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# RESEARCHING THE TEACHING AND LEARNING OF MATHEMATICS II



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Prof. Dr. Brian Hudson
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# RESEARCHING THE TEACHING AND LEARNING OF MATHEMATICS II

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## INTRODUCTION

This is the second volume of papers arising from the MATHED Intensive Programmes Researching the Teaching and Learning of Mathematics. These papers arise from seminars and workshops that took place during the summers of 2004 and 2005. The second MATHED Intensive Programme was held in the Tolmin valley in Slovenia from 11th to 22nd August 2004, hosted by the University of Ljubljana. The third event was held in Vijlandi in Estonia from 3rd to 13th July 2005, hosted by Tallinn University.

The partner institutions taking part were Sheffield Hallam University (co-ordinator), Pädagogische Hochschule, Linz; Charles University of Prague; University of Helsinki, University of Riga, Oslo University College, Oxford Brookes University, Tallinn University and Aristotle University of Thessaloniki. The MATHED Intensive Programmes have been closely associated with the Socrates-Erasmus EUDORA Project.

The intensive programmes enabled participants to examine issues of policy and practice relevant to their work in mathematics education within an international context. The programmes aimed to enable participants to develop a critical focus on the nature of research into the teaching and learning of mathematics in an international context







and to develop theoretical approaches and methods appropriate to comparative research. This was intended to support the further development on the part of the participants of their understandings of the methodological complexities of research in mathematics education, to identify current themes in mathematics education and critically analyse their significance and to relate these outcomes to their own contexts.

The content of the intensive programme was based on the research interests of the participants and involved active participation by students and collaboration with peers from the outset. Each student led a workshop and/or seminar based on their current work and interests. In addition staff members led workshops on current issues in mathematics education research.

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# THE MATHEMATICS ACHIEVEMENT OF SLOVENE STUDENTS, BASED ON TIMSS DATA

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***Abstract***
*In the last few decades, student achievement has played a central role among the indicators used to evaluate the quality of education systems. Accordingly, recent education reform in Slovenia included the achievement of international standards of knowledge and skills as an important goal. This chapter examines to what extent this goal has been achieved in the non-reformed system. It describes the mathematics achievement of students in the final grade of the non-reformed compulsory education in Slovenia based on the data from the Third International Mathematics and Science Study (TIMSS). TIMSS data are also used to provide information about achievements of students from other European countries, which are taken as a point of reference for describing the Slovene achievement. Another point of reference is derived from the attainment targets in the reformed mathematics curriculum. The goal of these comparisons is to provide information to support the efforts to successfully implement the reforms.*

**Keywords:** Achievement, curriculum reform, TIMMS

**Introduction**
The political, social, and economic changes that occurred after 1991 urged Slovenia to reform its education system. These reforms encompassed the structure of the school system as well as the curricula of all school subjects. The goals for reformed education





explicitly stated that Slovene education: '… makes possible the achievement of internationally comparable standards of knowledge after the completion of the primary education …' (White Paper, 1996, 92)[1]. The reform is currently being implemented into the school system through a 10-year process of stepwise transformations and it is expected to be completed in the 2008/2009 school year.

Within this context the aim of curriculum development in Slovenia was to reduce curriculum content and increase integration among different school subjects with more emphasis on inter-disciplinary and conceptual knowledge. Throughout the political debate surrounding the reform in Slovenia, comparisons with other countries, and especially with those from the European Union, were emphasised. Given contemporary worldwide attention to achievement, among the important questions to be answered were: 'How well do Slovene students perform in comparison with students from other countries? Do they reach expected levels of achievement? What should be expected of the students?' The answers to these questions were sought through opinions of experts, experiences of teachers and others involved in education and, where available, data from empirical studies.

As part of the curriculum reform the attainment targets, also called the standards, were set for all school subjects at two levels in lower grades and at three levels in the two final grades, grades 8 and 9 (Učni načrt, 2002, translation: Curriculum Guide). When a student masters the set of contents labelled minimum standards (called Level 1 standard in this chapter) he or she receives a pass grade. The second level includes fundamental standards (called Level 2 standards). They are defined as the knowledge and skills that it is expected that the average student should attain, and represent the knowledge and skills that teachers should strive for their students to gain. The higher level standards (Level 3) represent the knowledge and skills to be expected of higher achieving students.

In the process of curriculum reform, two perspectives for describing Slovene achievement are embedded. The first is the intended curriculum for mathematics in Slovenia (what teachers are supposed to teach and students are supposed to learn). This chapter describes

the mathematics achievement of Slovene students in the non-reformed system in the light of the newly set standards. Analysing mathematics achievement of students just before the start of the implementation of the new curriculum will enable insight into 'the starting point' of the new curriculum. Future assessments carried out in Slovenia may utilise this information to find out whether improvement measures introduced in the reformed curriculum have had the desired effects on student achievement.

The second perspective from which Slovene mathematics achievement will be examined is a comparison with achievements of students from other European countries. The relevance of this perspective is also clear from Slovene policy documents. In order to identify areas of Slovene achievement in which improvements might be desired, countries with similar or higher overall achievements than Slovenia are considered relevant for comparison. The selection of these countries is described in more detail in the section on methodology. The relevance of comparisons of Slovenia with these countries is underlined by Belgium Flemish and the Netherlands being in the European Union for a number of years and by the very recent accession, in May 2004, of Hungary and the Slovak Republic, as well as Slovenia.

Corresponding to the two perspectives for describing Slovene mathematics achievement, two research questions are posed:

1. How well did Slovene students at the end of compulsory education in the non-reformed system in the late 1990s perform in mathematics when compared to the attainment targets in the reformed mathematics curriculum?

2. How well did Slovene students at the end of compulsory education in the non-reformed system in the late 1990s perform in mathematics when compared to the performance of students in other European countries?

There is no single 'right' perspective for describing student achievement. Using two different perspectives will provide more insight into students' achievements than a single one. Furthermore,







the differences between the results of the analyses based on each perspective can be examined. This is called convergence between the two perspectives. The analysis of convergence of the results from the two perspectives may highlight areas in Slovene achievement in which improvements might be desired from both perspectives. Furthermore, areas in which expectations in the intended curriculum might be considered too high or too low, taking into account achievements of students from other European countries, may also be highlighted. Kellaghan (1996) describes this function of international assessments as 'enlightenment'.

Data on student mathematics achievement from the Third International Mathematics and Science Study (TIMSS) will be used. The next section briefly presents the TIMSS database and the methodology used. In the third section the results of the analyses are presented and in the last section the main conclusions drawn from these results are given.

**Database and methodology**

The Third International Mathematics and Science Study (TIMSS, recently renamed Trends in International Mathematics and Science Study; Robitaille, et al., 1993; Mullis, et al., 2003), is conducted under the auspices of the International Association for Evaluation of Educational Achievement (IEA; see e.g. IEA, 1998). Up to the present, three TIMSS surveys have been carried out, in 1995, 1999, and 2003. At the time of the present study, TIMSS 2003 data were not yet available for analyses. To describe the performance of Slovene students at the end of compulsory education in the non-reformed system, the TIMSS 1999 data (IEA, 2001) will be used. The most important feature of TIMSS for the present study is that it is based on the curricula of participating countries. This enabled a link between student achievement (the attained curriculum) and the curriculum as is prescribed in the official documents (the intended curriculum). Through this link, areas in student achievement in which improvements might be desired can be identified and possible remedial actions developed.

TIMSS 1999 focused roughly on the end of compulsory education. This was grade eight in almost all countries (Mullis, et al., 2000). In

Slovenia, grade eight students were taken as the target population although they were, on average, slightly older than students in many other countries (Mullis et al., 2000). The TIMSS design required that a minimum of 150 randomly selected schools be sampled and in each school one whole class of students in the target grade. In Slovenia and the reference countries this yielded sample sizes of approximately 3000 students. The TIMSS mathematics achievement tests were developed through an international consensus involving input from experts in mathematics and measurement specialists (Garden, 1996). The aim of TIMSS instrument development was to have items that had maximum validity across participating countries and to test as wide a range of school mathematics curricula as possible. The items underwent an iterative development and review process, including the pilot testing. Every effort was made to help ensure that the tests represented the curricula of the participating countries and that the items did not exhibit any bias towards or against particular countries.

**Measuring attainment targets in the curriculum**

Since the TIMSS achievement tests were not designed specifically to describe the Slovene intended curriculum for mathematics, the mutual coverage between the intended curriculum and the TIMSS items needed to be examined in order to provide evidence of the appropriateness of the test for measuring and linking the intended and the attained curricula in Slovenia. For each attainment target in the curriculum, a Slovene mathematics curriculum specialist indicated whether it was covered by the items in the TIMSS tests. The percentage of attainment targets at Levels 1 and 2 that were covered was 77 % for the TIMSS 1999 test. The main topics that were not covered in the TIMSS tests were geometry topics about circles, triangles, constructing angles and triangles, parts of three-dimensional geometry including the Pythagorean theorem, and simplifying symbolic expressions using properties of operations. These topics are mainly covered in the final grades of compulsory education. Also, very few attainment targets at Level 3 were covered. The TIMSS items can be seen as covering the 'general' part of the curriculum reasonably well, while they do not cover its 'specialised' part.







The reverse issue was also addressed. For each TIMSS item, three mathematics curriculum specialists indicated whether it was covered by the attainment targets in the reformed mathematics curriculum and at which level. The percentage of TIMSS 1999 items that were covered was 97 %. These percentages for the mutual coverage between the test and the attainment targets were deemed sufficiently large to enable meaningful examination of Slovene achievement on the basis of TIMSS items, at least at Levels 1 and 2. The measures for item coverage were given values 'Level 1', 'Level 2', and 'Level 3'.For each item, the median of the three measures was taken as the measure that was used in this study. The reliability of these expert judgments was 0.72 [percent? And of what?] which was deemed acceptable (e.g., Wolf, 1994) considering that attainment targets are a novelty in Slovene mathematics education.

In the process of allocation of items to the levels of the standards, it was found that in addition to the five items for which the standards in the intended curriculum could not be determined, two items had problems with translation and only six items were allocated at Level 3. These items were excluded from further analysis.

**Selection of countries for reference points**

The second reference point for describing Slovene mathematics achievement was based on achievements of students from several other European countries. As mentioned, in order to identify possible areas for improvement in Slovene mathematics education, countries with similar or higher overall achievement than Slovenia were selected. Further, since Slovenia exhibited stability in achievement in the period between 1995 and 1999, countries were selected that had similar or higher overall achievement in both TIMSS measurements in the late 1990s, indicating stability in their comparison with Slovenia. The countries were selected based on their IRT scores for average achievements reported in Beaton et al. (1996) and Mullis et al. (2000). Four countries were selected in this way: Belgium Flemish, the Netherlands, Hungary, and the Slovak Republic. The average achievement in Belgium Flemish was significantly higher than in Slovenia on this scale in both TIMSS data collections, while the average achievements in the other three countries were similar to Slovenia.

**Assessing the correspondence of achievement in Slovenia with the selected reference points**

In order to describe the correspondence of mathematics achievement of Slovene students with the standards in the intended curriculum, TIMSS items were used as the link between the two sets of curricula. Similarly, the TIMSS items were used to link the curricula attained in Slovenia and in the reference countries. However, while the achievements were directly comparable in the form of the percentages correct the measurements on the intended curriculum were carried out in a non-numeric form. To make them comparable with the Slovene achievements, they were operationalised into the 'intended percentage correct'. This was done on the basis of general descriptions of the differences between the three levels of the attainment targets as follows: for items allocated at Level 1, the expected percentage correct was set to 75 %; and for items allocated at Level 2, the expected percentage correct was set to 50 %.

This operationalisation is, of course, a very simple model for the expected percentage correct (or intended difficulty) for individual items. Items that are used to measure student achievement may and should vary in their intended difficulty as well as in other characteristics. However, in the absence of explicit guidelines of how these levels of the attainment targets should be operationalised in order to assess whether they have been achieved, this operationalisation was deemed a sufficient approximation. It was also considered plausible by the mathematics curriculum experts who allocated the TIMSS items to the levels of the attainment targets.

Once the target and reference measures were calculated in the form of comparable scores, correspondence between the two was assessed. When the estimate for Slovene achievement was significantly higher than the reference point, taking into account the standard errors of the estimates, this was taken as an indication of a strength in Slovene achievement. Generally, the level of significance was taken at 0.05 using Bonferroni adjustment for multiple comparisons. In the cases where it was significantly lower, this was taken as an indication of a weakness. In the remaining cases, Slovene achievement was described as corresponding with the standards.







When the reference point was constructed on the basis of the achievements of students from the four reference countries, four reference measures were computed as the average percentage correct estimates of achievement in these countries and their standard errors. In comparison with each of these countries it was indicated whether Slovene achievement was significantly higher or lower. For this a usual test of significance was used (t-test with Bonferroni adjustment and significance level of 0.05). In this way, comparisons with each of the four reference countries were examined. However, since no particular country was taken as 'the most important' for comparisons with Slovenia, it was judged that there were indications of strengths or weaknesses in Slovene achievement if significantly higher or lower achievement in Slovenia was observed in at least two comparisons.

**Results**

Following the design of the research questions, this section is organised into three subsections. The first presents the results of analysis of correspondence of Slovene mathematics achievement with the attainment targets in the reformed curriculum. The second presents the results of analysis of correspondence of Slovene mathematics achievement with achievements of students in other European countries. The third discusses the convergence of the results between the two reference points.

**Correspondence of Slovene mathematics achievement with the attainment targets in the reformed curriculum**

The results of analysis of correspondence with the attainment targets by these content areas are presented in Table 1. Average scores (in the form of the average percentage correct) of Slovene students across content areas in the curriculum and across the levels of the standards are presented, as well as the score on the total TIMSS 1999 mathematics test. As shown, average scores of Slovene students varied across content areas and across levels of the standards. In one cell in Table 1 the results are not shown nor is the content area 'probability' due to insufficient numbers of items (a minimum of five items was deemed necessary for meaningful analysis). It can be observed from Table 1 that correspondences of scores with the standards varied across content areas. When looking at the overall

level, a satisfying correspondence was observed for most content areas. On items about 'natural numbers', 'algebraic expressions', and 'data representation', average scores of Slovene students were significantly higher than the standards. Average scores in the content areas: 'meaning of rational numbers' and 'geometrical shapes' were lower than the standards.

**Table 1**. Mathematics achievement of Slovene students in 1999 by content areas and for the total test[2]

| | Overall | | Level 1 standard | | Level 2 standard | |
|---|---|---|---|---|---|---|
| | Number of items | Average percent correct | Number of items | Average percent correct | Number of items | Average percent correct |
| Natural numbers | 9 | 75 (0.8) ▲ | 8 | 77 (0.8) | 1 | - |
| Meaning of rational numbers | 20 | 64 (0.8) ▼ | 15 | 66 (0.8) ▼ | 5 | 59 (1.4) ▲ |
| Operations with rational numbers | 19 | 59 (1.0) | 8 | 70 (1.2) ▼ | 11 | 52 (1.0) |
| Algebraic expressions | 25 | 61 (0.9) ▲ | 6 | 69 (1.1) ▼ | 19 | 58 (0.9) ▲ |
| Functions and proportionality | 21 | 54 (0.8) | 5 | 69 (0.9) ▼ | 16 | 50 (0.9) |
| Geometrical shapes | 23 | 55 (0.9) ▼ | 8 | 55 (1.3) ▼ | 15 | 55 (0.9) ▲ |
| Measurement | 18 | 59 (0.8) | 7 | 78 (0.9) ▲ | 11 | 48 (0.9) |
| Data representation | 15 | 71 (0.8) ▲ | 9 | 82 (0.6) ▲ | 6 | 56 (1.3) ▲ |
| Total test | 153 | 61 (0.7) | 69 | 70 (0.7) ▼ | 84 | 53 (0.8) ▲ |

▼ Average percent correct is significantly lower than the standard
▲ Average percent correct is significantly higher than the standard
() Standard errors are presented in parentheses
- Data are not shown due to insufficient number of items

The correspondences in Table 1 differed between the two levels of the standards for most content areas. As expected, scores were higher at Level 1 than at Level 2 in most content areas ($p<0.05$)[4] , except in 'geometrical shapes' where they were identical. However, except for the content areas 'natural numbers', 'measurement', and 'data representation', students' scores at Level 1 were lower than the standards with the lowest difference of 20 percentage points in 'geometrical shapes'. As explained in the methodology section, the intended achievement at Level 1 is 75 percentage correct and at Level 2 50 percentage correct. In many content areas, average scores of Slovene students at Level 2 were higher than the standards and in







no content area at this level were they lower. In quite a few content areas weaknesses at Level 1 seem to be compensated by strengths at Level 2.

The average score of Slovene students on all items in the TIMSS 1999 achievement test was 61 percent which corresponded with the standards. This could be interpreted as showing that overall achievement of Slovene students in mathematics is satisfactory when compared to the attainment targets in the curriculum.

When considering the different levels of the standards, it can be observed that the average score at Level 2 was lower than at Level 1. The results in Table. 1 indicate that while scores in Slovenia corresponded with the standards at the level of the overall test, there seem to have been weaknesses at Level 1 and strengths at Level 2.

Correspondence of Slovene mathematics achievement with the achievements of students in other European countries
To assess the comparability of Slovene achievement with the achievement of students in other European countries, average percentage correct scores of these students on the overall mathematics test and in content subdomains were compared. The results are presented in Table 2.

**Table 2.** Mathematics achievement of Slovene students in 1999 by content areas and in total compared to the reference countries

| | Number of items | Slovenia | Belgium Flemish | Netherlands | Hungary | Slovak Republic |
|---|---|---|---|---|---|---|
| Natural numbers | 9 | 75 (0.8) | 80 (0.9) ▲ | 75 (2.0) | 76 (0.9) | 80 (1.0) ▲ |
| Meaning of rational numbers | 20 | 64 (0.8) | 71 (0.7) ▲ | 74 (1.8) ▲ | 68 (1.0) ▲ | 66 (1.1) |
| Operations with rational numbers | 19 | 59 (1.0) | 62 (1.1) | 53 (2.3) | 61 (1.0) | 61 (1.3) |
| Algebraic expressions | 25 | 61 (0.9) | 63 (1.0) | 53 (2.0) ▼ | 63 (1.0) | 63 (1.2) |
| Functions and proportionality | 21 | 54 (0.8) | 58 (0.8) ▲ | 51 (1.8) | 51 (0.9) | 53 (1.2) |
| Geometrical shapes | 23 | 55 (0.9) | 62 (1.0) ▲ | 54 (2.0) | 50 (1.1) ▼ | 59 (1.3) ▲ |
| Measurement | 18 | 59 (0.8) | 61 (0.8) ▲ | 63 (1.4) | 64 (0.8) ▲ | 61 (1.1) |
| Data representation | 15 | 71 (0.8) | 75 (0.8) ▲ | 73 (1.8) | 68 (1.0) | 69 (1.0) |
| TIMSS 1999 | 153 | 61 (0.7) | 67 (0.8) ▲ | 61 (1.8) | 62 (0.9) | 63 (1.1) |

In italic, estimates of average percent correct are reported
▼ Average percent correct is significantly lower than in Slovenia
▲ Average percent correct is significantly higher than in Slovenia
() Standard errors are presented in parentheses

When we 'zoom in' on the strengths and weaknesses in Slovene achievement by comparison with other European countries, it can be observed that Slovene scores corresponded with the scores in the reference countries in most content areas. The greatest difference can be found in comparison with Belgium Flemish. An overview of Table 2 across countries and across content areas reveals that Slovene achievement was significantly lower than at least two reference countries in the content areas: 'natural numbers', 'meaning of rational numbers', 'measurement', and 'geometrical shapes'. It seems plausible to argue that, especially in the content area: 'meaning of rational numbers', more general weaknesses in Slovene achievement existed. Considering the 'European dimension' emphasised in the Slovene policy documents, further improvements might be focused on these areas.

**Convergence of the results when compared to the two reference points**

By using two reference points, two descriptions of Slovene achievement in mathematics were obtained. Table 3 presents a summary of the results for locating strengths and weaknesses in Slovene achievement in content areas of the Slovene curriculum. As shown, there were similarities and differences in the descriptions of Slovene achievement between the two perspectives. From both perspectives weaknesses in Slovene achievement were observed in the content areas: 'meaning of rational numbers' and 'geometrical shapes'. Content areas 'algebraic expressions', 'functions and proportionality', 'operations with rational numbers', and 'data representation' can be described as satisfactory from both perspectives.

**Table 3.** Summary of correspondence of achievement with the two reference points

| | Correspondence with the attainment targets | Correspondence with the reference countries |
|---|---|---|
| Natural numbers | ▲ | ▼ |
| Meaning of rational numbers | ▼ | ▼ |
| Operations with rational numbers | ○ | ○ |
| Algebraic expressions | ▲ | ○ |
| Functions and proportionality | ○ | ○ |
| Geometrical shapes | ▼ | ▼ |
| Measurement | ○ | ▼ |
| Data representation | ▲ | ○ |

▲ Area could be described as strong compared to the reference point
○ Area could be described as corresponding with the reference point
▼ Area could be described as weak compared to the reference point







The differences between the two perspectives emerged in the content area 'measurement', in which Slovene achievement seemed to correspond with the standards but it was lower than in two of the reference countries. Similarly, average scores in 'natural numbers' were higher than the standards, but were lower than in two of the reference countries. These results reveal the importance of understanding the standards in the reformed Slovene mathematics curriculum in the light of achievement of students from other countries.

**Conclusion**

In this chapter, the problem of describing Slovene mathematics achievement to serve the needs for comparative international information as an input into the process of curriculum reform, and its implementation, has been addressed. This adds to other studies (e.g., Magajna, 2000), in shedding more light on to the areas in which improvements might be desired and be possible. Two research questions were formulated, giving two perspectives for the description of achievement. The first examined the correspondence of Slovene achievement in mathematics with the attainment targets in the reformed curriculum and the second with the achievements of students from four other European countries: Belgium Flemish, the Netherlands, Hungary, and the Slovak Republic. These correspondences were examined at the level of the overall mathematics domain and in content subdomains of the curriculum.

The most general finding in this study is that mathematics achievement of Slovene students in the non-reformed system corresponded with the attainment targets in the reformed curriculum. However, this is at the level of overall mathematics achievement only. Detailed analyses revealed variation in the correspondences. When looking at achievements at Level 1, weaknesses in student achievement were observed. These were largely compensated with strengths at Level 2. There seem to be deficiencies in the knowledge and skills of Slovene students that are inappropriate from the perspective of the intended reformed curriculum. In order to achieve the goals of the reform, attention must be focused on the knowledge and skills at Level 1, that is on knowledge and skills that nearly all students should master.

Further analyses, carried out for different content subdomains in the Slovene curriculum, revealed that the contrast between Level 1 and Level 2 occurred in most of these areas. The complementary comparisons with other European countries revealed also that improvements might be desired in the content areas of: 'meaning of rational numbers' and 'geometrical shapes'.
This study was basically descriptive. It provided information on the strengths and weaknesses observed in Slovene students' mathematics achievement when compared to the relevant reference points. By linking student achievement in the non-reformed system to the reformed curriculum it provides information on the point from which the reformed curriculum is starting. This information may be also used in possible future studies to examine whether the desired effects of the reforms can be observed.

The results in this chapter pointed out the importance of in-depth analyses of the national results in an international context. For example, the international TIMSS reports (Beaton et al., 1996; Mullis et al., 2000) showed that Slovenia is among the higher achieving European countries. However, in more detailed comparisons it was found that there is room for improvement.

This study also showed the importance of describing student achievement from several perspectives. While the intended curriculum is an important perspective for student achievement, it was shown that there might be differences between the intentions in the national curriculum and what it is possible to achieve as observed in other countries. This function of international assessments is termed 'enlightenment' (Kellaghan, 1996). It may be that the intentions in the curriculum are too high, however, it may also be that they are too low. Using the results of this study, Slovene educators are in a position to understand better student achievement and at the same time the attainment targets that were set in the reformed curriculum. This study was not an evaluation of the attainment targets but aimed at providing information on what they mean and how can they be used. Through measurements of student achievements and analyses of their correspondence with the standards, as well as with other reference points, these standards may be refined further in terms of wording and content as well as in terms of intended levels of







achievement. Although the perspectives used in this study certainly do not reflect all possible views from which student achievement could be described, they provide a wealth of information to help Slovene educators in their efforts to improve Slovene mathematics education and its outcomes.

**Footnotes**:
[1] The authors used the term primary education to describe the compulsory part of the Slovene education system. Sometimes the terms basic or elementary education are used. In this study, the term compulsory education will be used.
[2] In all tables in this chapter, rounded estimates are presented, while tests of significance were carried out using non-rounded values. This may cause some inconsistencies in the tables.
[4] The significance of this difference is not indicated in Table 1 to avoid confusion with indications of whether the scores correspond with the standards. This significance can be determined using t-test with the standard error that is presented in Table 1. The same holds for similar comparisons in other tables in the remainder of this chapter.

**References**
Beaton, A.E., Mullis, I.V.S., Martin, M.O., Gonzalez, E.J., Kelly, D.L., and Smith, T.A. (1996) Mathematics Achievement in the Middle School Years: IEA's Third International Mathematics and Science Study (TIMSS). Chestnut Hill, Mass: Boston College.
Garden, R.A. (1996) Development of the TIMSS Achievement Items, in D.F. Robitaille, and R.A. Garden (eds), TIMSS Monograph No.2: Research Questions and Study Design, pp. 69–80. Vancouver, Canada: Pacific Educational Press.
International Association for Evaluation of Educational Achievement (IEA) (1998) IEA Guidebook 1998: Activities, Institutions and People. Amsterdam: Author.
IEA (2001) TIMSS 1999 International Database. Chestnut Hill, Mass: Boston College.
Kellaghan, T. (1996) IEA Studies and Educational Policy. Assessment in Education, 3, pp.143–60.
Magajna, Z. (2000) Obravnava primanjkljajev matematičnega znanja slovenskih učencev v učnem načrtu slovenske osnovnošolske matematike [Addressing Slovene students' lack of mathematical

skills in the elementary school mathematics syllabus]. Sodobna Pedagogika, 2, pp.162–80.
Mullis, I.V.S., Martin, M.O., Gonzalez, E.J., Gregory, K.D., Garden, R.A., O'Connor, K.M., Chrostowski, S.J., and Smith, T.A. (2000) TIMSS 1999 International Mathematics Report: Findings From IEA's Repeat of the Third International Mathematics Science Study at the Eighth Grade. Chestnut Hill, Mass: Boston College.
Mullis, I.V.S., Martin, M.O., Smith, T.A., Garden, R.A., Gregory, K.D., Gonzalez, E.J. Crostowski, S.J., and O'Connor, K.M. (2003) TIMSS Assessment Frameworks and Specifications (2nd edn.). Chestnut Hill, Mass: Boston College.
Robitaille, D.F., Schmidt, W.H., Raizen, S.A., McKnight, C.C., Britton, E., and Nicol, C. (1993) TIMSS Monograph No.1: Curriculum Frameworks for Mathematics and Science. Vancouver, Canada: Pacific Educational Press.
Učni načrt: program osnovnošolskega izobraževanja: matematika. [Curriculum guide: the program of elementary education: mathematics] (2002) Ljubljana, Slovenia: Ministry of Education and Sports.
White Paper on Education in the Republic of Slovenia (1996) Ljubljana, Slovenia: Ministry of Education and Sports.
Wolf, R.M. (1994) The Validity and Reliability of Outcome Measures, in A.C. Tuijnman, and T.N. Postlethwaite (eds), Monitoring the Standards of Education: Papers in Honor of John P. Keeves, pp.121–32. Oxford: Pergamon Press.

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# RESEARCHING THE CONTEXT IN MATHEMATICS

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***Abstract***
*The chapter is an analytical account of conceptualisations of the context in mathematics. It is not intended to comprise all possible conceptualisations; it attempts to be pragmatic and to focus results that were found to be relevant to mathematics class practices. These conceptualisations of context are related to other subject domains, for example the cultural environment, and to different theories of connecting mathematics to the context: common modelling, ethnomathematics, and activity theory. Furthermore they are based on different views of the nature of mathematics. To illustrate these ideas, a small research topic comparing the treatment of the context in two mathematics textbooks for Slovenian vocational schools is described.*

**Key words:** activity theory, context, ethnomathematics, modelling, word problems

**Introduction**
Context is a commonly used word in mathematics education, yet its meaning is often unclear. In this author's experience most teachers understand contextualisation of mathematics as 'using words from outside mathematics in a mathematics class' and they believe that contextualisation of mathematics is mainly important for two reasons: 1. for motivational purposes, and 2. for learning to apply mathematics to real-world situations. Such an understanding of





contextualisation appears rather narrow and perhaps naive, for the role of context in mathematics is multifarious: the context is not just something from which data are taken and to which mathematics can be applied – the context may be essential in helping students to build mathematical knowledge, to relate mathematical knowledge properly to their beliefs, their values, to their cultural background and foreground as well as to their professional activity.

This article is an analytic account of conceptualisations of the context in mathematics. It is not intended to include all possible conceptualisations of the context, rather it attempts to be pragmatic and to focus on understandings that the author finds to be relevant to practices in mathematics classes. Three areas of contextualisation of mathematics will be covered: subject domains, culture, and activity. Related to these are three different ways of connecting mathematics to the context: common modelling, ethnomathematics, and activity theory. Note that the three areas of contextualisation also reflect different views of the very nature of mathematics, but this issue will not be pursued here.

**Subject-domain as context**
Greeno (1991, 174) defines a subject-domain as "a structure of facts, concepts, principles, procedures and phenomena that provides resources to the cognitive activity of knowing, understanding, and reasoning". Mathematics, seen as a formal body of knowledge, is a subject-domain, and so are physics, economics, and perhaps even knitting. Usually, we think of mathematical knowledge as a cognitive structure, consisting of representations and procedures to manipulate them, which is (in a way) a subset of mathematics as a subject-domain.
Context is often interpreted as an environment in which the notion of a concept is related to its meaning, i.e. as a general set of conditions that organise the meaning of a notion at the linguistic level. In mathematics education, however, the term 'context' is often used in a specific way in the sense of 'applying words from outside mathematics' (Sträßer, et al., 1989). The underlying assumption is that there is a formal, abstract, 'decontextualised' mathematical subject-domain, in which the mathematical objects have meaning, and a real-world environment or other subject-domains, which

are referred to as 'the context'. Mathematics is thus understood as a body of knowledge in which meanings are detached from the experiential world. Putting mathematics in context, as in the case of school-mathematics tasks given in context, is understood as relating – often just at a semantic level – the meanings of the mathematical objects outside mathematics.

There is no doubt that students sometimes apply school-learnt mathematical procedures or ideas to situations outside the mathematics class. Yet there is plenty of evidence that children often do not activate real-world knowledge in solving school-mathematics word problems. Many pupils, for example, claim that if we pour together into a container 1 litre of water at 40◎C and 1 litre of water of 80◎C the resulting mixture will be 120◎C (Verschaffel, et al., 1994). In fact, an analysis of typical school-mathematics tasks given in context (usually as word problems) shows that the intended links between the situation described and the supposed mathematical meanings can be quite artificial, sometimes even whimsy. DeLange cites an extreme example of a story problem quoted by Pollak:

Given that two bees can gather nectar from 100 hollyoak blossoms in 30 minutes and assuming that each bee works eight hours a day, five days a week, how many blossoms do these bees gather nectar from in a summer season of fifteen weeks? (DeLange, 1996, p. 67)

One cannot but agree with Lave (1992) that such problems (and indeed most story problems in school-mathematics textbooks) are not aimed at elaborating intrinsic connections between abstract mathematics and real-world contexts but are a socioculturally established way of grounding the abstract mathematical concepts in familiar terms for pupils who have only a concrete grasp of mathematics. More precisely, the context (in mathematics) sometimes serves as a general motivator, and sometimes as a facilitator, by suggesting thinking in terms of well known representations instead of abstract entities (Boaler, 1993). Sierpinska (1995) claims that everyday contexts in school-mathematics are not intended to be authentically everyday so that pupils learn how to use mathematics in everyday life – their function is to be authentically mathematical so that they help pupils to learn abstract mathematical ideas. Boaler (1993), on the other







hand, believes that contexts in school mathematics promote the perception of common underlying structures that enable pupils to use mathematical knowledge in different contexts.

The common way of applying mathematics (as a subject domain) to other subject domains is mathematical modelling (see Figure 1). It presupposes two distinct systems of meanings: the system of mathematical meanings and other systems of meanings, usually referred to as real-world or some subject-domain. The two are, in theory, autonomous, but there is some sort of parallelism between them. Basically, given a real-world phenomenon the idea is to set a 'mathematical equivalent' (e.g. a formula or a set of differential equations), which is elaborated 'mathematically' without any reference to the original model. Then the results obtained are translated back into the real-world context and evaluated.

Such dualistic reasoning is part of the epistemological tradition of the Western scientific community and has significant didactical implications. The most important one is perhaps the importance and reasonableness of learning pure, 'decontextualised' mathematics. Second, the mathematical reasoning in the second phase of modelling is, in theory, done without reference to the original situation. Third, the link between mathematics and the real-world context is quite artificial, for what matters is that the appropriately interpreted outcomes of the calculations fit the experimental data. For students it is, as a rule, disturbingly difficult to find the links, through reasoning, between a real-world phenomenon and its mathematical model.

Applying mathematics by modelling is definitely successful in science and is deeply rooted in school practice. Perhaps the modelling paradigm is most clearly expressed in the idea of the technique curriculum, which is based on mathematical procedures, methods, skills, rules and algorithms (Bishop, 1988). Since the 'linking' part of the modelling is beyond the grasp of most students, they simply learn a number of models (application-related formulas, equations, etc.) and essentially all their effort is put into learning to apply the established mathematical model. In other words, the majority of students are supposed to learn pure (i.e. decontextualised) mathematics in order to use the techniques they have learned in various modelling

situations. Yet nowadays there is plenty of evidence that when solving mathematical problems in out-of-school situations people only rarely resort to school learnt modelling, instead they use job-specific techniques and methods (e.g. Noss and Hoyles, 1996). The fact that mathematics students do a lot of exercises (usually stated as word problems) on modelling does not necessarily mean that they are able to 'mathematise' real-world situations. Very often they just '"respond to word problems according to stereotyped procedures assuming that the modelling situation is 'clean'" (Greer, 1993). Thus, a lot of effort is put into finding ways of enabling the students to learn how to model.

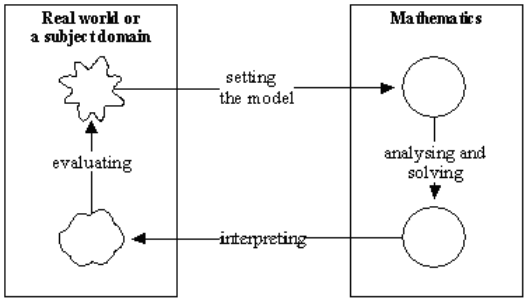


Figure 1. Mathematical modelling

**Culture as context**

From this perspective mathematics is not 'a subject domain of universal validity' but rather a culture dependent interpretational means. Mathematics is still considered to be a body of knowledge, but this knowledge is a social category, which is socially distributed, transmitted by social means, and is inextricably linked to other elements of human culture. Bishop (1988) conceives mathematics as a cultural product which has developed in societies as a result of universal activities: counting, locating, measuring, designing, playing, and explaining. Mathematics is, in this sense (a part of) the symbolic technology of culture. In this sense mathematics differs from culture to culture. Western mathematics is thus not universal, it is perhaps just the most widespread and well known, but traditional societies, non-Western cultures and even various cultural groups in







the Western world have their own mathematics, coherent with their culture.

Thus mathematics is not something 'decontextualised' for it is embedded in its cultural context. The links between mathematics and various cultural constituents are often hardly discernible. However, they become apparent or even disturbing if the mathematics of one culture is learnt in another cultural environment. The resulting tensions may be simply ignored (e.g. assuming that mathematics is a culture-free knowledge area) or may be taken into consideration at various levels in the curriculum, teaching style, and language.

Boaler (1993) points out that a mathematical class is by itself a cultural entity in its own right and thus that mathematical knowledge and cognition cannot be separated from it. On the other hand 'school mathematics' should acknowledge that (practical) solutions developed by various cultural groups (outside school) to real-life mathematical problems are also mathematical. According to Boaler, both of these links should be considered if we want pupils and students to learn mathematics that is both meaningful to them and also useful in general situations.

If mathematics is understood as a cultural phenomenon, it is worthwhile to take into account that culture, and mathematics in particular, can be experienced on different levels – a fact that has been pointed to by Sierpinska (1994, 161). It is based on the theoretical considerations of C. T. Hall, who has identified three such levels distinguished by specific ways of transmission of knowledge, emotional relations, forms of communications (Sierpinska, 1994). From this perspective the mathematics of mathematicians (as a cultural product) is not all technical, that is explicit and rationally justified or explained. The way proofs are written, the way problems are attacked, the expected rigour in argument, etc. belong to the informal level. And there is also a level of mathematical beliefs, unquestioned facts (not axioms) taken as self-evident.

Ethnomathematics is just another approach to relating mathematics to its cultural milieu. Gerdes (1994) speaks of ethnomathematics as "the cultural anthropology of mathematics and mathematical education".

During the last two decades the concept of ethnomathematics has diversified into at least three directions: 1. as the mathematics of various cultural groups, e.g. Australian aboriginal cultures or the street vendors in Brasilia; 2. as an emancipatory movement in former colonial societies, opposing world welfare injustices and the supposed superiority of Western culture; 3. as an educational theory (described below).

Ethnomathematics as an educational theory stresses the need to base mathematics education on the students' cultural background, thus using their out-of-school experience, extracting mathematical ideas from the cultural environment and embedding them into it. It recognises that each cultural setting, including underprivileged minorities or non-Western populations, possesses knowledge for coping with 'mathematical' challenges (counting, measuring, etc.). Organised learning in school should take account not only of such mathematical practices, but also consider the related jargons, codes and styles of reasoning (D'Ambrosio, 1991). However, some authors are opposed to such simplified linking of (school) mathematics and ethno-culture. Vithal and Skovsmose (1997), from the standpoint of critical education, claim that school mathematics should train students' minds in order to enable them to cope with today's technological society, that school mathematics should give the students a tool to improve the life of all of society – it should not be seen as an essentially neutral 'defrosting' of mathematical ideas from traditional artefacts.

**Activity as context**

A third view of mathematical knowledge focuses on (mathematical) actions carried out in actual situations. Based on the theory of activity originated by Vygotsky and developed by Leont'ev and later by Engeström, Wertsch and others, actions are the constitutive basis of a human mind (Wertsch, 1985). The actions people do, according to this theory, are not the result of applying some sort of abstract (decontextualised) knowledge to specific situations, but are instead inextricably linked to the situations in which they are learnt and executed. Adding scores while playing a card game and adding numbers while practising in a mathematics class are perhaps mathematically isomorphic problems but considered as actions







they are certainly not. Often the same person solves apparently equivalent mathematical problems differently in various situations, using different methods, with different success rates, etc. It this sense the knowledge appears to be situated. Several studies confirm that there is a gap between school-learnt mathematical knowledge and mathematical practices carried on elsewhere. Nunes, Schliemann, and Carraher (1993), for example, studied the mathematical behaviour of children selling candies or other goods on the streets of Brazilian towns. They were, in general, successful in solving the mathematical tasks in selling situations, less successful in (apparently) isomorphic tasks given as word problems, and even less successful in performing equivalent 'decontextualised' school-like calculations. Another well known study that confirmed the existence of a schism between mathematical practices in various activities is the Adult Mathematics Project (Lave, 1988). Here is an illustrative example. In one of the tasks the participants (housewives) were asked to choose the best value-for-money buy given several articles of the same type and quality with respective weights and prices. When the problem was given in the form of school-like exercise (performed in a home setting) the participants, on average, were able to identify the best buy in 59% of cases while in a supermarket setting in a real situation the success score was a remarkable 98%.

Mathematical tasks may be given 'in abstract', but the actions undertaken in order to solve them always occur in specific situations. These indicate the connotations of the task, add many additional restrictions, and explicit and implicit assumptions which imply specific ways of acting. It therefore seems inappropriate to view individuals' mathematical actions as applications of formal, abstract knowledge learnt at school to concrete situations. Instead, it appears that mathematical actions undertaken in specific situations result from situation-specific learnt knowledge, consisting of specific strategies, methods, tool-related procedures (see, for example, Carraher et al., 1987). Noss and Hoyles (1996) claim that the process of abstraction that produces mathematical knowledge should be seen more as synthesising diverse situational links than as disconnecting or decontextualising from them. They use the expression situated abstraction to underscore the idea that abstraction develops in an activity through reflection.

If knowledge is assumed to consist of actions, then context should be considered as the environment in which the action occurs. This environment is sometimes seen simply as 'the situation' in which the action or learning happens; however, the proper environment of an action is the activity in which the action occurs. Activity is a socioculturally defined context in which human functioning occurs (Wertsch, 1985). Examples of activities are school-learning, playing, working-on-a-job, gambling, etc. Activity as a context, according to the Vygotskian perspective, affects cognition on two levels. On a sociocultural history level it provides tools, methods, schemes, etc. for cognitive activity and for solving problems. On the level of immediate social interactional context it structures the cognitive activity. Being engaged in an activity means interacting with, or even adopting, the socially shared cognitive tools, methods, strategies, rules, ways of social interacting, and behaviour that are part of the activity and are coherent with the aims of the activity – and all this (context) situates the knowledge (actions). Work as activity is a particularly important case of a context of mathematical actions, for one of the aims of learning mathematics at school is its use in pupils' and students' future professional life. Until a few decades ago work mathematics was considered essentially an application of the mathematics learnt at school, but nowadays we are aware of a considerable discontinuity between school and work mathematics – though there is a certain (not yet well understood) interplay between them. Millroy (1992), for example, made an ethnographic study of a group of carpenters, which nicely illustrates how the mathematical methods and even concepts in work situations differ from those learnt at school. Working as an apprentice in a carpentry workshop she discovered that mathematical cognition in such situations has an important social connotation and is inextricably linked to other job-related practices, conventions, tools and social relations in the particular activity. For example, she was taught by her teacher about perpendicularity in relation to drilling holes perpendicular to a given plane, together with the technique of drilling the hole and 'feeling' the perpendicularity in the tool and so interactively adapting the drilling direction.







**An illustrative application**

In school-mathematics, as we have seen, context may be used with different meanings (depending on the perception of what mathematics is) and for different purposes. In this section we shall consider the role of context in mathematics education in vocational schools in Slovenia. We shall illustrate this by reporting a small-scale study, which formed the diploma thesis by Maja Lebar, a student at the Faculty of Education in Ljubljana. In the study two textbooks used in mathematical courses in Slovenian vocational schools were considered. One of the textbooks (referred to as Textbook I) is more 'mathematically oriented', the author put a lot of emphasis on nicely elaborated examples, on systematic elaboration, on developing 'mathematical ways' of thinking. The other textbook (referred to as Textbook II) is more 'student oriented', the authors tried to simplify elaborations and to gain the students' attention with appealing illustrations related to (at least apparently) real-life situations.

**Table 1**. Categories used in the classification of the task

| The relation between mathematics and the context used is not considered in the textbook (so that the students presumably did not relate their experience to mathematics) | |
|---|---|
| 0 There is no non-mathematics context | The exercise relates strictly to considered (mathematics) topics. The exercise also relates to other mathematics topics. |
| 1a The context serves purely for motivational purposes | The context is mentioned but is not related to the mathematical content. |
| 1b The context is artificial, whimsy | The context is far from students' experience. The students presumably just read the data and did not consider the situation itself. The students presumably had not and would not encounter the situation described. The context is clearly impossible in real life. In real situations the described task cannot or should not be solved with the given data. The context used does not help in understanding mathematics nor does mathematics1b |

| | |
|---|---|
| 1c The context is authentic but the students presumably do not relate it to mathematics | The students will presumably not relate the context and mathematics. |
| The context relates mathematical knowledge and student's experience The relation between mathematics and the used context is important and is considered in the textbook or is evident (thus the students presumably relate their experience and mathematics) | |
| 2a The experience helps students to build mathematical knowledge | The context is authentic and the students are encouraged to relate it to the mathematics. The tasks considered are well known to the student who can solve them by common sense. |
| 2b Mathematics enables a new interpretation of students' experience | The students can meaningfully interpret their experience with mathematical concepts. The mathematics considered gives a new dimension to students' experience. Mathematics helps in understanding a context that has not been experienced by the student but is nevertheless well known to them. |
| 2c The context and the mathematics under consideration complement each other | The students' experience and mathematical thinking are intertwined. A complex task has to be solved by resorting to the students' experience. |
| 3 Mathematics is related to values, emotions, tradition or other elements of cultural identity | The text refers to values, emotions, tradition or other elements of cultural identity. |
| 4 The context is related to a (professional) activity | The solution of a problem or the interpretation of the result requires knowledge about a (professional) activity, e.g. conventions, norms, social relations between participants. In the exercise various elements of a professional activity play a prominent role. The solution of the task depends on specific assumptions, suppositions or circumstances related to a professional activity. |

Here is a brief description of the methodology that was used to compare the role of the context in the two textbooks. First, the unit of analysis had to be chosen. Since both textbooks are based on exercise-like segments (e.g., exercises, examples), they were set as units of analysis (note that precise rules for tackling compound exercises







had to be set). Second, possible roles of the context were specified with related criteria for classification (Table 1). Third, the units of analysis (i.e. exercises, examples) were categorised according to the classification developed (Table 2). Part of the units of analysis was reviewed independently by two experts to check the appropriateness of the classifications used.

**Table 2.** Numbers of types of exercises in the two textbooks

| | Textbook I | | Textbook II | |
|---|---|---|---|---|
| | No. | % | No. | % |
| 0 There is no non-mathematics context | 457 | 73.83 | 688 | 79.81 |
| 1a The context serves purely for motivational purposes | 3 | 23.91 | 5 | 19.03 |
| 1b The context is artificial, whimsy | 128 | | 140 | |
| 1c The context is authentic but is not related to mathematics | 17 | | 19 | |
| 2a The experience helps the student to build mathematical knowledge | 3 | 1.29 | 3 | 0.81 |
| 2b Mathematics enables a new interpretation of students' experience | 3 | | 3 | |
| 2c The context and the mathematics complement each other | 2 | | 1 | |
| 3 Mathematics is related to other elements of cultural identity | 0 | 0.00 | 0 | 0.00 |
| 4 The context is related to a (professional) activity | 1 | 0.16 | 2 | 0.23 |
| Other | 5 | 0.81 | 1 | 0.12 |

The figures in Table 2 speak for themselves. Though the textbooks differ in many respects they relate mathematics to the context in a similar way. The contextualisation of mathematics is done, basically, by 'mentioning words from outside mathematics'. Both textbooks very rarely explicitly use students' everyday or professional experience to help them build their mathematical knowledge or, vice versa, use mathematics to gain better understanding of the world. The textbooks also do not relate mathematical knowledge to

cultural values or students' emotive sphere, and essentially they also do not consider mathematics as part of other professional activities. Obviously, teachers in vocational schools may treat the examples and exercises in the textbooks in a different way – they may, for example, emphasise and elaborate the links between mathematics and the context mentioned in the exercises. The table merely shows that this is not done in the textbooks themselves. As a matter of interest it is worth noting that the new curriculum for vocational schools in Slovenia strongly emphasises the intrinsic links between mathematics, students' experience and cultural environment, and professional activities.

**References**

Bishop, A.J. (1988) Mathematical Enculturation: A Cultural Perspective on Mathematics Education. Dordrecht: Kluwer Academic Publishers.
Boaler, J. (1993) The Role of Contexts in the Mathematics Classroom: Do they Make Mathematics More "Real"? For the Learning of Mathematics, 13(2), pp. 12–17.
Carraher, T.N., Carraher, D.W. and Schliemann, A.D. (1987) Written and Oral Mathematics. Journal for Research in Mathematics Education, 18(2), pp. 83–97.
D'Ambrosio, U. (1991) Ethnomathematics and its Place in the History and Pedagogy of Mathematics. in M. Harris (ed.) Schools, Mathematics and Work. London: The Falmer Press. pp. 15–25.
DeLange, J. (1996) Using and Applying Mathematics in Education, in Bishop, A.J., Clements, K., Keitel, C., Kilpatrick, J. and Laborde, C. (eds) International Handbook of Mathematics Education. Dordrecht: Kluwer Academic Publishers. pp. 49–97.
Gerdes, P. (1994) Reflections on Ethnomathematics. For the Learning of Mathematics, 14(2), pp. 19–22.
Greeno, J.G. (1991) Number Sense as Situated Knowing in a Conceptual Domain. Journal for Research in Mathematics Education, 22(3), pp. 170–218.
Greer, B. (1993) The Mathematical Modelling Perspective on Wor(l)d Problems. Journal of Mathematical Behavior, 12, pp. 239–50.
Lave, J. (1988) Cognition in Practice. Cambridge: Cambridge University Press.







Lave, J. (1992) Word Problems: A Microcosmos of Theories of Learning,, in Light, P. and Butterworth, G. (eds) Context and Cognition: Ways of Learning and Knowing. Hillsdale NJ: Laurence Erlbaum Associates, pp. 74–92.
Millroy, W.L. (1992) An Ethnographic Study of the Mathematical Ideas of a Group of Carpenters. Reston VA: National Council of Teachers of Mathematics.
Noss, R. and Hoyles, C. (1996) Windows on Mathematical Meanings: Learning Cultures and Computers. Dordrecht: Kluwer Academic Publishers.
Nunes, T., Schliemann, A.D. and Carraher, D.W. (1993) Street Mathematics and School Mathematics. Cambridge: Cambridge University Press.
Sierpinska, A. (1994) Understanding in Mathematics. London: The Falmer Press.
Sierpinska, A. (1995) Mathematics: "in Context", "Pure", or "with Applications"? For the Learning of Mathematics, 15(1), pp. 2–15.
Sträßer, R., Barr, G., Evans, J. and Wolf, A. (1989) Skills Versus Understanding. Zentralblatt fur Didaktik der Mathematik, 21(6), pp. 158–68.
Verschaffel, L., DeCorte, E. and Lasure, S. (1994) Realistic Considerations in Mathematical Modelling of School Arithmetic Word Problems. Learning and Instruction, 4, pp. 273–94.
Vithal, R. and Skovsmose, O. (1997) The End of Innocence: A Critique of 'ethnomathematics'. Educational Studies in Mathematics, 34, pp. 131–57.
Wertsch, J.V. (1985) Vygotsky and the Social Formation of Mind. Cambridge, MA: Harvard University Press.

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# CAN WE COMPARE LIKE WITH LIKE IN COMPARATIVE EDUCATIONAL RESEARCH? – METHODOLOGICAL CONSIDERATIONS IN CROSS-CULTURAL STUDIES IN MATHEMATICS EDUCATION

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***Abstract***
*The literature claims that one of the main problems that tends to arise in cross-national comparative analysis is that of equivalence – how to study the same problem in different societies and cultures. The core issue is to compare like with like, in order to develop a deeper understanding of the phenomena under study and to be able to generalise. In this paper it is argued that it is also necessary to study anomaly, recognising that anomalies can contribute to a better, albeit more complex, picture of the processes that are being studied.*

**Keywords:** Comparative education research, cross-cultural studies

**Introduction**
In recent decades, and at first stimulated by the United Nations and its agencies, international and interdisciplinary collaboration has been encouraged, and international comparisons increasingly used in the social sciences by researchers and policy makers. This trend is particularly notable in the area of mathematics education. Freudenthal argued in 1975 that:

In no branch of teaching have international contacts been closer and more numerous than in mathematics; no subject matter has known





as much international comparing and influential research than mathematics. (Freudenthal, 1975, p. 130)

International activity continues to be evident in the numbers of participants attending the four-yearly International Congress on Mathematical Education (ICME) which now brings together mathematics educators from almost one hundred countries. Whilst the seemingly ongoing internationalisation of mathematics education offers opportunities for international and comparative research, there are also greater challenges in its undertaking. Clarke (2003) offers the following.

Challenges confronting the international research community require the development of test instruments that can legitimately measure the achievement of students who have participated in different mathematics curricula, research techniques by which the practices, motivations, and beliefs of all classroom participants might be studied and compared with sensitivity to cultural context, and theoretical frameworks by which the structure and content of diverse mathematics curricula, their enactment, and their consequences can be analysed and compared. (Clarke, 2003, p. 144)

Critical ideas on comparative studies in mathematics education
There have been numerous international studies and comparisons of attainment in mathematics education over the past twenty years. The First International Mathematics Study (FIMS) was carried out in 1964, with twelve countries participating (Husén, 1967). At the time Freudenthal (1975) had already stated that cross-national comparisons are not valid without considering curricular aspects. He argued that a country's success depended to a large extent on the degree to which the test instrument (what is tested and how it is tested) was aligned with the mathematics curriculum of the particular country.

The Second International Mathematics Study (SIMS) was carried out between 1980 and 1982, and twenty countries participated. Taking into consideration the curricular criticism, SIMS differentiated between the intended, the implemented and the attained curriculum (for description see Travers and Westbury, 1989). In 1992 Westbury suggested that 'the lower achievement of the United States is the

result of curricula that are not as well matched to the SIMS tests as are the curricula of Japan' (Westbury, 1992, 18). Thus he raised a common concern, i.e. that such studies might – albeit unintentionally – measure little else than the alignment between the test instrument and the curriculum of the particular country (Clarke, 2003).

In 1995 the Third International Mathematics and Science Study (TIMSS) was carried out, with over 40 countries participating (for description see, for example, NFER, 1997 or Beaton and Robitaille, 1999). Alerted to the problems of earlier studies, the researchers paid particular attention to curricular organisation.

Combining sequence data on the flow of mathematics curricula (that is, the grades at which topics were typically covered) with document-based data on the commonly intended topics at the three key grades, revealed a picture of considerable cross-national diversity. This again suggests caution in interpreting mathematics attainment results. (Schmidt, et al., 1997, p. 19)

Alongside TIMSS, a series of smaller accompanying studies drew upon additional sources of data, amongst them a videotape classroom study which analysed 8th grade lessons in the USA, Germany and Japan (Stigler and Hiebert, 1997). Clarke highlights what he regards as one of the most important results of conducting the video studies.

One of the most powerful outcomes of large-scale video studies such as this has been the demonstrated potential of the video data to sustain multiple analyses. (Clarke, 2003, p. 165)

This was, for example, reflected in the study of Kawanaka, Stigler and Hiebert (Kawanaka, et al., 1999) who analysed teacher questioning. They concluded that:

Teaching and learning, as cultural activities, fit within a variety of social, economic and political forces in our society. Every single aspect of mathematics education, from a particular teacher behaviour to national policy, must be considered and evaluated within a socio-cultural context. (Kawanaka et al., 1999, p. 103)







More recently, the OECD Programme for International Student Assessment (PISA) included 'mathematical literacy' as one of its foci (see for example, www.pisa.oecd.org).

Keitel and Kilpatrick (1999) give probably one of the most condemning verdicts of large-scale comparative international studies, and TIMSS in particular, when they criticise the 'rationality and irrationality of international comparative studies'. They refer to fundamental problems in those studies, such as 'comparing the incomparable', 'many visions, many aims, one test', and 'problems of comparing curricula across educational systems', already cited by other authors (Bracey, 1997; Husen, 1983; Westbury, 1989). Their final verdict is that:

TIMSS threatens to poison for some time the waters of educational policy, as politicians and researchers scramble to take advantage of what TIMSS allegedly says about the teaching and learning of mathematics in their country. … [International comparative studies] rest on the shakiest of foundations – they assume that the mantle of science can cover all weaknesses in design, incongruous data and errors of interpretation. They not only compare the incomparable, they rationalise the irrational.
(Keitel and Kilpatrick, 1999, p. 254)

Beside those large-scale studies (mainly conducted by the IEA), there have been numerous small-scale, sometimes entirely qualitative, perhaps ethnographic, or sometimes 'mixed', studies.

The growing interest in the use of qualitative research methods has been a reflection of epistemological critiques of the positivist traditions of social science that have, until recently, dominated educational research, in particular in mathematics education. Numerous authors have emphasised the advantages of case studies to survey research in comparative international research. Some even argue that those studies can inform policy.

Studies such as Spindler (1987), Tobin et al. (1989), or Shimahara and Sakai (1995) provide powerful insights into the way teacher practice and belief is shaped within different national contexts. Comparative

studies that provide this level of ethnographic detail hold much potential for educational policy. (Letendre, et al., 2002, p. 23)

Keitel and Kilpatrick state:

… the treatment of school mathematics curricula in international comparative investigations is a story of increased efforts to take aspects of curriculum complexity into account. It is also, however, a story of persistent failure to probe sufficiently below the surface of, and to challenge assumptions about, what is to be understood as curriculum. (Keitel and Kilpatrick, 1999, p. 242)

Keitel and Kilpatrick argue that internationally equivalent instruments need to be developed, that is instruments that have 'internationally-comparable' relations to the national curricula. Otherwise international studies would continue, in the words of Torsten Husén, to 'compare the incomparable'. They criticise that the present standard of measurement (a common set of tasks organised by content) leaves no allowance for 'different aims, issues, history and context across the mathematics curricula of the systems being studied', which leads to the situation, in their view, that 'no one really addresses how well the students in a system are learning the mathematics curriculum that their system has offered them' (Husén, 1983, 243).

The critiques of positivism argue that there is a fundamental difference between the study of natural objects and human beings, in the sense that human beings interpret situations themselves and give meanings to them. Some authors go so far as to argue that any worthwhile sociological explanation must be related to the actual ways in which people themselves interpret their situations. This has major implications for the conduct of research in the sense that it requires researchers to observe and interact/communicate with the subjects of their research. Blumer asserts that:

… one would have to take the role of the actor and see his world from his standpoint. This methodological approach stands in contrast to the so-called objective approach …, namely that of viewing the actor and his action from the perspective of an outside, detached observer







… the actor acts towards his world on the basis of how he sees it and not on the basis of how that world appears to the outside observer. (Blumer, 1971, p. 21)

It also reflects the belief that curricular activities are 'culturally embedded'. Cogan and Schmidt (1999) suggest that 'not only is teaching an activity embedded in culture, but so is what is taught' (p. 77). Kawanaka et al. (1999) also point to the 'cultural' side of classroom practices and emphasise that:

Teaching and learning, as cultural activities, fit within a variety of social, economic and political forces in our society. Every single aspect of mathematics education, from a particular teacher behaviour to national policy, must be considered and evaluated within a socio-cultural context. (Kawanaka et al., 1999, p. 103)

How do researchers understand different pedagogies in the light of different educational 'cultures'? If they believe that pedagogy is 'culturally embedded', what are the cultural, intellectual and philosophical underpinnings that influence teaching in different countries? These and more questions have to be researched and answered, if the aim is to develop an understanding of curricular processes in different countries.

Furthermore, reducing methodology to a series of techniques, carried out in specific stages, as is often done in quantitative studies (for example, conduct of interviews, design of questionnaires), plays down the importance of social processes and the context of research (Vulliamy et al., 1990). Burgess (1984) argues that this misrepresents the nature of social scientific enquiry and the practice of research. He suggests that:

Recent developments in research methodology indicate that 'methodology' involves a consideration of research design, data collection, data analysis, and theorising together with social, ethical and political concerns of the social researcher. In short, research is no longer viewed as a linear model but as a social process. … Accordingly, questions now need to be raised about the actual problems that confront researchers in the course of their investigations and some

consideration needs to be given to the ways in which techniques, theories and processes are developed by the researcher in relation to the experience of collecting, analysing and reporting data. (Burgess, 1984, p.2)

For qualitative researchers, the social nature of the research process is likely to be more obvious from the outset, given the usually extended communication periods between the researcher and the researched. For example, educational ethnographers, such as Woods (1986), or case-study researchers, such as Walker (1985), tend to draw more on personal experiences than quantitative researchers. This reflects the extent to which research procedures differ according to the context of their use (Vulliamy et al., 1990).

**The search for equivalence**

Warwick (Warwick and Osherson, 1973) compares the choice of the research method to an accessory lens for a camera, where it depends on the type of picture that the researcher wants (broad panoramic or intense concentration on detail). In this light the comparative perspective serves as a systematic setting in which to evaluate a theory formulated in terms of common factors, and the comparative focus becomes central when an attempt is made to come to terms with social reality in a specific setting and when differences are of interest for their impact on society (Hantrais et al., 1985).

Warwick and Osherson (1973) consider the core issue in comparative research methods that is equivalence – 'how to study the same problem in different societies and cultures'. They regard this as the 'central theoretical and methodological question raised by cross-societal comparisons' (p. vii). Furthermore, they explore the various facets of equivalence. Conceptual equivalence refers to the question of whether the concepts under study have equivalent, or any, meaning in the cultures which are being considered. Some concepts have meaning in many but not all cultures and settings. A major challenge of comparative research is to provide conceptual understandings that have equivalent, though not necessarily identical, meanings in the settings under study. Probing of subjects/participants for their meaning and long periods in the field (in order to get to know the







context in which subjects/participants are working) are two of the possible 'solutions' suggested by the literature.

Another problem is that of equivalence of measurement, which involves the challenge of developing equivalent indicators for the concept under study. Theoretically applicable concepts may differ in their salience for the culture as a whole, or respondents might be unwilling or unable to discuss sensitive topics. Warwick and Osherson (1973) suggest that the question format should be flexible (as in semi-structured interviews).

There is also the problem of linguistic equivalence. Linguistic equivalence refers to the problem of translation itself. Warwick and Osherson (1973) assert that the issue of linguistic equivalence is inseparable from the theory and concepts guiding the study, the problem chosen and the research design. They offer suggestions with a view to setting the problem within the broader framework of conceptualisation and research design. Amongst them are: that the research problem should be salient to the cultures involved; that the primary emphasis in translation should be on conceptual equivalence ('comparability of ideas'); that many problems of translation could be avoided by advance familiarity with the cultures under study and that conceptual-linguistic equivalence can be improved through extensive pre-testing in the local culture, in particular qualitative pre-tests. Basically, as Warwick and Osherson (1973) point out, conceptual equivalence, equivalence of measurement and linguistic equivalence are 'tightly interwoven and should be treated as a single fabric'.

We have argued elsewhere (Pepin, 2000 and 2002), and exemplified in other studies (Pepin, 1999a and b; Pepin and Haggarty, 2001; Haggarty and Pepin, 2002), in which ways a qualitative approach can contribute to comparative research design, and how to address the problems of equivalence. In those papers it has been asserted that, whichever methodology is employed, using research strategies cross-nationally highlights problems of culture, language and communication, which infuse all aspects of the research. Cross-national studies have to grapple with language and communication problems intensively at the stage of formulating problems and defining the meaning of concepts and interpreting findings. The advantage of qualitative

studies, we have argued, lies in their in-built potential for establishing conceptual equivalence by their underpinning epistemology which has implications for the conduct of the research and the strategies used. Researchers usually stay in the field for long periods of time, which gives them the opportunity to become 'enculturated' and probe for meanings, which in turn is likely to help in understanding the context under study, counter threats to the validity of the findings and help to establish conceptual equivalence.

**A different view of 'comparing'**

To develop a coherent and 'complete' picture of research design and methodology in comparative international research education is a difficult undertaking. It could be argued that this is due to various factors and we suggest reasons why this might be the case (adjusted from Banathy, 1991):

(1) a disconnected or 'piecemeal' approach to research design;
(2) the apparent failure to integrate new ideas from other research areas (i.e. science);
(3) the fragmented discipline-by-discipline study of education;
(4) reductionist approaches to research design;
(5) thinking within existing boundaries.

The reader might usefully ask for a brief explanation of the above-mentioned factors and why they seem valid reasons. However, we would like to refer the reader to the ideas explained below which relate to some of the above-mentioned reasons (and leave the rationale for a later stage). We would like here to concentrate on the main intention of this chapter, which is to look at the place of studying 'anomaly' in comparative education research design.

The word 'anomaly' is described in the dictionary (Hanks, McLeod and Urdang, 1979) as a 'deviation from the normal or usual order' or 'irregularity'. The word 'anomalous' comes from the Late Latin 'anomalus' or Greek 'anomalos' which means 'uneven, inconsistent' (as 'homos' means 'one and the same'). In statistical language it is likely to mean the 'outliers', the 'odd one out' from the rule or law or theory that was established.







Whilst acknowledging the advantages of qualitative research designs in cross-national comparative studies, the main purpose of this chapter is to show that it is not enough to consider qualitative approaches in our endeavour to achieve equivalence. In fact, we argue that it is not sufficient to consider comparing 'like with like', indeed it might be impossible. We propose that it might be beneficial to study 'anomalies', that is cases that do not compare with others, in order to 'define' the boundaries of our developing theories and thus help to deepen our understanding.

Discovery commences with the awareness of an anomaly, i.e. with the recognition that nature has somehow violated the paradigm-induced expectations that govern normal science. It then continues with a more or less extended exploration of the area of anomaly. And it closes only when the paradigm theory has been adjusted so that the anomalous has become the expected. (Kuhn, 1970, pp. 52, 53)

Doll (1986) contends that a mechanistic view of education has dominated western thought right through to the twentieth century, and that this worldview underpinned a positivist view of knowledge. Central to this approach, he asserts, is an emphasis on analysis to identify independent elements or parts that equate to variables in experiments. Only over the past 20–30 years has another paradigm, namely postmodernism, become a popular way of thinking, contradicting the notion of objective reality and promoting 'pragmatic doubt' (Doll, 1993, 61). Thomas (1998) provides an interesting account of the 'myth of rational research'.

In its flight from 'positivism' educational enquiry still cleaves to a faith in the ordered and the rational. Educationists continue to believe in an order, accessible via rational inquiry and ordered reflection, governing human affairs and thought. This belief has … unwelcome consequences. First, it promotes the notion that certain rationalistic ingredients are obligatory in research: a technology of inquiry is thus constructed and maintained. Consequently, inquiry (even interpretative inquiry) is formulaic; it follows predictable ruts and leads often to uninteresting findings. Second, a belief in the ordered mind leads to a faith in certain models of mind, and in 'personal

theory' which can be developed via particular and orthodox methods of finding out …(Thomas, 1998, p. 141)

It appears that the most dominant paradigm for educational thought over the last century has been the mechanistic view and ways of thinking, influencing ideas of teaching and learning (Hoban, 2002) and theories of their measurement in research design.

The mechanistic worldview assumes that reality can be observed, explained and predicted. According to Doll (1993) the legacy of a mechanistic worldview is a narrow-minded methodology for conducting research that produces prescriptive knowledge, and subdivided and disconnected parts to be taught to learners.

It is Newton's metaphysical and cosmological views – not his scientific ones – that have dominated modern thought for so long, providing a foundation in the social sciences for causative predictability, linear ordering, and a closed (or discovery) methodology. These, in turn, are the conceptual underpinnings of scientific (really scientistic) curriculum making. (Doll, 1993, p. 34)

Schön (1983) called this way of thinking 'Technical Rationality' (p. 21). He claimed that it evolved from positivism and called it 'positivist epistemology of practice' (p. 31). Furthermore, he contends that university researchers need to acknowledge the 'complexity, uncertainty, instability, uniqueness, and value-conflict' (p. 39) of work settings. 'Discrete' knowledge, 'discretely' researched and 'packaged' in separate boxes, presents a technical view of education and does not encourage a way of thinking that considers how aspects of education have an effect on one another. In addition, presenting educational knowledge in discrete packages can lead to a misrepresentation of the complexity evident in the phenomena that are being researched, and the ways they can be understood.

Thus, it is proposed that the technical view of knowledge and discovery appears to be an attempt to force phenomena of teaching and learning, and their investigation, into conceptual and separate boxes supplied by the research community. We argue that research should also proceed without such boxes, i.e. by studying anomalies,







whatever the element of arbitrariness in their historic origins and, occasionally, in their subsequent development.

This sense of arbitrariness, however, does not mean that the researchers studying 'without such boxes' could practise without some set of considered beliefs. Questions like the following have to be answered when studying an anomaly:

1. What are the 'fundamental entities' of which this case is composed?
2. What are the links/relationships between the entities within the case?
3. How does this anomalous case interact with the other 'rule-like' cases?
4. In which ways does this case help to understand the boundaries of the developing theory, and indeed the theory itself?

Feyerabend (1993) suggests that we may use 'counterrules': hypotheses that contradict well-informed theories. He advises us to proceed counterinductively. This, of course, gives rise to various questions, in particular to those of whether and under what conditions it is appropriate/useful/reasonable to use counter induction rather than induction. He claims that:

The evidence that might refute a theory can often be unearthed only with the help of an incompatible alternative. … Also, some of the most important formal properties of a theory are found by contrast, and not by analysis. A scientist who wishes to maximise the empirical content of the views he holds and who wants to understand them as clearly as he possibly can must therefore introduce other views; that is, he must adopt a pluralistic methodology. (Feyerabend, 1993, pp.20, 21)

This rings true with much of what comparative education can offer as potential. We can consider the very act of comparing, for example. Rather than being a second-order activity, comparison is, arguably, central to the very act of knowing and perceiving and therefore likely to be essential to social scientific analysis (Hantrais and Mangen, 1996).

Feyerabend (1993) contends that we must compare ideas with other ideas (rather than with 'experience') and must try to improve rather than discard the views that have failed in the competition.

Knowledge so conceived is not a series of self-consistent theories that converges towards an ideal view; it is not a gradual approach to truth. It is rather an ever increasing ocean of mutually incompatible alternatives, each single theory, each fairy-tale, each myth that is part of the collection forcing the others into greater articulation and all of them contributing, via this process of competition, to the development of our consciousness. (Feyerabend, 1993, pp. 21)

**Conclusions**

International and comparative research in education, in particular in mathematics education, has been commonly wedded to large-scale achievement studies that are, allegedly, generalisable and can inform policy-makers. In terms of the general features of the research design, it is claimed that one has to compare 'like with like', one has to establish equivalence. These studies have been criticised by many authors, in particular on the basis that the research design does not take account of the cultural embeddedness of the phenomena under study. This has been addressed by researchers who have implemented more qualitative approaches into their design.

In this paper it has been argued that it is not enough to implement qualitative approaches in comparative research design. Indeed, we believe that an important aspect of developing 'comparative' theory is to consider the study of 'anomalies', in order to help us to develop a deeper understanding of the phenomena and delineate the boundaries of our theories.

**References**

Banathy, B.H. (1991) Systems Design of Education: A Journey to Create the Future, Englewood Cliffs, NJ: Educational Technology Publications.

Beaton, A.E. and Robitaille, D.F. (1999) An Overview of the Third International Mathematics and Science Study, in G. Kaiser, E. Luna and I. Huntley (eds) International Comparisons in Mathematics Education (pp.30–47). London: The Falmer Press.







Blumer, H. (1971) Sociological Implications of the Thought of George Herbert Mead, in Cosin, B. and Dale, I.R. (eds), School and Society (pp.11–17). London: Routledge and Kegan Paul/Open University Press.

Burgess, R.G. (1984) The Research Process in Educational Settings: Ten Case Studies. Lewes: The Falmer Press.

Burgess, R.G. (1985) Strategies of Educational Research: Qualitative Methods. Lewes: The Falmer Press.

Clarke, D. (2003) International Comparative Research in Mathematics Education, in Bishop, A., Clements, M.A., Keitel, C., Kilpatrick, J. and Leung, F.K.S. (eds) Second International Handbook of Mathematics Education, Part One, Dortrecht/Boston/London: Kluwer Academic Publishers.

Cogan, L.S. and Schmidt, W.H. (1999) An Examination of Instructional Practices in Six Countries, in Kaiser, G., Luna, E. and Huntley, L. (eds), International Comparisons in Mathematics Education. (68–85). London: The Falmer Press.

Crossley, M. and Vulliamy, G. (1984) Case-study Research Methods and Comparative Education. Comparative Education, 20, pp.193–207.

Feyerabend, P. (1993) Against Method, 3rd edn, London: VERSO.

Freudenthal, H. (1975) Pupils' Achievement Internationally Compared – The IEA, International Studies in Mathematics, 6, pp.127–86.

Haggarty, L. and Pepin, B. (2002) An Investigation of Mathematics Textbooks and Their Use in English, French and German Classrooms: Who Gets an Opportunity to Learn What?. British Educational Research Journal 28 (4).

Hanks, P., McLeod, W.T. and Urdang, L. (1979) Collins Dictionary of the English Language, 2nd edn, London: Collins.

Hantrais, L., Mangen, S. and O'Brien, M. (eds) (1985) Doing Cross-national Research. Birmingham: Aston Modern Languages Club.

Husén, T. (1967) International Study of Achievement in Mathematics: A Comparison of Twelve Countries (Vols. 1 and 2). New York: Wiley.

Husén, T. (1983) Are Standards in US Schools Really Lagging Behind Those of Other Countries? Phi Delta Kappan, pp.455–61.

Kawanaka, T., Stigler, J.W. and Hiebert, J. (1999) Studying Mathematics Classrooms in Germany, Japan and the United States: Lessons from TIMSS Videotape Study, in G. Kaiser, G., Luna, E. and Huntley,I.

(eds), International Comparisons in Mathematics Education (pp. 86–103) London: The Falmer Press.
Keitel, C. and Kilpatrick, J. (1999) The Rationality and Irrationality of International Comparative Studies, in Kaiser, G. Luna, E. and Huntley, I. (eds), International Comparisons in Mathematics Education (pp. 241–56) London: Falmer Press.
Kuhn, T.S. (1970) The Structure of Scientific Revolutions, Chicago, Ill.: University of Chicago Press.
Letendre, G., Baker, D., Akiba, M., Goesling, B. and Wiseman, A. (2002) Teachers' Work: Institutional Isomorphism and Cultural Variation in the US, Germany and Japan. Educational Researcher, 30(6) pp. 3–15.
NFER (1997) Third International Mathematics and Science Study. Second National Report (S. Harris, W. Keys, and C. Fernandes), Part 1. Slough: NFER.
Pepin, B. (1999a) Mobility of Mathematics Teachers Across England, France and Germany. European Educational Researcher, Volume 5 (1), pp. 5–14.
Pepin, B. (1999b) The Influence of National Cultural Traditions on Pedagogy: Classroom Practices in England, France and Germany, in Leach, J. and Moon, B. (eds), Learners and Pedagogy (pp. 124–35) London: Sage Publications.
Pepin, B. (2000) Reconceptualising Comparative Education: The Case of International Studies in Mathematics Education, Pedagogy, Culture and Society, 8 (3).
Pepin, B. and Haggarty, L. (2001) Mathematics Textbooks and their Use in English, French and German Classrooms: A Way to Understand Teaching and Learning Cultures, Zentralblatt for the Didactics of Mathematics, 33 (5).
Pepin, B. (2002) Methodological Issues of Cross-National Comparisons: Efforts to Establish Equivalence in a Cross-National Study of Mathematics Teachers' Work in England, France and Germany, in Fries, A., Rosenmund, M. and Heller, W. (eds) Comparing Curriculum Making Processes (in the series "Explorations"), Zurich: Peter Lang.
Schmidt, W.H., McKnight, C.C., Valverde, G.A., Houang, R.T. and Wiley, D.E. (1997) Many Visions, Many Aims, Volume 1: A Cross-National Investigation of Curricular Intentions in School Mathematics. Dordrecht: Kluwer Academic Publishers.







Shimhara, N. and Sakai, A. (1995) Learning To Teach in Two Cultures, New York: Garland Publishing.
Spindler, G. (1987) Cultural Dialogue and Schooling, in Schoenhausen and Roseville: A comparative analysis. Anthropology and Education Quarterly, 18(1) pp. 3–16.
Stigler, J. and Hiebert, J. (1997) Understanding and Improving Classroom Mathematics Instruction: An Overview of The TIMSS Video Study. Phi Delta Kappan, 79 (1) pp.14–21.
Stigler, J.W. and Perry, M. (1988) Cross Cultural Studies of Mathematics Teaching and Learning: Recent Findings and New Directions, in. Grouws, D.A, Cooney, T.J., and Jones, D. (eds), Perspectives on Research on Effective Mathematics Teaching (pp. 194–223). Reston, VA: National Council of Teachers of Mathematics.
Thomas, G. (1998) The Myth of Rational Research, British Educational Research Journal, 24 (2).
Tobin, J., Wu, D. and Davidson, D. (1989) Preschools In Three Cultures: Japan, China and The United States, New Haven, CT: Yale University Press.
Travers, K.J. and Westbury, I. (1989) The IEA Study of Mathematics, I: Analysis of Mathematics Curricula, Oxford: Pergamon Press.
Vulliamy, G., Lewin, K. and Stephens, D. (1990) Doing Educational Research in Developing Countries: Qualitative Strategies, London: The Falmer Press.
Walker, R. (1985) Doing Research: A Handbook For Teachers. London: Methuen.
Warwick, D. and Osherson, S. 1973). Comparative Research Methods: An Overview. Englewood Cliffs: Prentice-Hall.
Westbury, I. (1992) Comparing American and Japanese Achievement: Is the United States Really a Lower Achiever? Educational Researcher, June-July (pp. 18–24).
Woods, P. (1986) Inside Schools: Ethnography In Educational Research. London: Routledge and Kegan Paul.

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# EXPLORING CHILDREN'S CONCEPTIONS OF ZERO IN RELATIONSHIP TO OTHER NUMBERS

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The paradoxes posed by an innocent number (zero), rattling even this century's brightest minds and threatening to unravel the whole framework of scientific thought. … It (zero) provides a glimpse of the ineffable and the infinite. That is why it has been feared and hated – and outlawed. (Seife, 2000, p. 2)

***Abstract***
*This paper is part of a larger project to investigate children's conceptions of zero. It arose from the researcher's specific interest in noting, during many years in primary education, that zero created its own mathematical problems for children and for teachers. Examples, provided by teaching staff, illustrated where zero created cause for concern in various areas of children's mathematics. These examples mirrored this researcher's experiences. When it is admitted that researchers tend to leave out the zeros knowing that they cause problems (Guedj, 1996; Suydam and Dessart, 1976) then it is not surprising that there is a noticeable lack of research on children's understanding of zero. This paper concentrates on one aspect of the research findings; children's reaction to zero in relationship to other numbers.*

**Keywords:** conceptions of zero, primary education







**Introduction**
Schools for this study were selected on the basis that the children would be affected by internal and external factors. Within the school there would be a variety of teaching styles and mathematical activities influenced by a range of professional interpretations by teachers with their own mathematical ability and philosophical viewpoint. In their editorial article, on 'Effective Educational Research', Riding and Wheldall saw the value of pure research carried out with atypical subjects; however, they were 'pressing for more research with normal children in normal schools' (Riding and Wheldall, 1981, 6).

Five 'normal' primary schools were selected. The schools were situated within a square mile, in northern England. The questionnaire, previously piloted in a different school, was given to the Year 6 (Y6) children (aged 10–11 years) in the five selected schools, in July 2002. In total there were 100 returns. On analysis no one school showed any significant, singular result, indeed quite the opposite in that there were many common features.

In his work on children's errors in algorithms, Engelhardt acknowledges his research shortcomings in that he did not include talking to the child. He felt that examining only written performance without the opportunity to investigate a given error further, greatly increased the possibility of misjudging a pupil's erroneous approach (Engelhardt, 1977). While the questionnaire produced valuable quantitative data its limitations as a method of data collection needed to be addressed. The opportunity to gain insight into children's thinking came from the use of interview-tasks undertaken in one of the original five schools; Research School A in July 2003, with twenty children in Y6 (10–11 year olds). Throughout the tasks the children were encouraged to explain and illustrate their answers. In turn this provided an opportunity for the researcher to observe each child and to ask probing questions. Experience has led the researcher to be in full agreement with Ginsberg that errors are seldom capricious or random. He goes on to say:

Typical children's errors are based on systematic rules … Children's faulty rules have sensible origins. Usually they are distortions or

misinterpretation of sound procedures (Ginsberg, in Byers and Erlwanger, 1977, p. 274).

Here Ginsberg is speaking of errors; this researcher also believes that the correct answer can also be the result of distortion and misinterpretation. The children were encouraged to explain all answers, whether or not the researcher knew them to be correct.

Exploring children's number ordering – including zero
The introduction of The National Numeracy Strategy (NNS), (DfEE, 1999) is said to have led to significant changes in primary mathematics teaching in England (Earl et al., 2003). One of the NNS recommendations, pertinent to this research paper; is the display and use of a number line, including the zero number symbol, in each classroom.

The reasons for including ordering numbers were to gain insight into how the children viewed zero within the number order and to explore their understanding of zero's relationship to other numbers, but without the involvement of any calculations or algorithms.
The sets of numbers used were:
• small single digits
• simple fractions plus zero
• decimals plus zero

The same series of numbers, in the same order as in the questionnaire, were presented to the children in the Y6 task-interview situation. This meant there was a comparison between the raw and the collated data from the questionnaire and that collected from the interview-task. In the interview-task each of the numbers was written on a piece of card and the child was asked to read each symbol. If the child could not read the symbols and did not appear to be familiar with fractions or decimals then the researcher did not continue with the relevant question. The interview-task allowed participants to expand on their answer, to provide an explanation as to how the answer was reached. It also allowed the researcher to ask questions in order to gain clarification. Whatever the answer, correct or not, the child was asked for an explanation.







The numbers and the order presented in the questionnaire and in the interview-tasks were:

Table 1. Interview tasks

| | | | | | |
|---|---|---|---|---|---|
| (a) | 1/4 | 1/2 | 2 | 1 | 0 |
| (b) | .3 | .4 | 0 | .5 | .1 |
| (c) | 1/4 | 3/4 | 1/4 | 0 | 1/2 |
| (d) | 0.4 | 5 | 1.2 | 8 | 0 |
| (e) | 3 | 0 | 5 | 4 | 7 |
| (f) | 8 5 7 1 0 4 3 2 9 6 | | | | |

**Collating and classifying the data**
When classifying the answers there was an acute awareness that the results contained the child's depth of understanding of fractions and decimals as well as that of zero. An example of this could be seen in some of the answers given to question (c) ordering:
¼ ¾ ¼ 0 ½

**Table 2.** Options

| | |
|---|---|
| option 31…... | 0 ¼ ½ ¾ |
| option 32….. | 0 ½ ¼ ¾ |
| option 33….. | 0 ¼ ¾ ½ |
| option 34…... | 0 ¾ ¼ ½ |
| option 35….. | 0 ¾ ½ ¼ |

The mis-ordering of the fractions and decimals had to be taken into consideration when isolating the child's positioning of zero from the other factors. Options 31 to 35 (Table 2) place zero at the beginning of the order sequence. This appeared to be the important element as far as this study was concerned. The answers in Table 2 were classified as 'zero then fractions'; it was this element, regardless of the fraction order, which was noted as significant.

In the process of the explanation some children changed their mind and re-ordered the cards. The numbers shown in brackets are the first answers. There was no apparent pattern in the change, either with individual children or with specific explanations. Some changes were from correct to incorrect answers, some from incorrect to correct. In many of the sections there is strong connection between the percentage of answers from the questionnaire and the first answer given in the task-interview. This is especially noted in the first fraction and the first decimal ordering: Table 4 and Table 8 In the second fraction and second decimal question: Table 5 and Table 9 the connection is not as close. The researcher felt that this was due to the task interview children explaining their reasons and often noting their error. This then had an effect on their approach, later in the interview, to the second, related question. In this second question some children made reference to the first question and gave the same rationale. While the comparison of questionnaire and interview-task results in the second question are not as strong they are still noteworthy.

**Data analysis of the results**
The analysis was undertaken in three sections:
a) Fractions and Zero
b) Decimals and Zero
c) Single Digit Numbers and Zero

In relation to Fractions and Zero two ordering questions contained fractions.
**Question A** – fractions and whole numbers plus zero
**Question C** – simple fractions plus zero

**Question A** ½ ¼ 2 1 0
Two frequent answers, in both the questionnaire and the task-interview were:

**Table 3.** Frequent answers to Question A

| | Questionnaire | Task-interview |
|---|---|---|
| (zero/fractions/whole numbers) | 43% | (45%) 60% |
| (fractions/zero/whole numbers) | 48% | (40%) 25% |







The strong link between the questionnaire and task-interview first answers is reassuring from the aspect of methodology selection. However, it is not possible to know the rationale for the questionnaire answers. It cannot be assumed that there is a connection between their reasoning and the reasons given by the children in the task-interview group.

Reviewing the rationale for the first set of answers:

**Table 4.** Rationale for first set of answers to question A

| | Questionnaire | Task-interview |
|---|---|---|
| (zero/fractions/whole numbers) | 43% | (45%) 60% |
| (fractions/zero/whole numbers) | 48% | (40%) 25% |

Of the 60% of children in the task-interview who gave the answer as zero/fractions/whole numbers the reasons given by 70% of these children involved comparing the size of zero with the other numbers.

- zero is lower than 1, zero is a whole one lower than one
- ¼ and ½ are more than zero
- 0 is nothing, ¼ is something, zero isn't anything
- ¼ is more than nothing

Some of the explanations given during the task-interview were similar but the outcome, the number order, differed. One rationale, that zero is a **whole number**, was used by two children to explain

a) why 0 went before the fractions **0 ¼ ½ 12**, zero is a whole number, zero has no parts to it
b) why 0 went after the fractions ¼ ½ 0 1 2, these ¼ ½ are parts of a whole, 0 is a whole number

Analysis of the rationale for the second set of answers:

**Table 5**. Rationale for second set of answers to question A

| | Questionnaire | Task-interview | |
|---|---|---|---|
| **(zero/ fractions/whole numbers)** | **43%** | **(45%)** | **60%** |
| (fractions /zero/whole numbers) | 48% | (40%) | 25% |

Observing the children ordering the cards the initial reaction of many children was to place **fractions/zero/whole numbers.** The explanations were not that children were putting zero after the fractions but that they saw **0 as being before 1.** Putting zero next to one was often the initial reaction but, as seen in the chart, on reflection a number of children changed their order to zero then fractions.

- you have to have 0 in front of 1
- it goes 0 then 1, then 2
- because 0 is more than ¼, zero comes before one
- zero is after ½ and zero is the next smallest, then you get 1, like with 0,1,2,3

The three children who gave answers other than option 1 and option 3 (see Table 5) also used the zero before one rationale.

- 0 1 2 ½ ¼ 0 always goes first, before 1 and then 2
- 0 ¼ 1 ½ 2, zero is before 1, but one quarter ¼ comes in between.
- 0 1 ¼ ½ 2 zero is before 1 then ..reads ¼ as one and a quarter, ½ as one and a half, then 2

**Question C** ¼ ¾ ¼ **0** ½

The frequent answers, in both the questionnaire and the task-interview were classified as:

**Table 6**. Children's explanations for question C

| | Questionnaire | Task-interview | |
|---|---|---|---|
| zero then fractions | 52.5% | (75%) | 80% |
| fractions then zero | 44.4% | (25%) | 20% |







Looking at the children's explanations for the first of these classifications, zero then fractions, 81% of the reasons given by the children involved **comparing the size of zero to the other numbers:**

- zero is nothing [child draws the fractions as fractions of a circle] you've got none of the circle there
- because these ¼ ½ ¾ are more than zero
- zero has no parts to it

The other reasons for putting zero then the fractions were:

- zero always goes first
- because 0 is not a number. I'm putting it at the front but it could go at the end
- Not sure where the zero goes

The children's explanations for the second classification were most revealing.

**Table 7**. Children's explanations for question C

| | Questionnaire | Task-interview | |
|---|---|---|---|
| zero then fractions | 52.5% | (75%) | 80% |
| **fractions then zero** | **44.4%** | **(25%)** | **20%** |

While question A included fractions and single digits (½ ¼ 2 1 0) question C contained only simple fractions plus zero (¼ ¾ ¼ 0 ½). The reason for this was to remove the temptation for the child to put 0 and then to put 1. Nevertheless the most common reason given for putting the 0 after the fractions was linked to the **number line order of zero, one, two, three,** etc.

- zero is more than the fractions, it starts the whole numbers, zero, one, two, three ….
- ¾ is the smallest, then we go bigger up to zero, then one, two and we count up.
- zero always goes first. When you have loads of numbers then 0 (pointed to the symbol on the card), you always write this first. It's like zero, one, two, three, four, five, six, seven, eight, nine, ten. You always say zero first.

The other explanations for putting the fractions then zero made reference to the '**nothingness**' of zero:

- Nought is just nothing (puts the zero after the fractions) – it could go anywhere. (Why?) because 0 is not a number. I'm putting it at the front but it could go at the end (Why?) – because it is nothing, so it can go anywhere. It doesn't mean anything so it doesn't matter where it goes.
- Not sure where the zero goes. (Why not?) – I could put it there (Moves the card to the end of the sequence). (Could you put it anywhere else) Yes, there (points to the space between two fractions), or there (points to the space between two other fractions).(Why could it go in all these different places?). It's nothing so it can go anywhere – it's nothing so it makes no difference where you put it.
- Nought is just nothing (puts the zero after the fractions) – it could go anywhere. (Why?) Because it is nothing. (Here the 'nothing' was said with great emphasis.

These children saw zero, the symbol 0, as being nothing, of no importance. Thus it does not matter where it goes or even if it is left out.

The following questions relate to Decimals and Zero:

**Question B .3 .4 0 .5 .1**
The frequent answers, in both the questionnaire and the task-interview were classified as:

**Table 8**. Frequent answers to question B

| | Questionnaire | Task-interview |
|---|---|---|
| Zero then decimals | 49.5% | (45%) 55% |

While explanations of the decimal and zero sequence sometimes threw light on a child's understanding of zero the also linked closely with the child's understanding of decimals. It proved difficult to differentiate between the two. What was of interest was the reference to zero being a whole number, particularly as this was used as the main reason for thinking the decimals were smaller than zero:







- .1 .3 .4 .5 0 because zero is a whole number
- 0 .1 .3 .4 .5 zero comes first then you have zero point 1 (0.1) then you go up to 0 on its own. The number before the point is a whole one. A whole one, what do you mean? A whole number like one and two.

**Question D 0.4 5 1.2 8 0**
The top three order of frequency of answers, in both the questionnaire and the task-interview, were:

**Table 9**. Frequency of answers to question D

| | Questionnaire | Task-interview |
|---|---|---|
| zero then decimals | 60.8% | (60%) 55% |
| **Zero amongst the decimals and whole numbers** | **28.9%** | **(30%) 35%** |
| decimals then zero | 2% | 5% |

It was surprising to note the high number of responses which placed zero amongst decimals (0.4, 0, 1.2, 5, 8) with 0 coming after 0.4. Here the question of whether zero was **a whole number** was again raised; these were not the same children who had voiced their thoughts on this issue in the fractions sequencing.

- puts zero then 0.4, changes to 0.4 then zero. (You don't seem too happy with the answer?) I don't know if the zero is a whole number or not. (Points to the 0 card.)
- 0.4 then zero, changes to put zero then 0.4, (Are you unsure?) I think that's right, it depends if zero is whole one. (A whole one? Can zero be one? – child laughs.) I mean a whole number.
- zero then 0.4, then changes to 0.4 then zero, says 0.4 (nought point four) is smaller than zero. Why did you change? Well, I thought that zero is a whole number so a decimal must be smaller.

The following questions relate to Single Digit Numbers and Zero

**Question E 3 0 5 4 7**
**Table 10**. Responses to question E

| Results of Ordering Numbers – Question E | | |
|---|---|---|
| Question E 3 0 5 4 7 | Questionnaire | Task-interview |
| Total number of responses | **96** | **20** |
| option 56…..0 3 4 5 7 | **92** | **20** |
| Spoiled answers | 4 | |

**Question F 8 5 7 1 0 4 3 2 9 6**
**Table 11**. Responses to question F

| Results of Ordering Numbers 7 Question F | | |
|---|---|---|
| Question F 8 5 7 1 0 4 3 2 9 6 | Questionnaire | Task-interview |
| Total number of responses | **96** | **20** |
| option 62…..0 1 2 3 4 5 6 7 8 9 | **92** | **20** |
| Spoiled answers | **4** | |

The ordering of the single digits produced 100% correct answers (excluding the spoiled answers). When the children were asked for their reasons for placing zero 0 3 4 5 7 and 0 1 2 3 4 5 6 7 8 9 the answers were all connected with the number order:

- You say zero, one, two, three (etc)
- It's that order on the number line

In the interview-task sessions each child was asked "Could you put the zero card in any other place?" In question E (Table 10) the answer was always "no". However, in question F (Table 11) where all the single digits were involved, three children said "yes".

Two children said that the zero (0) could go after the 9. This gave 1234567890, the two explanations were:

- because when we were in Mrs ~ class we had a number line on the wall that had 0 at that end.







- I've seen 1234567890 in lots of places – it's on the computer keyboard.

One child said that
• You could leave it out [child removed the 0 card] or it could go anywhere [child moved the 0 card to different places between the other digits. Asked why the child explained that] because zero is worth nothing it doesn't matter where it goes or whether you put it in or not. [Asked if you could do this with other numbers the child said] No, because they're something and zero is nothing.

**Summary and discussion of the main points**
The teaching and learning of early number is a complicated, multifaceted task. Part of this is the learning of the number order, that is to recite the number words (zero), one, two, three, four, etc. and the knowing of the symbol order (0) 1, 2, 3, 4, etc.

Most mathematical educators are reluctant to discuss memory when considering mathematical learning and understanding, Gagne (1970) being one of the few exceptions. This researcher agrees with Morris (1981) that reliance upon memory can have deleterious effects, as reliance upon memory adds significantly to 'mathematics anxiety', especially when memorisation replaces understanding as this often results in confusion. Freedmont (in Byers and Erlwanger, 1971) goes to one extreme when he described rote learning as one of the time-honoured enemies of effective mathematics learning. At the other extreme is Krutetskii (1976) who considers mathematical memory to be one of the abilities which distinguish mathematically capable from mathematically incapable students.

This researcher's experience is in keeping with Byers and Erwanger (1985) as they see the discrepancies between theory and practice, in that teachers in mathematics, far from ignoring memory, are very cognisant with the problems it presents. Classroom teaching and learning of mathematics departs significantly from what theorists have proposed. The main differences may be summed up in two words: repetition and practice.

The importance of memory for doing mathematics, from the lowest computations to the more sophisticated proofs is almost self-evident. The most crucial question is not whether memory plays a role in understanding mathematics but what it is that is remembered and how it is remembered by those who understand it – as well as those who do not. (Byers and Erwanger, 1985, p. 261)
When learning the number order and the number symbol sequence, memory is of paramount importance. The 1 to 10 range of number words is learned by rote with exposure to the sequence of number names and the experience of moderate amounts of sequence production activities (Fuson and Hall in Byers and Erwanger, 1985; Maclellan in Thompson, 1997). As a result many children can recite the number sequence to 100 by the time they are about six years of age (Maclellan, 1997). The learning of the number and order of the number names is dependent upon aural and verbal memory while the learning of the number sequence of the number symbols, the number line order, relies upon visual memory. The marrying of the number names and number symbols does require considerable effort 'because nine words and symbols have to be associated, and none of them is predictable from any of the others' (Wigley 1997, 116). Interesting Wigley speaks of nine symbols, one can only assume that he was not including zero.

As far back as 1883 Galton referred to the visual number line, 'persons who are imaginative almost invariably think of numerals in some form of visual imagery' (Thompson, 1990, 116). A hundred years later, Ernest (1983) found that, as a result of a questionnaire given to teacher training college staff, 65% had an internalised number line. All but 5% of these were straight-line number forms which he deduced were '… possibly stimulated by the greater use of graded rulers and physical number lines since Galton's time'. (Thompson, 1990, 116)

Two decades later many schools are following the NNS recommendation that each classroom displays a number line, including the zero number symbol. While the interview-tasks were conducted in a room other than the classroom, each of the research classrooms did have a number line on the wall. All the number lines began with 0 and there was a left to right ordering:

0 1 2 3 4 5 6 7 8 9 …







As has been seen in the ordering of fractions and zero a number of children placed the fractions in such a way as to preserve the 0, 1, 2, number order (as with, ½ ¼ 1 2 3). Zero was seen as a symbol with a specific place on the number line, next to the number one. An area of tension was created with the children who wanted to preserve this number order relationship but at the same time they used reasoning which would put other numbers between 0 and 1. This reasoning often contained a reference to the size of zero; its nothingness or to whether zero was 'a whole number'. These two aspects are discussed in the following sections.

Children, throughout the ordering tasks, raised the question as to whether or not zero was a whole number. To try to gain further insight into the children's understanding the Y6 task-interview children were put into random groupings of four/five children for an informal discussion. The debate centred round how one recognised whole numbers, where they were to be found and what their attributes were. The following statements represented the 'commonly agreed' opinions of the children.

What are whole numbers?

- Full ones, not part of something like fractions.
- Like the numbers on a number line …but negative numbers are not whole numbers… because the minus sign meant that they are less than zero.

The discussion moved on to the topic of fractions and whole numbers.

- Whole numbers are full numbers not bits like fractions.
- Whole numbers appear in front of fractions.
- You write it big in front of the fraction like one and a half, or three and a half. (The child wrote 1½ or 3½ ). But you don't write zero and a half. (The child wrote 0½, the other children laughed.)

The children were asked why you didn't write zero and a half? After a pause the responses were:

- You just don't. It's silly.
- You can cut whole numbers into pieces but you can't do that with zero. You can't cut nothing.

- You can't have half of nothing.

The children were undecided as to which aspects of zero to 'stress' and which to 'ignore' in which situation. Their explanations were logical within the limits of their knowledge and allowing for their ideas of keeping to the conventions they had learned, including the way to say, to do, to write and to order numbers. By association, if zero were a 'whole number' then '0', would be expected to behave like the other single digit number symbols. But they found areas where the '0' sign was not used while 1, 2, 3 … were. Within each group the children's reasoning moved from the abstract consideration of number symbols to the practical, concrete consideration of 'zero's worth'. The 'nothingness of zero' is considered in the final section.

This paper has concentrated on the children's ordering of number symbols and in their explanations of their number orders the term, 'worth nothing' was frequently used. The train of thought of a few children was reminiscent of the riddle of assumptions – a bird has wings, a bird can fly, a penguin has wings so a penguin can fly. This transferred to zero-as-zero is worth nothing, nothing is worthless and of no value, if it is worthless it is of no significance, if it is of no significance it has no effect, if it has no effect then it can be ignored. This concept of insignificance is expressed in the ordering of single digit numbers and with ordering fractions and zero. There is a tension between the children's conceptions and use of the symbol for zero and the 'nothingness' of the empty set which is seen when the abstract '0' symbol is explained in concrete terms using the problematic' zero language'. To quote Rotman, zero serves as:

… the site of an ambiguity between an empty character . . . and a character for emptiness: a symbol that signifies nothing. (Rotman, 1993, p. 26)

There is no intention, within this study, to assume generalisations beyond the area of data collection contained in the research. The aim is not to test a hypothesis but to undertake an exploratory investigation of an as-yet-uncharted area of student experience; the data being used to look at the nature of attitudes and trends. This paper reports on part of an ongoing research project containing other







areas of inquiry, to be reported elsewhere where. Early analysis of the data, from children aged 3 to 10, suggest that the notions of zero as 'empty', 'worthless' and 'insignificant' are high profile features.

**References**

Angheleri, J. (ed.) (2001) Principles and Practices in Arithmetic Teaching: Innovative Approaches for the Primary Classroom. Buckingham: Open University Pres

Aubrey, C. (ed.) (1994) The Role of Subject Knowledge in the Early Years of Schooling. London: The Falmer Press.

Bell, C.W. (1980) Arithmetic Teacher, 28, p. 21.

Blake, R. and Verhille, C. (1985) For the Learning of Mathematics, 5, pp. 35–47.

Byers, V. and Erlwanger, S. (1985) Educational Studies in Mathematics, 16, pp. 259–81.

Cagne, R.M. (1970) Conditions of Learning. Sussex: Holt, Rinehart and Winston.

Brown, A.J. and Dowling, P.C. (1998) Doing Research – Reading Research – A Mode of Interrogation for Education. London: The Falmer Press.

Catterall, R. (1999) The Number Pack – Practical Pre-School. Leamington Spa: Step Forward Publishing Limited.

Clark, A. and Moss, P. (2001) Listening to Young Children; The Mosaic Approach. London: National Children's Bureau for the Joseph Rowntree Foundation.

Cohen, L. and Manlon, L. (1994) Research Methods in Education. London: Routledge.

Cook, T.D. and Reichardt, C.S. (eds) (1979) Qualitative and Quantitative Methods in Evaluation Research. London: Sage.

Dickinson, L., Brown, M. and Gibson, O. (1984) Children Learning Mathematics: A Teacher's Guide to Recent Research. East Sussex: Holt, Rinehart and Winston.

Department of Education and Employment (DfEE) (1999) The National Numeracy Strategy for Teaching Mathematics Framework for Teaching Mathematics from Reception to Year 6. London: DfEE.

Dolan, C. (ed.) (1998) The Development of Mathematical Skills. Hove, East Sussex: Psychology Press.

Donaldson, M. (1987) Children's Minds. London: Fontana Press.

Earl, L., Levin, B., Leithwood, K., Fullan, M. and Watson, N. (2001) Watching and Learning 2; OISE/UT Evaluation of the Implementation of the National Literacy and Numeracy Strategies. Toronto: OISE/UT.
Engelhart, J. (1977) British Journal of Educational Psychology, 47, pp. 149–54.
Ernest, P. (1983) Mathematical Education for Teaching, 4, pp. 51–53.
Geldman, R. and Gallistel, C.R. (1978) The Child's Understanding of Number. Cambridge, MA: Harvard University Press.
Guedi, D. (1998) Numbers: The Universal Language (London: Thames and Hudson).
Hughes, M. (1986a) Children and Number: Difficulties in Learning Mathematics, Oxford: Basil Blackwell.
Hughes, M., Desforges, C., Mitchell, C. and Carre, C. (2000b) Numeracy and Beyond: Applying Mathematics in the Primary School. Buckingham: Open University Press.
Inhelder, B. and Piaget, J. (1967) The Early Growth of Logic. New York: Norton.
Kaplan, R. (1999) The Nothing That Is: A Natural History of Zero,. New York: Penguin.
Landau, E. (1954) Foundation of Analysis, The Arithmetic of Whole, Rational, Irrational and Complex Numbers. New York: Chelsea Publication Company.
Lappan G. (ed.), (1987) Arithmetic Teacher, 35, pp. 42–44.
Lee, K. (ed.) (2000) Childhood Cognitive Development – Essential Reading. Oxford: Blackwell.
Maclellan, E (1997) The Importance of Counting, in Thompson, I. (ed.) (1997) Teaching and Learning Early Number. Milton Keynes: Open University Press.
Mant, A. (1983) Leaders We Deserve. Martin Robinson, London.
Menninger, K. (1992) Number Words and Number Symbols, New York: Dover Publications, Inc.
Oesterle, R.A. (1959) The Arithmetic Teacher, 6, pp. 109–11.
Orton, A. (1987) Learning Mathematics: Issues, Theory and Classroom Practice. London: Cassell.
Pogliani, L., Randic, M. and Trinajstic, N. (1998) International Journal of Mathematical Education in Science and Technology, 29, pp. 729–44.







Riding, R. and Wheldhall, K. (1981) Educational Psychology, 1, pp. 6–11.
Rotman, B. (1993) Signifying Nothing: The Semiotics of Zero. Stanford, California: Stanford University Press.
Seife, C. (2000) Zero: The Biography of a Dangerous Idea. London: Souvenir Press.
Shuard, H. and Rothery, A. (eds) (1984) Children Reading Mathematics. Oxford: The Alden Press.
Suydam, M.N. and Dessart, D.J. (1976) Classroom Ideas from Research on Computational Skills. Reston, Virginia: National Council of Teachers of Mathematic
Thompson, I. (ed.) (1997a) Teaching and Learning Early Number. Milton Keynes: Open University Press.
Thompson, I. (ed.) (1999b) Issues in Teaching Numeracy in Primary Schools. Buckingham: Open University Press.
Thompson, S.V. (1990) Educational Psychology, 10, pp. 140–49.
Wigley, A. (1997) Approaching Number Through Language in Thompson, I. (ed.) Teaching and Learning Early Numbe. Milton Keynes: Open University Press.
Wheeler, M.M. and Feghali, I. (1983) Journal for Research in Mathematics Education, 14, pp. 147–55.

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full agreement with Ginsberg that errors are seldom capricious or random. He goes on to say:

Catterall, R. (1999) The Number Pack – Practical Pre-School. Leamington Spa: Step Forward Publishing Limited.
Clark, A. and Moss, P. (2001) Listening to Young Children; The Mosaic Approach. London: National Children's Bureau for the Joseph Rowntree Foundation.
Cohen, L. and Manlon, L. (1994) Research Methods in Education. London: Routledge.
Cook, T.D. and Reichardt, C.S. (eds) (1979) Qualitative and Quantitative Methods in Evaluation Research. London: Sage.

Dickinson, L., Brown, M. and Gibson, O. (1984) Children Learning Mathematics: A Teacher's Guide to Recent Research. East Sussex: Holt, Rinehart and Winston.
Department of Education and Employment (DfEE) (1999) The National Numeracy Strategy for Teaching Mathematics Framework for Teaching Mathematics from Reception to Year 6. London: DfEE.
Dolan, C. (ed.) (1998) The Development of Mathematical Skills. Hove, East Sussex: Psychology Press.
Donaldson, M. (1987) Children's Minds. London: Fontana Press.
Earl, L., Levin, B., Leithwood, K., Fullan, M. and Watson, N. (2001) Watching and Learning 2; OISE/UT Evaluation of the Implementation of the National Literacy and Numeracy Strategies. Toronto: OISE/UT.
Engelhart, J. (1977) British Journal of Educational Psychology, 47, pp. 149–54.
Ernest, P. (1983) Mathematical Education for Teaching, 4, pp. 51–53.
Geldman, R. and Gallistel, C.R. (1978) The Child's Understanding of Number. Cambridge, MA: Harvard University Press.
Guedi, D. (1998) Numbers: The Universal Language (London: Thames and Hudson).
Hughes, M. (1986a) Children and Number: Difficulties in Learning Mathematics, Oxford: Basil Blackwell.
Hughes, M., Desforges, C., Mitchell, C. and Carre, C. (2000b) Numeracy and Beyond: Applying Mathematics in the Primary School. Buckingham: Open University Press.
Inhelder, B. and Piaget, J. (1967) The Early Growth of Logic. New York: Norton.
Kaplan, R. (1999) The Nothing That Is: A Natural History of Zero,. New York: Penguin.
Landau, E. (1954) Foundation of Analysis, The Arithmetic of Whole, Rational, Irrational and Complex Numbers. New York: Chelsea Publication Company.
Lappan G. (ed.), (1987) Arithmetic Teacher, 35, pp. 42–44.
Lee, K. (ed.) (2000) Childhood Cognitive Development – Essential Reading. Oxford: Blackwell.
Maclellan, E (1997) The Importance of Counting, in Thompson, I. (ed.) (1997) Teaching and Learning Early Number. Milton Keynes: Open University Press.
Mant, A. (1983) Leaders We Deserve. Martin Robinson, London.







Menninger, K. (1992) Number Words and Number Symbols, New York: Dover Publications, Inc.
Oesterle, R.A. (1959) The Arithmetic Teacher, 6, pp. 109–11.
Orton, A. (1987) Learning Mathematics: Issues, Theory and Classroom Practice. London: Cassell.
Pogliani, L., Randic, M. and Trinajstic, N. (1998) International Journal of Mathematical Education in Science and Technology, 29, pp. 729–44.
Riding, R. and Wheldhall, K. (1981) Educational Psychology, 1, pp. 6–11.
Rotman, B. (1993) Signifying Nothing: The Semiotics of Zero. Stanford, California: Stanford University Press.
Seife, C. (2000) Zero: The Biography of a Dangerous Idea. London: Souvenir Press.
Shuard, H. and Rothery, A. (eds) (1984) Children Reading Mathematics. Oxford: The Alden Press.
Suydam, M.N. and Dessart, D.J. (1976) Classroom Ideas from Research on Computational Skills. Reston, Virginia: National Council of Teachers of Mathematic
Thompson, I. (ed.) (1997a) Teaching and Learning Early Number. Milton Keynes: Open University Press.
Thompson, I. (ed.) (1999b) Issues in Teaching Numeracy in Primary Schools. Buckingham: Open University Press.
Thompson, S.V. (1990) Educational Psychology, 10, pp. 140–49.
Wigley, A. (1997) Approaching Number Through Language in Thompson, I. (ed.) Teaching and Learning Early Numbe. Milton Keynes: Open University Press.
Wheeler, M.M. and Feghali, I. (1983) Journal for Research in Mathematics Education, 14, pp. 147–55.

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# TRANSFORMATION IN A SERIES OF LINEAR IMAGES

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***Abstract***
*The purpose of the study was to see if children could complete linear patterns which had been translated, reflected or rotated, to consider which they found easier and whether abstract or picture images generated a different response. The study was carried out with 100 children between the ages of 7.6 and 11.6 years in one junior school 7–11 year olds). The results of the study showed that translation patterns were most successfully completed, followed by reflection and then rotation. Picture images were more successfully completed than abstract design images.*

**Keywords:** Linear patterns, primary education, reflection, rotation, translation

**Introduction**
In mathematics there have been advocates of pattern (Whitehead, 1925; Sawyer, 1955; Devlin, 1997) but it appears there is more than one interpretation of the term. These three mathematicians refer to the fundamental structures of mathematics when they talk about pattern. They see pattern underpinning mathematics in a similar way to the notion of 'equivalence'. Without pattern, it would not be possible to make sense of situations, extrapolate or predict outcomes. Parallel to this is a more limited use of the term pattern in mathematics, referring to various strands which are different types of pattern, such





as reflective symmetry, or number sequences. It is possible that the primary mathematics curriculum has become very fragmented into such topics which involve pattern. It is unlikely that the teacher has a sense of pattern underpinning mathematics and approaches much mathematics as unconnected topics (Askew et al. 1997).

Of the two elements operating in the generation of a mathematical pattern, the first could be called the 'image'. This is the visual unit of the pattern. It might be a series of beads:

red, green, blue, red, green, blue, red, green, blue……

Here the image or unit is a series of three beads, 'red, green, blue'. In this case the image is repeated in a linear fashion. The second element of a pattern in mathematics is to describe the action or 'function' of the pattern. This could be referred to as the rule. In the case of the beads the function is a linear translation of three places to the right.

Sometimes the pattern is not so obvious because the image does not repeat. The function is the pattern and as such it changes the image:

2, 4, 6, 8, 10, 12, 14……

Here the function is to add 2 (+2) but in doing so we get a different image. It is still a pattern because there is a constant relationship and the sequence can be extended and predictions made. Children need to understand the regularity of the number system to accept that this is a pattern.

Both these examples are linear. It is possible to develop patterns in two and three dimensions. It is possible to make interesting 2D and 3D structures, but they must be classified as designs unless there is a predictable element of repeat or growth present.

Reflective symmetry patterns are easy to see and create (Bruce and Morgan, 1975) but they are probably the hardest to describe. One could consider they grow outwards from a point, line or plane:

12, 11, 10, 9, 8, 7, 8, 9, 10, 11, 12

in this case starting at 7 and using a +1 function/rule and growing in two directions.

To meet the situation where more than one image is presented and there is a relationship or regularity present which can be perceived then the pattern can be extended or described in mathematical terms. A simple definition of pattern could be:

> A sequence of elements with a perceived rule for development.

It is very important that an element of repeat is involved either by applying the rule or in the use of the image generated by the rule. This allows for prediction to occur and allows mathematicians to express the situation in terms of a mathematical relationship.

**Pattern Seeking in Mathematics**

For pattern seeking to be successful it is necessary to recognise the structure within a situation. For example, in a pattern of beads; red, red, blue, blue, red, red, blue, blue, it is important to realise that two colours, red and blue, are being used and that there are pairs of them alternating. From this information one could reproduce the pattern or construct one with a similar structure and different images.

All mathematical patterns have structure, but so too have many designs. The difference is that a mathematical pattern can have a rule applied to it which absorbs the structure and allows one to continue the pattern. 112233 is a structure. If one applies a rule it then becomes a mathematical pattern. For example, one rule might be: create a new image, double it and continue in a linear fashion when applied, a mathematical pattern of 112233‡‡ might acceptably occur. Another rule might be: repeat the unit of 112233, which would then generate 112233112233. A third alternative rule might be: add one to the previous number and then repeat it, which would generate 11223344 …

To carry out the above actions a child would need to apply pattern seeking strategies to establish both the structure and the rule. This could involve counting or use of operations, or use of verbal cues. Prior knowledge of possibilities would enable a rule to be







created. Creating a rule in a new situation would require some transfer of mathematical knowledge. Seeing structure is noticing something which holds a design or pattern together. To continue the mathematical pattern requires recognition and sometimes statement of the rule. It is possible that children could recognise structure and continue a rule without verbalising the situation.

**Pattern in image sequences**

In this study the children were expected to work out the pattern contained in a linear sequence of three images and then select the next correct image and manipulate it into a fourth space. In contrast to the standard spatial IQ tests using pencil and paper the children were able to select a card and physically match and manoeuvre it before placing it at the end of a series of similar images. There was an expectation that the children would recognise the structure of the pattern. In addition, the children were required to continue the pattern. The images repeated but the rule was not overtly stated. This was the key to the task, seeing the hidden regularity and continuing the rule. Presmeg (1992, 605) refers to this as the 'regularities and commonalities' found in pattern imagery. Gibson (1968, 284–86) in his modified theory of perception recognises the need to respond by:

- isolating external invariants
- learning the affordance of objects
- detecting the invariance of events
- the development of selective attention

These factors would be strongly relevant to a child selecting cards and deciding the appropriate orientation. Focusing on the image, having a perception of the orientation of the image, recognising the rule by which the image moves and remaining concentrated on the relevant factors.

Three types of transformation were used in this study, reflection, translation and rotation. Sheppard and Cooper (1982) found when working with computer image transformation, that inverted images (reflections about a horizontal axis) were easily matched. On their computer screen the 3D images would distort to maintain perspective

but inversion retained its points of reference. In this study none of the images distorted so all images would retain their points of reference but children might confuse inversion with 180 degree rotation. Bruce and Morgan (1975) found that reflective symmetry in repetition is more easily recognised than non-reflective repetition (translation, rotation) although Sheppard and Cooper (1982) found with the computer images that reflection took longer to match than rotation. In contrast Mach (1959) noticed the immediate recognition of translated shapes. Corcoran (1971) suggests that rotating a symmetrical image is easier than rotating an asymmetrical image. He also suggests some people find rotation difficult because they visualise the image strongly in a vertical position. This indicates that children might find translation patterns easy to transform and rotating symmetrical images easier than asymmetrical.

A second aspect of the study was to see if children manipulated picture or abstract images more easily. This question developed from the Orton presentation (1993) where one image seemed impossible to mentally rotate until it gained the 'identity' of being rather like a duck. This allowed it to be moved as a whole. Corcoran (1971) suggested that responses will vary depending on whether the image is in outline or blocked in, also whether it fits with previous experience. Haber (1971) also supported feature detecting, as did Gibson (1950). Corcoran (1971) is a supporter of the theory of the mind extracting features from a situation rather than recording the complete image. He suggests that responses will vary depending on whether the image is in outline or blocked in. There is the possibility that images might contain cues or features which would enable the image to be more easily rotated. The children in this study will be asked which image they found the most difficult and which the easiest to transform.
Nola (1997) offers six types of knowing which could prompt thinking about children's depth of understanding of pattern. Paraphrased and applied to pattern they are:

1. A person knows a direct object (e.g. that is a pattern)
2. A person knows how to do something (e.g. draw a reflection, a skill)
3. A person knows how to explain (how a pattern works)







4. A person knows why something works (e.g. can explain the rule behind the images)
5. A person knows that something happens (e.g. that it can be a reflection, etc…)
6. A person knows what something is (e.g. it is a reflection pattern), (Nola, 1997, p. 62)

Types 2, 3 and 4 appear to require far more intricate and active knowledge. Knowing the name of something allows identification only. For viable use to occur when learning about pattern there is a need to reach the knowing of 2, 3 and 4.

**Method of data collection and rationale**
The study was carried out with 100 children between the ages of 7.6 and 11.6 years in one junior school (7–11 year olds). This was a one-to-one task with a book containing the patterns and cards which had to be selected and placed correctly on the book (Figure1). Each child was asked to spread out 20 cards. There were pairs of cards which were similar but not identical. The children were then shown 10 linear series of three cards and a space. They were presented with these one at a time and they had to select one card and place it in the space with the correct orientation. Only one card was correct and that had to be place in one of four rotational positions. The researcher recorded correct and incorrect responses. At the end the child was asked which they considered the hardest and which the easiest pattern to complete. This was also recorded. This, therefore, did not require children to mentally hold and move images from one location to another. It also allowed for some trialling to take place when doing the matching. This practical approach meant that the skill level was lower but the understanding of the pattern sequence was captured.

Individual interviews were selected as the best method of collecting data as this allowed for a hands on approach. A task based, practical data collection was selected as this obviated the need for written responses or reading and therefore allowed the children to focus more strongly on the problem presented to them and demonstrate their understanding without needing to read or explain their reasoning. The ability to apply knowledge indicates a certain degree of security

in that knowledge. It also meant that the data collection could be extended further down the primary age group.

A small scale survey/interview style was chosen because this was considered to be a particularly appropriate method to use with young children. The material was visual. There was no writing required. The children were in a one-to-one situation or a two-to-one situation with the researcher. It was difficult for them to copy. They had the opportunity to talk and ask questions if they wished. The presence of other children nearby could have been considered reassuring although none of the children were shy to the point of not participating which was a positive factor. It is possible that this was due to the presence of practical materials and tasks.

The additional advantage of seeing individuals or pairs was that it was possible as a researcher to observe closely the children's responses and ask questions to 'probe' as Brown and Dowling (1998) would say. As one of the purposes is to try and build a picture of developmental understanding, opportunities to prompt and probe rendered the data more useful. 'How a person reasons is not open to direct inspection' (Brown and Dowling, 1998, 61), so one is therefore forced to make assumptions. This would place any such responses in the realm of qualitative data. The tabulated results could be regarded as a 'small-scale survey' and have been treated as such. It is recognised that the size of the survey can only allow for conclusions of an indicative nature to be drawn.

The data is considered to be reliable as similar results are likely to be achieved with other groups of children using the same data collection methods. The validity of the study in terms of getting children to continue a pattern, indicating whether abstract or picture images are more easily managed and discerning the different responses for translation, reflection and rotation were considered to be secure. The perception of the particular rule in rotational situations was questionable, as results will indicate.







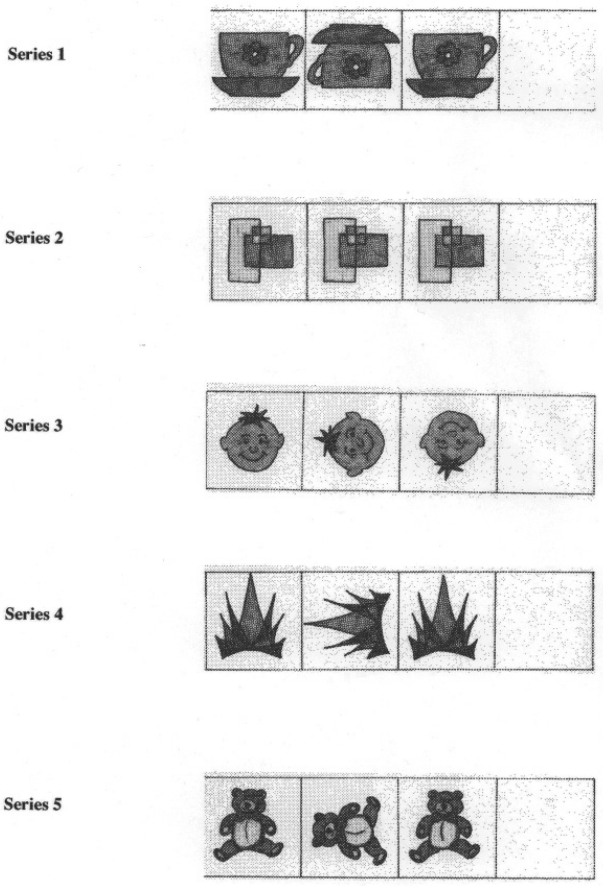



**Figure 1** **Series one to five of the linear transformation task**

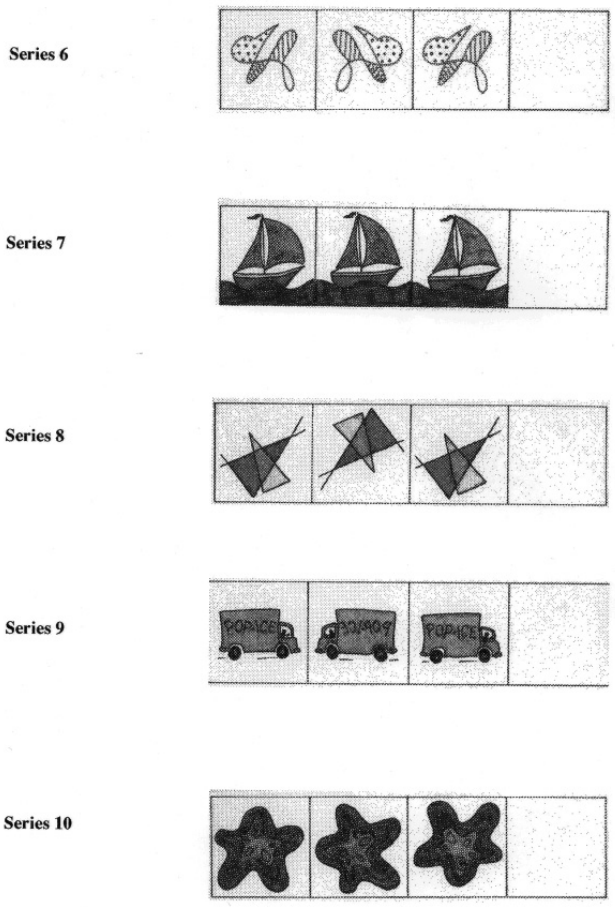



**Figure 2** **Series six to ten of the linear transformation task**







**The instructions were:**
Turn over all 20 cards and spread them out.
There is one card that fits a particular way round.
You have to look at the others (pointing to the sequence) to see which way round it goes.
Look closely at what is happening.

Occasionally a second comment was made:
You can try them and see.

For each of the ten series there were two cards which had designs similar to the given images. One only was correct and this needed to be placed the right way round. Given that the children were going to select one of the two probable cards as each pattern was very different, there were eight possible placements from the two cards. A tick in the recording table indicated a correct placement. The children were then asked which they thought was the hardest and which the easiest pattern. The linear patterns offered included translations, reflections and rotations. Some of the images offered were simple pictures and others were abstract designs. One of the designs was symmetrical, four were asymmetrical.

**Table 3**.To show details of the linear transformation patterns

| | |
|---|---|
| Series 1 | A picture image of a cup and saucer with a single flower motif. In the series the cup is seen the right way up and then rotated through 180 degrees. The third image is the cup in the original position. It is anticipated the fourth cup will be rotated 180 degrees again. The selection is likely to be made between the image and its reflection. |
| Series 2 | A straight lateral translation of an abstract image of rectangles. Selection is likely to be made between the image and its reflection. |
| Series 3 | A picture image of a cartoon face which has been rotated 90 degrees anticlockwise in each position. A further rotation of 90 degrees is anticipated. Selection is likely to be made between a face set square on a card and one place diagonally (45 degrees). |
| Series 4 | A symmetrical abstract crown image. The image is rotated 90 degrees anticlockwise and then in the third space returned to the original. It was expected that the final space would be a repeat of the second space, a rotation of 90 degrees anticlockwise. Selection is likely to be made between a crown with a blue band and a crown with a purple band. |

| | |
|---|---|
| Series 5 | Exactly the same rule as series 4 but the image is a teddy bear. The selection is likely to be made between a teddy bear with a red bow tie and a teddy bear with a blue bow tie. |
| Series 6 | An abstract asymmetrical image. The second image is a reflection, the third a reflection again which returns it to the original image. The selection is likely to be made between the original image or its reflection. |
| Series 7 | A lateral translation of a sailing boat image. The selection is likely to be between the original image and a reflection. |
| Series 8 | An abstract image of triangles set on the diagonal with a rotation of 180 degrees each time. The selection is likely to be between the original image and its reflection. |
| Series 9 | A picture image of a lorry with pop-ice inscribed on it. The series is reflections. The selection made is likely to be between the original image and a reflection. |
| Series 10 | An abstract 'blob' rotated 90 degrees anticlockwise. The selection made is likely to be between an original card and the image placed diagonally on the card (45 degrees). |

**Results**

It was observed that children responded to the task with varying degrees of speed. It was noticeable that many of the oldest group (age 10.6–11.6 years) selected the two relevant cards and compared them side by side. Here we see the comparison strategy (Fellows, 1968).This was not something the younger children did. In fact, it was observed that a few of the younger children failed to scan all the cards when they were looking for a match.

Several of the children described the rotation in terms of the cards facing in all four directions. One child used the four compass points to explain what was happening. When asked, the child said they had recently done work on compass points in the class. This is an example of transferring previous experience to the new situation (Corcoran, 1971). Experience of symmetry and pattern work was mixed which is fairly normal for a junior school which followed the Peak published mathematics scheme (Nelson).

As soon as the children completed the task they were asked which they thought was the easiest and which the hardest series to do. They could look through the cards if they wished. Correct and incorrect







responses were recorded (Table 2) as well as the children's final decision on easy and hard pattern (Table 5).

**Table 2**. To show correct responses to linear transformation patterns by age group

| Year group out of | s1 | s2 | s3 | s4 | s5 | s6 | s7 | s8 | s9 | s10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 6 (10.6 – 11.6) 25 | 25 | 25 | 13 | 16 | 22 | 17 | 25 | 17 | 25 | 14 |
| 5 (9.6 – 10.5) 25 | 25 | 25 | 12 | 18 | 21 | 21 | 25 | 20 | 25 | 3 |
| 4 (8.7 – 9.5) 25 | 24 | 25 | 5 | 11 | 17 | 15 | 25 | 15 | 24 | 6 |
| 3 (7.6 – 8.6) 25 | 24 | 22 | 5 | 13 | 19 | 16 | 25 | 12 | 22 | 1 |
| Total (out of 100) | 98 | 97 | 37 | 58 | 79 | 69 | 100 | 64 | 96 | 24 |

This would indicate an overall order of difficult using correct responses (out of 100):

**Table 3**. To show overall correct responses to linear transformation patterns

| series | correct responses |
|---|---|
| 7 | 100 |
| 1 | 98 |
| 2 | 97 |
| 9 | 96 |
| 5 | 79 |
| 6 | 69 |
| 8 | 64 |
| 4 | 58 |
| 3 | 35 |
| 10 | 24 |

Examining the results in terms of overall success rate, this is very high for some of the series. Image 7, the boats, a simple translation scored 100% correct response. This would support Mach's view that translation is the easiest of symmetries to work with (Sheppard and Cooper, 1982). But a 98% correct response is given to series 1, a tea cup picture with a 180 degree rotation. High in the performance is also series 2, a translation of abstract rectangles and series 9, a reflection of vans. Commonality is not clear. In the top four results there are both pictures and abstract designs and all three forms of symmetry. Poor performance occurred in series 3, a rotating picture of a cartoon face and series 10 a rotating abstract splodge.

If the order of successful performance data is placed against the type of response required and whether the design is a picture image or abstract image the following table is generated:

**Table. 4** To show type of design and nature of transformation matched to number of correct responses

| series | picture/abstract | transformation |
|---|---|---|
| | | |
| 7 | picture | translation |
| 1 | picture | rotation 180 degrees |
| | | |
| 2 | design | translation |
| 9 | picture | reflection |
| 5 | picture | rotation 90 degrees & return |
| | | |
| 6 | design | reflection |
| 8 | design | rotation 180 degrees |
| 4 | design | rotation 90 degrees & return |
| 3 | picture | rotation 90 degrees |
| 10 | design | rotation 90 degrees |

From this it can be seen that two factors appear to be operating here. The results would seem to suggest that using a picture to generate a pattern is very helpful. At the same time series 3 is a picture and had







a poorer success rate. The children found it hard to continue a series which required a 90 degree rotation in one direction, as these three series (3, 4 and 10) are at the bottom of the table. Even the picture did not help. When considering rotations as a 180 degree turn this seems to be easier. The two translations are high scorers, confirming Mach's view (1982). The reflection does seem to be easier than most rotations (Bruce and Morgan, 1975) but is not as well completed as the picture rotations. Series 5 initially seems to score higher than one would expect for a 90 degree rotation. It is the image of the teddy bears. There is the possibility of a third influence entering the equation, that of Corcoran (1971) familiarity.

The series 5 has interesting results when the table for children's opinion of 'easiest' and 'hardest' is examined below.

**Table 5.** To show which series were thought to be hardest and easiest by the children

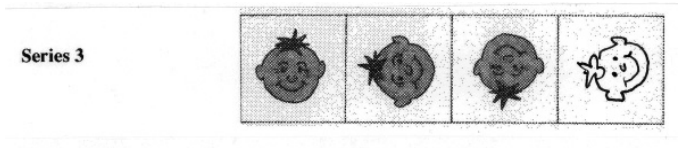
| series | hardest | easiest |
|---|---|---|
| 1 | 1 | 41 |
| 2 | 1 | 1 |
| 3 | 3 | 5 |
| 4 | 7 | 0 |
| 5 | 0 | 13 |
| 6 | 28 | 1 |
| 7 | 0 | 16 |
| 8 | 2 | 1 |
| 9 | 2 | 21 |
| 10 | 51 | 0 |
| no opinion | 5 | 1 |

The series identified as easiest were 1, 5, 7 and 9 which are all picture images. This included translation, 180 and 90 degree rotation and reflection. This indicates that pictures were considered easier to manipulate than designs. The unexpected success in performance is series 2 which is a translation/design which children did not considered easiest but performed well on. It is possible that this was

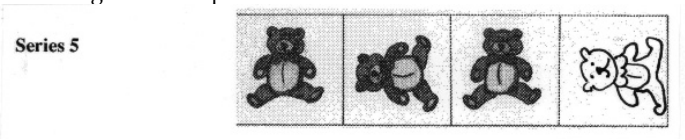
because it was a translation. Both translations have high success rates.

Looking at the series which children judged the hardest, series 10 and 6 dominate (Table 5). 10 is a design/90 degree rotation and 6 is a design/reflection. Children's judgement matches their performance with series 10. The series 6 reflection lies in the middle order of success. If translations are relatively easy and rotations are relatively hard, one would expect reflections to lie in the middle order. Children appear to be consistent when selecting pictures as easier. They have also identified designs as the hardest.

From observation of the children's approach to series 3 and 10, which gave the poorest results many placed the correct card in a similar position to space two so that the series read:




original, 90 degrees anticlockwise, 180 degrees anticlockwise, and the return to 90 degrees clockwise. At first this seems incorrect, especially if one is anticipating a continuing anticlockwise rotation. But earlier in series 4 and 5, the image had rotated 90 degrees and returned 90 degrees. This series could be viewed in a similar way; rotate 90, return 90, rotate 90, return 90 to original. This would give a four image unit of repeat too.




This then calls into question the validity of the results on the rotation patterns. From this we can see that three rotations are not sufficient to be unambiguous.







**Discussion**

The performance of the children throughout the age groups was high. Some of the youngest children tailed off on the rotation situations, but there were good performances in all age groups. The top two age groups have met rotation on at least one occasion in their published mathematics scheme (Peak, Nelson).

From observation, most children appeared more confident in the picture situations and very confident in the translation situations. When viewing the abstract images they did not speak, they seemed to be searching for reference points in the designs as suggested by Gibson (1968) and Haber (1971). This was particularly evident with series 6. Many seemed to arrive at a satisfactory classification and then rotate the card. Far more problems were experienced with series 10 where it seemed many of the children did not find a satisfactory cue and were left hazarding a guess. Those who seemed to use the red dot as a cue appeared to have more success. This could be an indication of using a distinctive feature within the design. These statements are speculation based on observed response and would be worth further investigation.

Corcoran (1971) suggested that blocked images would be easier to manipulate than outlines. Of the abstract images in the study they were mostly blocked with colour. The exception was series 6 which had a line infilled lightly with pattern. This was one with which children had poorer results than one would expect from a reflection. Corcoran also suggests it is easier to rotate symmetrical shapes. The only image that fell into this category was series 4 with the abstract 'crown' image. Results indicate children were reasonably successful with this series but their opinion of the series was 'difficult'.

None of the children showed uncertainty about the task or confusion in placing the cards. Only a few children selected the wrong card. On most occasions this appeared to be because they had failed to spot an alternative card or they had not registered the detail such as the colour of the bow tie or the crown. This would suggest that one or two had low scanning or low observation skills or even poor colour discrimination. This would support Fellow's view of failure strategies (1968).

Four strategies observed which seemed to facilitate correct solutions were: scanning the whole selection of choices, observing detail within a design, holding the options next to each other before selection of the card and placing the choice in position and 'seeing' if it fitted.

In the context of continuing a linear pattern of spatial images the study showed several trends. Picture images were more successfully manipulated than abstract design images. The children also identified the series with the picture images as the easier tasks. Results showed that the abstract designs were harder to continue unless it was in the context of a translation pattern. The children themselves convincingly chose two of the abstract designs as the hardest series to complete. The images of spatial patterns might have an effect on the success rate of continuing patterns. If the findings of the Gestalt school (Haber, 1971, Bruce and Green, 1986) are applicable then certain images will be easier to manipulate than others. In a simple form, this could be pictorial images compared with design images. If the features in the design are not distinctive they might be difficult to manipulate (Orton, 1993).

In terms of symmetry, translation patterns were the most successfully completed patterns. Reflection patterns fell between translation and rotation apart from a 180 degree rotation of a picture (Figure 1, series 1: teacups). The 90 degree rotations were the least successfully completed series. Responses were more varied and it would have been useful to have collected exactly what the incorrect responses were although observational data has been discussed earlier. There are strong indications that children had not developed a strong sense of rotational rules within linear patterns and were therefore not recognising it as a linear pattern. But also the task, by offering only three images, allowed for an interpretation of inverting positions 1 and 3 and repeating positions 2 and 4. There is a need to revisit rotation contexts to establish what rule children are applying.

Observation of the strategies employed by the children offered possible ways of improving the success rate of the children. These included scanning all the materials, paying attention to detail, using matching strategies before making decisions and trialling solutions. Pattern seeking requires the skills of: recognition of the same pattern,







recognition of similar structures, categorising patterns by property, generalising the properties sufficiently to enable transfer to new situations, and being able to conjecture or predict what might happen next by identifying a rule and applying it. When Bird (1991) observed 5 and 6 year-olds at work on problem solving, she amassed a list of 96 skills and strategies which she considered the children had used. From this list we can see some of the skills mentioned here: searching for/finding patterns and relationships, conjecturing, extrapolating, manipulating, deciding on rules, recognising equivalence, patterning, generalising, using rules, transforming, classifying, structuring and comparing.

These are skills which need to be learned and refined. For example checking is a strategy which leads to more consistent results. Closely aligned to checking is a child's perception of accuracy. As Gibson and Gibson (1950) observed, 'similar' was acceptable to the children when asked to identify the 'same' pattern. There might be a need to clarify that 'same' means 'identical' in mathematics. Fellows (1968) found children failing to match images because they would not orientate them, or not take sufficient notice of the features, or not compare carefully enough, or not carry through the task. The positive form of any of these might be considered strategies which will support successful work with pattern images. Vurpillot (1968) found that the younger children were weaker at scanning and comparison.

Mason (1989), indicates it is important to direct students' attention to the significant features that allow for abstraction to take place. This is particularly important at a later stage of pattern work when formulae are being sought to generate any stage of the pattern (Orton and Frobisher, 1994). If patterns remained as a collection of different images in the mind, eventually a large collection would amass. These could be drawn out of memory for matching purposes. However, not many patterns are the same. This bears some of the same characteristics as investigations and problem solving and the difficulty of transfer. It has been suggested that recognition of properties and generalisation enable transfer. In research, it has been difficult to establish that regular transfer takes place in problem solving because problems can be so very different mathematically (Scott, 1977; CASE, 1988; CAME, 1997). These projects have met with

varying degrees of success. Patterns too, can be very varied. Transfer will probably depend on a range of mathematical knowledge as well as an awareness of structures and rules. For example, knowing that rotation is about the amount of turn allows the child to focus on this property when working with any rotation pattern.

Mason, Burton and Stacey (1986) consider that when students generalise, they focus on aspects common to many examples and ignore other features. Generalisation is the action of drawing out information from a situation which can be used elsewhere. For example, the fact that there is a repeat of the unit in a particular linear translation could be a generalisation made about other linear situations using different materials. For the generalisation to occur the structure and/or the rule of the pattern has to be recognised and stored in memory. This implies a more sophisticated level of understanding (Nola, 1997) based on the recognition of the mathematical properties of a situation.

It is possible to continue some patterns with only a limited amount of understanding about their structure and rule. Reflecting on the six types of knowing proffered by Nola (1997) it might be that a child could say, 'It's a mirror pattern' (reflection). Here they are showing they know what something is (number 6 in Nola's list), but it does not necessarily mean they know how to explain the reflection pattern (number 3), or why it works (number 4). Teachers need to be aware of what depth of understanding a child has and what depth of understanding they expect from the child and their ability or inability to express their understanding of the situation in words.

**References**

Adhami, M., Johnson, D. and Shayer, M. (1997) Does 'CAME' work? Summary Report on Phase 2 of the Cognitive Acceleration in Mathematics Education, CAME, Project, in Proceedings of the Day Conference of the British Society for Research into Learning Mathematics (Bristol, November 1997).
Askew, M., Brown, M., Rhodes, V., Johnson, D. and Wiliam, D. (1997) Effective Teachers of Numeracy (Final report) London: Kings.
Bird, M. (1991) Mathematics for Young Children – An Active Thinking Approach, London: Routledge.







Brighouse, A., Godber, D. and Patilla, P. (1981–5) Peak Mathematics, Walton-on-Thames: Nelson.
Brown, A., Dowling, P. (1998) Doing Research/Reading Research, London: The Falmer Press.
Corcoran, D. (1971) Pattern Recognition, Harmondsworth: Penguin.
Devlin, K. (1997) Mathematics: the Science of Pattern, New York: Scientific American Library.
Fellows, B. (1968) The Discrimination Process and Development, Oxford: Pergamon.
Gibson, J. (1950) The Perception of the Visual World, Cambridge, Mass: Riverside.
Gibson, J. (1968) The Senses Considered as Perceptual Systems, London: Allen and Unwin.
Haber, R. (1971) Where are the Visions in Visual Perception. Chapter 3 in Imagery: Current Cognitive Approaches (S. Segal ed.), New York: Academic Press.
Mach, E. (1959) in The Analysis of Sensation (107) quoted in Mental Images and Their Transformation Sheppard, R. and Cooper, L., Cambridge, Mass: MIT Press.
Mason, J. Burton, L. Stacey, K. 1986 Thinking Mathematically, London: Addison-Wesley.
Mason, J. (1989) Mathematical Abstraction as the Result of a Delicate Shift of Attention For the Learning of Mathematics 9, 2 June 1989, pp. 2–8.
Nola, R. (1997) Constructivism in Science and Science Education: A Philosophical Critique, Science and Education 6, 1–2, pp. 55–83.
Orton, A. Frobisher (1996) Insights into Teaching Mathematics, London: Cassell.
Orton, J. (1993) Pattern in Relation to Shape The Proceedings of the British Congress on Mathematical Education (G. Wain ed.), University of Leeds, July 1993.
Presmeg, N. (1992) Prototypes, Metaphors, Metonymes and Imaginative Rationality in High School Mathematics Educational Studies in Mathematics, Vol 23, 6, pp. 595–610.
Sawyer, W. (1955) Prelude to Mathematics (Pelican series), Harmondsworth: Penguin.
Scott, N. (1977) Enquiry Strategy and Mathematics Achievement, Journal for Research in Mathematics Education 8, pp. 132–43.

Sheppard, R., Cooper, L. (1982) Mental Images and Their Transformation, Cambridge, Mass: MIT Press.
Whitehead, A. (1925) An Enquiry Concerning the Principles of Natural Knowledge, Cambridge: Cambridge University Press.
Vurpillot, E. (1968) The Development of Scanning Strategies and their Relation to Visual Differentiation, Journal of Experimental Child Psychology 6 5, pp. 622–50.

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# DESIGNING A LEARNING ENVIRONMENT FOR TEACHING PRIMARY MATHEMATICS: ICT USE IN MATHEMATICS EDUCATION FROM TEACHERS' PERSPECTIVES

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***Abstract***
*This chapter is based on a plan of a research in mathematics education, especially in ICT use in primary mathematics from the point of view of teachers. It outlines the way in which design research can be used as a form of educational research focused especially on teaching mathematics and using ICT in mathematics education. Many computer tools and resources have been developed for school activities generally, as well as especially for mathematics education. However, although Finnish schools have high standards of equipment applications in use, ICT is not as common as some authorities would like it to be. One aim of this research is toestablish teachers' attitudes and perceptions towards technology and the use of ICT in mathematics education. Experiences and opinions of teachers are central. Another aim is to develop a model for a learning environment for teaching mathematics that combines e computer resources with elements of the traditional learning environment, and also takes into consideration teachers' opinions. The focus is on the background and methodology of the research. The current state of research is also described by focusing on problem analysis, which includes need assessment and the clarification of constraints.*

**Keywords:** design research, ICT use, mathematics education, teacher attitude and perception, usability





**Introduction**
Over several years, development of technology has been enormous and it has become more common to use information and communication technology (ICT) in education. Many kinds of applications, digital learning materials and digital learning environments have been developed for school activities generally and also especially for mathematics education. The revision of the framework curriculum for comprehensive schools in Finland was published in January 2004. One of the objectives emphasised in mathematics education was the development of mathematical thinking and ICT use in supporting learning processes (NBE 2004). Information technology has an important role in modern learning environments.

Although Finnish schools have high standards of equipment and use several educational applications, ICT is not as common as some authorities would like it to be. For example, it might be surprising that the students representing schools known to use ICT intensively were not using it as an integrated part of their everyday schoolwork. (Hakkarainen et al. 2000) On the other hand, using ICT in a particular subject should not be emphasised just for its own sake. Firstly, it is important to think carefully about what ICT can or cannot offer. Secondly, it is central to consider how children learn, according to educational research. Thirdly, relationships between ICT, the discipline of the subject domain and the aims of the particular school subject, should all be examined carefully when applying ICT in education. (Haydn 2003)

An important part of teaching primary mathematics is to concretise contents and to use manipulatives in order to help in understanding and learning mathematical concepts. This issue has to be taken into consideration while considering what kind of possibilities digital learning materials and environments offer for teaching mathematics. In this research one aim is to develop a learning environment for teaching mathematics which combines the elements of real and virtual learning in a suitable way for teaching primary mathematics.

This research also focuses on teachers' attitudes, skills and perceptions of ICT use in mathematics education. Besides the need for different educational technology applications, it is important that teachers

and students master ICT well and are also willing to use ICT in school activities. The teacher has a central role in school activities as a decision maker and has a great impact on implementations of mathematics education. The teacher should be able to create a learning environment with his/her students which provides a suitable setting for mathematical learning and in which ICT use is integrated with mathematics in meaningful ways. There are several studies on the use of technology in education, examining for example, teachers' and students' technological skills (Baylor and Ritchie 2002), attitudes and perceptions toward technology (Carey et al. 2002; Christensen 2002; Ruthven et al. 2004) and also finding out the ways technology has been used in education (Hakkarainen et al. 2000; Ruthven and Hennessy 2002). Here the focus is on teachers' perceptions and attitudes toward ICT, as well as on technological skills evaluated by the teachers themselves. Those themes comprise the basis for a model of a learning environment of teaching and learning mathematics using ICT meaningfully and with suitable support.

This chapter focuses on the background concepts of the research, on methodology and on the present state of research. Four frames of reference are introduced: computer-based education and ICT use in education; the concept of a learning environment; usability of ICT; and the teacher's role in teaching mathematics especially with the help of computer tools and resources. The design research approach is one way to discover the characteristics of ICT use in primary mathematics and to take teachers' attitudes and perceptions into consideration as users' opinions in the development of a model for a technological learning environment (see Henneman 1999; Sugar 2001; Edelson 2002). At the beginning of the research the first two phases: design procedure and problem analysis, are central, so that the review of the background concepts and of the methodological setting is emphasised (see Edelson 2002).

**Theoretical Background**

The background to the research is based on four frames of reference that affect the research and the problem analysis. The first is the concept of computer-based education and use of information and communication technology (ICT) in education. The focus must be clarified carefully as definitions of computer-based education vary







greatly. Secondly, as the aim is to develop a learning environment which integrates technological solutions as part of teaching and learning mathematics, the concept of the learning environment is central. The third concept is usability of ICT in education. Pedagogical usability is a most important criterion for teachers in evaluating educational solutions from technology in the context of mathematics. The fourth concept is teachers' pedagogical thinking and teachers' roles as decision makers in school activities. Teachers' attitudes towards and perceptions of technology as well as teachers' technological competence, impact on implementations in mathematics education. On the other hand, conceptions of and attitudes towards mathematics rather than technology are emphasised in practice (Pietilä 2002).

**Computer-based education and the use of ICT**

Definitions of ICT in education vary greatly. As this research focuses on using computer tools and resources in mathematics education in general, and also on attitudes and perceptions towards technology, the definition of ICT use in education needs to be clarified. The uses of technology and ICT in education may be single digital learning units or virtual learning environments with many technological tools. Digital learning material can be defined as digitally-published, computer-based learning material (e.g. CD-ROM or learning tasks involving the use of ICT). The common feature of digital learning material is that it aims at particular objectives of learning and can be used through its own operating system. A virtual learning environment is an application in which unit(s) of learning material can be placed, and which has tools for studying and learning processes. There are two kinds of environments, those with units of digital learning material and those with characteristics and tools for learners and teachers (Nokelainen 2004). There are also computer-based, technological applications and tools which cannot be included in the definitions above. Some were produced initially for other activities, but are now used in education. In this chapter the focus is on all kinds of technological applications which are in some way computer-based and both known to, and available for, teachers.

Reeves describes and evaluates different forms of computer-based education (CBE) according to fourteen pedagogical dimensions (Reeves

1997, 2). This model suggests that these pedagogical dimensions can possibly be used as criteria for comparing, understanding and evaluating different implementations of computer-based education. According to Reeves (1997, 2) the dimensions are concerned with those aspects of the design and implementation of CBE that directly affect learning. These fourteen pedagogical aspects are:

1) epistemology: theories about the nature of knowledge ranging from objectivism to constructivism;
2) pedagogical philosophy: approaches to teaching and learning ranging from a strict instructivist philosophy to a radical constructivist one;
3) underlying psychology: dimensions related to the basic psychology underlying CBE, ranging from behavioural to cognitive psychology;
4) goal orientation: dimensions related to the degree of focus represented by a programme ranging from sharply-focused to unfocused;
5) experiential value: a continuum ranging from abstract to concrete;
6) the teacher's role: pedagogical roles ranging from didactic to facilitative;
7) flexibility: a continuum of programme flexibility ranging from unchangeable to easily modifiable;
8) the value of errors: a continuum ranging from errorless learning to learning from "trial and error";
9) origin of motivation: a dimension ranging from intrinsic to extrinsic motivation;
10) accommodation of individual differences: a continuum that ranges from non-existent to multi-faceted;
11) learner control: a dimension of control ranging from complete program control to unrestricted learner control;
12) user activity: related to learning environments ranging from mathemagenic (access to various representations of content) to generative (engaging learners in the process of creating, elaborating and representing knowledge);
13) co-operative learning: ranging from a complete lack of support for cooperative learning to inclusion of cooperative learning; and







14) cultural sensitivity: ranging from non-existent to integral (Reeves 1997).

As these are all criteria for evaluating and describing the particular technological application, they seem to be useful for this research. The most central described by Reeves (1997) are pedagogical philosophy; goal orientation; experimental value; teacher role; flexibility; and user activity. It is important to notice that the focus is not only on computer-based education (CBE), as important as CBE is in teaching and learning mathematics in a learning environment with computer tools and resources in a traditional way. All the points described are related to the characteristics of the traditional learning environment, which also happen to define the characteristics of the technology used in mathematics education.

**Learning environment with ICT use**

As the aim of the research is to develop a model of a learning environment which integrates computer tools and resources into teaching and learning mathematics, the concept of a learning environment is central. The term learning environment related to technological applications is usually linked to computer networks and the WWW. In this research a learning environment is understood as a combination of traditional real classroom practices and technological (computer based) applications used especially for teaching and learning mathematics. The learning environment can be understood as those factors that define the context of the learner's studying and learning (Lattu 2003, 23). It seems to be external to, and something to be absorbed by, the learner before it can become internal to a learner. The educator's role in creating the learning environment is however emphasised. The teacher acts as a supervisor to motivate learners to use the possibilities offered and also controls the activities by reshaping the environment during the studying and learning process. (Lattu 2003, 24–25)

The development of technology has had an effect on educational aims and on learning environments (Haydn 2003; NBE 2004). ICT use has become one of the central characteristics of modern education. Now there are different kinds of technological solutions in education and new ways of supporting learning in schools. Finland like other

industrialised countries has invested in ICT and therefore the standard of equipment is high. Yet despite the high profile of ICT in education, it seems to be quite demanding to find practical and meaningful ways to integrate ICT into school practice. Certainly not all teachers are using ICT as a routine part of their teaching (Hakkarainen et al. 2000; Ruthven and Hennessy 2002; Haydn 2003). There are several reasons which either encourage or discourage teachers in using ICT in mathematics lessons. Forgasz (2002) has found four critical factors: computer access; teachers' confidence and skill levels; availability of appropriate software; and technical support. One issue here is that even those teachers who do integrate ICT into their lessons are sceptical of its advantages for enhancing learning.

Besides the exigencies of ICT use, mathematics as a school subject has specific characteristics to be taken into consideration. According to Haydn (2003, 4–6), three propositions can be made for integrating ICT better into teaching activities within the real world of teaching and learning. Those can also be applied in mathematics education. The first proposition is the need to think carefully about exactly what ICT can or cannot offer. ICT shouldn't be seen as an educational miracle. As it is mathematics education under discussion, the benefits and dangers of ICT in teaching mathematics and for those who are teaching mathematics in school must be carefully examined. Designing ICT applications with collaboration between teachers and producers is one solution. The second proposition is the need to think about technology in the context of how children learn in general. It is also central to find out how the characteristics of the new technology fit with the nature of mathematics the kind of learning strategies that will be emphasised. The third proposition is the need to think about the relationship between ICT, the discipline of mathematics and the purposes of school mathematics. Much of the discussion about ICT and education has been non-subject-specific. According to Haydn (2003, 5), even if there are applications that can have possible uses in all subjects, there are also those which have more potential for enhancing learning in some subjects than in others.

Teachers' attitudes should be explored in discovering the characteristics of suitable learning environments with ICT. Teachers are those who have the greatest impact on what takes place in







classrooms. Ruthven and Hennessy (2002) have developed a model of computer use in education. The study hints at a gradual mechanism whereby teachers initially view technology through the lens of their established practice, and employ it accordingly. There were several themes (according to the teachers' views), e.g. productivity, progress and participation as well as sufficiency, which should be considered while creating the learning environment with computer tools and resources.

**Usability of ICT**

The third concept that is part of the framework behind the research design is usability of information and communication technologies in the educational field. There are several systems for defining usability. (Nielsen 1993; Henneman 1999; Nokelainen 2004) It can be measured by, e.g. usability evaluations, questionnaires and focus groups. Usability, or ease of use, may be defined as the effectiveness, efficiency and satisfaction with which specified users achieve specified goals in particular environments. Usability goals should be set in these three areas. Aspects of may include ease of installation, ease of learning, user support, productivity, user errors, and ease of customisation. (Henneman 1999, 137–38) Henneman (1999) defines usability in general, not particularly in an educational context. As this research is aimed at finding out teachers' technological perceptions and also attitudes toward computers, their professional knowledge in the use and evaluation of ICT in mathematics education should be emphasised (Da Ponte et al. 2002). Criteria of usability can also be divided into the pedagogical and technical criteria of usability (Nokelainen 2004). As the focus of the research is on teaching and learning mathematics, the characteristics of pedagogical usability can be used for evaluating the educational solutions of technology.

Pedagogical usability can be defined according to ten different themes (Nokelainen 2004; see Reeves 1997):

- i) learner control: Does a learner have a feeling that he/she is controlling the learning situation? Is he/she able to focus on essential things according to objectives of learning?

- ii) learner activity: How much do the ICT tools activate learners? How problem-centred is the computer-aided learning environment?
- iii) cooperative learning: How do ICT tools support the learners and give opportunities to cooperate?
- iv) goal orientation: What are the objectives/goals for learning?
- v) applicability: In what way are the contents of the material related to real-life situations? How are the skills for real life enhanced?
- vi) added value: What kind of new applications or new approaches do ICT tools offer for mathematics education?
- vii) motivation: What kinds of motivation arise from the use of ICT ? How does the use of ICT motivate learners?
- viii) valuation of previous knowledge: How are the skills and previous knowledge of learners taken into account)
- ix) flexibility: How flexible are the ICT tools which are used? Is there a opportunity of varying tasks individually?
- x) feedback: What kind of feedback do learners get?

As Haydn (2003) emphasises, the subject domain and content of mathematics should be taken into consideration while evaluating the use of ICT in mathematics education. However, the criteria for usability defined by Nokelainen (2004) and the characteristics of computer-based education described by Reeves (1997) are on a relatively general level. It is important to note that the characteristics of usability should not be used for judging computer tools and resources without taking a particular content and learning situation into account. The general features of mathematics education should be considered in the light of the criteria listed above. Only then should mathematics teachers use those criteria as tools for planning their lessons and for evaluating technological applications..

**Teachers' pedagogical thinking and conceptions of computers**

The fourth viewpoint is teachers' pedagogical thinking and their role as decision makers in school activities. This can be considered as a main tool of evaluation and creation of the learning environment suitable for mathematics education. In the Finnish school system teachers are decision makers who have a great impact on the implementation of mathematics education within the curriculum.







Teachers have the responsibility for choosing educational materials, in this case computer tools and resources. Other specific issues related to teachers' roles, are their view of mathematics and their conceptions of technology particularly as part of mathematics education.

The pedagogical thinking of teachers in general has been examined in different contexts and in relation to different goals. Research has varied in terms of context and research subjects (e.g. Alexandersson 1994; Alexandersson 1995; Oser and Baeriswyl 2001). The pedagogical thinking of teachers has been analysed in several models (e.g. Van Manen 1977; Patrikainen 1997; Kansanen et al. 2000). The concept of interaction is central to teaching. Interaction is part of the educational process and there is no educational process without interaction or values (Kansanen 1993, 53–54; Kansanen et al. 2000, 27–28). In order to understand teachers' actions, it is crucial to examine the concepts guiding the teacher, for example consciousness and reasoning. Pedagogical thinking can be defined as the making of decisions according to the situation and there is always reasoning behind the decision. The decision-making is reflective; there is time to think through different alternatives; the teacher does not have to put the decision into practice immediately. In the light of this research teachers will have time to evaluate the source material. (Kansanen 1993, 51) The teacher is a decision maker in his/her work and s/he gives reasons for acting in a certain way through pedagogical thinking. The degree of consciousness in the teacher's thinking about purposiveness can vary: from being a total technician to being an independent decision maker. A teacher can develop his/her work through pedagogical thinking. (Kansanen 1993, 61)

On the other hand, conceptions of and attitudes toward mathematics are emphasised in practices (Pietilä 2002). Teachers' thinking reflects a view of mathematics, that is a set of beliefs and attitudes towards mathematics that develops with exposure to different experiences. Their own views of mathematics consist of their knowledge, beliefs, conceptions, attitudes and emotions about 1) themselves as learners and teachers of mathematics and 2) mathematics and its teaching and learning. (Pietilä 2002, 23–24) Motivational factors and meaningfulness in the learning process are central to both students' and teachers' opinions about learning materials. Beliefs, attitudes

and practices in teaching and learning mathematics are significant determinants of the way teachers view their role as educators as well as the way in which students view their role as learners. On the other hand the method of teaching and learning affects attitudes towards mathematics as it also does those of student teachers (e.g. Utsumi and Mendes 2000, 241; Macnab and Payne 2003, 55). Especially interesting is to discover the conceptions teachers have of mathematics education, of ICT use and of ICT use especially as a part of mathematics education.

Attitudes and perceptions of teachers toward computer tools and resources have been highlighted in research. These, together with their technological competence, will have an impact on implementation in the classroom. Naturally willingness to try, experience in, and knowledge of, different approaches will affect whether the technological applications will be chosen or not and also how they will be used. Positive attitudes and teachers' willingness to integrate ICT in teaching and learning mathematics are crucial to bring about the educational change related to technological development. According to Carey (2002, 224), the extent to which nations can exploit the potential economic, social and educational benefits of ICT is dependent upon individual citizens' perceptions of and attitudes toward technology; their access to computers; and their experiences in using those technologies. Research suggests that the greater the access and usage, the more positive the attitudes towards the technology (Christensen 2002; Hong and Koh 2002).

The findings described above reveal only one mechanism related to attitudes to technology. As the teacher's role is central, attitudes towards technology must also be emphasised in teacher education (e.g. Da Ponte et al. 2002; Hazzan 2002). Teachers should be aware of their attitudes and also of arguments for and against the use of computers that might influence their choosing to employ them in mathematics lessons. A further point is also that student teachers might support positive changes in schools if they are motivated enough and can resist discouragement from veteran teachers. (Hazzan 2002) Attitudes of teachers and of students affect each other so positive attitudes from students might also affect school practices (Christensen 2002). Som student teachers are suspicious and fearful







of ICT. It is therefore important to make them more confident users of ICT and help them to develop a positive relationship with technology. Future teachers should also take a critical attitude towards technology. Professional knowledge of using ICT and technological skills are also crucial for teachers. They should be able to integrate ICT within their goals and objectives for mathematics teaching. They should certainly be confident of their technological competence. (Da Ponte et al. 2002) In this research, there are examples of models to enhance and support use of ICT in education in the research field of teacher education. The studies described above reveal mechanisms that could be considered while creating models in other research fields.

Baylor and Ritchie (2002) have examined the factors facilitating teacher skill, teacher morale and perceived student learning in a technology-using classroom. They found that the degree of teacher openness to change is a critical variable; teachers who are open to change appear more easily to adopt technologies and thus their technical competence increases. Unfortunately, it is difficult to influence teachers' openness to change. The other predictive feature is the level of technology leadership and support for professional development. The support of teachers, rather than of policy-makers, seems to be of greater importance.

**Method**

Besides an overview of the theoretical background, a further aim of this paper is to describe the design research approach as a method especially in this educational research. Design research explicitly exploits the design process as an opportunity to advance the researcher's understanding of teaching, learning and educational systems. (Edelson 2002, 106–7) Here I hope to clarify this procedure research, indicating especially the present juncture of the research.

**Design research approach as an educational research method**

Educational researchers are increasingly incorporating design into their research activities as one form of educational research. However, what is meant by design research varies between researchers just as does the name for this particular approach (design experiment, design research, design-based research, design studies or user-

design research). The focus of these design efforts also varies quite dramatically; for example, from medical to technological research. (Henneman 1999; Kim et al. 2001; Collings and Pearce 2002; Edelson 2002; Kelly 2003) The research described here is focused on teachers and especially on teaching primary mathematics with computer tools and resources. As one aim is to investigate attitudes and perceptions toward computers as well as the technological skills of teachers, and another is to develop a model for a learning environment suitable for teaching mathematics with ICT, the design research approach seems to be a good methodological choice (cf. Collings and Pearce 2002). This kind of design research is both descriptive and prescriptive, with the intention both to improve as well as describe the use of ICT in mathematics education from the teachers' points of view. An important characteristic of design research is to eliminate the boundary between design, development and research (Edelson 2002; The Design-Based Research Collective 2003).

Five features differentiate design experiments (design research) from other methodologies. (Cobb et al. 2003; the Design-Based Research Collective 2003) First, the purpose of design research is to develop a class of theories about the process of learning and the means designed to support that learning. The central goals of designing learning environments and developing theories or 'prototheories' of learning are intertwined. Second, a design research process is essentially iterative. Research takes place through continuous cycles of design, enactment, analysis, and redesign. Third, designs must lead to shareable theories that help communication between educators and designers. Fourth, research must indicate how designs function in authentic settings. Fifth, design research should produce solutions directly applicable in ordinary teachers' classrooms. These last features are related to the methods used in the research process. They must not only document the success and failure but also focus on interactions that refine our understanding of the learning issues involved.

The process of design might seem complex. There are several descriptions of design research (cf. Henneman 1999; Nelson et al. 2000; Sugar 2001) but in this paper we will use the one that Edelson (2002) has described. Edelson's model is quite simple and is a good







basis for a methodological overview. According to Edelson (2002), within any design a researcher or a research team has to make three sets of decisions iteratively that determine the results of the process. The first set of decisions is the design procedure, the second is the problem analysis and the third are the design solutions. The process of design research is not, however, meant to be straightforward so decisions will not be specified all at once (Edelson 2002; cf. Henneman 1999). The design procedure specifies the processes and the people involved in the development of a design. In other words, these solutions answer the question of how the design process will proceed. When specifying the expertise and process required, the goals and constraints of the design determine the decisions. When considering the needs and opportunities the design will address, the problem analysis will be characterised. The problem analysis specifies the goals or needs that the design is intended to address, together with challenges, constraints and opportunities presented by the design context. This typically evolves over the course of a design process incorporating information from a variety of sources. In the design solution the resulting design is described. The question is what form the resulting design will take. In solution construction, designers often decompose a complex design problem into manageable components. A researcher has to take on considerable challenges and constraints and exploit the opportunities (Edelson 2002, 108–9).

In this research project, the user-centred approach is one of the approaches (Henneman 1999, 135–36; Sugar 2001). Teachers' attitudes and perceptions towards computers, as well as their computational skills, are the central part of problem analysis in the learning environment with technological applications (cf. Da Ponte et al. 2002). Design research with three sets of decisions to be made is the structure of the research. Teachers' opinions will be taken into consideration especially within the phases of design procedure and problem analysis. Constraints and resources are described in the initial design procedure. Firstly the relevant forms of expertise are considered. The researcher and teachers as educators represent this. Secondly, the processes (e.g. data collection and testing) to be used during the research process have to be decided. Two methods are used: a survey and interviews augmenting the survey data. While developing the model of a learning environment to enrich the use

of ICT observations will also be made. Thirdly the study setting is planned and the schedule of the project thought out. The researcher's role is thus an active one whilst observing and interviewing the teachers.

Another set of decisions to be made is related to problem analysis. The research problems as well as the goals of the research are described below. The design of the learning environment for improving the use of ICT is intended to be developed partly based on teachers' conceptions. As a preliminary study has already taken place the research problems have already been specified. Accordingly teachers' own conceptions and attitudes have a central role. Teachers' opinions about, and conceptions of, ICT use in mathematics vary greatly. This results from their own technological skills and experiences in using ICT. Teachers were not thinking about ICT directly related to mathematics education but more generally and quite subjectively.

As the research is still new the outlines of the design solution that will take place are not clear yet. The design solutions will be a model of the learning environment using ICT in mathematics education. Teachers' opinions in the preliminary study suggest that one of the most difficult problems preventing their use of ICT is the lack of pedagogical support. Another problem is that teachers are still not familiar with the use of ICT in their classrooms and technological applications are not a part of the traditional primary mathematics classroom. The model that will be developed here should be prescriptive and it is planned to test it with teachers in their classrooms.

**Research focus**

The aim of this research is the better understanding of teachers' technological skills, attitudes and perceptions towards technology and using ICT in mathematics education. The experiences and opinions of teachers are the central. Data will be collected by questionnaires and supplementary interviews. A further aim is to develop a model of the learning environment in a way that combines computer tools and resources with elements of the traditional learning environment. The supporting system for teachers is one part of the model. The







development process will be carried out in response to the viewpoints of teachers.

At the outset the problem analysis has to be made. The problems are divided into two categories according to the aims of the research. They will be specified step by step during the research process; a typical procedure for this kind of design research (Edelson 2002; Carr-Chellman and Savoy 2004). The problems are based on two themes: teachers' using ICT in mathematics education and the development of a model of the suitable learning environment for primary mathematics education with ICT use.

How do teachers use ICT in their mathematics lessons?

- What kind of attitudes and perceptions do teachers have toward computers and ICT use in general?
- What are their reasons to use or not to use ICT?
- How do teachers see the role of ICT in primary mathematics education?
- How do teachers evaluate their confidence in using ICT?

What are the features of the suitable learning environment for primary mathematics with ICT use?

- What are the factors of the learning environment emphasised by teachers which are relevant in order to improve the integration of ICT use in mathematics education?
- How is it possible to give teachers the support they need with the help of the model?

**Research subjects**

As this research is focusing on teachers and teaching primary mathematics, the role of teachers will be central. The teacher is a decision maker who strongly influences the way the learning environment of mathematics teaching will be designed. His or her attitudes towards, and perceptions of, both mathematics and ICT use as well as their experiences in teaching mathematics, affect the implementation of mathematics education. During the research process a survey examining the attitudes, perceptions and skills of teachers evaluated by themselves will be completed by a sample of teachers (N=200) (see Francis et al. 2000). All research subjects will

be elementary school teachers with different kind of attitudes, skills and experiences. It is also important to be aware that the schools will not be chosen according to standards of equipment or technological culture. As the aim is gain information about the mechanisms of, and understand the reasons behind, teachers' actions, the interviews will take place with those key participants (N=10) who are willing to engage intensively in the process of development.

**Validity and reliability of research**

In assessing research methods it is important to uphold the concepts of qualitative analysis: credibility, transferability, dependability and confirmability (Lincoln and Guba 1985; Tynjälä 1991). In the design process of this research, issues related to validity and reliability should be taken into consideration. Methodological triangulation is one way to solve the problem of credibility. Triangulation is based on the threefold aspect of observations, interviews and quantitative data based on a survey. Data collection must be sufficiently extended to give a credible picture of the phenomena and to ensure that the researcher is familiar with the research area. However, the researcher has an active role in the process and s/he may have an effect on action in the classroom which could decrease the credibility of the research. Transferability depends on how similar the researched context and the application context are. A researcher has to describe and report the process carefully. By making assumptions explicit the researcher makes it easier to evaluate the research. Dependability relates to elements which affect the circumstances of the research. Few elements may be standardised well enough to affect the process. However, it is hard to standardise the way the researcher's approach will affect the attitudes and opinions of the research subjects. Confirmability is a characteristic of the research data as well as the researcher. The problem with qualitative research is that there is not just one true reality but many possible points of view. Subjectivity is a part of qualitative research and that is why the researcher's preconceptions and pre-understanding should be reported carefully.

Special features related to design research must also be considered. It is important to define the research subject clearly. It has to be clear to the reader what the real research object is. The researcher must describe the process of data analysis and its steps delicately







enough for the reader to consider the transferability of the results to other contexts (cf. Stake 1997). Ordinary design efforts can be augmented to yield useful research results in four ways. Firstly, the work has to be connected to both research findings and perspectives. Secondly, the elements of design that typically remain implicit must be made explicit. Thirdly, formative evaluation is critical because it can identify inadequacies in the problem analysis. Fourthly, through the process of generalisation a design researcher develops domain theories, design frameworks, and design methodologies (Edelson 2002, 116–17).

**Present juncture of research**

As the research process is in its early stages, preliminary interviews and observations (N=3) have only just been completed. The initial research problems and study design have been specified according to the preliminary study. The research is now more focused on teachers' roles in using ICT in mathematics education and on their needs related to ICT use. The teachers, who have all been informally interviewed and observed for two lessons, wanted to emphasise the importance of teachers' professional knowledge and also the need for technological support in schools. They talked especially about a need for pedagogical support and about the motivational problems many teachers have. All those teachers who took part in the preliminary interviews were motivated to use ICT in their lessons and were also quite innovative. The main reason for this was that the teachers were willing to engage in the research process and they believed that improving ICT use in schools was important. As the research focus is on teachers, it is important to complete the survey with teachers with different kind of attitudes and professional knowledge.

**References**

Alexandersson, M. (1994) Metod och medvetande [Method and Consciousness]. Göteborg Studies in Educational Sciences 96. Göteborg: Vasastadens Bokbinderi AB.

Alexandersson, M. (1995) The Missing Content in Teaching: Focus on what Teachers Reflect Upon While They are Teaching, in P. Kansanen (ed.) Discussions on Some Educational Issues VI (pp. 5–22) Research report 145. Helsinki: Yliopistopaino.

Baylor, A. and Ritchie, D. (2002) What Factors Facilitate Teacher Skill, Teacher Morale, and Perceived Student Learning in Technology-using Classrooms? Computers and Education 39 (2002), pp. 395–414.
Carey, J., Chisholm, I. and Irwin, L. (2002) The Impact of Access on Perceptions and Attitudes Towards Computers: An International Study. Education Media International 39 (3–4), pp. 223–35.
Carr-Chellman, A. and Savoy, M. (2004) User-Design Research, in D. Jonassen (ed.) Handbook of Research on Educational Communications and Technology, 2nd Edition (pp. 701–16). New Jersey: Lawrence Erlbaum Associates.
Christensen, R. (2002) Effects of Technology Integration Education on the Attitudes of Teachers and Students. Journal of Research on Technology in Education 34 (4), pp. 411–33.
Cobb, P., Confrey, J., diSessa, A., Lehrer, R. and Schauble, L. (2003) Design Experiments in Educational Research. Educational Researcher, 32 (1), pp. 9–13.
Collings, P. and Pearce, J. (2002) Sharing Designer and User Perspectives of Web Site Evaluation: a Cross-Campus Collaborative Learning Experience. British Journal of Educational Technology 33, pp. 267–78.
Da Ponte, J., Oliveira, H. and Varandas, J. (2002) Development of Pre-Service Mathematics: Teachers' Professional Knowledge and Identity in Working with Information and Communication Technology. Journal of Mathematics Teacher Education 5, pp. 93–115.
Design-Based Research Collective, The (2003) Design-Based Research: An Emerging Paradigm for Educational Inquiry. Educational Researcher, 32 (1), pp. 5–8.
Edelson, D. (2002) Design Research: What We Learn When We Engage in Design. The Journal of the Learning Sciences 11 (1), pp. 105–21.
Forgasz, H. (2002) Teachers and Computers for Secondary Mathematics. Education and Information Technologies 7 (2), pp. 111–25.
Francis L., Yaacov, J. and Jones, S. (2000) The Reliability and Validity of the Hebrew Version of the Computer Attitude Scale. Computers and Education 35 (2000), pp. 149–59.
Hakkarainen, K., Ilomäki, L., Lipponen, L., Muukkonen, H., Rahikainen, M., Tuominen, T., Lakkala, M. and Lehtinen, E. (2000) Students' Skills and Practices of Using ICT: Results of a National







Assessment in Finland. Computers and Education 34 (2000), pp. 103–17.
Haydn, T. (2003) Introduction, in T. Haydn and C. Counsell (eds) History, ICT and Learning in the Secondary School (pp. 1–10). London: RoutledgeFalmer.
Hazzan, O. (2002) Prospective High School Mathematics Teachers' Attitudes toward Integrating Computers in Their Future Teaching. Journal of Research on Technology in Education 35 (2), pp. 213–25.
Henneman, R. (1999) Design for Usability: Process, Skills, and Tools, Information-Knowledge-Systems Management 1, pp. 133–44.
Hong, K-S. and Koh, C-K. (2002) Computer Anxiety and Attitudes toward Computers among Rural Secondary School Teachers: A Malaysian Perspective. Journal of Research on Technology in Education 35 (1), pp. 27–48.
Kansanen, P. (1993) An Outline for a Model of a Teacher's Pedagogical Thinking, in P. Kansanen (ed.) Discussions on Some Educational Issues IV (pp. 51–65) Research report 121. Helsinki: Yliopistopaino.
Kansanen, P., Pitkäniemi, H., Byman, R. and Hulkkonen, H. (2000) Pedagogical Thinking in a Student's Mind, in P. Kansanen (ed.) Discussions on Some Educational Issues IX (pp. 27–51) Research report 211. Helsinki: Yliopistopaino.
Kelly, A. (2003) Research as Design. Educational Researcher, 32 (1), pp. 3–4.
Kim, S., Brock, D., Orkand, A. and Astion, M. (2001) Design Implications from a Usability Study of GramStain-Tutor. British Journal of Educational Technology 32, pp. 595–605.
Lattu, M. (2003) School from the Teachers Perspective. Research Report 246. Helsinki: Yliopistopaino.
Lincoln, Y. and Guba, E. (1985) Naturalistic Inquiry. Beverly Hills: CA: Sage Publications.
Macnab, D. and Payne, F. (2003) Beliefs, Attitudes and Practices in Mathematics Teaching: Perceptions of Scottish Primary School Student Teachers. Journal of Education for Teaching 29 (1), pp. 55–68.
NBE, (2004) Perusopetuksen opetussuunnitelman perusteet. [Framework Curriculum for the Comprehensive School.] National Board of Education. Helsinki: Opetushallitus.
Retrieved 1.3.2004, from http://www.oph.fi/info/ops/po_16_1_versio.doc

Nelson, W., Bueno, K. and Huffstutler, S. (2000) If You Build It, They Will Come. But How Will They Use It? Journal of Research on Computing Education 32, pp. 270–86.
Nielsen, J. (1993) Usability Engineering. Boston: Academic Press.
Nokelainen, P. (2004) Digitaalisen oppimateriaalin käytettävyyden arvioinnin kriteerit [Criteria for Usability of Digital Learning Material]. Presented in Digital Learning Seminar in May 2004. Retrieved 24.5.2003 from http://dll.hamk.fi/uusidll/pdf/pn_strategiaseminaari_2004.pdf
Oser, F. and Baeriswyl, F. (2001) Choreographies of Teaching: Bridging Instruction to Learning, in V. Richardson (ed.) Handbook of Research on Teaching, 4th edition (pp. 1031–65) Washington DC: American Educational Research Association.
Patrikainen, R. (1997) Ihmiskäsitys, tiedonkäsitys ja oppimiskäsitys luokanopettajan pedagogisessa ajattelussa [Conception of Man, Conception of Knowledge, and Conception of Learning in Primary Teachers' Pedagogical Thinking.] Research report 36. Joensuu: University of Joensuu, Faculty of Education.
Pietilä, A. (2002) Luokanopettajaopiskelijoiden matematiikkakuva: matematiikkakokemukset matematiikkakuvan muodostajina [Pre-service Elementary Teachers' Views of Mathematics: The Role of Mathematics Experiences in Forming the Views of Mathematics.] Research report 238. Helsinki: Yliopistopaino.
Reeves, T. (1997) Evaluating What Really Matters in Computer-Based Education. Retrieved 24.5.2003 from http://www.educationau.edu.au/archives/cp/REFS/reeves.htm
Ruthven, K. and Hennessy, S. (2002) A Practitioner Model of the Use of Computer-Based Tools and Resources to Support Mathematics Teaching and Learning. Educational Studies in Mathematics 49, pp. 47–88.
Ruthven, K., Hennessy, S. and Deaney, R. (2004) Incorporating Internet Resources into Classroom Practice: Pedagogical Perspectives and Strategies of Secondary-School Subject Teachers. Computers and Education.
Stake, W. (1997) Case Study Methods in Educational Research: Seeking Sweet Water, in R.M. Jaeger (ed.) Complementary Methods for Research in Education 2nd edition (pp. 401–14) Washington D.C.: American Educational Research Association.







Sugar, W. (2001) What Is So Good about User-Centered Design? Documenting the Effect of Usability Sessions on Novice Software Designers. Journal of Research on Computing in Education 33 (3), pp. 235–50.
Tynjälä, P. (1991) Kvalitatiivisen tutkimusmenetelmän luotettavuudesta [About Reliability of Qualitative Research Methods.] The Finnish Journal of Education (Kasvatus) 22, pp. 387–98.
Utsumi, M. and Mendes, C. (2000) Researching the Attitudes Towards Mathematics in Basic Education. Educational Psychology 20, pp. 237–43.
Van Manen, M. (1977) Linking Ways of Knowing with Ways of Being Practical. Curriculum Inquiry 6, pp. 205–28.

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# INVESTIGATING CLASS TEACHERS' PEDAGOGICAL THINKING AND ACTION IN MATHEMATICS EDUCATION: THEORETICAL AND METHODOLOGICAL OVERVIEW

*Sanna Patrikainen, University of Helsinki*

***Abstract***
*Finnish students have been successful in international comparative research e.g. in mathematics. But mathematics education is criticised because of the behavioristic teaching tradition. A socioconstructivistic curriculum and the new school system for all grades from 1 – 9 also offers new challenges to Finnish teachers. The purpose of this study is to describe the quality of teachers' pedagogical thinking and action. In this research I will investigate how teachers teach mathematics in practice and how they give reasons for their pedagogical decisions. The aim is to make visible the didactics of mathematics through the thinking of class teachers and to conceptualise it. The theoretical framework of this study consists of two parts: teachers' own pedagogical thinking and the relationship between general and subject didactics. The six participants in this study are class teachers from the first to the sixth grades. The data has been gathered by observing and by using stimulated recall interviews. This chapter is a short introduction to and overview of my doctoral dissertation. First the objectives of this study, reasons for them and the connection between the objectives and the theoretical framework are presented. Then the methodological solutions from this study will be discussed. The opportunities of describing teachers' actions and understanding and the thinking of teachers in classroom situations are considered and presented with an example of data.*





**Key words:** classroom research, mathematical beliefs mathematics education, pedagogical thinking, stimulated recall method video observation

**Introduction**
Finnish students have qualified highly in recent international comparative research, e.g. in mathematics learning (TIMMS 1999, PISA 2000). On the other hand, mathematics education is still criticised because of a behavioristic teaching tradition which fosters mechanical practices instead of facilitating the development of diverse mathematical thinking. These two extreme views are contradictory about the quality of mathematics teaching and learning. In consequence, learning how mathematics is taught at schools is of great importance.

One reason for the criticism of mathematics education could be the beliefs and conceptions within the field, which have an effect on the teacher's pedagogical thinking. The current Finnish curriculum is based on a socioconstructivistic conception of teaching and learning. The change in curricular thinking from a behavioristic approach to a socioconstructivistic model should also initiate change in mathematics teaching practices which will develop along the same lines as a teacher's pedagogical thinking.

This chapter is based on my master's thesis of the same name (Patrikainen, 2001, 2003). In that thesis the quality of the class teacher's pedagogical thinking and action in mathematics education was compared with the goals of the Finnish curriculum. Attention was especially drawn to how teachers taught in practice and on what kind of mathematical beliefs the teacher's pedagogical thinking is based.

According to the results of the study, all teachers shared common traits of pedagogical action. One was to divide the mathematics lesson into three phases: orientating, teaching and practising. There were also some differences in action related to the use of teaching methods and to the speed of progress in the teaching-studying-learning process.

Another important result noticed was the division of the teacher's pedagogical thinking into general didactical, and subject didactical, thinking. General didactical thinking included ideas related to maturing as a person and to the organisation of the context of the teaching-studying-learning process. However the subject didactical thinking model focused on the subject, and its aspects, important skills, teaching, learning, and evaluation. The mechanistic practice of basic counting skills and the importance of understanding were emphasised in the teachers' mathematical beliefs. In practice, these beliefs occurred in partial inconsistency with each other.

**Objectives of the research**

The purpose of this study is to describe the quality of teachers' pedagogical thinking and action in mathematics education. It will investigate how teachers teach mathematics in practice and what reasons they give for their pedagogical decisions. The aim is to depict the didactics of mathematics through the class teacher's own thinking and to conceptualise it.

Class teachers are expected to have a thorough knowledge of their subject, in addition to pedagogical skills and theoretical knowledge of teaching and learning. A socio-constructivistic learning environment demands closer control of discipline and its methods than a behavioristic learning environment. In a new Finnish school system for all classes from first to ninth it is possible that subject teachers could work as class teachers.

In the light of these new challenges it is important to investigate the class teacher's thinking, especially in specialised subject disciplines. The teacher's pedagogical thinking in mathematics itself is seldom investigated. The way that teachers think in this area is part of what this study tries to illuminate. It should be possible to reveal, in part, what kind of mathematics teachers class teachers are.

**Research problems**

The goals of this study are approached in the form of the following questions:







1. How is the class teacher's pedagogical action represented in practice in mathematics education?
1.1 Into what kind of pedagogical episodes are the mathematics lessons divided?
1.2 What kind of structure does the action of the individual teacher take?
1.3 What kind of common traits and differences are represented in individual teacher's actions?

2. How is the class teacher's pedagogical thinking represented in practice in mathematics education?
2.1 Into what categories is the class teacher's pedagogical thinking divided?
2.2 In which ways are the didactical reasons and beliefs of the subject represented in the teacher's pedagogical thinking?
3. What set of concepts could be used in describing pedagogical thinking and action in mathematics?

4. How does the class teacher's pedagogical thinking and action in mathematics compare to the thought behind socioconstructivistic curricular thinking?

**Theoretical framework**

The theoretical framework of this study consists, on the one hand, of the teacher's pedagogical thinking and, on the other hand, the relationship between general didactics and subject didactics. The main concepts are discussed both at a general educational level and from the viewpoint of mathematics education.

**The teacher's pedagogical thinking**

The starting point is that a teacher's pedagogical action is guided by his/her pedagogical thinking which is based on a personal set of beliefs and practical theories (see Clark and Peterson, 1986; Kansanen, 1995, 1999; Kansanen, et al. 2000; Kosunen, 1994; Thompson, 1992).

Attention is drawn especially to the mathematical beliefs of the teacher's thinking (see Pehkonen, 1995, 1998, 1999) and to their effect on teaching situations (see Pehkonen and Törner, 1996; Lindgren, 1998; Kupari, 1999; Thompson, 1992). The teacher's personal beliefs

are parallel to the behavioristic and constructivistic teaching and learning theories which also have a background effect on mathematics education.

The general and subject didactical goals of the Finnish curriculum, based on current socio-constructivistic curricular thinking, are considered to be the ideal model of mathematics education. Changes in beliefs and teaching practice are seen as part of a teacher's professional development (see Pehkonen, 1994; Patrikainen, 1997).

**The relation between general didactics and subject didactics**

According to Kansanen and Meri (1999), general didactics and subject didactics are usually in opposition to each other, along with their respective background disciplines. These writers refer to Klafki (1994), who has written that the relation between general didactics and subject didactics is not hierarchical by nature, but rather reciprocal. Both deal with the same problems in which a certain subject has its own characteristics. The main difference between them is in the extent to which their solutions and discussions can be generalised.

In this study, teaching is understood broadly as a teaching-studying-learning process. Kansanen and Meri (1999) stated that the elements of the teaching-studying-learning process could be described, according to Herbart, with the didactic triangle, in which the most usual approach is to view the relationship between the teacher and the students: a pedagogical relationship (see Figure 1). In a didactical sense the student's relationship to the content is the most essential because the whole instructional process aims at achieving the aims and goals stated in the curriculum. The teacher's task is to guide this relationship, called a didactic relationship by Kansanen and Meri, according to Klingberg (1995). Therefore, the didactic relationship means a relationship to another relationship. First, there is a relationship between the student and the content, which represents itself as visible studying and invisible learning. Secondly, the teacher has a relationship to the relationship between the student and the content. Kansanen and Meri (1999) emphasise that concentrating on the relationship between the student and the content is the core of a teacher's profession.







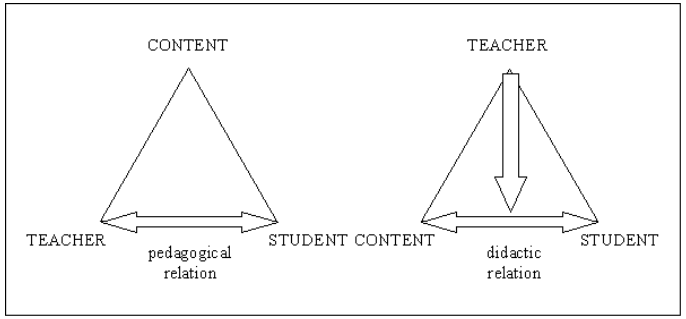


**Figure 1**. The pedagogical relationship and the didactical relationship in the didactic triangle
(Kansanen and Meri 1999)

Kansanen and Meri (1999) state that it is not possible to organise the didactic relationship universally nor to define technical rules for it. Every teacher is supposed to consider his/her own actions and decide how to cope with the relationship to the students' studies. This means that every teacher has a didactics of his/her own, and, according to Kansanen and Meri (1999), this comes close to Elbaz's (1983) concept of the practical theories of a teacher or Kansanen's (1999) teacher's pedagogical thinking.

**The research process**

This study is qualitative because the aim is to find out how teachers teach, what they think about mathematics in practice, and what kinds of reasons are given for their pedagogical decisions. The starting point is to describe a real life situation, where the research subject is depicted as comprehensively as possible. The purpose is to discover, or uncover, reality, not to verify statements already known.

At least two fundamentally different approaches are used in making qualitative analysis and interpretation. The first is the inductive approach, in which the data is analysed without theoretical pre-assumptions, and the second is the deductive approach, in which previous theories are exploited and proved. In this study the abductive approach is used, instead of pure induction and deduction, in which

the researcher has a basic principle, which guides the observation. The theory can be used as a support, not for verification, but as a source of ideas. However, this research will proceed mainly based on the collected data, and the final theory and research questions will develop alongside the process and during its final stage.

**The teachers participating in the research**

In qualitative research it is important to find informants who clearly display the phenomenon being researched. At times it is beneficial to choose only extreme examples, but ordinary cases can be investigated as well. Above all, the researcher should find the best informants for his/her specified study. In my thesis mentioned earlier the teachers could be characterised as 'ordinary teachers'. Only one in six had specialised in mathematics.

In the primary research here the subjects (6) are class teachers from all grades from first to sixth. They have proved to be teachers who actively want to discuss issues and develop themselves in their profession. They also want to apply the knowledge they have studied in their training as teachers to practical work and to train themselves further as well. Through studying these teachers the aim is to illustrate versatile mathematics education and the pedagogical thinking behind it.

**Study design**

The focus of this research, mathematics education, is a qualitative phenomenon by nature and is therefore examined mostly with qualitative methods (see Figure 2). The research material has been gathered by observing and by using stimulated recall interview. By using several methods the comparability of the research tends to improve.

One of the characteristics of qualitative research is participation, which means that both the subjects and the researcher him/herself are involved in the research process. When the researcher tries to preserve the phenomenon for research as it is, without manipulation, the intention is said to be to reach the subject's own point of view. In this study, the research material will be gathered in natural, real,







teaching situations which highlight, with the methods used, the teachers' own points of view.

Every teacher participating in the study will be observed by videotaping the mathematics lessons during one teaching process, which is estimated to last two to three weeks per teacher. Differing from previous research studies, mathematics education itself is investigated in real classroom situations in which the straight and direct information about the teachers' actions are received. Because the aim here is to portray the teachers' pedagogical thinking as it relates to teaching and learning situations and to find out the reasons teachers themselves give for their pedagogical decisions, the teachers are interviewed after the videotaping using the stimulated recall method.

The data gathered will be analysed by using the method most suitable for qualitative research. In qualitative analysis, the researcher must consider what the best method is for her/his own study. In this study, videotapes, which have captured the mathematics lessons, are transcribed into written form as descriptions of the events in the lessons. A set of concepts is constructed from the theory which describe the teaching-studying-learning process. With this set of concepts the action of each teacher is described in the form of lesson profiles. These lesson profiles are compared with each other to highlight the similarities and differences between individual teachers and her/his peers.

Stimulated recall interviews, which represent the teacher's pedagogical thinking, are also transcribed into written form. After that, they are categorised and analysed from the data bases. Through analysis, the thinking underlying the didactics of mathematics is portrayed. Through conceptualisation, a view is constructed of how subject didactics in mathematics are represented in the class teacher's pedagogical thinking.

Finally, the connection between thinking and action is analysed and compared to the thought behind the socio-constructivistic curricular thinking.

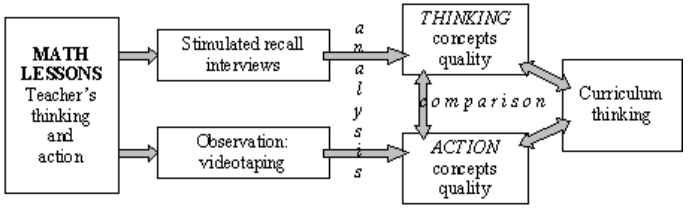


**Figure 2**. Study design

Video observation
Observation is one of the basic data gathering methods in scientific research. By observing it is possible to investigate both human activities and the context in which these activities take place. The other salient methods, interviewing and survey, give us information about what people think, feel, and perceive around them. But by observing we can make sure what really happens and discover if people are really acting as they say they do.

Since the 1960s, the nature of observational studies has developed from quantitative, strictly structured procedures and analyses to relatively undetermined data gathering and analysis processes typical of qualitative research. Over the past few decades observation methods have improved even more because of the development of technology. The use of video and computers has enabled investigation of complex and layered teaching and learning situations in great detail, along with other additional methods like stimulated recall interview.

**Characteristics of video observation**
Video is a suitable tool for gathering visual and aural information because it easily captures diverse behaviour and complex interactions in detail. The use of video recordings also enables researchers to re-examine data again and again and from different point of view. Pirie (2001, 346) refers to Erickson (1992, 205) who notes that video recordings are used especially when 'the distinctive shape and character of events unfolds moment by moment, during which it is important to have accurate information on the speech and non-







verbal behaviour of particular participants in the scene'. Cobb and Whitenack (1996) emphasise the importance of social context where the action being observed occurs. Videotaping produces data where the context is in evidence all the time.

Paterson et al. (2003, 6–7) mention that some researchers have designated the use of video recordings as 'participant observation' because subjectivity is a characteristic aspect of videotaped data. Video observing requires constant decision making about what to observe and record and it produces a selective description of the phenomenon being studied which is based on the researcher's preconceptions and the objectives of the study. Pirie (1996, 553) points this out. She writes that: 'all research is to some degree subjective. We see what interests us; we look with a purpose. The field notes we take are already an interpretation of the phenomenon that we study. We rationalise as best we can the value of the data we gather and the worthlessness or irrelevance of that which we do not.'

As mentioned above one of the features of videotaped data is its selective nature. When the data is gathered the researcher has to make another significant decision concerning the form in which the data will be analysed. Some researchers prefer working with videotapes and others transcribe video into written form. According to Pirie (1996, 555; 2001, 349) working with either videotapes or transcripts is not intrinsically better or worse, but it is different and each has its merits and demerits.

Powell et a.l (2003, 410–11) remind us that it is impossible to write an exact transcript of verbal and non-verbal interactions captured on videotape. However it is possible to produce written descriptions which are close approximations and representative enough for particular research purposes. Powell et al. (2003, 422–23) prefer the use of transcripts in their research because they are a permanent record and the researcher can consider more permanently the meaning of specific utterances. When the research is reported, transcripts can also provide evidence of findings in the participants' own words. Unlike Powell et al., Pirie (1996, 555–56; 2001, 349) works exclusively with the videos when analysing the data. In her opinion the written word is less conducive to discovering new insights than

**Appendix 4**
Lerman and Tsatsaroni (2004)

Theory type

| | PME | | | | ESM | | | | JRME | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 90-95 | | 96-01 | | 90-95 | | 96-01 | | 90-95 | | 96-01 | |
| | No. | % | No | % | No | % | No | % | No | % | No | % |
| Traditional psychological & mathematics theories | 49 | **73.1** | 49 | **60.5** | 52 | **63.4** | 49 | **51.6** | 34 | **54.8** | 44 | **57.9** |
| Psycho-social, including re-emerging ones | 8 | **11.9** | 8 | **9.9** | 8 | **9.8** | 19 | **20.0** | 4 | **6.5** | 10 | **13.2** |
| Sociology, Sociology of Ed, socio-cultural studies & Historically orientated studies | 2 | **3.0** | 8 | **9.9** | 3 | **3.7** | 11 | **11.6** | 1 | **1.6** | 6 | **7.9** |
| Linguistics, social linguistics & semiotics | 0 | **0.0** | 2 | **2.5** | 1 | **1.2** | 5 | **5.3** | 2 | **3.2** | 6 | **7.9** |
| Neighbouring fields of Maths Ed, science ed and curriculum studies | 1 | **1.5** | 0 | **0.0** | 0 | **0.0** | 0 | **0.0** | 1 | **1.6** | 0 | **0.0** |
| Recent broader theoretical currents, feminism, post-structuralism and psychoanalysis | 1 | **1.5** | 0 | **0.0** | 8 | **9.8** | 1 | **1.1** | 0 | **0.0** | 1 | **1.3** |
| Philosophy/ philo of mathematics | 0 | **0.0** | 3 | **3.7** | 0 | **0.0** | 3 | **3.2** | 1 | **1.6** | 1 | **1.3** |
| Ed theory and research | 2 | **3.0** | 0 | **0.0** | 1 | **1.2** | 1 | **1.1** | 2 | **3.2** | 0 | **0.0** |
| Other | 0 | **0.0** | 0 | **0.0** | 1 | **1.2** | 1 | **1.1** | 2 | **3.2** | 0 | **0.0** |
| No theory used | 4 | **6.0** | 11 | **13.6** | 8 | **9.8** | 5 | **5.3** | 15 | **24.2** | 8 | **10.5** |
| Total | 67 | | 81 | | 82 | | 95 | | 62 | | 76 | |

# GRAPH COMPREHENSION OF PRIMARY SCHOOL STUDENTS

*Ioannis Michalis, Aristotle University of Thessaloniki*

***Abstract***
*This study discusses graph comprehension, which is defined as the abilitys of graph readers to derive meaning from graphs. It is classified into three levels: reading the data (i.e. literal), reading between the data (i.e. interpretation, interpolation) and reading beyond the data (i.e. prediction, extrapolation). The preliminary results of a pilot research project on the graph comprehension of primary school students (grades 4, 5 and 6) are presented. Students were given a questionnaire, comprised of ten graphs each with six questions, covering the three comprehension levels. The sample consisted of 280 primary school students from the urban area of Thessaloniki and nearby cities.*

**Keywords:** data analysis, graph comprehension, primary school, statistics

**Introduction**
Over the past fifteen years, documents on mathematics education reform (NCTM, 2000; DFES, 2001; AEC, 1994; M.E., 1993) have highlighted the importance of including statistics and data analysis throughout the school mathematics curriculum.

It is suggested that the teaching of statistics and data analysis in school mathematics must be based on the statistical investigation





process (Graham, 1987), which typically involves four stages: posing a question, collecting the data, analyzing the data, and interpreting the results; a fifth stage could be added, which is communicating the results (Kader and Perry, 1994).

It is important to look at students' understanding as related to concepts linked to this process in order to establish a better learning environment and to design the appropriate instructional material. Located at the core of that process is the understanding of the graphical representations or visual displays of data. As Shaughnessy et al. (1996) state:

the current meaning of data analysis emphasizes organizing, describing, representing, and analyzing data, with a heavy reliance on visual displays such as diagrams, graphs, charts and plots (p. 205)

From an early age we meet different kinds of graphs, charts, thematic maps, and cartograms in educational as well as everyday situations. Graphs are commonly used to depict mathematical functions, display data from social and natural sciences, and specify scientific theories in textbooks and other print media in and out of the classroom (Shah and Hoeffner, 2002). Skills in the critical reading and interpretation of data, presented in these visual forms, are a necessity in our highly technological society and represent one of the major components of quantitative literacy. Reflecting that view, The National Research Council (1990) suggests:

Most obvious, perhaps, is the need to understand data presented in a variety of different forms and displays…. Citizens who cannot properly interpret quantitative data are, in this day and age, functionally illiterate.

Therefore, we need to know much more about how students (and teachers) think about, and comprehend graphical representations.

The graphical presentation of data has a considerable history, from the work of William Playfair (1759–1823) to the contemporary innovations of John Tukey (Wainer, 1990). Playfair is credited with

the invention of the best known graphs; while EDA (Exploratory Data Analysis) is a relatively new area of statistics, where the data are explored by graphing techniques (Tukey, 1977). This approach/ philosophy has given a surge to the use of graphical representations because such graphical techniques are much used in analyzing data. The focus is on meaningful investigation of data sets with multiple representations and little probability theory or inferential statistics.

**Graph definition and forms/types**

Graphs have the potential of showing more than the values in the data table. They can provide an overview of the whole data set and can highlight specific characteristics that are not visible from the numbers. They display the data and tell the truth (although they may sometimes attempt to persuade); encourage comparison of different pieces of data; pack large amounts of quantitative information into a small area; reveal the data at several levels of detail; provide impact; communicate with clarity, precision and efficiency; serve a defined purpose of discovery, understanding and presentation; and are more closely integrated with statistical and verbal descriptions of the data.

Kosslyn (1994) defines a graph as 'a visual display showing one or more relationships between numbers', while Fry's (1984) description is: 'a graph is information transmitted by the position of point, line or area on a two-dimensional surface' (p. 5). James and James (1992) in their Mathematics Dictionary define a graph as: 'a drawing which shows the relation between certain sets of numbers …Used to convey a better idea of the meaning of the data than is evident directly from the numbers'. (p. 189)

The characteristics and features of the most common graphs and plots used widely in newspapers, magazines and reports, and mentioned in all curriculum documents around the world – including the traditional forms of graphs and the new plotting techniques (Curcio, 2001) – are presented below.

The **picture graph** (picture chart, pictogram, pictograph or pictorial graph) is used to display discrete or categorical data using ideographs or symbols to depict quantities of objects or people. Rectangular







axes define the field of display. In its simplest form the ideograph is in one-to-one correspondence with the item it represents. In more complex displays, a many-to-one correspondence between the items represented and the ideograph requires the use of a legend. In this type of display, sometimes the size and the type of ideographs mislead the audience either on purpose or accidentally.

The **bar graph** (bar chart) is also used to display discrete or categorical data. Within rectangular axes that must be labelled, the heights of rectangular bars of uniform width are proportional to the quantities they represent. The bar graph may be set up either horizontally (usually when the labels of the categorical data are long) or vertically (the most common form). Discrete stratified data (from particular groups) may be compared in double or multiple bar graphs.

The **histogram** contains grouped data arranged like a vertical bar graph. Only one set of data may be represented, the class intervals (groups) must be equal, and both axes must contain a numerical scale.

A **line graph** is used to display continuous data (change over a period of time, or time-series data). The axes that intersect at a common point (usually zero) are labelled and they define the field for the display. On each axis, the units of division must be consistent. The graphed points are connected by line segments. Two or more sets of continuous data may be graphed on the same set of axes to create a double or multiple line graph allowing the comparison of the data sets.

A **circle graph** (area graph, pie chart, pie diagram or pie graph) is used when data are to be compared to a whole or to different parts of a whole. The field of display of a circle graph is defined by the circumference of a circle. Sectors of the circle created by line segments from the centre correspond proportionally to fractional parts of the unit being analyzed.

A **line plot** is created by representing numerical data in the form of x's on a number line (like a primitive bar graph). This visual display is easy to construct. The shape and the spread of the data are revealed

while having access to each individual datum. The median and the mode are easily identified. The number of items plotted usually does not exceed twenty-five.

A **stem-and-leaf plot** provides a display of data that is created by separating the digits in the data based on their place value. In a regular stem-and-leaf plot, two columns identified by place value are established to list each digit. The digit with the higher place value is the stem, and the digit with the lower place value is the leaf. This type of plot usually contains more than twenty-five data entries and up to two hundred and fifty. Similar to the line plot, it is easy to construct, and the shape and the spread of data are revealed while having access to individual datum. The median and the mode can be easily identified. Back-to-back stem-and-leaf plots can be constructed to compare two sets of related data.

A **box plot** (box-and-whiskers plot) uses five summary points: the lower extreme, the lower quartile, the median, the upper quartile and the upper extreme. The plot has one axis in the form of a number line arranged either horizontally or vertically. A rectangle is used to represent the middle 50% of the data. The median is represented by a line segment that partitions the rectangle to show the spread of the upper and lower quartiles, proportional to the size of the rectangle. The extremes are represented by points, and they are connected to each end of the rectangle and are identified in terms of the interquartile range. This plot is used when analyzing more than 100 pieces of data. Unlike the line plot and the stem-and-leaf plot, individual data are not identifiable.

According to Kosslyn (1989, 1994) almost all graphs have similar structural components (graphic constituents). The similarity is not in appearance, but in function, in the role that these constituents play in how information is represented in a display and respectively how it is interpreted.

The **framework** of a graph (axes, scales, grids, reference markings) gives information about the measurement being used and the data being measured. The most common framework has an L shape with the horizontal line depicting the category axis (the data being







measured) and the vertical line the value axis (providing information about the measurement being used). Picture graphs, line graphs, bar graphs, histograms and line plots have that L-shaped framework. Box plots use a variation of that shape, while stem plots and tables have T-shaped frameworks. Others, like pie graphs, have a framework based on polar coordinates (Fry 1984).

The **specifiers** are visual dimensions which are used to represent data values. These might be the bars on a bar graph, the lines on a line graph etc.

**Labels** are another component of a graph that name the type of measurement being made, or the data to which measurement applies, including the title of the graph.

Lastly, the **background** of a graph includes any colouring, grid, and pictures over which the graph may be imposed. The background serves no essential role in communicating the information conveyed by a chart or graph.

**Graph Comprehension**

One of the reasons why graphs are so pervasive is that they seem to make quantitative information easy to understand. Research in statistics education, however, shows that graphs are difficult to interpret for most people:

The increasingly widespread use of graphs in advertising and the news media for communication and persuasion seem to be based on an assumption, widely contradicted by research evidence in mathematics and science education, that graphs are transparent in communicating their meaning. (Ainley, 2000, p. 365).

We need to define the processes and the factors that influence graph comprehension in order to understand why it is difficult for people to interpret graphs. In that sense we have to focus on graph comprehension as reading and interpreting graphs; as the graph reader's ability to derive meaning from graphs (Friel et al., 2001).

In general, comprehension of information in written or symbolic form involves three kinds of behaviours (Jolliffe, 1991) that seem to be related to graph comprehension, namely:

- Translation
- Interpretation
- Extrapolation/interpolation

To translate between graphs and tables, one could describe the contents of a table of data in words or interpret a graph at a descriptive level, commenting on the specific structure of the graph (Jolliffe, 1991). To interpret graphs one can look for relationships among specifiers in a graph or between a specifier and a labelled axis. To extrapolate, which is considered to be extension of interpretation, requires stating not only the essence of the communication but also identifying some of the consequences; one could extrapolate by noting trends perceived in data by specifying implications.

Following that model the NCTM principles and standards for school mathematics suggest that students in early elementary school should be able to use graphs to identify what quantities are the highest and lowest and to make comparisons between two single data points. Late elementary students should be able to aggregate data and evaluate the whole picture presented by a graph. Middle school students should begin to develop ideas of statistical inference and high school students are expected to use graphs to make inferences (NCTM 2000).

These standards provide, above all, a framework for identifying a hierarchy of question types for the graphs used in real world tasks. The simplest questions require identification of specific data points or comparison of two or more data points and involve only the extraction of explicit information. For this type of question, the desired information is explicitly represented in the graph and the graph reader is required only to locate and read the specific data point. A more difficult question, requiring aggregation of data, is an integration question. The graph reader must read off multiple data points and then integrate the information using some kind of mental operation (i.e. determining trends). The most difficult questions are







those that require the user to make inferences and predictions. The desired information is not explicitly represented, so the graph reader must use extrapolation in order to extract it.

Gal (1998) suggested two similar types of question in interpreting information in tables and graphs: literal-reading questions that involve reading the data, or reading between the data and opinion, and questions that focus on reading beyond the data. The latter types require eliciting and evaluating opinions (rather than facts) about information presented in representations.

Three levels of graph comprehension have emerged according to relevant research (Bertin, 1983; Curcio, 1987; McKnight, 1990; Carswell, 1992; Wainer 1992):

- An **elementary** level focused on extracting data from a graph (locating, translating)
- an **intermediate** level characterized by interpolating and finding relationships in the data as shown on a graph (integrating, interpreting), and
- an **advanced** level that requires extrapolating from the data and analyzing the relationships implicit in a graph (generating, predicting).

Curcio's (1987) terminology refers to the above three levels as: read the data, read between the data and read beyond the data. Shaughnessy et al. (1996) described a fourth aspect as looking behind the data, involving consideration of the context within which data sets arise. Friel, Curcio and Bright (2001) in an extended review of graph comprehension research have identified some critical factors influencing graph comprehension, including:

1. Purposes for using graphs; there are two kinds of uses for graphs, for analysis and communication. Graphs help us summarize data: describe measures of interest, discover relationships or test models that are based on our beliefs about how the world works. This is an aspect that appears to be related to the school curriculum. The second use is communicating our observations to others. That is, pictures that intend to convey information about numbers and relationships

among numbers. These graphs often called 'presentation graphics' and are typically more comprehensive than analytical graphs.

2. Characteristics of tasks in graph perception: The process of visual decoding – which dimension associated with graphs (line, area, position etc.) should be employed to facilitate graph use. Cleveland and McGill (1984) have identified ten graphical perception tasks that form the basic perceptual judgments that a person performs to decode visually quantitative information encoded on graphs (specifiers). For example, finding positions on a common aligned scale (bar graph) can be processed more easily and more accurately than determining area (pie chart). The nature of judgment tasks – for example, comparison judgments (between absolute lengths of bars) and proportional judgments (comparing individual slices with the whole in pie charts). These tasks require point reading, or integration of information across data points such as computations, making comparisons and identifying trends. For example, the effect of contextual setting – a graph's visual characteristics (syntax) and the graph's context (semantics). Because data are usually from real world contexts, a graph reader must be able to describe, organize, represent, and analyze and interpret data taking into account the contextual frame of the data.

3. Characteristics of the discipline: spread and variation – data reduction and scaling. Data type and size of data set – as stated above the use of specific graphs is determined by the type and size of the data. Another factor is the graph complexity – simple graphs should be used early in the school while more complex ones later during middle school.

4. Characteristics of graph readers; familiarity with the context, domain knowledge, general learner characteristics (general intelligence).

Similar classes of factors affecting the interpretation of graphs have been stated by other researchers (Shah and Hoeffner, 2002) in reviewing graph comprehension research: 'The characteristics of the visual display; the viewers' knowledge of graphical schemas







and conventions; the content of the graph and the viewer's prior knowledge and expectations about the content'

**Pilot research**

Based on that theoretical framework, we conducted research into graph comprehension of primary school students (4th, 5th and 6th grades) to serve as a pilot to my final research.

We visited three schools – one in the city of Thessaloniki, the other in an industrial area in the outskirts of Thessaloniki and the third in the small town of Chalkidiki – and we asked the students to complete a questionnaire (see Appendix). The sample consisted of 280 students (100 from 4th grade, 95 from 5th grade and 85 from 6th grade, 132 boys and 148 girls) who were given the questionnaire and asked to complete the tasks without any further instruction in one hour.

The questionnaire included 10 graphs (2 picture graphs, 2 bar graphs, a line plot, a line graph, a pie chart, a stem-and-leaf plot, a box plot and a histogram) covering topics that were real or familiar to children such as the height of children, books read by children, monthly temperature, number of letters in names, newborn babies' weight, birthdays etc. Most of the graphs were followed by six questions (in a multiple choice format) intended equally to reflect on the three levels of graph comprehension, as stated above. In some graphs there were more than six questions and open-ended.

We will discuss here the preliminary results of 4 out of the 10 tasks (i.e. bar graph, line plot, line graph and box plot). Relevant research (Curcio, 1987; Pereira-Mendoza and Mellor, 1991; Bright and Friel, 1998;) indicates that students experience few difficulties with 'read the data' questions, but they make errors when they encounter 'read between the data' questions. Such errors may be related to mathematics knowledge, reading/language errors, scale errors, or errors in reading the axes. 'Read beyond the data' questions seem to be even more challenging. Students must make inferences from the representations in order to interpret the data, for example, to compare and contrast data sets, to make a prediction about an unknown case, to generalize to a population, or to identify a trend.

The results of our pilot study seem to be more or less similar. In general, there were few problems with the literal reading of graphs as shown in the table below:

**Table 1:** Success rate of the read-the-data questions

| Graph | Success rate | Graph | Success rate | Graph | Success rate | Graph | Success rate |
|---|---|---|---|---|---|---|---|
| Bar 1 | 94% | Line-plot 1 | 34% | Line-graph 1 | 70% | Box plot 1 | 93% |
| Bar 2 | 77% | Line-plot 2 | 91% | Line-graph 2 | 88% | Box plot 2 | 65% |
| | | | | Line-graph 3 | 91% | Box plot 3 | 42% |

The first question was 'what does this picture tell you', which requires translation of the labels and titles of the graphs and as seen in the table had a very good success rate in bar graph, box plot and line graph. The line plot had no clear title, a fact that seemed to be very confusing for the students. The question here was open-ended so 34% of the students answered correctly at a very descriptive level while 54% were incomplete in their answers, focusing on the labels ('the names of a class'). Similarly, in the line graph 1 question 26% of the students gave incomplete answers mostly combining the labels and disregarding the title although there was a definite one. The answers to these two questions from one student are quite interesting. He wrote for the first one 'it is a pyramid of X' and for the other 'it is a mountain'. A graph is a sign, and a sign is defined as something that stands for something else or for someone (a student in our case) (Baker, 2004). The first something is mainly an inscription on paper and the second something is a mental construction, a mathematical or statistical object. It is acknowledged that a sign is always interpreted as referring to something else within a social context.

The second (and third in two graphs) question(s) required locating some specific information explicitly shown in the graph. In the bar 2 question 20% of the students answered '100 millimeters'', obviously a reading/language error, while in box plot questions 2 and 3 25% and 22% respectively chose the lower and upper quartile to answer. The students (and teachers) had never seen a box plot before and







that seem to have influenced their answers because they hadn't had the domain knowledge (Freedman and Shah, 2002) or the graph schema (Pinker, 1990). Students encountered more problems with the interpretation questions.

**Table 2:** Success rate of the read-between-the-data questions

| Graph | Success rate | Graph | Success rate | Graph | Success rate | Graph | Success rate |
|---|---|---|---|---|---|---|---|
| Bar 3 | 98% | Line-plot 3 | 81% | Line-graph 4 | 68% | Box plot 4 | 56% |
| Bar 4 | 61% | Line-plot 4 | 11% | Line-graph 5 | 82% | Box plot 6 | 37% |
| | | Line-plot 6 | 54% | Line-graph 6 | 62% | | |
| | | | | Line-graph 7 | 32% | | |

The third question on the bar graph had the highest success rate of all in the test, while in the fourth almost 11% of the students again confused the measurement unit and the rest (28%) made computational or scale errors. In the line plot 3 question 5% of students answered 13 (the highest number in the x axis). The question that seemed to be frustrating for almost all students was the line plot 4. It had the lowest success rate and 30 different answers with 10% of them being 6, which comes from adding the Xs on 4, 5, 11 and 12 letters. In the sixth question 13% of students answered, none confusing the axes. On the other hand, students seemed to be better in the interpretive questions on the line graph, although this could not be seen in the success rate mainly because most of the answers were only partly correct: in the line graph 4 and 5 questions 21% didn't include dec and 8% didn't include sep in their answer whereas in line graph 7 question, 48% mention either feb/dec or jul/aug. In the line graph 7 question 30% of students were wrong; mostly answering descriptively 'the weather is very warm during summer and cold during winter, or July is the warmest and January is the coldest month'.

Students encountered major problems in the read beyond the data questions where they needed to infer or predict from the data

represented to the graphs. Bar graph questions 5 and 6 were in that category and students had a 57% and 72% success rate respectively. Even though the last one was more than obvious, 20 % of the students answered that 'Chara is of average (normal) height for her age'. To the more predictable question of line plot (5), 54% of the students were correct with fair explanations ('most of the student's names have 8 letters'), while some pointed out that the name of the new student would have 3, 4, 5, 11, 12, or 13 letters (22% all together), because there were few names with so many letters. A noticeable 6% answered 28, confusing the language of the question.

The answers of students to the (open-ended) read beyond the data questions of the line graph and box plot show us why we should use that kind of question and avoid the multiple choice format. In the line graph students were asked to predict the lowest and highest temperature of a typical day in May bearing in mind that the monthly average temperature was 19.6 degrees Celsius. Nineteen percent of the students used the lowest and the highest temperature of the graph, that is 5 and 26, while 10% answered 18 to 20. There were a number of students (almost 26%) who answered, correctly using temperatures scales from 9 to 16 for the lowest and 20 to 29 for the highest. An interesting fact was that only few numbers were integers. The majority of students used decimal numbers like (18.6 to 21.5). One girl didn't answer, explaining that after all, she wasn't a meteorologist.

The responses of students to the same question in the box plot were even more interesting because they had to explain the answer. There were 24 different answers in a scale of 500 **grammes** to 5500 **grammes**. Many of the students (19%) correctly used the median (3100) to predict the weight of a heavy newborn brother or sister. Fourteen percent and 8% used the low and the upper quartile (2800 and 3700), while 5% and 4% used the two extremes (1500 and 4500). The most common explanation was 'because I weighed that much too' but there were a lot of explanations that need to be further examined (with an interview perhaps). 'I think it will be born at about 1500 grammes because my mom doesn't eat much', or '4000 grammes because we all are overweight in our family', or '1500, because my mom is very thin'.







**Discussion**

The elementary mathematics curriculum in Greece that refers to data analysis seems to adopt a superficial approach. Specifically, under the topic of collection of data (ΥΠΕΠΘ, 1997), the curriculum suggests: 'with appropriate activities students are expected to be capable of collecting, organizing, interpreting and presenting data and interpreting graphical displays (grade 4); To interpret and construct graphical displays, to describe the concept and to find the Mean and to make estimations (grades 5 and 6)'. In the new revised national curriculum (Government Gazette, 1376, 2001) some indicative and fundamental concepts of the so called interdisciplinary approach are added only by a brief reference: 'variation, system, organization, space-time, unit-whole, similarity-difference, probability'.

Mathematic textbooks (which are the same for all students throughout the country) include 8 teaching units, to cover these optimistic suggestions, during all six years of primary school. As a result, primary school students in our country have very limited domain knowledge and familiarity with graphs and visual representations of data. Nevertheless, this primitive situation is helpful in determining some basic processes and features that take place during graph comprehension.

Most of the students in our research faced their work with joy and interest. They didn't even take their break in order to finish the task and there were no complaints at all. They asked for some explanations but we preferred not to help, urging them to look at the picture and answer what they understood. The fourth graders were struggling to finish the task and many of them didn't make it, mainly because they were delayed by computations and language understanding. Interestingly, there were no major differences at grade levels even though sixth graders were slightly better in most of the questions. In some cases there were sex differences (for example, in bar 5 question – 62% to 53% – and in bar 6 question – 64% to 78%) something that has to be further examined. In general, girls were better at completing the tasks, with better success rates in most questions.

**References**

Australian Education Council (1994) Mathematics – A Curriculum Profile for Australian Schools. Carlton, Vic: Curriculum Corporation.
Bertin, J. (1983) Semiology of Graphics (2ndedn, trans W.J. Berg). Madison: University of Wisconsin Press. (First published 1967).
Bright, G.W., Friel, S.N. (1998) Graphical Representations: Helping Students Interpret Data, in S.P. Lajoie (ed.), Reflection on Statistics: Agendas for Learning, Teaching, and Assessment in K-12, pp. 275–95. Mahwah, NJ: Erlbaum.
Carswell, C.M. (1992) Choosing Specifiers: An Evaluation of the Basic Tasks Model of Graphical Perception. Human Factors, 34, pp. 535–54.
Cleveland, S.W., Mcgill, R. (1984) Graphical Perception: Theory, Experimentation, and Application to the Development of Graphical Methods. Journal of the American Statistical Association, 79, pp. 531–54.
Curcio, F.R. (1987) Comprehension of Mathematical Relationships Expressed in Graphs. Journal for Research in Mathematics Education, 18, pp. 382–93.
Curcio, F. (2001) Data Representation, in Grinstein, L.S. and Lipsey, S.I. Encyclopedia of Mathematics Education. New York: Routledge, pp. 184–86.
Department of Education and Skills (2001) The National Numeracy Strategy Framework. London: HMSO.
Freedman, E.G. and Shah, P. (2002) Toward a Model of Knowledge-Based Graph Comprehension. Paper presented at Diagrammatic Representation and Inference, Second International Conference, Diagrams 2002, Callaway Gardens, GA, USA.
Friel, N.S., Curcio, R.F., Bright, W.B. (2001) Making Sense of Graphs: Critical Factors Influencing Comprehension and Instructional Implications. Journal for Research in Mathematics Education, 32 (2), 124–58.
Fry, E. (1984) A Theory of Graphs for Reading Comprehension and Writing Communication. New Brunswick: NJ: Rudgers University (ERIC Document Reproduction Service No: ED 24528).
Gal, I. (1998) Assessing Statistical Knowledge as it Relates to Students' Interpretation of Data, in S.P. Lajoie (ed.), Reflection on Statistics:







Agendas for Learning, Teaching, and Assessment in K-12 (pp. 275–295). Mahwah, NJ: Erlbaum.
Graham, A. (1987) Statistical Investigations in the Secondary School. New York: Cambridge University Press.
James, G. and James, R. (1992. Mathematics Dictionary. 5th ed., New York: Van Nostrand Reinhold.
Jolliffe, F.R. (1991) Assessment of the Understanding of Statistical Concepts, in D. Vere-Jones (ed.), Proceedings of the Third International Conference on Teaching Statistics, 1, pp. 461–66). Voorburg, The Netherlands: International Statistical Institute.
Kader, G., Perry, M. (1994) Learning Statistics with Technology. Mathematics Teaching in the Middle School, 1, pp. 130–36.
Kintsch, W. (1988) The Use of Knowledge in Discourse Processing: A Construction-Integration Model. Psychological Review, 95, 163–82.
Kosslyn, S.M. (1989) Understanding Charts and Graphs. Applied Cognitive Psychology, 3, pp. 185–226.
Kosslyn, S.M. (1994) Elements of Graph Design. New York: W.H. Freeman.
National Council of Teachers of Mathematics (2000) Principles and Standards for School Mathematics. Reston, VA: NCTM.
National Research Council (1990) Reshaping School Mathematics: A Philosophy and Framework for Curriculum. Washington, D.C.: National Academy Press.
Mcknight, C.C. (1990) Critical Evaluation of Quantitative Arguments, in G. Kulm (ed.), Assessing Higher Order Thinking in Mathematics (pp. 169–85). Washington, DC: American Association for the Advancement of Science.
Ministry of Education (1993) The New Zealand Curriculum Framework. Wellington:
Learning Media. http://www.minedu.govt.nz/web/downloadable/dl3526_v1/MATH_NZC.pdf
Pereira-Mendoza, L., and Dunkels, A. (1989) Stem-And-Leaf Plots in the Primary Grades. Teaching Statistics, 13, pp. 34–7
Pereira-Mendoza, L., and Mellor, J. (1991) Students' Concepts of Bar Graphs – Some Preliminary Findings, in D. Vere-Jones (ed.), Proceedings of the Third International Conference on Teaching Statistics, 1, pp. 150–57. Voorburg, The Netherlands: International Statistical Institute.

Tukey, J. W. (1977) Exploratory Data Analysis. Reading, MA: Addison-Wesley.
Shah, P. and Hoeffner, J. (2002) Review of Graph Comprehension Research: Implications for Instruction. Educational Psychology Review, 14(1), pp. 47-69.
Shaughnessy, J.M., Garfield, J. and Greer, B. (1996) Data Handling, in A.J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, C. Laborde (eds), International Handbook of Mathematics Education. Part 1, pp. 205–37. Dordrecht, The Netherlands: Kluwer Academic Publishers.
Wainer, H. (1990) Graphical Visions from William Playfair to John Tukey. Statistical Science, 5 (3), pp. 340–45.
Wainer, H. (1992) Understanding Graphs and Tables. Educational Researcher, 21(1), 14–23.

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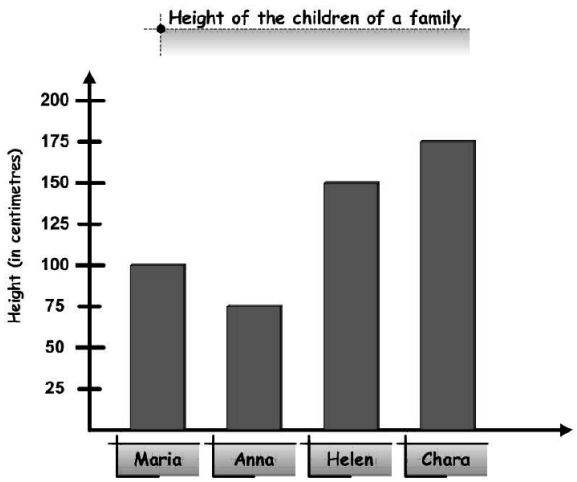
**Appendix**

**Bar graph** (Adapted from Curcio (1987))







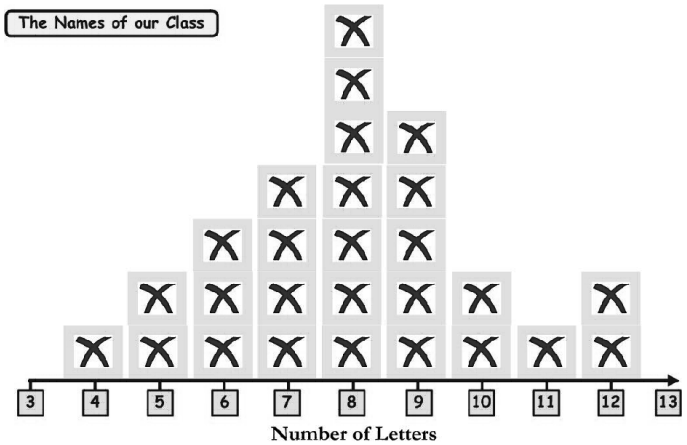



View the picture carefully and try to answer the questions below

1. What does this picture tell you?
α. The weight of four children of a family
β. The grades of four children of a family
γ. The height of four children of a family
δ. The age of four children of a family
2. How tall is Maria
α. 75 centimeters
β. 100 millimeters
γ. 125 millimeters
δ. 100 centimeters
3. Who was the tallest
α. Maria
β. Anna
γ. Helen
δ. Chara
4. How much taller was Helen than Anna
α. 25 centimeters
β. 50 centimeters
γ. 75 millimeters
δ. 75 centimeters
5. If Maria grows 5 centimeters and Anna 10 centimeters who will be taller, and by how much?
α. Maria by 20 centimeters
β. Anna by 20 centimeters
γ. Maria by 5 centimeters
δ. Anna by 5 centimeters

6. If Chara is 5 years old, which of the following is a correct statement?
α. Chara is much too short for her age
β. Chara could never be that tall for her age
γ. Chara is of average height for her age
δ. Chara is thin for her age

**Line plot**




View the picture carefully and try to answer the questions below

1. What does the picture tell you?

……
……
……
……
……

2. How many children have 9 letters in their names?

……
……
……

3. How many children are in the class?

……
……
……







4. What is the sum of the letters of the 4 smallest names?

……
……
……

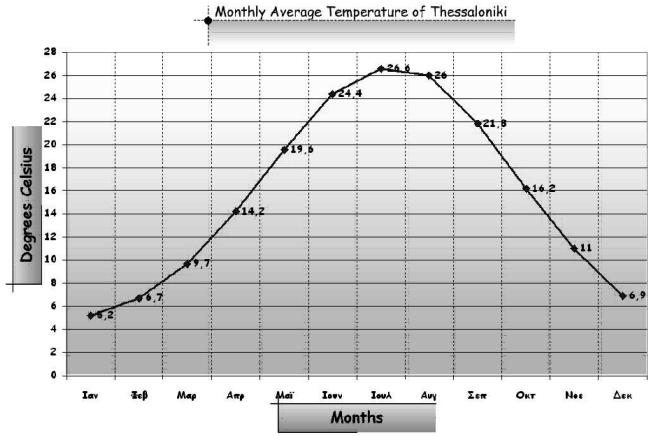
5. If a new child came to the class, how many letters do you believe that his/her name would have? Explain why.

……
……
……

6. How many children have more than 8 letters in their names?

……
……

**Line graph**




**View the picture carefully and try to answer the questions below**

1. What does the picture tell you?

……
……
……

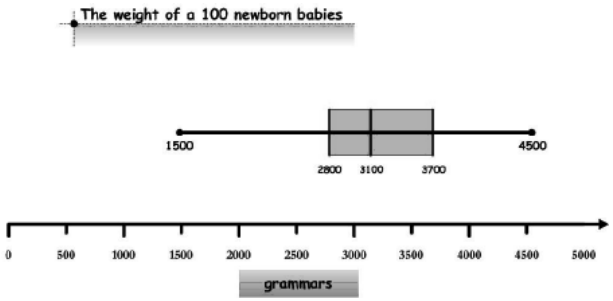
2. Which is the coldest month of the year
……
……....
3. Which is the warmest month of the year
.....

4. For which months is the average temperature below ten degrees?
.....
.....

5. For which months is the average temperature above twenty degrees
.....
.....
6. What is the difference between the warmest and the coldest month
.....
.....
7. Which months have almost the same average temperature
.....
.....
8. Could you predict the lowest and the highest temperature of a typical day of May having in mind that the monthly average temperature of May is 19,6?

Lowest: ……. Highest: …….

**Box plot**










**View the picture carefully and try to answer the questions below**
1. What does the picture tell you?
……
……
2. According to the picture how much is the lightest baby's weight?
α. 3100 grammes
β. 3700 grammes
γ. 2800 grammes
δ. 1500 grammes
……
3. According to the picture how much is the heaviest baby's weight?
α. 3700 grammes
β. 3100 grammes
γ. 4500 grammes
δ. 5000 grammes
……

4.According to the picture, which of the following is a correct statement?
α. Half of the babies are born with a weight below 2800 grammes
β. Half of the babies are born with a weight above 3700 grammes
γ. Half of the babies are born with a weight between 2800 and 3700 grammes
δ. Most of the babies are born with a weight below 2800 grammes
5. If a new brother or sister of yours was born tomorrow how much do you think that he or she would weigh keeping in mind the information provided by the picture?
……
…… Why?
6. What is the likelihood that a new born baby will weigh 2800 to 3700 grammes?
α. 1/2. β. 1/4
γ. 1/3 δ. 1/5

**Graph components**

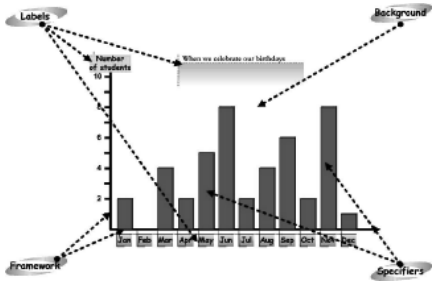
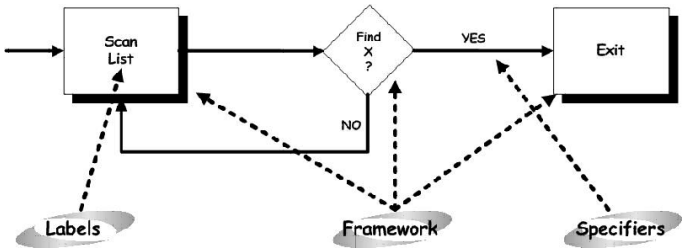



Chart components



# KNOWLEDGE AND CONCEPTS OF RATIONAL NUMBERS HELD BY ELEMENTARY SCHOOL TEACHERS

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***Abstract***
*The growing literature of the last twenty-five years on rational numbers is concerned with the difficulties that students have in this area, and the way that teaching and learning must be connected. The multi-conceptual character of rational numbers, and the lack of their development in elementary schools, give them a central position among the many sources of misconceptions and deficiencies. The four subconstructs of rational numbers that Kieren introduced – part-whole and measure, ratio, quotient and operator – are widely accepted. The lack of knowledge and misconceptions of teachers are examined here, as they are an important factor in the teaching process, which within with the Greek curriculum, gives little attention to the development of these subconstructs, thus leading to deficiencies in learning.*

**Keywords:** In-service education, primary teaching, rational numbers, teachers' knowledge

**Introduction**
The definition of rational numbers as ordered pairs of integers (a, b), with b ≠ 0 and the construction of Q as an infinite quotient field of the integers is not useful in elementary schools, because the introduction of the rational numbers should be 'rich in connections among symbols, models, pictures, and context' (Cramer et al., 2002, p. 112).





The most common symbol that we use for the representation of a rational number is the fraction. As Freudenthal (1983, p. 134) argues: 'Fractions are the phenomenological source of the rational number – a source that never dries up.'

Over the past 30 years many things have been written about rational number concepts as being 'a formidable learning task' (Behr et al., 1983, pp. 92–93) and a serious obstacle in the mathematical development of children. 'When fractions and rational numbers as applied to real-world problems are looked from a pedagogical point of view, they take on numerous 'personalities' (Behr et al., 1992, p. 296).

In one of the latest approaches, Behr et al. (1992, p. 298) stated: 'It would appear, then, that five subconstructs of rational number – part-whole, quotient, ratio number, operator, and measure – which have to some extend stood the test of time, still suffice to clarify the meaning of rational number', which seems to agree with Kieren, Vergnaud and Freudenthal. These subconstructs are:

1. Part-Whole
2. Measure
3. Ratio
4. Quotient, and
5. Operator

Taking into account the theory that Dienes proposed, we see clearly why knowledge and conceptualization of all the previous subconstructs are indispensable. The Perceptual Variability Principle that Dienes (1967 in Post and Reys, 1979) asserted suggests that conceptual learning is maximized when children are exposed to a concept in a variety of physical contexts or 'suits of clothes'. His Mathematical Variability Principle, also, asserts that if a mathematical concept is dependent upon a certain number of variables, the systematic variation of these is a prerequisite for effective learning of the concept.

With these thoughts, and within the previous analysis, Behr et al. (1992, p. 315) identified five areas where mathematics curricula of elementary and middle schools are 'deficient':

1. Composition, decomposition, and conversion of units.
2. Operations on numbers from the perspective of mathematics of quantity.
3. Constraint models.
4. Qualitative reasoning
5. Variability principles.

A characteristic of the Greek educational system is that, as curriculum is constructed from the central educational service (Pedagogical Institute) and in the vast majority of cases is followed 'blindly' by teachers, these deficiencies pass on to the instruction. A recent study (Chassapis and Katsarkas, 2005) evidences that.

**Rational numbers and recursive knowledge**

Kieren (1993) characterized personal rational number knowledge in terms of four kinds of knowing: ethnomathematical (E), intuitive (I), technical-symbolic (TS), and axiomatic-deductive (AD) (see Fig. A), and he gives the following descriptions:

'The first kind of knowing (E) is the kind that children, or adults for that matter, possess because they have lived in a particular environment' (p. 66). Intuitive (I) knowing of fractional numbers entails the use of thinking tools, imagery, and the informal use of fraction language. […] Technical-symbolic (TS) knowing is knowing that is the result simply of working with symbolic expressions involving fractional or rational numbers. […] Finally, axiomatic-deductive (AD) is knowledge derived through logically situating a statement in an axiomatic structure.'

Kieren (1993) in collaboration with Pirie, proposed a more enhanced model of understanding as a dynamic, nonlinear process that they regard as part of a recursive theory of mathematical understanding which is shown in Figure B. A feature of this model is that, 'the outer level does not just appear and then become linked. It is in some way already coherent with what has gone before' (p. 74). This model







suggests that 'someone can function with fractions at a symbolic level in a way that shows no connection to or awareness of previous, more intuitive ways of knowing. […] But such "blindness" […] can cause difficulty. In forming an image or in formalizing, one may develop an outer level of functioning that may be "wrong" even though it is wrapped around "correct" intuitive knowing' and also "In knowing with understanding learners fold back to the inner level of knowing"' (pp. 74–75).

**The context of the research**

As Post et al. (1988) assert: 'We know from pilot investigations (for example, Lesh and Schultz, 1983; Post et al., 1985) that many of the same misunderstandings and "naive conceptualizations" that we have identified in youngsters also are prevalent among teachers (p. 200). This gives us the fuel for our research.'

1. Translations among several modes of representation.

'When physical materials are used in instruction, they should provide a concrete representation, or embodiment, of a mathematical principle. When different, yet appropriate, concrete materials are used to develop the same mathematical idea, a "multiple embodiment" is provided' (Post et al., 1979, p. 353). The use of a single embodiment or a single mode in instruction causes misunderstandings and deficiencies.

2. Perceptual distractors.

'The extent to which a child [and teacher] is able to resolve conflicts between visual information and their logical-mathematical thinking is viewed as one of several important indicators of how solid or tenuous is the child's understanding of the rational-number concept in question' (Behr and Post, 1981, p. 8). For example, the necessary, but not directly used, marks on a number line may act as perceptual distractors (Behr and Post, 1981).

3. Order, equivalence and quantitative notion of rational number.

Behr et al. (1983) observed that children's ability to acquire a quantitative notion of rational number is crucial to the development of other rational number concepts. 'One measure of children's quantitative notion of rational number is their ability to perceive the relative size of the rational numbers in a pair or a larger set' (Behr et al., 1984, p. 324).

4. Partition behavior and unit formation.
The concept of partitioning or dividing a region into equal parts, or of separating a set of discrete objects into equivalent subsets, is fundamental to an understanding of rational numbers. The concept of a unit underlies the concept of a fraction. Formation and reformation of units is important in constructing many rational number subconstructs. Partition behavior can give us an insight in how units are formed.
5. Recursive understanding.
Commenting on the results of the second NAEP, Carpenter and his colleagues pointed out that 'Students appear to be learning many mathematical skills at a rote manipulation level and do not understand the concepts underlying the computation' (Carpenter et al. 1980, p. 47 in Post et al., 1982). Teachers have to act, during instruction, at all levels of Kieren's model in order to assist childrens' efforts, which can be done only if they have a firm grasp of the rational number concept.

**Research questions**
Are teachers able to:
1. make translations among several modes of rational number representation?
2. resolve conflicts between visual information and their logical-mathematical thinking?
3. perceive the relative size of the rational numbers in a pair or a larger set?
4. make formation and reformations of units through several partition behaviors?
5. use all levels of recursive understanding during instruction?

**Method of research**
For our research we have adopted a version of the method used by Post et al. (1988) in his assessment of intermediate teachers' knowledge of rational number concepts. Thus, our research has three parts.

Part I includes a pilot-research questionnaire with 28 tasks which examines: general knowledge for the quotient field, equivalence, dominant subconstruct, fraction as an entity, fraction operations, choice of unit, partitive and quotitive division, embodiments







with perceptual distractors, proportionality, connections between representations, recursive knowledge, and decimals.

Part II includes interviews with a number of teachers who took part in Part I. Subjects are male and female teachers in equal proportions, with a wide range of length of service. The questions are declarative of the answers they have given to the questionnaire and extend to other areas as well.

Part III includes a small questionnaire with four tasks. Each task asks teachers to solve one problem or correct a false answer to a problem and then to explain thoroughly how they are planning to teach it in their class.

**Results and interpretations**
Part I was concluded in June 2005. Thirty questionnaires were collected. The subjects were in-service teachers in Greek elementary schools (K-6 to K-12) from Macedonia (Northern Greece).

A statistical analysis of the responses gives the following first results:

Years of service and sex of the subjects play no significant role in their responses. Almost half of the subjects (53%) cannot recognize the term 'rational number', although all of them know the term 'fraction'. Part-whole is the dominant interpretation of the fractions. The explanation given for the meaning of three-eighths uses (100%) this interpretation. 53% of the subjects uses an area ('pizza-circle' or rectangle) for the unit representation and 33% don't use a graphical representation at all. None uses the number line. When asked to recognize a fraction from an area with 10 parts, 4 of them shaded, all recognize 4/10 (or equivalent) or 6/10. None recognizes 2/3 (ratio) or 21/2 (conceptualizing the shaded area as the unit of measure) or 12/3. 73% give a false response to Noelting's 'orange juice' test (1979, in Behr et al., 1983), interpreting a ratio as a part-whole situation.

Quite a large number probably don't have a firm concept of the rational numbers. 31% cannot overcome the perceptual distractors of non-related lines in a 'pizza-circle', and define it wrongly or they can't

define the fraction of an indicated part of it. Success in estimating the results of addition and subtraction of proper fractions and location on a number line ranges from 37% (6/7+2/3) to 47% (11/12-8/9) and finally to 50% (4/9+3/7).

Two-thirds of the subjects recognize the fraction as an entity, but seem not to have a firm knowledge of it. 60% cannot compare two fractions without using the formal procedure of making them homonyms (which was not allowed in that specific task). 53% cannot define correctly what the denominator shows.

In the task of finding the unit region of a given square with area 9/4, 46% cannot solve the problem at all and 21% use rulers and algebraic equations. Only 33% use partitioning methods successfully. Partitioning is a problem to 37% of the subjects when they try to divide 11 pizzas between 9 children, although the majority use higher forms of partitioning.

The vast majority of the subjects prefer to use formal solutions to the given tasks. 57% use a formal solution to a division school problem of fractions and only 17% a graphical solution. This percentage increases (23%) when the problem is not customary. In spite of this, 24% cannot solve a customary K-11 division problem. In addition, 37% don't know how to calculate the division of fractions, and only 13% can give an explanation for the procedure of this calculation. 17% of them seem to have whole number ideas about fractions and almost 20% use additive procedures in equivalence and order tasks.

**Conclusions and discussion**

Results from Part I of our research indicate that in-service teachers don't have clear ideas about rational number concepts. This seems to be in agreement with previous research (see Post et al., 1988).

Most of the teachers use a single mode of representation in their instruction ('pizza circle') which leads to certain interpretations that seem to dominate their own thinking too. Most of them solve 'successfully' word problems using formal procedures and operations. This seems to be in accordance with the school curriculum and the way that schoolbooks represent rational numbers, thus leading to







the deficiencies that we have mentioned above. Behr et al. (1992) state that: 'Based on our own and others' research in the content domain of multiplicative structures, we have come to the realization that many of the limited, alternative (or mis-) conceptions that children and some adults (teachers of middle grades, for example) have about many multiplicative concepts result from deficiencies in the curricular experiences provided in school. […] In the absence of counter-experience or "high level" mathematical education, these limited conceptions remain into adult life' (p. 300). An indication that gives support to the above is that 43% of the subjects in our research view 1/2 as their favourite fraction, which is, as Jack (a 12 year old child in Kieren (1993)) described it: the fraction that 'I have known the longest. I have known it since before I went to school' (p. 56).

The previous indications need to be clarified and examined further through interviews and more specified questions. Part II and III of our ongoing research will we hope provide some declarative remarks to our first results.

**References**

Behr, M., Harel, G., Post, T., and Lesh, R. (1992) Rational Number, Ratio, and Proportion, in D. Grouws (ed.), Handbook of Research on Mathematics Teaching and Learning (pp. 296–333). New York: Macmillan.

Behr, M., Lesh, R., Post, T.R., and Silver, E.A. (1983) Rational Number Concepts, in R. Lesh and M. Landau (eds), Acquisition of Mathematical Concepts and Processes (pp. 91–126). New York: Academic.

Behr, M., and Post, T. (1981) The Effect of Visual Perceptual Distractors on Children's Logical-Mathematical Thinking in Rational Number Situations, in T. Post and M. Roberts (eds), Proceedings of the Third Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (pp. 8–16). Minneapolis: University of Minnesota. (also available online 25/6/2005 http://education.umn.edu/rationalnumberproject/81_1.html)

Behr, M., Wachsmuth, I., Post, T., and Lesh, R. (1984, November) Order and Equivalence of Rational Numbers: A Clinical Teaching Experiment. Journal for Research in MathematicEducation, 15(5), 323–41. (also available online 25/6/2005 http://education.umn.edu/rationalnumberproject/84_2.html)

Chassapis, D. and Katsarkas, A. (2005) Fractional Number References in Real Situations: The Elementary Schoolbooks Case, in Chassapis, D. (ed.) Social and Cultural Dimensions in Mathematics Education (Proceedings of 4th Two-days of Conversation for Mathematics Instruction, 19 and 20 March 2005, Aristotle University of Thessaloniki, (pp. 171–78). Thessaloniki: Copy City. (in Greek)

Cramer, K.A., Post, T.R. delMas, R.C. (2002) Initial Fraction Learning by Fourth- and Fifth-Grade Students: A Comparison of the Effects of Using Commercial Curricula With the Effects of Using the Rational Number Project Curriculum. Journal for Research in Mathematics Education, 33(2) 111–44.

Freudenthal, H. (1983) A Didactical Phenomenology of Mathematics (pp vii–209). Dordrecht, Netherlands: D. Reidel.

Kieren, T.E. (1993) Rational and Fractional Numbers: From Quotient Fields to Recursive Understanding, in Carpenter, T P., Fennema, E., Romberg, T.A. (eds), Rational Numbers – An Integration of Research (pp. 1–10). Hillsdale, NJ: Lawrence Erlbaum Associates.

Post, T., Behr, M., and Lesh, R. (1982, April) Interpretations of Rational Number Concepts, in L. Silvey and J. Smart (eds), Mathematics for Grades 5–9, 1982 NCTM Yearbook (pp. 59–72). Reston, Virginia: NCTM. (also available online 25/6/2005 http://education.umn.edu/rationalnumberproject/82_1.html)

Post, T., and Reys, R.E. (1979) Abstraction Generalization and Design of Mathematical Experiences for Children, in K. Fuson and W. Geeslin (eds), Models for Mathematics Learning. (pp. 117–39). Columbus, OH: ERIC/SMEAC. (also available online 25/6/2005 http://education.umn.edu/rationalnumberproject/79_2.html)

Post, T., Harel, G., Behr, M., and Lesh, R. (1988) Intermediate Teachers' Knowledge of Rational Number Concepts, in Fennema, et al. (eds), Papers from First Wisconsin Symposium for Research on Teaching and Learning Mathematics (pp. 194–219). Madison, WI: Wisconsin Center for Education Research. (also available online 25/6/2005 http://education.umn.edu/rationalnumberproject/88_11.html)

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# THE DISCOURSE OF PRIMARY SCHOOL TEACHERS FOR ASSESSMENT IN MATHEMATICS

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***Abstract***
*Considering language as discourse, thus as a non-neutral means of communication, this chapter attempts to analyse the primary school teachers' discourse of assessment in mathematics. Dominant assessment discourses in pedagogical literature are examined, and official texts of curricula and relevant circulars are analysed, focusing on the assessment of primary school mathematics. Our analysis shows differences in positions and assessment practices of teachers, an appeal to a variety of discourses for their justification, as well as contradictions and ambiguities within the official discourse, which give rise to various tensions.*

**Key words:** critical discourse, curriculum, mathematics assessment, teachers' assessment practices

**Introduction**
Adopting the theoretical premises of Halliday (1985) as well as Critical Discourse Analysis (C.D.A.) by Fairclough (1989; 1995) and Kress (1989), this study aims to investigate the picture of assessment constructed within primary school teachers' discourses, as well as the role assigned to the teacher. The picture of assessment thus constructed is examined as regards the meaning attributed to it, the functions and purposes it serves and the forms and criteria





highlighted. The language features of texts that contribute to it are also examined.

In order to comprehend the texts that teachers produce to assess students in mathematics, it is necessary to examine the frame of discourses within which those texts are produced (Fairclough, 1989). In our study, due to the limitations of the proceedings, we only consider the pedagogical literature related to assessment in education and examine official texts on assessment, focusing on mathematics at primary school.

tThe ways in which teachers justify their positions together with the practices they use, the ways they position themselves within the official discourse and any tensions they may be experiencing are also considered.

**Discussion of assessment in general**
**The concept of assessment, aims, forms and criteria**
Reviewing the assessment literature we notice a convergence of opinions on the following criteria. Assessment is the systematic process (and not the accidental observation) of students which asks for data to be collected, for the collected information to be interpreted, and for the existence of concrete criteria and predetermined objectives.

In contrast to the traditional discourse of performance assessment, which is only interested in the products of learning and not in the internal thought processes that students follow, more recent approaches (NCTM, 1995, 2000) see performance in a broader sense, including elements such as creativity, thought process, attitudes, thus broadening the concept of assessment to make it a dynamic process that supports learning in a complex manner, rather than simply as a means of quantifying students' traits (Matsagouras, 2004).

The main criteria of school performance assessment that appear in the literature are those of: the average classroom student, the student's position with respect to the classroom norm (Gauss curve), the curriculum and self-improvement (see Guskey, 1994, Hiotakis, 1997).

Various forms of assessment appear in the literature to which different functions are attributed, expressing differing perceptions of the concept of assessment. Prevailing amongst them is the distinction between formative and summative assessment. Scriven was the first to make this distinction (Tyler, 1967). Researchers recognize the difficulty of formative and summative assessment co-existing (Torrance, 1993), as they come from different theories of learning. Researchers also point to the priority that should be given to formative assessment in the classroom (Black and Wiliam, 2003).

Many researchers suggest that assessment results should be formulated in recording of comments and/or in a descriptive level, rather than assigning marks (Black and Wiliam, 2001).

The traditional theories of learning and the view of learning as a knowledge construction process within specific contexts (Resnick, 1989) tend to be questioned nowadays. This has resulted in a shift from the assessment model that looks at the performance of individuals in context-free situations towards the model that is related to the assessment of knowledge and skills within authentic situations that have meaning for the students (Maclellan, 2001).

Researchers recognize the difficulty of applying new assessment methods to contexts where the dominant traditional discourse of assessment conflicts with the alternative methods, thus preventing their use for the purposes of promoting learning (Broadfoot, 1998: 447) and creating tensions (Lyons, 1998).

**The necessity of assessment**

The functions of assessment are more generally placed in a complex framework of discourses that oppose each other. These discourses put forward arguments for or against the necessity of assessment.

We found the following six dominant discourses:
the socio-economic discourse in favour of assessment, the sociological against, the pedagogical-psychological in favour, the pedagogical-psychological against, the educational-political in favour and the methodological discourse against the assessment.







The socio-economic discourse in favour highlights the social functions of school assessment. The sociological discourse against the institution of assessment focuses on the examinations and marking, arguing that these contribute to the social categorisation of students, its argument being based on a wealth of sociological research.

The pedagogical-psychological discourse in favour is aligned to the school's pedagogical mission. It is in tension with the pedagogical-psychological discourse against, which is mainly focused on examinations and marking, on how these are employed and on their effects on the student's personality from the pedagogical and psychological point of view.
The educational-political discourse in favour focuses on the educational system, aiming to improve it.

The methodological discourse against focuses on the ineffectiveness of assessment means and methods and on the subjectivity of assessor.

**Methodology**

We considered the texts of Presidential Decrees[1] and circulars currently in effect (P.E. 8/1995 mi 121/1995, F7/228/T1/1561/15–11–96) and related to mathematics assessment in Greek primary schools and mathematics textbook supplements for the teacher. The teachers' view of mathematics' assessment was examined through the transcripts of semi-structured interviews of six teachers.

A method based on Halliday's Functional Grammar (Halliday, 1985) and on the interpretative techniques of Fairclough's Critical Discourse Analysis (Fairclough 1992a, 1989, Kress, 1989, Hodge and Kress, 1993) was used to analyse the official texts and the teachers' texts. Thus characteristics of the language of these texts were examined and the functions – 'ideational', 'interpersonal' and 'textual' – (Halliday, 1973) that they carry for the speaker/author and the reader/listener were interpreted.

The 'ideational' aspects of texts to be analysed relate to the picture constructed for mathematics assessment. They were analysed primarily through the examination of types of processes (Halliday,

1985:101–31) that take place in the discourses under examination and are related to the task of assessment, the type of logical subjects, ie the human or inanimate actors of these processes (ibid: 32–37). The presence of a human being in a text or the absence thereof, and the use of inanimate abstract nouns as actors of the processes, were examined via the use of passive voice and nominalisations that relate to social and ideological aspects of language (Fairclough, 1989, 1992a, Hodge and Kress, 1993).

The teacher's relationship to assessment, with the person to whom the teacher's discourse is addressed, the roles constructed for the task of assessment and the degree of the teacher's autonomy concerns the 'interpersonal' function of language. It was analysed through the use of personal pronouns, modes and the text's modality (Halliday; 1985: 86, Fairclough, 1989:129, 1992a:159 on modality, and Kress, 1989, on their interpretation).

The structure of official texts as a whole and the type of text that is related to the 'textual' function were examined through the 'themes' (Halliday, 1985) that dominate the text.

Drawing on the theory of C.D.A. by Fairclough, 'member resources' were also examined which the producers of texts draw on, in order to justify their positions and practices.

**The official discourse of mathematics assessment**

Examining the official discourse of assessment in Presidential Decrees, relevant circulars and curricula currently in effect in Greece, we notice that there is an agreement between official discourse and teachers' discourse regarding the pedagogical dimension of assessment and its objectives. Official texts draw on the researchers' discourse to justify their positions, yet at the same time considerable tension is apparent when it comes to applying pedagogical principles of assessment in practice, through the use of school textbooks. We notice contradictions between the official texts of Presidential Decrees and the teachers' supplement. While in official texts of P.D. there is talk of continuous process with diagnostic and feedback purposes, in the teacher's supplement assessment appears to be considered as a process separate from the other educational processes, since







assessment is planned in two phases: while the students work with their textbook and at the end of each main unit. The primary means of assessment is the book's predetermined 'assessment criterion' that is, a test in student's textbook at the end of each unit.

Today in the assessment system for the primary school in Greece, the numerical assessment coexists with the descriptive assessment. These forms of assessment contradict each other, since the descriptive assessment clearly has pedagogical aims and is linked to the individual norm-referenced assessment and individual-specific teaching, whilst the numerical assessment grades and categorizes students and follows the classroom norm promoting comparison and competition. With regard to the descriptive assessment and the criteria of assessment which the official texts suggest, explicit instructions to teachers on how to apply them in practice are not provided.

In the teacher's textbook there are contradictions. For example, the teacher's textbook emphasizes the process of problem solving that student follows and the same time proposes assessment tasks which require that students demonstrate their knowledge and skills. Methods of comprehension and assessment of the process that the student follows are not described.

Within the official texts, we see two dominant discourses in conflict with each other. On one hand we find the pedagogical discourse which is encoded in words and phrases of pedagogical content, and on the other hand the educational-political discourse which expresses the government's intentions for education and is encoded in the syntax of official texts.

The main characteristics of official texts are similar to those of scientific texts: tendency to impersonal expression that adds objectivity to the statements in the text, absence of personal syntax achieved with the use of passive syntax and nominalisations, absence of actors, use of processes that primarily concern definitions and relationships, use of simple present in the indicative that expresses the modality of certainty, specialized technical/scientific vocabulary (Kress, 1989).

In the teacher's supplement, the teacher is constructed as the one who needs information, protection from insecurity and haphazard planning, and as the executor of the proposed instructions and predetermined activities. The diversity and the individuality of students are not taken into account, as the existence of a single school textbook and the universally applicable curriculum show. This contradicts the official educational policy that highlights its assignment of priority to the provision of equal educational opportunities and the non-selective and non-competitive character of assessment.

**The teachers' discourse of mathematics assessment**

Analysis of the teachers' discourse showed that all teachers agree with the official texts and the researchers' discourse as regards the concept and objectives of assessment that make assessment a necessity. Research findings by Philippou and Panaoura (2000), as well as Brown (2004), agree with the teachers' opinion about the diagnostic and formative role of assessment.

Teachers consider that a fundamental obligation is to consult official texts on assessment. The degree of freedom allowed by the educational system to deviate from the curriculum is interpreted in various ways by the teachers. They take the initiative of the usage of assessment results according to the students' needs, modifying the material to be taught, the teaching objectives or the proposed teaching time for each unit. They also differentiate from each other as regards the use of assessment criteria. Earlier research also indicates the different interpretation of official policy by teachers (Yung, 2002). The analysis showed that teachers don't take into account 'assessment criteria' mainly because students tend to prepare for them at home. The teachers make up their own tests for student assessment.

Teachers seem to use the same forms and methods of assessment. They do not apply alternative methods because they lack information on how to use them and because of lack of time. Research suggests that there is a difficulty in applying alternative assessment methods (Black and Wiliam, 1998, Kahn, 2000) and recognizes the tensions this causes to teachers (Broadfoot, 1998, Lyons, 1998).







The analysis also showed that teachers do not apply descriptive assessment, arguing that they lack information on how to apply it and, again, due to lack of time. They mark written work and tests despite the fact that official instructions forbid it. This practice agrees with Mavromatis's findings (1997).

Most teachers appear to believe in the counterbalancing role of the teacher for students who come from difficult family backgrounds and this agrees with findings by Zbainos and Hallam (2002).

All teachers agree with the official discourse regarding the main objectives of mathematics education and the importance that should be given to the process of problem solving that students follow. However, when assessing, the majority of teachers appears to focus more on results and on the existence of a single right answer or right path to a solution, rather than on the process and variety of student answers, hinting at an absolutist view of mathematics (Ernest, 1996). This inconsistency reflects the contradiction that exists between the value which the official discourse places on the solving process and the concurrent requirement that students demonstrate skills and precision of results, as required by the practical discourse of the textbook.

All teachers recognize the lack of the assessment's objectivity due to the subjective criteria of each teacher. Variations are mainly to be found regarding (1) the individual characteristics of students and the kind of criteria that influence the assessment, vis a vis the need to make students aware of them, (2) their view of the role of marking and (3) their view of the ideal way to express assessment results.

With regard to the assessment criteria put forward by official texts, most participants agree on taking into account the effort spent, performance in written tests, daily orals, all round participation in the classroom and the student's degree of interest. Homework and student behaviour at school (which are criteria defined by the official discourse) appear not to influence the assessment of most teachers. An overview of related research verifies the variety of assessment criteria that teachers employ and the different prioritisations applied (Gipps et al., 1996, Philippou and Christou, 1997, Zbainos

and Hallam, 2002, Pilcher, 1994). Variations in assessment practice spotted by the analysis is confirmed by research that supports the idiosyncratic nature of assessment, which is based on the teacher's beliefs, knowledge and experience (Webb, 2004, Cizek et al., 1995) and shows that different teacher views lead to different assessment practices, even when the curriculum and assessment criteria are predefined to a large extent (Watson, 1999).

Within the teachers' discourse the role constructed for the teacher is that of the local enforcer who is obliged to follow the official policy on assessment, yet, at the same time, ought to take into account his/her students' needs. He/she takes the initiative in modifying, to a degree, the proposed assessment practices, depending on the students' individual situation, yet taking care not to stray far from the general frame of assessment that the official discourse shapes. The role of participants as teachers/educators and as enforcers of the official discourse's instructions appears to create tension which teachers address with the following strategies: they yield to the external authority or are influenced, each to a greater or lesser extent, by the official policy on assessment practices and adapt them, more or less, to the particularities of the class (on tension, see: Baker and O' Neil, 1994, Morgan, 1998, Black, 2001, Brown, 2003).

By investigating the discourses that influence teachers' practices and the sources of their arguments, it became apparent that teachers mainly draw on the official discourse and the discourse of researchers with respect to the concept, objectives and necessity of assessment. As regards marking, they draw on the sociological and pedagogical-psychological discourse (for or against marking). A small percentage argues based on common sense and their personal experience, except for one participant who mainly draws on her personal experience.

As for the main linguistic features used in the interview transcripts, which are linked to 'ideational' and 'interpersonal' functions of language, not only similarities but also essential differences were observed. Specifically these were: the processes that dominate are mostly 'material' and 'mental' processes (Halliday, 1985); the choice of a certain type of process puts the speaker in the position of the







person who considers that the activities to which they refer involve processes of this type.

With regard to the type of the human-subject of the processes in use (where stated explicitly) – excluding cases in which the student is the human actor – we note the following four categories:

1. A specific person, the interviewee herself/himself talking in the first singular, implying his/her personal involvement in the activity stated in the text or that the views expressed are personal opinions for which he/she assumes responsibility. We couldn't claim that the use of the pronoun 'I' carries any ideological significance (Fairclough, 1989).
2. The general participant 'you' used to claim commonality of experience between transmitter and receiver (ibid.: 180), whereas in other cases in order to share responsibility for the transmitter's statements with the general participant 'you' or to make the receiver feel that the practices and convictions of the transmitter also constitute convictions of the general participant 'you' (ibid.).
3. The teacher, whose use as actor implies the speaker's distancing from the task of assessment.
4. The teachers' community which, in Greek, is revealed by the ending of verbs in the first plural, and whose use indicates that the speaker speaks on behalf of the teachers' community to which he/she belongs, which guarantees the authority of his/her statements or shows the diffusion of responsibility to this community (see Kress, 1989).

There exists minimum usage of inanimate subjects and, where present, they refer to the impersonal authority of the government. More generally, within interview texts, there appears some inconsistency in the use of personal pronouns, while the dominant type of human actor varies between texts.

The usage of passive voice and of nominalisations (infrequent) also varies. They are mostly used in the discourse of two participants who mainly use as their actor the general participating 'you'. As a result, the relationship constructed between speaker and listener

and between speaker and assessment varies substantially among texts according to the degree of familiarity with the listener and the distance or 'ownership' of the task of assessment. It is worth noting the case of a text in which the absence of the first person singular and the high usage of passive syntax and nominalisations contribute to the creation of a formal and impersonal relationship between speaker and listener and express his/her tendency to distance himself/herself from his/her statements, as well as from the task of assessment. A second text also shows the speaker's tendency to distance his/herself and evade responsibility for statements concerning personal assessment practices; in this text, the speaker does not employ the first person singular, thus avoiding a personal involvement in the task of assessment.

The use of the simple present (indicative) is dominant in all texts, attaching the modality of certainty which, combined with the use of certainty or uncertainty modals, varies among texts. In most texts, modals of uncertainty are used almost as frequently as modals of certainty, implying a lack of certainty for numerous statements by interviewees.

**Conclusions**

The differences between teachers' assessment practices shown by analysis of their discourse highlight the complexity of the assessment process and raise issues of social justice for the students (Watson, 1999).

Teachers draw on different, sometimes contradictory, discourses for the justification of their positions and practices. This renders obvious the need for teachers to strengthen their awareness of the way in which particular discourses position them and their awareness of assessment practices towards which these discourses direct them. So doing they can realize which discourses, values, beliefs and practices they identify with, resist those with which they do not agree and face effectively the tensions that they experience.

Additionally, contradictions and ambiguities within the official discourse were highlighted as well as inconsistency between revised objectives and suggested assessment practices. All these require that







positions and practices should be submitted to scrutiny and that teachers' training programmes on assessment and how to apply alternative methods should be planned.

**Footnotes**:

[1] Author's note: A Presidential Decree is a legislative instrument widely used by the Hellenic Republic, whereby the President of the Hellenic Republic signs off a specific item of legislation, thus putting it into effect.

**References**

Baker, E.L. and O'Neil, H.F. (1994) Performance Assessment and Equity: A View from the USA. Assessment in Education 1(1), 11–26.

Black, P. (2001) Dreams, Strategies and Systems: Portraits of Assessment Past, Present and Future. Assessment in Education 8(1), 65–85.

Black, P. and Wiliam, D. (1998) Assessment and Classroom Learning. Assessment in Education 5(10), 7–73.

Black, P. and Wiliam, D. (2001) Inside the Black Box: Raising Standards through Classroom Assessment, London: King's Assessment for Learning Group, King's College. Available online at: http://ngfl.northumberland.gov.uk/keystage3ictstrategy/Assessment/blackbox.pdf

Black, P. and Wiliam, D. (2003) In Praise of Educational Research: Formative Assessment. British Educational Research Journal 29(5), 623–38.

Boud, D. (1986) Implementing Student Self Assessment. Sydney: HERDSA.

Broadfoot, P. (1998) Records of Achievement and the Learning Society: A Tale of Two Discourses. Assessment in Education: Principles, Policy and Practice 5(3), 447–77.

Brown, G. (2003) 'Teachers' Instructional Conceptions: Assessment's Relationship to Learning, Teaching, Curriculum, and Teacher Efficacy'. Paper Presented to the Joint Conference of the Australian and New Zealand Associations for Research in Education. Auckland, NZ: AARE/NZARE. Available on line: http://www.aare.edu.ak/03pap/bro03022.pdf

Brown, G. (2004) Teachers' Conceptions of Assessment: Implications for Policy and Professional Development. Assessment in Education 11(3), 301–18.
Cizek, G.J., Fitzgerald, S., Shawn, M. and Rachor, R.E. (1995) Teachers' Assessment Practices: Preparation, Isolation and the Kitchen Sink. Educational Assessment, 3, 159–79.
Ernest, P. (1996) The Nature of Mathematics and Teaching, Philosophy of Mathematics Education Newsletter 9, POME. UK: University of Exeter.
Fairclough, N. (1989) Language and Power. Harlow: Longman.
Fairclough, N. (1992) Discourse and Social Change. Cambridge: Polity Press.
Fairclough, N. (1995) Critical Discourse Analysis: The Critical Study of Language. Harlow: Longman.
Gipps, C., McCallum, B., and Brown, M. (1996) Models of Teacher Assessment among Primary School Teachers in England. The Curriculum Journal, 7(2)
Guskey, T. (1994) Making the Grade. Educational Leadership, 52(2).
Halliday, M.A.K. (1973) Explorations in the Functions of Language. London: Edward Arnold.
Halliday, M.A.K. (1985) An Introduction to Functional Grammar. London: Edward Arnold.
Hiotakis, St. (1997) Abolition of Examinations and Degrees in the Primary School: Expectations and Reality, in Hiotakis, St. (ed.), The Contestation of Examinations and Degrees in School. Athens: Grigoris, 2nd edition, pp. 48–85.
Hodge, R. and Kress, G. (1993) Language as Ideology (2nd ed.). London: Routledge and Kegan Paul.
Kahn, E. A. (2000) A Case Study of Assessment in a Grade 10 English Course. The Journal of Educational Research, 93, 276–86.
Kress, G. (1989) Linguistic Processes in Socialculture Practice (2nd ed.). Oxford: Oxford University Press.
Lyons, N. (1998) With Portfolio in Hand: Validating the New Teacher Professionalism. New York, NY: Teacher College Press.
Maclellan, E. (2001) Assessment for Learning: The Differing Perceptions of Tutors and Students. Assessment and Evaluation in Higher Education, 26(4), 308–18.
Matsagouras, H. (2004) Theory and Practice of Teaching. Strategies of Teaching. Athens: Gutenberg, 5th edition.







Mavrommatis, Y. (1997) Understanding Assessment in the Classroom: Phases of the Assessment Process—the Assessment Episode. Assessment in Education, 4(3), 381–400.
Morgan, C. (1998) Writing Mathematically. The Discourse of Investigation. London: Falmer Press.
NCTM (1995) Assessment Standards for School Mathematics. Reston VA: National Council of Teachers of Mathematics.
NCTM (2000) Principles and Standards for School Mathematics. Reston, VA: National Council of Teachers of Mathematics.
Philippou, G. and Christou, K. (1997) Cypriot and Greek Primary Teachers: Conceptions about Mathematical Assessment. Educational Research and Evaluation, 3(2), 140–59.
Philippou, G. and Panaoura, A. (2000) Teachers' Conceptions of Primary Education of Mathematics Assessment. Modern Education, 110.
Pilcher, J. K. (1994) The Value Driven Meaning of Grades. Educational Assessment, 2(1), 69–88.
Resnick, L. (1989) Introduction, in Resnick, L (ed.), Knowing Learning and Instruction. New Jersey: Lawrence Erlbaum Associates.
The Ministry of Education (1995) Presidential Edict 121/95: Completion of Provisions of Primary School Students' Assessment.
The Ministry of Education (1995) Presidential Edict 8/95: Assessment of Primary School Students.
The Ministry of Education (1996) Circular F.7/228/G1/1961/15–11–1996. Students' Assessment.
Torrance, H. (1993) Formative Assessment: Some Theoretical Problems and Empirical Questions. Cambridge Journal of Education 23: 333–43
Tyler, R.W., Gang, E.R.M and Scriven, M. (1967) Perspectives of Curriculum Evaluation. Chicago: Rand McNally.
Watson, A. (1999) Paradigmatic Conflicts in Informal Mathematics as Sources of Social Inequity. Educational Review, 51(2), 105–15.
Webb, D. (2004) 'Classroom Assessment as a Research Context: Variations on a Theme of Pedagogical Decision Making'. Overview presentation for ICME 10 – TSG 27: Internal Assessment. Available on line: http://www.icme_organisers.dk/tsg27/papers/00_Overview_Webb_fullpaper.pdf
Yung, B.H.-W. (2002) Same Assessment, Different Practice: Professional Consciousness as a Determinant of Teachers' Practice in

a School-based Assessment Scheme. Assessment in Education, 9(1), 97–117.
Zbainos, D. and Hallam, S. (2002) Marking in the Primary School: Towards the Creation of a Factors and Practices' Model that Influences It. Modern Education, 13, 35–47.

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# MATHEMATICS EDUCATION AS A SCIENTIFIC DISCIPLINE: *IMPLICIT ASSUMPTIONS AND OPEN QUESTIONS*

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***Abstract***
*Considerable development of various activities associated with, or considered as constitutive of, mathematics education has taken place during the latest thirty years. In spite of this the scientific object of mathematics education remains, to a great extent, vague. In consequence, the constitutional conceptual system and the fundamental research methodologies of mathematics education also remain ambiguous, since the scientific object, the conceptual system and the privileged research methodologies of a discipline are simultaneously constructed and developed through reciprocal determinations, wherein each one presupposes the existence of the other two. As a result, the epistemic status of mathematics education as a discipline remains in question, and its contribution to the solution of problems arising in the real world of mathematics classrooms is doubted. Relevant questions are posed in this chapter and possible answers are traced by highlighting three interconnected aspects. First, the historical conditions under which mathematics education has been founded both as a discipline and as a research field, second, the fundamental characteristics of the relationship between mathematics education theory and practice which has been developed, and third its essential interdisciplinary nature.*

**Keywords:** History of mathematics education, interdisciplinarity, theory and practice





**Introductory remarks**
In considering mathematics education as a discipline, what is its scientific object? More precisely, what object of study is ascribed explicitly to mathematics education or is deduced implicitly by the relevant activities, publications, conferences and research programs of researchers and educators of the community?

At first glance, the answer to this question seems obvious and easy: the learning and teaching of mathematics. I claim, however, that the answer is neither obvious nor easy. This is because this answer and any similar answer assumes that the 'phenomenon of learning mathematics' has quite distinct characteristics from the phenomenon of learning any other subject matter. On the same basis, the teaching of mathematics is assumed to be a different activity from the teaching of any other subject matter, being an application of that branch of knowledge that has emerged from the study of learning mathematics. Such assumptions, however, may be considered neither self-evident nor as being categorically accepted.

In consequence, the scientific status of mathematics education may be questioned, and, by analogy, the scientific status of any other discipline which is included in the class of the so-called 'educational sciences', since all these disciplines are founded on the same assumptions about particular learning phenomena.

This chapter raises questions and attempts to outline answers, although fragmented and incomplete, adopting an epistemological perspective grounded on the philosophical contributions of Luis Althousser as they have been further developed and exemplified in the epistemology of science by the Greek philosopher Aristidis Baltas (1990).

The main points are summarised here. According to Baltas' analyses (Baltas, 1990), each scientific discipline has its own object of inquiry, which is constructed by the discipline itself. At the same time this object of inquiry, being its scientific object, constitutes a founding component of the discipline. Each scientific discipline constructs its own object, selecting, defining and describing real world phenomena, re-constituting conceptually and incorporating into its scientific

object. This constructive process makes up simultaneously the scientific object of the discipline and the discipline itself. This process presupposes a conceptual system, and it produces this conceptual system by contributing to its development. The same holds for the using and formation of a research methodology appertaining to the particular discipline. The object of inquiry, the conceptual system and the research methodology of a discipline are, therefore, being constituted at the same time by a process of reciprocal determinations and controls, wherein each of the constituents presupposes the existence of the others. This construction process makes its own scientific object relatively autonomous from the object of any other discipline and builds up to the meaning of its fundamental concepts in relative autonomy. These two autonomies of a discipline, i.e. the autonomy of its subject matter and the autonomy of the meanings of its fundamental concepts, define the boundaries of its interdisciplinary relations.

**The emergence of mathematics education as a scientific discipline**

As indicated by Baltas (1983), the formation of each scientific discipline constitutes: 'an especially complex process, which involves and brings into action many different and dissimilar elements of the whole social practice in particular ways. Changes in material production and in the life of people, technical achievements and innovations, novel social experiences, tensions and ruptures in practical ideologies, changes caused to theoretical philosophical diversifications and conflicts are accumulated, interwoven and create a point of critical concentration, where the partial pressures on the known are condensed, mutually reinforced and lead to its destruction. This process of destruction is at the same time the process which creates the scientific object of a discipline, constitutes its conceptual system and instals the research methodologies appertaining to the discipline. Namely, this is the process which constructs, in a unified movement, all the constituent elements of the discipline, a process which is known, after Bachelard, as an "epistemological break"'.

Mathematics education as a scientific practice and as a research field has emerged in the same era as, and to a great extent by way of, the radical school mathematics reforms of the 1960s. These reforms had been inspired and guided by the Organisation for







European Economic Cooperation (OEEC) – later on Organization for Economic Cooperation and Development (OECD) as a response of the USA and its allies to the technological challenge of the Soviet Union posed by the launching of Sputnik on October 1957. This was a challenge directly related to the Cold War. During the 1960s distinguished mathematicians, mathematics education experts and educational decision-makers from the USA and many Western European countries met in seminars and conferences and worked out a plan for the radical reform of school mathematics objectives, curricula, text-books, and in-service preparation of teachers, known afterwards as the 'New Mathematics' movement. (Fehr, 1961a, 1961b, 1964). UNESCO played an important role in these reforms with publications and seminars aiming at the dissemination of 'new school mathematics' in other countries besides the USA and Europe (UNESCO, 1966).

Two pioneer publications were established, the journal Educational Studies in Mathematics was founded in 1968 and two years later the Journal for Research in Mathematics Education. These two journals have heavily influenced the formation of mathematics education as a discipline. At the same time many primary and secondary school mathematics projects, which were developed by various foundations and institutes (Nuffield-UK, Alef-Germany, Analogue-France, Wiskobas-Holland, School Mathematics Study Group-USA and others) have had a considerable impact on the conditions of the foundation of mathematics education as a discipline.

In this societal and political framework, and directly related to the activities of planning and implementation of school mathematics reforms, mathematics education has emerged as a distinguished scientific practice and therefore as a discipline and a research field. In following years, mathematics education has been developed as a scientific discipline and has been established in the academy. The foundation of university departments, provision of undergraduate and postgraduate courses in almost every university around the world, research and development projects, foundation of national and international associations, organization of numerous conferences, publication of books and journals devoted exclusively

to mathematics education, etc., are all circumstantial evidence of this development.

The scientific object of mathematics education has, therefore, been constructed by adopting and applying a 'scientific' view to the phenomenon of 'teaching mathematics' in formal education. That is, altering the view of a phenomenon, under the pressure of socio-political necessities, has produced the scientific object of mathematics education. Until then, the prevailing view approached mathematical thinking and as a consequence mathematics learning as an individual aptitude or as an innate capability. Therefore, mathematics teaching addressed particular individuals, considered to be more or less mathematically gifted, and its contribution to mathematics thinking was considered as more supportive than crucial.

Summing up, mathematics education emerged in a context created by a new view of first the teaching and then the learning of mathematics. However, the formation of mathematics education in these terms implied from its very beginning an ambiguity in the relationships between its theory and its applications, as well as in the relationships that both of them establish with educational policy.

Finally, in my view, the aspects of application took priority and prevailed over the aspects of theory in the construction of the scientific object of mathematics education, a fact that has moulded accordingly its conceptual system as well as its research methodologies.

**The relationships between theory and applications of mathematics education**

An inherited ambiguity exists in the relationships between theory and applications of mathematics education, as well as in the relationships that both maintain with educational policy. In my view, this ambiguity is a result of the interdisciplinary status of mathematics education and the impact of the socio-political conditions under which it has emerged as a scientific discipline.

The research questions in mathematics education and the inquiries which set them off are, as a rule, arising from, and referring to, mathematics classrooms. Their study is not primarily justified by







theoretical requirements but on the grounds of their usefulness in promoting effective mathematics teaching, i.e. in particular applications of mathematics education. These applications are mostly recommended by occasional and specific needs of mathematics education policy-making and not by the demands of mathematics education theory development. Most of the research papers published in journals or presented at conferences, particularly in the USA, justify their aims with reference to objectives or needs arising from educational policy documents or induced by school mathematics curricula and evaluation frameworks. At the same time, as reported by Lerman and Tsatsaroni (2004), more than 85% of the articles published during 1990–2001 in Educational Studies in Mathematics and in Journal for Research in Mathematics Education construct, project or promote an explicit pedagogical model for teaching mathematics.

Thus, applications of mathematics education generated by educational policy-making gain a clear priority over issues arising from its theoretical evolution. This predominance of applications over theory has two clear implications. First, a reduction in the scientific objectives of mathematics education (and as a result a one-sided development of the conceptual system), and second, an establishment of a particular state of affairs concerning the legitimization of the resulting knowledge which, in turn, informs and shapes its content.

To a great extent, the criterion for the legitimization of knowledge produced by mathematics education is not considered to be its truth, however defined, but rather its potential usefulness in mathematics educational policy-making. So, mathematics education is constantly called to account either for employing not scientific but educational norms or for adopting criteria imposed by prevailing educational policies with consequent distortions and insufficiencies in its theoretical evolution.

The assignment of a primary role to theory in its relationship to the applications of mathematics education, as well as the attainment of complete independence from relevant ideological influences, remain an ideal. These are, amongst other requirements, the necessary

conditions for the formation of mathematics education as a scientific discipline, as an interdisciplinary subject.

**The interdisciplinarity of mathematics education**

As mentioned earlier, mathematics education has been formed and evolves on an interdisciplinary basis. The boundaries of its interdisciplinarity are defined by two relative autonomies. The autonomy of its scientific object against the scientific objects of other disciplines and the autonomy of meanings of its fundamental concepts against the meanings, which these concepts possibly have in the empirical world or in the conceptual system of any other discipline (Baltas, 1983).

Assuming that the phenomenon of learning mathematics, and as a consequence the activity of teaching mathematics, have quite different characteristics from the phenomenon of learning and the activity of teaching any other disciplinary knowledge, mathematics education emerged as a scientific discipline adopting and transforming concepts, at first, from three well-established disciplines. Mathematics defines the learning and teaching content, psychology describes the phenomena of human learning at the individual level and sociology approaches the phenomena of learning and the activities of teaching as socially and culturally embedded processes.

However, although social functions are assigned to the teaching of mathematics in formal schooling and its outcomes, almost nothing social is attributed to the phenomenon of learning mathematics itself. The approach to learning mathematics is mainly delimited at an individual level by the mainstream of mathematics education (e.g. Chassapis, 2002; Kilpatrick, 1992; Lerman et al.; 2003, Lubienski and Bowen, 2000; Reyes and Stanic, 1988; Secada, 1992). Social approaches from various standpoints to the study of learning mathematics, individually or collectively, are always a requirement, considering little about the occasional acknowledgements of social discriminations that are produced or are justified by processes of learning of mathematics (Chassapis, 2004).

In addition, the autonomy of the mathematics education object of inquiry against the objects of mathematics and the objects of







psychology is always a requirement. It is alsonecessary, although to a lesser extent, to consider the autonomy of the constitutional conceptual system of mathematics education in relation to these disciplines, as well as against the current teaching empiricism.

The approach to mathematics education as a particular application of mathematics, has many supporters influenced by a philosophy of Platonism in mathematics. From the standpoint of Platonism – and because of its epistemological premises concerning mathematical knowledge – an approach to the phenomenon of learning mathematics in terms which would permit or even facilitate its autonomy as a scientific object seems to be, almost, unfeasible. An approach to the learning of mathematics which adopts a Platonism for mathematics, privileges the mathematics curricula as an object of inquiry, closely associated with issues concerning the status of mathematics as a teaching subject in formal schooling. Moreover, the teaching of mathematics is considered as an activity inherently determined by the discipline of mathematics itself, which self-determines its didactics.

Another perspective, leading to similar outcomes, includes various dominant trends in psychology, such as behaviorism or cognitivism, which claim for themselves the phenomenon of learning mathematics as an object of scientific inquiry. They utilise conceptual sets of mathematics, and more broadly fundamental aspects of mathematical thinking, as a preferred field for applying and testing their theories for human learning or cognitive functioning. As a result, they incorporate the phenomenon of learning mathematics in their field of applications, questioning indirectly the founding assumption of mathematics education: that the phenomenon of learning mathematics has quite distinct and particular characteristics from the phenomenon of learning any other scientific subject matter. E. Thorndike's book (1922) The Psychology of Arithmetic is an early exemplar of such an approach, which adopts the standpoint of behaviourism. Further books promote approaches which adopt perspectives on the learning of mathematics inspired by cognitive psychology (for example, Dehaene, 1997, The Number Sense: How the Mind Creates Mathematics).

As mentioned above, the ambiguity of disciplinary boundaries of mathematics education, as well as the relativity of its scientific object, which inevitably results from such an ambiguity, is an indispensable characteristic of any interdisciplinary relationships, as analysed by Baltas (1983). Furthermore, an indispensable characteristic of the interdisciplinarity of mathematics education is the production of new knowledge, which meets the requirements of its theoretical evolution as a scientific discipline and, at the same time, the production of new knowledge, which is clearly useful and directly applicable to mathematics teaching in schools. These expectations are contradictory in several ways, a fact that explains many of the characteristics of the relationships between theory and applications of mathematics education.
Summarising, and to great extent simplifying, Baltas' analysis (Baltas 1983) the production of scientific knowledge, which is a requirement and a result of the theory development in any discipline, mathematics education included, is primarily emanated and promoted by 'internal' processes to mathematics education. Questions and problems which arise are posed and defined, mostly, in the context and in the terms of the conceptual system of mathematics education, while their answers and solutions are subjected to the research methodologies of the discipline and to the broadly accepted standards of the scientific practice.

On the other hand, the production of clearly useful and directly applicable knowledge aims at the solution of practical problems of school mathematics education. A practical problem in mathematics teaching and learning in schools arises, primarily, in a social context by processes 'external' to mathematics education and, as with every phenomenon of everyday life, is many-sided. So, the solution of a problem in school mathematics may not be approached in terms which are determined uniquely or exclusively by the discipline of mathematics education or even in strictly scientific terms, since a multitude of social factors interfere. Consequently, any solution of a practical problem of school mathematics education is not necessarily subject to formal scientific standards and actually constitutes a multitude of options or a field of potential solutions. Any one solution finally selected is not absolutely and uniquely included in







the domain of mathematics education, being an outcome of broader social interests and relationships (Baltas, 1983).

**Concluding comments**

Despite the fact that activities related to mathematics education have exhibited a remarkable development since the 1960s, the scientific object of mathematics education considered as a scientific discipline remains vague and in consequence its constitutive conceptual system remains fuzzy. Both are constituted at the same time by a process of reciprocal determinations, wherein each of the constituents presupposes the existence of the others. As a result, the scientific status of mathematics education is questioned and its essential contribution to school mathematics teaching and learning is thrown into doubt. Recognising this problem and its negative outcomes, the International Commission on Mathematics Instruction (ICMI) – an old organisation involved in mathematics education issues expressing mainstream options – raised issues concerning the 'identity' of 'mathematics education as a research domai' in a study conference held in 1994 at the University of Maryland, USA. The proceedings of this conference (Sierpinska and Kilpatrick, 1998) simply record and confirm the problem posed in this paper. The scientific object, fundamental concepts, theoretical frameworks, research practices and fields of applications of mathematics education are defined in different terms – in most cases incompatible or/and opposing amongst themselves – in the 33 papers of the internationally known experts in mathematics education research, in the five working groups' reports. This is also the case in the editors' summary, characteristically entitled Continuing the Search (Sierpinska and Kilpatrick, 1998, vol. 2, p. 527–48).

Adopting the analysis of interdisciplinarity developed by Baltas (1983, p. 39–44), I suggest that we have to accept that mathematics education constituted on an interdisciplinary basis, and on the aforementioned assumptions concerning the phenomena of learning and teaching mathematics, is inherently characterised by two features. First: a fluidity in the definition of its scientific object in connection with fundamental aspects of these phenomena and, second: an ambiguity in the demarcation of its scientific object against related disciplines and in particular mathematics, psychology and sociology,

which include in their objects of study the same or different aspects of learning and teaching mathematics.

In conclusion, both the autonomy of the scientific object of mathematics education and the autonomy of the meanings of the concepts which constitute its conceptual system are in question. However, a question is posed and discussed taking into account the interdisciplinarity of mathematics education which imposes a 'scientific indeterminacy' (Baltas 1983) in its approaches to the practical problems of learning and teaching mathematics. Therefore, the scientific status of mathematics education will remain an object of an on-going, interesting, discussion.

**References**

Balibar, E. (1978) From Bachelard to Althusser: The Concept of 'Epistemological Break', Economy and Society, 7(3), 207–37.
Chassapis, D. (2002) Social Groups in Mathematics Education Research: An Investigation into Mathematics Education-Related Research Articles Published from 1971 to 2000, in P. Valero and O.Skovsome (eds) Proceedings of 3rd International Mathematics Education and Society Conference, vol.1, Copenhagen: Centre for Research in Learning Mathematics, Danish University of Education, 273–81.
Chassapis, D. (2004) The Objects of Mathematics Education Research: Spotting, and Commenting on, Characteristics of the Mainstream, Paper presented at the Erasmus Summer School on Researching the Teaching and Learning of Mathematics (MATHED2004), Tolmin: Slovenia.
Dehaene, S. (1997) The Number Sense: How the Mind Creates Mathematics, New York, Oxford: The Oxford Press.
Fehr, H.F. (ed.), (1961a) New Thinking in School Mathematics, Paris: OEEC.
Fehr, H.F. (ed.), (1961b) Synopses for Modern Secondary School Mathematics, Paris: OEEC.
Fehr, H.F. (ed.), (1964) Mathematics Today, Paris: OECD.
Kilpatrick, J. (1992) A History of Research in Mathematics Education, in D. Grouws (ed.), Handbook for Research on Mathematics Teaching and Learning, New York: Macmillan, 3–380.







Lerman, S., Xu, G., and Tsatsaroni, A. (2003) A Sociological Description of Changes in the Intellectual Field of Mathematics Education Research: Implications for the Identities of Academics. Proceedings of the British Society for Research in Learning Mathematics, 23(2), 43–8.
Lerman, S. and Tsatsaroni, A. (2004) Making Meaning in Mathematics Education Research as Social and Discursive Practice, in D. Chassapis (ed.) Picture, Figure and Discourse in School Mathematics Teaching, Proceedings of the 3rd Conference on Mathematics Teaching Issues, Thessaloniki: Aristotle University of Thessaloniki.
Lubienski, S. T. and Bowen, A. (2000) Who's Counting? A Survey of Mathematics Education Research 1982–1998, Journal for Research in Mathematics Education, 31, 626–33.
Μπαλτάς, Α. (1983) Για το οικολογικό κίνημα: επιστήμη και διεπιστημονικότητα, Ο Πολίτης, 63, 30–46.
Μπαλτάς, Α. (1990) Επιστημολογικά, Για την ιστορία μιας επιστήμης, Εκδόσεις Ο Πολίτης, Αθήνα.
Reyes, L.H., and Stanic, G.M.A. (1988) Race, Sex, Socioeconomic Status, and Mathematics. Journal for Research in Mathematics Education, 19, 26–43.
Secada, W.G. (1992) Race, Ethnicity, Social class, Language, and Achievement in Mathematics, in D.A. Grouws (ed.), Handbook of Research on Mathematics Teaching and Learning, New York: Macmillan, 623–60.
Sierpinska A. and Kilpatrick J. (eds), (1998) Mathematics Education as a Research Domain: A Search for Identity, An ICMI Study, Vols. 1 and 2, Dordrecht: Kluwer.
Thorndike, E. (1922) The Psychology of Arithmetic. New York: Macmillan
UNESCO (1966) New Trends in Mathematics Teaching, Paris: UNESCO.

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