first time that a continuous process $(Z_s)_{s \ge 0}$ with $Z_0 = y$ hits c, and Y is a three dimensional Bessel process. Since Y satisfies the stochastic differential equation $dY_s = dw_s + Y_s^{-1} ds$ (w is standard Brownian motion), we have for $y, d > 0$, that $\tau_{y+d}^y(Y) \le \tau_{y+d}^y(w)$. And consequently

$$\mathbb{P}(\tau_{y+d}^y(Y) > z) \le \mathbb{P}(\tau_{y+d}^y(w) > z) = \mathbb{P}(\tau_d^0(w) > z) = \mathbb{P}(\tau_1^0(w) > z/d^2) < d/\sqrt{z}.$$

For the last inequality, we used Fact 3. Thus

$$\mathbb{P}(\gamma_r^+ - \beta_r^+ > z) \le \mathbb{P}(\tau_{2k_1 \log r}^0(w) > z) < \frac{2k_1 \log r}{\sqrt{z}}. \tag{25}$$

We will show the same bound for $\beta_r^+ - \alpha_r^+$. The Poisson process of excursions away from zero for $(W_t - \underline{W}_t)_{0 \le t \le \beta_r^+}$ has characteristic measure $n(\,\cdot \cap \{e : h(e) < 1\})$ where n is the characteristic measure for the Poisson process of excursions for $(W_t - \underline{W}_t)_{t \ge 0}$. For an excursion e, we will denote by $\ell(e)$ its lifetime, and by $h(e)$ its height. If $(w_t)_{t \ge 0}$ is Brownian motion with $(e_t)_{t>0}$ the process of excursions for $(w_t - \underline{w}_t)_{t \ge 0}$ (parametrized with the inverse local time corresponding to the local time process $(-\underline{w})_{t>0}$) , then , on the event $[\alpha_r^+ < \infty]$, the length $\beta_r^+ - \alpha_r^+$ equals in law to

$$\sum_{0 \le t \le 2k_1 \log r} \ell(e_t) 1_{h(e_t) < 1}.$$

This is less than $\sum_{t \le 2k_1 \log r} \ell(e_t)$ which equals $\tau_{-2k_1 \log r}^0$. Thus, we get as before

$$\mathbb{P}([\beta_r^+ - \alpha_r^+ > z] \cap [a_r^+ < \infty]) < \frac{2k_1 \log r}{\sqrt{z}}.$$

Take $r = r_n = \exp(n^a), z = (\log r_n)^c/2$ where $c > 2$, in (25) to get

$$\mathbb{P}(\gamma_{r_n}^+ - \beta_{r_n}^+ > (\log r_n)^c/2) < n^{-a(c-2)/2} 4k_1.$$

Of course, $\mathbb{P}([\beta_{r_n}^+ - \alpha_{r_n}^+ > (\log r_n)^c/2] \cap [a_{r_n}^+ < \infty])$ has the same bound. For any $c > 6$, there is an $a \in (0, 1/2)$ with $-a(c-2)/2 < -1$. For this choice of a, it holds $\sum_{n=0}^\infty n^{-a(c-2)/2} < \infty$.

Thus a.s. eventually we have $\beta^+_{r_n} - \alpha^+_{r_n} < (\log r_n)^c/2, \gamma^+_{r_n} - \beta^+_{r_n} < (\log r_n)^c/2$. We use here and in the following that $a^+_r < \infty$ for large r, because of (24).

Now take an $r > 0$ large. There is a unique n so that $r_n < r \le r_{n+1}$. Then $\beta^+_r = \beta^+_{r_n}$ or $\beta^+_r = \beta^+_{r_{n+1}}$ because in the interval $(r_n, r_{n+1}]$ there is at most one jump for β^+_r (this is included in the proof of Lemma 12). If $\beta^+_r = \beta^+_{r_n}$, then $k_1 \log r < k_1 \log r_{n+1} = k_1(n+1)^a = (1+n^{-1})^a k_1 \log r_n < 2k_1 \log r_n$. So $\beta^+_r - \alpha^+_r < \beta^+_{r_n} - \alpha^+_{r_n} < (\log r_n)^c < (\log r)^c$, and similarly for $\gamma^+_r - \beta^+_r$.
If $\beta^+_r = \beta^+_{r_{n+1}}$, then

$$\beta^+_r - \alpha^+_r \le \beta^+_{r_{n+1}} - \alpha^+_{r_{n+1}} < \frac{1}{2}(\log r_{n+1})^c = \frac{(n+1)^{ac}}{2} = \frac{(1+n^{-1})^{ac}}{2}(\log r_n)^c < (\log r)^c$$

for large n. And similarly for $\gamma^+_r - \beta^+_r$. We do the same on the negative side with $\alpha^-_r - \beta^-_r$, $\beta^-_r - \gamma^-_r$. So that, a.s. for all large r

$$\max\{\beta^+_r - \alpha^+_r, \gamma^+_r - \beta^+_r, \alpha^-_r - \beta^-_r, \beta^-_r - \gamma^-_r\} < (\log r)^c \tag{26}$$

Observation 3: $W(\beta^-_r), W(\beta^+_r)$ are not close.

We claim that a.s. for all big r, we have $|W(\beta^+_r) - W(\beta^-_r)| \ge 3k_1 \log$. Indeed

$$\mathbb{P}(|W(\beta^+_r) - W(\beta^-_r)| \le 6k_1 \log r) = \mathbb{P}\Big(|\frac{W(\beta^+_r)}{r} - \frac{W(\beta^-_r)}{r}| < \frac{4k_1 \log r}{r}\Big) \le \frac{8k_1 \log r}{r}$$

because $-W(\beta^+_r)/r, -W(\beta^-_r)/r$ are i.i.d. exponentials with mean 1, so that their density is bounded above by 1. And since $\sum_{n=0}^\infty (\log r_n)/r_n < \infty$, it follows that a.s., for big n, we have $|W(\beta^+_{r_n}) - W(\beta^-_{r_n})| > 6k_1 \log r_n$. Similarly we show that a.s. for big n it holds $|W(\beta^+_{r_n}) - W(\beta^-_{r_{n+1}})| > 6k_1 \log r_n$, $|W(\beta^+_{r_{n+1}}) - W(\beta^-_{r_n})| > 6k_1 \log r_n$. And with similar arguments as above, we show that

$$|W(\beta^+_r) - W(\beta^-_r)| > 3k_1 \log r \text{ for all big } r\text{, a.s.} \tag{27}$$

Assume that $b_r = \beta^+_r$. Then clearly $\gamma_r = \gamma^+_r$, and we claim that also $\alpha_r = \alpha^+_r$ (recall the definition of α_r in (13)). The only way this can fail is if $W(\beta^-_r) < W(\beta^+_r)$ and $W^\#(\beta^+_r, \beta^-_r) < r$ (Otherwise (27) combined with $W(\beta^-_r) > W(\beta^+_r)$ or $W^\#(\beta^+_r, \beta^-_r) \ge r$ gives that $\alpha_r = \alpha^+_r$). But then $b_r = \beta^-_r$. So that (23) follows from (26).

To finish the proof of the lemma, it is enough to show that with probability 1, it holds

$$\log|b_r| > \log r \text{ for big } r. \tag{28}$$

Since for all r, it is $b_r \in \{\beta^+_r, \beta^-_r\}$, we will show this for β^+_r. First we note the following.

Claim: There is a constant C so that $\mathbb{P}(\beta^+_1 < x) \le C\sqrt{x}$ for all $x > 0$.

The claim needs a proof only for small x. The Laplace transform of β^+_1 is $(\sqrt{2\lambda}\coth\sqrt{2\lambda})^{-1}$, i.e., of the form $\lambda^{-1/2}L(\lambda)$ with L slowly varying function at ∞ (see Neveu and Pitman (1989), Lemma of §1). By a Tauberian theorem (Theorem 3 of §XIII.5 in Feller (1971)), it follows that $\mathbb{P}(b^+_1 < x) \sim x^{1/2}L(1/x)/\Gamma(3/2)$ for small x.

Now to show the analog of (28) for β^+_r, it is enough to show that, with probability 1,

$$\beta^+_r > \frac{r^2}{\log^4 r} \text{ for all big } r. \tag{29}$$

We will use again an interpolation argument. This time, the sequence $r_n = e^n$ for $n \geq 1$ is enough. Observe that, because of the above claim and scaling,

$$\mathbb{P}\Big(\beta_{r_n}^+ < \frac{r_n^2}{\log^3 r_n}\Big) = \mathbb{P}\Big(\beta_1^+ < \frac{1}{\log^3 r_n}\Big) < Cn^{-3/2}.$$

The first Borel-Cantelli lemma implies that, with probability one, $\beta_{r_n}^+ > r_n^2/\log^3 r_n$ for all big n. Now for $r > e$ there is a unique n such that $r_n < r \leq r_{n+1}$, and since $\beta_r^+ \geq \beta_{r_n}^+$, we get

$$\beta_r^+ \frac{\log^4 r}{r^2} \geq \beta_{r_n}^+ \frac{\log^3 r_n}{r_n^2} \frac{r_n^2}{r^2} \log r \geq \beta_{r_n}^+ \frac{\log^3 r_n}{r_n^2} \frac{r_n^2}{r_{n+1}^2} \log r = \beta_{r_n}^+ \frac{\log^3 r_n}{r_n^2} \frac{\log r}{e^2}.$$

With probability one, the last quantity is greater than $\log r/e^2$ for big r, which is greater than one for $r > \exp(e^2)$. This proves (29).

■

References

Andreoletti, Pierre (2006). Sinai's walk: a statistical aspect, Preprint. Available at arXiv.org, math.PR/0612209.

Brox, Th. (1986). A one-dimensional diffusion process in a Wiener medium, Ann. Probab. **14**, no. 4: 1206–1218.

Cheliotis, Dimitrios (2005). Diffusion in random environment and the renewal theorem, Ann. Probab. **33**, no. 5: 1760–1781.

Csörgö, M. and Révész, P. (1981). Strong approximations in probability and statistics, Academic Press, New York.

Dembo, A., Gantert, N., Peres, Y., and Shi, Z. (2007). Valleys and the maximum local time for random walk in random environment, Prob. Th. Rel. Fields **137**, no. 3-4: 443–474.

Durrett, Richard (1996). Probability: Theory and examples, Wadsworth Pub. Co, second edn.

Feller, William (1971). An introduction to probability theory and its applications, vol. 2, John Wiley and Sons, Inc., New York-London-Sydney, second edn.

Golosov, A. O. (1984). Localization of random walks in one-dimensional random environments, Comm. Math. Phys. **92**, no. 4: 491–506.

Hu, Yueyun (2000). Tightness of localization and return time in random environment, Stochastic Process. Appl. **86**: 81–101.

Hu, Yueyun and Shi, Zhan (1998). The limits of sinai's simple random walk in random environment, Ann. Probab. **26**, no. 4: 1477–1521.

Hu, Yueyun and Shi, Zhan (2000). The problem of the most visited site in random environment, Probab. Theory Related Fields **116**, no. 2: 273–302.

Le Doussal, P., Monthus, C., and Fisher, D. (1999). Random walkers in one-dimensional random environments: Exact renormalization group analysis, Physical Review E **59**, no. 5: 4795–4840.

Neveu, J. and Pitman, J. (1989). Renewal property of the extrema and tree property of the excursion of a one-dimensional Brownian motion, in Séminaire de Probabilités XXIII, Lecture Notes in Math., vol. 1372, 239–247, Springer, Berlin.

Révész, Pál (1990). Random walk in random and nonrandom environments, World Scientific Publishing Co., Inc., Teaneck, NJ.

Revuz, D. and Yor, M. (1999). Continuous martingales and Brownian motion, Springer, Berlin, third edn.

Shi, Zhan (2001). Sinai's walk via stochastic calculus, in F. Comets and E. Pardoux, editors, Milieux Aléatoires, Panoramas et Synthèses, vol. 12, Société Mathématique de France.

Tanaka, Hiroshi (1988). Limit theorem for one-dimensional diffusion process in brownian environment, in Stochastic analysis (Paris, 1987), vol. 1322 of *Lecture Notes in Math.*, 156–172, Springer.

Zeitouni, Ofer (2004). Random walks in random environment, in Ecole d'Eté de Probabilités de Saint-Flour XXXI-2001, Lecture Notes in Math., vol. 1837, Springer, Berlin.