# Representations of crossed products of Hilbert C*-modules

Maria Joiţa

*University of Bucharest, Romania*
[mjoita@fmi.unibuc.ro]

Hilbert $C^*$-modules are generalizations of Hilbert spaces by allowing the inner product to take values in a $C^*$-algebra rather than in the field of complex numbers. They are useful tools in theory of operator algebras, operator $K$-theory, Morita equivalence of $C^*$-algebras, group representation theory, the $C^*$-algebraic theory of quantum groups. The theory of Hilbert $C^*$-modules is very interesting on its own right.

Hilbert $C^*$-modules can be also regarded as generalizations of $C^*$-algebras. A Hilbert $C^*$-module $X$ over a $C^*$-algebra $A$ can be embedded in its linking algebra $\mathcal{L}(X)$ (that is, the $C^*$-algebra of all 'compact operators' on the Hilbert $C^*$-module $A \oplus X$). Many concepts as the notion of representation of a $C^*$-algebra on a Hilbert space or the notion of $C^*$-crossed product were extended in the context of Hilbert $C^*$-modules.

A *morphism of Hilbert $C^*$-modules* is a map $\Phi : X \to Y$ from a Hilbert $A$-module $X$ to a Hilbert $B$-module $Y$ with the property that there is a $C^*$-morphism $\varphi : A \to B$ such that

$$\langle \Phi(x), \Phi(y) \rangle = \varphi(\langle x, y \rangle) \text{ for all } x, y \in X.$$

If $H$ and $K$ are Hilbert spaces, then $L(H, K)$, the vector space of all bounded linear operators from $H$ to $K$, is a Hilbert $L(H)$-module with:

$$L(H, K) \times L(H) \ni (T, S) \to TS \in L(H, K)$$

and

$$L(H, K) \times L(H, K) \ni (T_1, T_2) \to \langle T_1, T_2 \rangle = T_1^* T_2 \in L(H).$$

*A representation* of $X$ on the Hilbert spaces $H$ and $K$ is a morphism of Hilbert $C^*$-modules, $\pi_X : X \to L(H, K)$.

Let $G$ be a locally compact group, $X$ a full Hilbert $A$-module and

$$Aut(X) = \{\Phi : X \to X; \Phi \text{ is an isomorphism of Hilbert} C^*\text{-modules}\}.$$

An *action* of $G$ on $X$ is a group morphism $g \to \mu_g$ from $G$ to $Aut(X)$ such that the map $g \to \mu_g(x)$ from $G$ to $X$ is continuous for each $x \in X$. An action $\mu$ of $G$ on $X$ induces an action $\alpha^\mu$ of $G$ on $A$ given by

$$\alpha_g^\mu(\langle x, y \rangle) = \langle \mu_g(x), \mu_g(y) \rangle \text{ for all } g \in G, \text{ for all } x, y \in X.$$

The vector space $C_c(G, X)$ of all continuous functions from $G$ to $X$ with compact support has a structure of pre-Hilbert $C^*$-module over $G \times_{\alpha^\mu} A$, with:

$$(\widehat{x} f)(s) = \int_G \widehat{x}(t) \alpha_t^\mu \left(f(t^{-1}s)\right) dt, \text{ for all } \widehat{x} \in C_c(G, X) \text{ and } f \in C_c(G, A)$$

and

$$\langle \widehat{x}, \widehat{y} \rangle (s) = \int\limits_G \langle \mu_{t^{-1}} (\widehat{x} (t)) , \mu_{t^{-1}} (\widehat{y} (ts)) \rangle \, dt, \text{ for all } \widehat{x}, \widehat{y} \in C_c(G, X).$$

The Hilbert $C^*$-module obtained by the completion of $C_c(G, X)$ with respect to the topology induced by the inner product is called *the crossed product* of $X$ by $\mu$ and it is denoted by $G \times_\mu X$.

Crossed product of Hilbert $C^*$-modules appears as imprimitivity bimodule. Suppose that $(G, \alpha, A)$ and $(G, \beta, B)$ are two $C^*$-dynamical systems such that the $C^*$-algebras $A$ and $B$ are Morita equivalent (this is, there is a full Hilbert $A$-module $X$ such that the $C^*$-algebras $K(X)$ and $B$ are isomorphic). If there is an $(\alpha, \beta)$-compatible action $\mu$ of $G$ on $X$, then the $C^*$-crossed products $G \times_\alpha A$ and $G \times_\beta B$ are Morita equivalent, and the Hilbert $C^*$-module which gives a Morita equivalence between $G \times_\alpha A$ and $G \times_\beta B$ is the crossed product of $X$ by $\mu$.

In this talk, after a review of some properties of Hilbert $C^*$-modules, we will discuss about representations of crossed product of Hilbert $C^*$-modules.

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