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Researching the Teaching and Learning of Mathematics: Proceedings of MATHED Intensive Programme 2003

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Dedication

On behalf of the participants of the MATHED 2003 Intensive Programme, I wish to dedicate this volume to the memory of John Gardiner (1943-2004) who was a great friend, colleague and research student and who will be sadly missed by the community of which he was a highly valued member.

Brian Hudson
Sheffield
December 2004

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Prof. Dr. Brian Hudson
Co-ordinator of MATHED 2003 Intensive Programme
Sheffield
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Researching the teaching and learning of mathematics

Introduction and overview

Brian Hudson, *Sheffield Hallam University and Umeå University*
Klaus Enser and Josef Fragner,
Akademienverbund – Pädagogische Hochschule des Bundes OÖ

> This view of mathematics is, above all, integrative … It stresses the acquisition of understanding by all … rather than the selection and promotion of an elite. It is a philosophy that simultaneously stresses erudition and common sense, integration through application, and innovation through creativity. Most important, it stresses the creation of knowledge. Against this ambitious view of mathematics, traditional school mathematics appears thin, lifeless and isolated.
>
> Romburg and Kaput, 1998, p. 7)

The above quote from Tom Romburg and Jim Kaput is cited by Leone Burton in her paper on research mathematicians as learners (Burton, 2001). She stresses the consistency between these views of learning and of research and highlights the way in which educators and mathematicians in her study 'stressed the need to site learning into a connected context where the fuel for the search is provided by the challenge and excitement of making new connections' (Burton, 2001, p. 596).

The papers included in this volume arose from the Socrates-Erasmus Intensive Programme MATHED 2003: 'Researching the Teaching and Learning of Mathematics' that was held at Pädagogische Hochschule, Linz, Austria between 2nd and 12th July 2003. This

brought together mathematics education doctoral students with tutors from eight universities across Europe for 10 days of intensive study on aspects of the current research in the field of mathematics education. The excitement involved in making new connections – social, pedagogical and mathematical – was an almost tangible aspect of the experience. We hope that the papers contribute to thinking about the teaching and learning of mathematics that is consistent with the view expressed by Tom Romburg and Jim Kaput. We believe that they do so.

The institutions taking part were Sheffield Hallam University and Umeå University(co-ordinator), Pädagogische Akademie des Bundes in Oberösterreich, Charles University, Prague, University of Helsinki, University of Riga, Oslo University College, Oxford Brookes University and Aristotle University of Thessaloniki. The MATHED Intensive Programme is closely associated with the Socrates-Erasmus EUDORA Project which aims to develop a European Doctoral Programme in Teaching and Teacher Education. The EUDORA Project is co-ordinated by Pädagogische Hochschule, Linz which hosted the event.

The intensive programme enabled participants to examine issues of policy and practice relevant to their work in mathematics education within an international context. The programme aimed to enable participants to develop a critical focus on the nature of research into the teaching and learning of mathematics in an international context and to develop theoretical approaches and methods appropriate to comparative research. This was intended to support the further development on the part of the participants of their understandings of the methodological complexities of research in mathematics education, to identify current themes in mathematics education and critically analyse their significance and to relate these outcomes to their own contexts.

The content of the intensive programme was based on the research interests of the participants and involved active participation by students and collaboration with peers from the outset. Each student led a workshop and/or seminar based on their current work and interests. All participants submitted draft papers in advance which were posted into the virtual learning environment. In addition staff members led workshops on current issues in mathematics education research.

The first chapter is based on the contribution of Hans Jørgen Braathe which involves the analysis of 'positionings' of student teachers' identities in their mathematical texts. This study is based on a communicational perspective on teaching and learning mathematics as a theoretical framework for analysing student teachers' texts and builds on the work of Sigmund Ongstad whose contribution is included in Chapter 6. This study applies concepts where the utterance's positionings are seen within a dynamic triadic understanding that all utterances have simultaneously cognitive, affective and social (activity directed) aspects. This 'triangulation' aims to give insight into how the student positions him/her self in relation to 1) the mathematics and the teaching and learning of mathematics, 2) his/her own emotions and experiences and 3) the classroom, teachers and others. Viewed in this light a specific focus of the project will be to apply this triangulation and to develop methods for analysing the influence of the development of students' identities as teachers of mathematics through their writing of mathematics educational texts.

Based on a Vygotskian perspective, Dimitris Chassapis highlights a central claim of the socio-cultural approaches to human cognition in Chapter 2. This is that children's mental functioning and development can be accounted for in terms of their engagement in culturally organised practices in which cultural tools play a crucial role. Such tools, mediating human practical and mental functioning, have been developed in a culture over extended periods of time and have become an integral part of human activity, functioning as 'carriers' of socio-cultural patterns and knowledge. By acting as mediators, cultural tools structure human practical activity and bring into play differentiated mental processes which in turn regulate and qualitatively transform that practical activity. Mediatory means, thinking processes, and human practical activities become functionally intertwined in their development, shaping each other in a dialectical interdependence. In this account, issues concerning the impact of material and symbolic mediatory means

upon mathematical concepts formation and skills development are presented and relevant research topics considered.

In Chapter 3, John Gardiner reflects a similar socio-cultural perspective to that of Dimitris. He reports on a research project with 11–14 year old pupils in the UK which studied mathematical meaning-making using dynamic geometry. Classroom observations from audio-recording and field notes are reported and related to short- and long-term factors identified by previous authors. Short-term factors, many concerned particularly with the use of dynamic geometry, are the importance of explanation in social proof, transparency of resources and technology, sense/meaning making, spontaneous/scientific concepts and the tension between proof and construction when using dynamic geometry. These immediate factors are seen to operate within a classroom environment which depends on a teaching and learning atmosphere generated over time by the teacher as guardian of the classroom community. In this area socio-mathematical norms, argumentation, and the idea of a whole class zone of proximal development (ZPD) have been identified as relevant factors. The study builds on the work of Peter Winbourne and Anne Watson who advance six aspects of the development of local communities of mathematical practice, and speak of an alignment of mathematical meaning-making which they call a 'telos'. The long and short-term factors identified in this study are seen to provide ways in which the reflective classroom teacher can analyse, and promote, local communities of practice, both in the use of dynamic geometry and other mathematical activity. The application of the Vygotskian methodology of 'tool-and-result' is a distinguishing feature of this study. John also makes connections with, builds on and extends the pedagogical and didactical frameworks outlined by Brian Hudson in Chapter 5 of this volume.

In his consideration of the meaning of the term 'school algebra' in Chapter 4, Jan Herman highlights a variety of interpretations and different emphases in various national contexts. He sets this against a background of a diversity of approaches in different national contexts involving algebra curricula organised around the ideas of 'expressing generality', 'equations and inequalities' and 'function'; 'work with systems of equations' and 'relations, functions and sequences'; 'pattern and relations' and 'variables and equations'; 'the study of systems of equations and inequalities' and 'the treatment of functions'. In some countries there is an emphasis on algebra as a means of 'expressing generality and patterns' whilst others introduce algebra within the context of problem situations by 'traditional word problems' and there tends to be more emphasis on symbolic manipulation. Some use more realistic modelling situations and there is less emphasis on symbolic manipulation. This paper is based on the outcomes of a workshop that was used in a formative way in the early stages of a PhD study that aims to illuminate and develop teaching and learning practices in school algebra.

In Chapter 5, Brian Hudson explores differences between traditions in relation to teaching and learning. Taking the Anglo/American curriculum tradition as the starting point, the paper seeks to highlight the ways in which the Central and Northern European tradition of Didaktik has offered a new dimension and fresh insights to the notion of reflective practice. In particular this chapter highlights the way in which the tradition of Didaktik offers tools for recognising and holding the complexity associated with teaching-studying-learning processes. Furthermore it explores the potential of these tools for the analysis and evaluation of the teaching-studying-learning of mathematics by seeking to make connections with other frameworks for analysis in the field. In considering the roles of technology in relation to the complex interrelationships between teacher, students and subject matter, researchers are confronted with an even greater degree of complexity than that faced in traditional settings. In his study, reported on in Chapter 3, John Gardiner made connections and built upon these frameworks in aiming to hold the complexity of the teaching-studying-learning situation rather than retreating from the challenge e.g. into a 'dyadic' mode of thinking, as discussed by Sigmund Ongstad in Chapter 6 or the 'Cartesian' dualism as highlighted by Chassapis in Chapter 2.

Sigmund Ongstad, in Chapter 6, presents a triadic framework for the study of communication in education. Utterance is seen as the basic unit for analysis and the triadic view implies that the form, the content and the use of the utterance are seen as mutual and reciprocal and hence inevitably triadic. Any main element of communication such as an utterance, a text, a genre, a discourse or

a context is principally seen as triadic within which communicators will have to communicate, both as utterers and interpreters. A model is presented, and an approach developed on the basis of the framework, analysis of positionings, is outlined briefly. Validity is concentrated on in particular. Hans Jørgen Braathe makes particular connections with these frameworks and these ideas are built on in the work discussed in Chapter 1.

In Chapter 7, Magdalena Prokopová considers possible tools for verifying whether there are parallels between the historical development of the public interpretation of a point and a straight line and their relation and possible cognitive developments examined in today's generation of pupils. The structure of this chapter consists of an historical analysis and an abstraction of the main points gleaned from that analysis, placed in a rough hierarchy of development. This is followed by the exploration of students' personal interpretations concerning the notion of a point and a straight line and their relation which, by considering the ages of the students, can also be put into a rough hierarchy of development. The two resulting hierarchies are subsequently compared and conclusions drawn.

The second Intensive Programme MATHED 2004 was held in the stunning location of the Tolmin valley in Slovenia during August 2004 and the proceedings from this will be published later in 2005. The third event in this series is planned for July 2005 and will be hosted by Tallinn Pedagogical University to be held in the picturesque and culturally rich location of Vijlandi in Estonia. Proceedings from this intensive programme will be published in 2006. As Tom Romberg and Jim Kaput stressed, there is a need to site the learning of mathematics into a connected context where the fuel for the search is provided by the challenge and excitement of making new connections. In reflecting on the experience of MATHED 2003, we think it is fair to say that all participants tasted the challenge and excitement of making new connections in several senses of the word – social, pedagagical, didactical and mathematical.

References

Burton, L. (2001) Research Mathematicians as Learners – and what mathematics education can learn from them, *British Education Research Journal*, 27, 5, p. 589-599.

Romberg, T. A. and Kaput, J. J. (1998) Mathematics worth teaching, mathematics worth understanding. In E. Fennema and T. A. Romberg (eds*) Mathematics Classrooms that Promote Understanding* (Mahwah, NJ: Lawrence Erlbaum).

Contact

Division of Education and Humanities, Faculty of Development and Society, Sheffield Hallam University
Email: B.G.Hudson@shu.ac.uk

Department of Interactive Media and Learning, Faculty of Teacher Education, Umeå University

Prof. Dr. Josef Fragner, Direktor, Pädagogische Hochschule, Linz
Email: Josef.Fragner@phlinz.at

Klaus Enser, MSc, Pädagogische Hochschule, Linz
Email: Klaus.Enser@phlinz.at <mailto:Klaus.Enser@phlinz.at>

Analysis of Positionings of Student Teachers' Identities in Their Mathematics Educational Texts

Hans Jørgen Braathe, *Oslo University College*

Abstract

The paper presents a doctoral project in Mathematics Education. The project takes a communicational view on teaching and learning as a theoretical framework for analysing student teachers' texts. I will apply concepts where the utterance's positionings are seen within with a dynamic triadic understanding that all utterances have simultaneously cognitive, affective and social (activity directed) aspects. The analyses of the students' texts are now seen as utterances within the teacher education genre. I will analyse the texts from 1) referential, 2) expressive and 3) addressive aspects. This 'triangulation' will give insight into how the student positions him/her self in relation to 1) the mathematics and the teaching and learning of mathematics, 2) his/her own emotions and experiences and 3) the classroom, teachers and others. Seen this way a specific focus of the project will be to apply this triangulation and to develop methods for analysing the influence of the development of students' identities as teachers of mathematics through their writing of mathematics educational texts. The analysis of positioning as a method is well suited to link to other methods for analysing texts and genres in mathematics education.

Introduction

Background

The current Norwegian curriculum for primary and lower secondary school, L97, holds out the active, autonomous and independent student as an ideal (Royal Ministry of Education, 1999). Such an ideal also has deep roots in Norwegian teacher education. At the same time both the primary school and teacher education demand an adaptation from the individual student. Like all institutions they have their own culturally determined discourses. To start in the primary school or to start in teacher education is to start learning new genres, new ways of communicating and behaving, i.e. becoming part of a new culture (Kvalbein 1999). In particular the students meet a number of subject- and educationally-related genres, for example theory-assignments, project-reports and exams. Such genres and norms change along with the current educational ideas, locally, nationally and internationally.

Theories and ideas about writing as a tool for reflection have increasingly become part of the thinking in teacher education. At Oslo University College, OUC, Faculty of Education, the final exam in the compulsory mathematics course for student teachers currently consists of two separate parts: a one week 'home' exam where four students in a group give a co-written presentation on a given task, and an individual oral examination. The explorative and activity-based mathematics teaching and learning as an ideology has gained a dominant position both within the primary schools and in teacher education. This is connected to current dominant theories within teaching and learning. These new aspects, both the norms within teaching and learning, and the forms of evaluation, demand more both from students, student teachers and teacher educators when it comes to communication: both *within* and *about* mathematics, both written and oral.

Learning as communication

Most students, when starting out on their teacher education, probably have a conception of themselves as a 'teacher' after completing their studies. This conception can be more or less vague, but will be influenced by the students' different backgrounds: teachers they have had during their own schooling, parents or others they know who are or have been teachers. In some cases they may even have experiences as substitute teachers. In every case their cultural background will be an influence.

During their teacher education it is presupposed and expected that they will develop an identity as teachers, and a view on students, school subjects and teaching/learning that will give them sufficient confidence and competence to enter the Norwegian primary and lower secondary schools. Thinkers like Bakhtin, Goffman, Foucault and Bourdieu provide evidence in support of the argument that individuals develop social identities through the (communicative) genres – written and oral – that society offers them. Empirical studies (Freedman 1992, Ivanic 1994, Ongstad 1997, Morgan 1998, Evans 2000) show how this is the case when students try out norms of writing and different genres they meet in universities and colleges. Ivanic shows, among others, how students construct discursive identities through their writing in dialogue with different 'voices' (Bakhtin 1986), from scientific literature they have read and written genres they have been in contact with. In research on students' writing in Norway, especially in the primary and secondary school, there has been focus on how children and young people develop their writing competence (Evensen and Hoel 1997), and on different text norms and contradictions present in the classroom and in the final exams (Smidt 1999, 2002, Berge 1996, 2002). Part of this research has looked especially at how students themselves interpret their own roles as writers, in relation to what they see as the school and the teachers' academic and educational norms (Smidt 1993).

That writing has become a central ideology of learning in Norway is clearly expressed in the goals for the subject Norwegian (mother tongue) that is radically new in L97:

> The teaching and learning in Norwegian has as one of its goals to make pupils conscious participants in their own learning process, provide them with insight into their own linguistic development, and enable them to use language as an instrument for increasing their insight and knowledge. (L97, p. 126, English version).

Such a goal will have consequences for all subjects in the teacher education. It will increase the focus on development and awareness of language and communication as important elements for learning.

Mathematics as a subject is problematic in the sense that it is 'tacit' about its own norms of learning and its different genres. However Morgan (1998) has examined students written assignments in mathematics in the secondary school in England focusing how students' writing and the teachers' reading are influenced by the explorative and activity based teaching and learning and the evaluation discourses that are dominating the classroom where the writing is taking place. One of her conclusions is that both the students' and the teachers' competence in communicating written mathematics is insufficient and insecure. She points to a requirement for focusing on 'learning to write mathematics'; it is not sufficient only to use writing to learn mathematics.

In the explorative mathematics activities in the theory lessons in their pre-service training the students have to explain mathematical patterns, connections and reasoning. By analysing these texts we can identify the students' attitudes and beliefs towards mathematics, and on teaching and learning (Morgan 2002). By focused dialogue with the student teachers, feedback on these texts could give reflections on communication of mathematics, i.e. the genres as contexts for learning. In the teacher education obvious contexts for learning mathematics are 1) the student's own experience of learning mathematics at school, their self image as mathematics learners, 2) the theories they meet at the teacher college, both in theory lessons and in the written material and 3) the practice teaching they meet during their teacher education.

In teacher education it is emphasised that the students shall connect to and reflect on their own experiences from school praxis in relation to the theory they read. Ongstad (1994) has in minor research study at Oslo University College, OUC, looked into how the student teachers in their educational writing in the subject Norwegian are reflecting on their use of genres across disciplines in their praxis in the primary school. However we still lack knowledge of the extent to which the student teachers are able to utilize educational writing to reflect on different practices.

One question to ask and explore will be in what way does students' writing in the different genre of mathematics educational texts, in dialogue both with praxis, theory and self-image, influence the development of their identity as teachers of mathematics?

Teacher Education in Mathematics

Egalitarism and 'Enhetsskole' as an educational ideology

The Norwegian schools are strongly influenced by ideologies associated with the principles of collective teaching and learning and equal rights for education. Sources of this strong and nationally shared *egalitarianism* can be traced in the political and cultural history of Norway (Braathe and Ongstad 2001). The egalitarian struggle for equal right to education in Norway resulted in what is called 'enhetsskole', which superficially can be translated as 'unity-school'.

There are two main different routes of teacher education in Norway. One is through the universities covering grade 8 to 13. The other is through the University Colleges that primarily educates grade 1-10 teachers. Through their broad certification the University Colleges keep up an important pattern connected to 'enhetsskolen', the principle of the class-teacher. One teacher can in principle have the same class through all years, in all subjects, including mathematics. Accordingly the Norwegian teacher education for primary schools gives right to teach any subject from grade 1 to 10. However mathematics as a subject in teacher education for primary school was not compulsory until 1992! In other words teachers teach mathematics in Norwegian primary schools with relatively low competence in mathematics. Official statistics have shown that more than 50% of teachers in primary school, who teach mathematics, have no mathematics in their teacher education (KUF 1996a). There is a significant danger that many of these teachers belong to the teachers who, according to the TIMSS study, organise their mathematics lessons leaning heavily on textbooks and other pre-produced material (Lie et al. 1997, Cogan and Schmidt 1999).

In the 1990s the will to change was influenced by the liberal right and its focus on quality (rather than equality) in school, especially being worried about Norway's international competitiveness in different subjects. The general political push for greater quality in the subject matter education in teacher education contributed to a teacher education reform in 1998 called LU98 (KUF 1998). In this current curriculum mathematics is strengthened to a compulsory half years study. Teacher education is still four years, and gives the right to teach all subjects from grade 1 to 10.

The student teachers, when they enter the teacher education, are likely to be led by subject *ideologies* that are in accordance with their teachers restricted background knowledge and low motivation for mathematics (Braathe and Ongstad 2001, Braathe and Kleve 1995). I will therefore in my work look into the possible ideologies and educational genres that influence the Norwegian mathematics classroom from which the student teachers have their experience.

Ideologies and Genres

We have in an earlier work (Braathe and Ongstad 2001) analysed educational ideologies and genres in the Norwegian mathematics classroom. Mathematics education often underlines that its goal is rational and reflexive *understanding*. Since time is scarce though and there are many students, the goal quite often is twisted during a course from understanding to *fast* understanding. By then it may even become clear that the goal is not really understanding for *all*, but just for some, since just a few can understand fast, and this favoured little group will mostly set the pace and conduct the progress. In a society like Norway these ideological twists may easily collide with an egalitarian ideology - is mathematics (collective) understanding for all or (individual) competition in understanding fast (Braathe and Ongstad 2001)? Whether educationalists in mathematics in Norway like it or not, most mathematics classrooms seem to convey these tensions: a Norwegian student in mathematics has to be slow *and* equal or bright *and* equal etc. Even if many teachers of mathematics may try to delimit competition, the tendency to create, to make visible or to evaluate and hence mark student differences seems to be at the heart of mathematics as an educational subject. Are the two ideologies didaktikally really compatible?

The evaluative nature of traditional mathematics classroom practice normally reveals most students' level of understanding. Results, and hence implicitly often even students as persons, are explicitly evaluated, making the rank order visible for everyone without really expressing it. This ideological problem has recently been lifted from the level of classroom to the national and even international level by the impact from international comparative studies, often reproduced uncritically by media. A common political and professional reaction is: *let's compete, nationally and internationally*. The counter question though is: *How can we allow competition in the classrooms when school at the end of the day is supposed to decrease social and other differences and to establish community?* In practice this creates an almost inevitable dilemma for any didaktiks of mathematics.

Perhaps the strongest ideology of mathematics education simply is to focus on the subject, not society, not context. Thus two important tendencies within most directions of the teaching of mathematics are: stress on (mathematical) thinking and stress on individual learning and understanding, in other words, ideologies of rationalism and constructivism/'auto-didaktism', the latter hinting at the tendency to overstate of the role of self-learning in institutionalised learning. Students are supposed to accept a silent didaktik contract in which they promise to give priority to explicit thinking and responsible, independent work (Mellin-Olsen 1987). A reflective, communicative and self-driven student then is not just a goal, it is even an ideological expectation, a premise for teaching. When this premise does not hold, ideologies will come to surface, and may or may not be negotiated explicitly within the classroom (Lemke 1990).

The students, on their part, may have created silent alliances since they are captives of a logic that forces them to stay in the system on a 'pseudo-voluntary' basis (Mellin-Olsen 1987). This alliance works as a *power basis* for negotiating values such as 'freedom' and 'entertainment'. Any 'imposed' teaching will have to, in some way or other, to wrestle with student ideologies of liberalism and info-tainment (Postman 1985). Thus there will be a tug of war between

teachers and students about how to melt social genres with genres of mathematics. Ideologies are resting in the contextual meeting places between mathematics (the subject), teaching (the teacher) and learning (the student). What is said directly (utterances) and what is said indirectly (genres and ideologies) have to be sufficiently 'fun', 'entertaining', and 'free' to engage and stimulate the students and solid enough to build up a progressive understanding of mathematics. Thus gate-keeping over teaching and learning genres functions as a contextual power basis for specific ideologies.

Ideologies are anchored in, or simply are central values, which, when they are not threatened, tend to be tacit. They are normally rooted in specific communicational communities and are conveyed through a system of genres or discourses. They may have a formal power basis, often controlled by particular communities.

Some suggested ideologies, their values, roots, carriers, genres and power basis

Ideology	**Central value**	**Expressed by**	**Genres, example**	**Power basis**
A. Egalitarianism	equality, lack of difference	society, nation	whole-class	laws, guidelines, trad
B. Rationalism	reflexivity, thinking	teachers, didaktik	logs, explanation	marks, future outcome
C. Autodidaktism	responsibility, independence	teachers,	didaktik studying	plan, marks
D. Performativism	doing, acting, working	students, society	plans, tasks	plan, evaluation
E. Competitionism	winning, ranking, quality	society, students	calculation race	future outcome
F. Liberalism	freedom, independence	students	free tasks, chat	the collective
G. Entertainism	fun, engagement	students	breaks/info-tainm.	the collective

Figure 1: From Braathe and Ongstad 2001

In this rather tentative overview the simplifying two main ideas are: 1) to make aware the *horizontal* connection, the relationship between ideology, genre/context and power, and 2) that there is a possible tension, inconsistence, blurring *between* ideologies (the *vertical* connection). I will underline that I see ideologies primarily as values that act as matters of fact and as presuppositions for participants of a discourse community through which the values will have strong impact on the final implication of utterances within that community. However ideologies are not just passive values, they have functions or effects, and these functions cannot be easily controlled. Hence they can develop both in a functional and a dysfunctional direction.

The student teachers have as their background the experience, to some extent, of this blurring of ideologies in the mathematics classroom during their schooling so far. At the same time these ideologies are also present in the literature the students meet and read during their study at the college (Ongstad and Braathe 2001). These will be seen as part of the context in the analysing of the students' texts. I will try to identify in the students' utterances the different 'voices' (Bakhtin 1986) that connect to these ideologies.

Theoretical Foundations

Semiotic genre theory – focus on communication

As a point of departure semiotic genre theory is chosen as a theoretical framework for investigating, analysing and describing phenomenon in mathematics education. Semiotics implies to study more than only language, to see all cultural expressions and utterances as meaningful signs. Semiotic genre theory takes communication as a fundamental concept, and therefore puts communication (in a wide sense) as the central element in the understanding and analysing of phenomenon concerning teaching and learning. Genres can be described as forms of communication. Basic concepts are *utterance*, *positioning*, *genre* and *ideology*. Utterance is a basic element. We communicate through utterances. An utterance, is simply put, any sufficiently closed use of sign that makes sense. We communicate by uttering and by giving utterances meaning. All utterances are uttered and interpreted in relation to expectations of genres, i.e. contexts that help us to understand the utterance. Genres are ideological, i.e. they give tacit premises for the participants' positioning in the communication.

Ongstad (1997, 2002a, 2002b) assigns to the utterance simultaneously a dynamic triadic character in the same way as Bakhtin (1986), Habermas (1988) and Halliday (1984). The utterance has simultaneously a referential aspect that connects the utterance to the *world*, an addressing (acting/use) aspect that connects it to *others* and an expressive (affective) aspect that connects it to the utterers' *self*. These aspects are in constant interaction. Positioning is an active process giving the utterance a meaning, both when one utters and when the utterance is interpreted. Positioning(s) becomes both a process (**-ing**) and a product (**-ings**) where the utterer positions themselves, the word and the others semiotically by the utterance.

Connections to other theories

The focus for analysis is about writing in a professional setting and will link up theoretically and methodically to analysing and interpreting texts in a wide sense. Text in this connection will also include mathematical text, i.e. mathematics as a semiotic sign system.

Positioning as a scientific concept has been used both more generally (Harre and van Langenhove 1999) and in mathematics education specifically. Walkerdine (1988) uses it within a post-structural theory as positing one self within a context, or inter-contextual positioning. Evans (1994, 2000) applies Walkerdine's concept in the development of his theory on students' positionings in different mathematical contexts/discourses. Morgan (1998) again uses both Walkerdine and Evans when she, in her analyses of mathematical texts from students in upper secondary school in England, analyses them to be positioned in different discourses: 'The adoption by a participant of different positions within these discourses may lead to different and possibly contradictory meanings being constructed from the same text' (Morgan 1998, s. 4).

Positioning as it is used by Ongstad (1997, 1999, 2002a), and partly by Smidt (2001, 2002), differs from the post-structural concept of positioning, but all have the basic view that the utterer (student) is positioned by and in the communication, and therefore is given possibilities and premises for the communication. Ongstad (2002b) and Ongstad and Braathe (2001) have applied positioning analysis as method for analysing mathematical educational texts. We will continue and develop methods of analysing within this rather wide framework of positioning analysis.

I will also use elements of Halliday's functional grammar in a similar way as Knain recently has done when analysing texts in science education, and Morgan within mathematics education (Halliday 1985, Morgan 1998 and 2002, Knain 1999).

This work will also use results from works within mathematics education that has studied the relation between language/communication/semiotics and mathematics (Pimm 1987, 1995; Durkin and Shire 1991; Brown 2001; Walkerdine 1998; Evans 2000; Rowland 2000).

The significance of communication in different mathematics education studies

I build on a social–cultural view on mathematics and mathematics education, a position I have been working from the last years (Braathe 1997, 1999, 2002). To study educational processes means from social–cultural theories point of view to interpret students' utterances related to possible social–cultural contexts they might position themselves into (Evans and Tsatsaroni 1994). In this work, with genre theory as a theoretical frame of reference, I will try to identify such social–cultural contexts in the students' mathematics educational texts. By analysing the different ways of communications that are being used, i.e. the genres, it will be tried to identify possible relations to the students' utterances and learning. In studies within mathematics education in Norway Stieg Mellin-Olsen's work and research, applying the communication theories of Bateson, can be seen as a contribution to a social–cultural approach to mathematics education (Mellin-Olsen 1987, 1991; Braathe, Kværnes and Ongstad 2000).

Focus, Objective and Methods

I have chosen to focus teaching and learning as communicative processes, actions of communication or utterances (Rommetveit 1972; Bakhtin 1986; Ongstad 1997, 2002b). All learning presupposes

some kind of communication – either it takes place as interaction between individuals, between individuals and groups, or it takes place through interaction in environments that combines theoretical studying with practical training. With Dewey one can claim that: 'Society exists through a process of transmission quite as much as biological life. This transmission occurs by means of communication of habits of *doing*, *thinking* and *feeling* from the older to the younger' (Dewey 1916:3; cited from Ongstad 2002a, Ongstad's emphases).

Utterance will both be the basic theoretical concept and also the object for analysing. The students' utterances and actions in praxis, in tutorial situations, tasks, lectures, discussions, books, project assignments, questions, and answers are all utterances that belong to the teaching/learning process.

I will have teacher education as the object of study. I will apply concepts where the utterance's positionings are seen with a dynamic triadic understanding that all utterances have simultaneously cognitive, affective and social (activity-directed) aspects. I will apply analysis of positioning on the students' texts that now are seen as utterances within the teacher education genre (Ongstad 1999, Ongstad and Braathe 2001). I will analyse the texts both from 1) referential, 2) expressive and 3) addressive aspects. This 'triangulation' may give insight into how the student (respectively) positions him/her selves in relation to 1) the mathematics and the teaching and learning of mathematics, 2) his/her own emotions and experiences and 3) the classroom, teachers and others. Seen this way a specific focus will be to apply this triangulation and to develop methods for analysing.

Another objective will be to gain a better understanding and knowledge of students' learning and development of teacher identity in different arenas, but with a focus on their own writing in mathematics education. This focus has been chosen from a point of view that presupposes that the educational genres, both oral and written, to which the student teachers are exposed will influence their professional development. This writing demands understanding and knowledge of the different genres on the part of the students. Another derived objective will be to develop better knowledge and conscience of this genre understanding in order to (later) give the students more confidence in communicating mathematics in the classroom. This knowledge will again be important for them as teachers of mathematics in primary school with its demands on communication, both through the norms for teaching and learning, and the forms of evaluation. I wish to develop new themes and questions that might possibly contribute to a more comprehensive and professional development of mathematics teachers. These will have an interdisciplinary profile. With the focus on communication I will to a considerable degree integrate elements from theories like mathematics education, language- and communication-theories, social-cultural theories, educational theories and not least semiotic genre theory. From this interdisciplinary point of departure I have ambitions to contribute to more than simply the field of mathematics education, but also to a more general professionalism within teacher education.

Ongstad discusses advantages and disadvantages with the analysis of positioning as a research method. One advantage he emphasizes is that this method is well suited to be combined with different methods of analysing texts and ways of understanding in different subjects (Ongstad 1999). This then gives me the opportunity to compare, by trying out and evaluating, other methods of analysing, for example Halliday's functional grammar and Adler's (2001) analysis within social practice theory.

This can be seen as a two-fold contribution. The first is the development of the analysis of positioning as a method, and the linking of this method to other methods for analysing texts and genres in mathematics education. The second is a contribution to better understanding of the education of teachers of mathematics for the primary school, and the education of teacher professionalism. I will focus particularly on the importance of genre competence for writing mathematics in teacher education.

Analysing Texts

Below are some examples of student teachers' texts in their compulsory mathematics course in teacher education at Oslo

University College. In my comments on and descriptions of the texts I take into consideration the discursive contexts or genres from which the texts are uttered. However I will finally analyse the texts with a focus on one aspect at a time; this triangulation will again be combined with linguistic methods for analysing texts.

The student teachers are in their first year in their teacher education, and most of them are in their early twenties. I will refer back to the tentative ideologies, genres and power basis of the mathematics classroom presented earlier, and also to the triadic aspects of how the utterance positions the student relative to 1) mathematics and the teaching and learning of mathematics, 2) their own emotions and experiences and 3) the classroom, teachers and others.

In the students' texts it can be possible to trace aspects of 1) the student's own experience of learning mathematics at school, that is, their self image as mathematics learners, 2) the theories they meet at the teacher college, both in theory lessons and in the written material and 3) the practice teaching they meet during their teacher education.

The following examples are from an ethnic Norwegian student, here called Kari. All examples are translated from Norwegian by me.

Autobiography

At the beginning of the academic year the students were asked to write up a short presentation of their experience as students of mathematics so far. They were also asked to give reasons for why they had chosen teacher education and their expectations for the mathematics they will meet at the College.

Kari: *'I remember that in the primary school mathematics was fun. Think it was because I was pretty good. It was always about going furthest in the book. We were a group always competing on this. The teacher set limits for how far we could work, by puting a mark in the books. No further than this, it meant. We still had our methods! I remember that my friend and I wrote with feeble pressure with pencil on the next pages. This way we could get a small advantage when the class started.'*

'Do you know? – tests were given after each chapter. It was great to do well!'

This utterance from Kari tells us about her experience of the mathematics lessons as competition, the teacher setting limits so that all can keep up with the speed of the class, and about the auto ideology of having the children working individually in their books.

I, as a reader, interpret that the expressive (*fun, it was great to do well*) and addressive aspects (*a group always competing, the teacher setting limits, get a small advantage when the class started*) are dominant in this utterance.

Kari: *'The first day in lower secondary we were told that we should start with blank sheets of paper. The first lesson I was still interested. The mathematics became unbelievably boring and soon became my "hatred" subject. Now I did not do more than we were supposed to, and became good in sneaking away. Don't think I saw any reason to learn this. I remember that many of the students asked the teacher about the reason for learning this. "It's the way it is", was the answer. I never could accept this answer and the mathematics became without value in my eyes. I only did what I had to and did as little as possible.'*

This utterance includes the referential aspect (*The mathematics became unbelievably boring. Don't think I saw any reason to learn this, the mathematics became without value in my eyes*). It also shows the students seeking for rationale, both for learning the subject and for the subject itself.

Kari: *'In upper secondary I got a very good maths teacher. For my part I think it was a bit late. I had already decided on what mathematics was for me. Such attitudes are not easy to alter. It was probably very obvious that I did not try very hard. It probably feels better to fail in something without any effort put into it than fail in something that you really have worked hard on. I think this was part of my philosophy. Anyway I tried to become part of the "mongo" math group, as we called it. I probably hoped not to be one of the worst in mathematics in the class. "Unfortunately" I was too good for that. Still did what I had to do and passed in a way. Anyway I was sure:*

mathematics was not for me, and I should never have it later! It was an unbelievable feeling at last not to have any more classes with these annoying numbers and calculations.'

In this utterance Kari is focusing on her mathematics teacher and the importance of the teacher for the students' experience of the teaching and learning. She also points to an aspect of school mathematics as a subject that ranks the students according to achievements, and that this ranking has prestige and influences their self image.

The three utterances show how Kari's experience of mathematics evolves over time and that she positions herself according to the context in which she sees the mathematics she is supposed to learn.

Finally she reflects on herself as a student teacher:

Kari: 'To choose to start in teacher education also became to choose again to have some more mathematics. In a way I feel that now I have got a distance to this feeling of "hate". I hope that I shall change my mind about the subject. I know that I once liked it very much though. I have also realised that this is something I must know and that I have a reason for learning it. At least I have decided that I really will try. Then I think first of all that I must try to be positive towards learning it. Earlier I lacked the liking of learning math. Hope I can manage to focus on this. If I can do that then I think it will turn out well.'

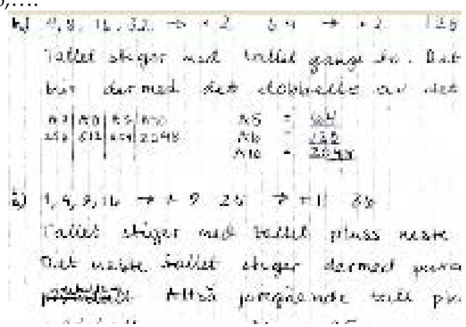
In this utterance I see how she actively positions herself; referentially towards mathematics and the teaching and learning of mathematics, her own feelings and experiences, and her actions for succeeding in mathematics as a student teacher.

Individual task on number series

After finishing the first semester the students got an individual mathematical task to solve within a week. The task was divided into two parts. The first part was to read and comprehend an introductory text about number sequences and figure numbers, in which there were some tasks to solve. The second part was about using figure numbers to create artistic decorations. The whole task, and the produced text, is on pure and applied mathematics. Below are some extracts from Kari's work.

The task illustrated below was to find and describe the pattern of the number sequences given and explicitly to find the 5th, 6th and 10th number in the sequences.

h) 4, 8, 16, 32,….
i) 1, 4, 9, 16,….



The figure above shows Kari's explanations of these tasks.

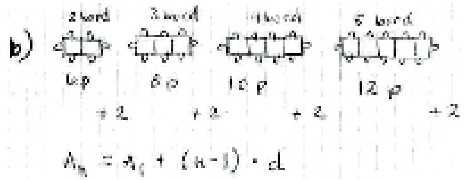
The written text in h) is: *'The number increases multiplied by two. The next number then becomes the double of the previous number'*.

The text in i) is: *'The number increases with the number plus the next odd number. The next number then increases parallelly with the odd numbers. That means the previous number plus the next odd number.'*

In both examples she uses an informal, nearly oral, genre in the beginning to describe how she is thinking by using the arrows and dots for illustrating the operations. The same is the case with the written texts that are also in an informal genre. In the last part she is setting up a table for the next numbers in the sequence. This informal, oral, genre could be seen as voices from her experience from school and also from the teaching of sequences at the teacher training college. The introductory text, and the textbooks she should read, use a more formal mathematical style. We also see that she has

got the answers correct, the square numbers, in i), but the written explanation is not correct.

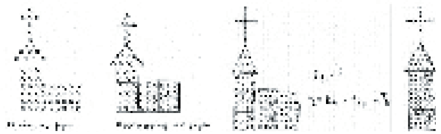
The next example is also about number sequences and is again one of the tasks from the introductory text. The problem is to find the number of persons you can seat around a table when you extend the table by adding a square to the end of the existing table.



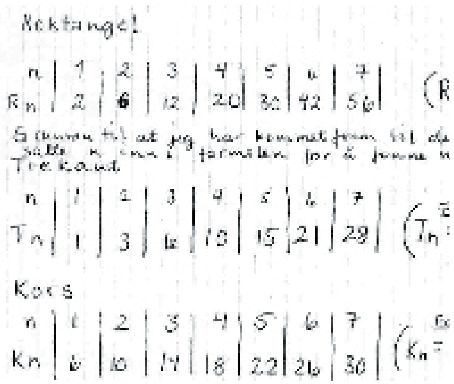
She is asked to give the number of people when she has 2 tables and 4 tables, and she is asked to write down the first five numbers in the number sequence. Finally she is asked to explain how many people can be seated around a table put together by n square tables.

Her text shows how she is thinking by drawing the tables and just by counting, writing up the number of persons around the table. She also shows that she has seen that the numbers in the sequence increase by two. To find the number for n tables she has probably taken the formula for finding the n-th number in an arithmetical sequence from the introductory text and put in the 4 for A_1 and 2 for the difference. Her text is without any explanatory text; all explanations are implicit in it. Here she is drawing both on her experience as a mathematics student in the class, and also from theory she has been reading. She is still not into the genre of the mathematics in the introductory text or able to explain her thinking in that genre.

The second part of this task was to use figure numbers to make artistic decorations. The introductory text introduced the students to triangle-, square- and rectangle-numbers, and also gave some examples of using these as building blocs for more complicated figures. The students were asked to draw three consecutive figures, and show how the different figure numbers construct the figure. It was supposed that tiles should construct the decoration, and the question then was how many tiles, as a number sequence, were necessary for constructing the different figures. This is Kari's drawing:



She draws a church: and with a cross, Ko_i, a triangle, T_i, and seven rectangles, R_i, constructs the church. She then presents the involved figure numbers in tables and as formulas.



Rektangel

n	1	2	3	4	5	6	7
R_n	2	6	12	20	30	42	56

Trekant

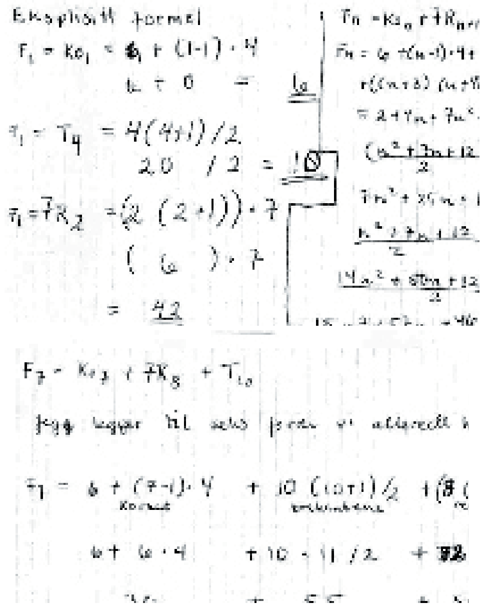
n	1	2	3	4	5	6	7
T_n	1	3	6	10	15	21	28

Kors

n	1	2	3	4	5	6	7
K_n	6	10	14	18	22	26	30

The written texts are: '*My reason for this is that I have put n into the formula to get the next rectangle*", and "*I then increased all the figures equally in relation to the formula. So that the figure expands proportionally*'.

Here she uses formulas for rectangle and triangle numbers given in the introductory text, but makes up her own for the cross. She makes it all her own by using first person. The cross will not evolve proportionally, so her concept of proportions is not fully developed. She continues to express the explicit formula for calculating the number of tiles in the figure:



All her calculations are correct, also the general formula for the n-th figure, but she does not use her general formula to calculate either the first or the seventh number in her number sequence of tiles. She goes all the way back to the three separate formulas to calculate the specific numbers. It seems as if the development of the general formula is just another exercise with no meaning besides itself. Just like in her experience from school.

Observations from praxis

The next text is written in the beginning of the second semester. It is written in connection with their praxis teaching in a primary school under the supervision of an experienced teacher. They had been to the praxis school for a period of four weeks in the first semester.

The task was given to a group and Kari was in a group with four other students. The task was divided in two parts; one individual task where the student observed one child of between 6 and 8, and one task where the whole group observed a mathematics class for children of class 1, 2, 3 or 4. The individual observations were to be presented as a summing up of the different individual observations in addition to the individual observations.

In this text I can expect to trace the first aspects of the didaktikal theories they meet at the teacher training college, both in theory lessons, in the written material and the practice teaching they meet during their teacher education. In addition I expect to hear the voices of the other group members and the praxis teacher in Kari's texts. The following is her introductory text.

Kari: 'Tuesday … the group and I went to the school [, I had arranged] to observe a student in 3de grade. I had planned beforehand to make a mathematical game, and I had prepared some question cards. The game, though, I wanted us to make together. By making and using this game I wanted to explore her competence in mathematics in an informal setting. In the first part of this description I want to refer to and quote some of the things she did and said. I will also describe what we did together and which instructions I gave her. In the next part I will show the results I got through the observations.'

The idea of using a game had been discussed in the theory lessons, and also in the group when they planned the observations. The group also decided to have the observed child develop the game together with the observing student.

She then has a long descriptive text, very informal, nearly like an oral genre, just describing what they did and what was said between them. This form was expected from them as part of the observation format. They were expected to try to describe episodes as they happened without putting any reflection or judgement into it. She then continues to analyse her observations:

Kari: 'I will now show some of the things I saw during the observation of this student. I will try to say something about this student's understanding of, and ability to express, mathematics. I will also see this in relation to

relevant reading. I observed a second-generation immigrant Indian girl. She was born in Norway and speaks fluent Norwegian. My primary aim for this observation was to try to understand some of the things that lie behind her way of thinking mathematically. I found it very difficult in that situation. The girl had never met me before and I knew very little about her. She wanted to impress me. When she did not know the answers right away she tried to disguise it, either by overhearing my questions or by hiding the question cards.'

In this utterance Kari is referring to the task she is about to do and the requirements on her to do the task right. She is supposed to relate her reflections to the readings at the theory lessons at the college, and she is supposed to say something about the mathematics understanding and the student's ability to speak mathematically. Kari is expressing her feelings about the situation, a situation that is new to her and where she feels insecure. She also describes some of the actions in their communications that are hard for her to handle, the girl's strategies. These strategies can again be linked to the competitiveness of the mathematics classroom.

Kari: *'The first task she did was to draw the game board. This was successful. She had a pre-understanding of how a game should look. She drew quadrangles and numbered them without difficulties. She also had a clear concept of what 'far away' should mean and drew the goal as far away as she could from the start. I also tested her conceptions of concepts like "small", "big", " more", "less" etc. [I can see that] She seems to have good control of these concepts from what she draws.'*

Kari, in the utterance above, focuses on how she sees the mathematical understanding through the girl's actions on the game board.

Kari: *'When we tried subtraction she got some problems. She clearly did not understand "minus" or how to "subtract". I tried different strategies and also put it as a formal subtraction for her. She still could not get the right answer. There might be many reasons for her not understanding. Maybe she did not understand what I was trying to say, or I explained it in a different way from what she was used to? At the end I tried to have her take away a star one by one. Then she understood what I was after and easily managed the tasks. Later I tried this out by using subtraction instead of addition when we threw the dices.'*

In this utterance Kari expresses her experience and feelings about the fragile communication with the girl. She has to take all her repertoire of strategies for explaining subtraction. She reflects on this communication and blames her self, which reflects her insecurity towards mathematics. She finds that the girl understands and is capable of doing the subtraction, but the way it is represented in the dialogue is not recognised by the girl. So Kari blames explicitly her own ability to communicate mathematics.

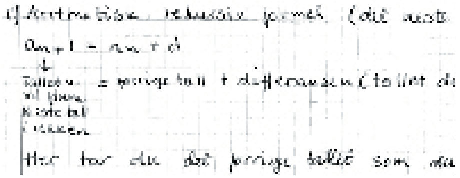
Compulsory individual test

As described earlier the final exam in the compulsory mathematics course for student teachers at Oslo University College, currently consists of two separate parts; a one week 'home' exam where four students in group give a co-written presentation on a given task, and an individual oral examination. As a condition for passing the final exam the students have to do an individual mathematics test, a school sitting for six hours. This test is held after one year of studying and contains both didaktikal and mathematical questions. In this test there was a task on number sequences:

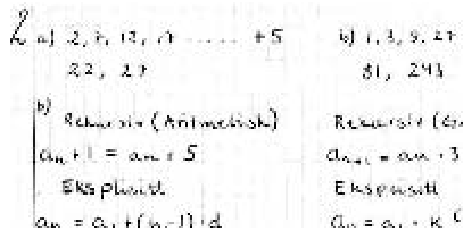
Task 2.
Two number sequences are given:

1) 2,7,12,17,….
2) 1,3,9,27,….

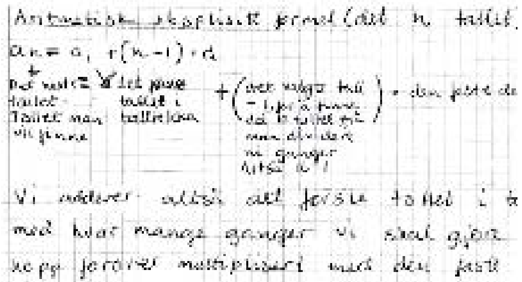
a) Find the next two numbers in the sequence.
b) Find the recursive and the explicit formula for the two sequences.
c) Explain why the formulas are correct.



Kari's texts on a) and b) are very similar to her texts from the individual task on number sequences referred earlier. Below is her answer to c):



To explain the recursive formula for the arithmetic sequence she writes: *'Here you take the previous number which you know and add the number you know is the difference.*



To explain the explicit formula for the arithmetic sequence she first explains the different parts of the formula. To explain (n-1) she writes: *'The chosen number –1, to find the 10th number one has to divide nine times. 10 - 1 then.'* Here she uses an example to explain the general case. To explain the formula she writes: *'We therefore add the first number in the sequence with the number of times we shall make a jump ahead multiplied by the difference.'* In line with her thinking, explaining the general with an example, in this utterance she is nearly tactile in her expression of the jumping ahead along the number sequence. This utterance gets its meaning from different genres as she is working herself towards the genre of the teacher education mathematics. She uses the inclusive "we", which is used in the mathematics texts she is meeting in the study, and she explains by examples which are used a lot both in educational texts and also in teaching sessions both at the college and in the practice schools. Her explaining of the formulas for the geometric sequences is similar.

Summing up

One of the questions I am asking and want to explore will be in what way students' writing in the different genre of mathematics educational texts, in dialogue both with praxis, theory and self-image, influence the development of their 'identity' as teachers of mathematics. The examples above show some extracts from such texts, and also focus elements from educational genres in the teacher education of mathematics for the primary school. Kari is an ordinary student teacher; her relation to mathematics has been mixed, but she is devoted to learn what is required of her to develop as a teacher. She is trying out the different genres and letting them gradually become part of her own communicative ability both within and about mathematics.

References

Adler, J. (2001) *Teaching Mathematics in Multilingual Classrooms* (Dordrect: Kluwer Academic Publishers).

Bakhtin, M. (1986) *Speech, Genre and Other Late Essays* (Texas: University of Texas Press).

Berge, K. L. (1996) *Norsksensorens tekstnormer og doxa. En kultursemiotisk og sosiotekstologisk analyse [The censor of Norwegians' text norms and doxa. A cultursemiotic and sociotextologic analysis.]* (Unpublished doctoral thesis NTNU: Trondheim).

Berge, K. L. (2002) Hidden Norms in Assessment of Students' Exam Essays in Norwegian Upper Secondary Schools, *Written Communication*, 19, pp. 452–457.

Brown, T. (2001) *Mathematics Education and Language. Interpreting Hermeneutics and Post-Structuralism* (Revised second edition. Kluver Academic Publisher).

Braathe, H. J. (1997) Matematikkundervisning, lærerutdanning og matematikkbevissthet [Mathematics teaching, teacher education and mathematics awareness]. In Sigurd Haga (ed.*) Vilkår for et verdig liv [Conditions for a dignified life]* (Festskrift til Edvard Befring: Oslo).

Braathe, H. J. (1999) *Didaktisering av matematikkfaget i Lærerutdanninga. Tradisjoner og Ideologier [Didaktization of Mathematics in the teacher education. Traditions and Ideologies]* (Conference report: Bodø University College).

Braathe, H. J. (2002) *Ideologier i skolematematikken [Ideologies in the school mathematics]* (Conference report Rapport from the conference 'Barn skaper matematikk [Children create mathematics]', Hurdal 2001).

Braathe, H. J. and Ongstad, S. (2000*) Equality vs. reason, competition and responsibility. 'Norwegian' blurrings of ideologies in mathematics classroom and curricula*. (Paper at AERA 2000 in New Orleans).

Braathe, H. J. and Kleve, B. (1995*) Evaluering av matematikkfaget ved Høgskolen i Oslo – Lærerutdanningen [Evaluation of the pre-service teacher education in Mathematics at Oslo College]* (Oslo: Report Oslo University College).

Braathe, H. J., Kværnes, L. and Ongstad, S. (2000) *Triads in mathematics education. Mellin-Olsen (1987) and semiotic genre theory– comparisons and examplifications* (Paper at the Gausdal seminar. (March.) Organized by B. Kleve, Faculty of Education. Oslo University College and B. Jaworski, Oxford Centre for Mathmatics, UK).

Braathe, H. J. and Ongstad, S. (2001) Egalitarianism meets ideologies of mathematics education – instances from Norwegian curricula and classrooms, *Zentralblatt Fur Didaktik Der Matematik* Vol 33 (5) 2001.

Cogan, L. S. and Schmidt, W. H. (1999) An Examination of Instructional Practices in Six Countries. In Kaiser, Luna and Huntley (eds) *International Comparisons In Mathematics Education* (London: Falmer Press).

Dewey, J. (1916) *Democracy and Education* (New York: The Macmillan Comp).

Durkin, K. and Shire, B. (Eds. 1991*) Language in Mathematics Education: Research and Practice* (Milton Keynes: Open University Press).

Evans, J. and Tsatsaroni, A. (1994) Language and Subjectivity in Mathematics Classroom. In Lerman, S. (ed.) *Cultural Perspective on the Mathematics Classroom* (Dordrecht: :Kluwer Academic Press).

Evans, J. (2000) *Adults' Mathematical Thinking and Emotions. A study of Numerate Practices* (London: Falmer Press).

Evensen, L. S. and Hoel, T. L. (eds. 1997) *Skriveteorier og skolepraksis Oslo [Theories of writing and school praxis]* (LNU/Cappelen: Oslo).

Foucault, M. (1970/1966) *The Order of Things: An Archaeology of the Human Sciences* (London: Tavistock).

Freadman, D. (1992) *An Alchemy of Genres. Cross-Genre Writing by American Feminist Poet-Critics* (Charlottesville and London: University Press of Virginia).

Gardiner, M. (1992) *The dialogs of critique. M.M. Bakhtin and the theory of ideology* (London: Routledge).

Gee, J. (1990) *Social Linguistics and Literacies. Ideology in Discourses* (London: Falmer Press).

Habermas, J. (1984) *The Theory of Communicative Action. (Vol 1)* (trans. T. McCarthy). (Boston: Beacon).

Halliday, M. A. K. (1985) *An Introduction to Functional Grammar* (London: Edward Arnold).

Harre, R. and Van Langenhove, L. (1999) *Positioning Theory* (USA: Blackwell Publishers Inc, USA).

Ivanic, R. (1994) I is for Interpersonal. Discoursal Construction of Writer Identities and Teaching of Writing, *Linguistics and Education*. 6, 315.

Knain, E. (1999) *Naturfagets tause stemme [The tacit voice of (the discipline of) Science.]* (Unpublished Ph.D. thesis. Oslo: Faculty of Mathematics and Science, University Of Oslo).

Kragh, H. And S. A. Pedersen (1991) *Naturvitenskapens teori. [The theory of Science.]* (Copenhagen: A. Busck).

Kvalbein I. A. (1999) *Lærerutdanningskultur og kunnskapsutvikling [Teacher education culture and development of knowledge]* (Ph.D. thesis, HiO-report 1999, 15, Oslo).

Kuf (1996) *Lærerutdanning. Mellom krav og ideal. [Teachereducation. Between demands and ideals.]* (Oslo: NOU (1996:22)).

Kuf (1998) *Rammeplan og forskrift for 4-årig allmennlærerutdanning [Curriculum and (administrative) regulation for the 4-year teacher education]* (Oslo: KUF).

L97: The Royal Ministry of Education, Research and Church Affairs (1999). *The Curriculum For The 10-Year Compulsory School in Norway*. (Oslo: KUF).

Lemke, J. L. (1990) *Talking Science. Language, learning, and values* (Norwood: Ablex).

Lerman, S. (1992) Reflective Practice. In J. A. Worski B and Watson A. (eds.) *Mentoring In Mathematics Teaching* (London: Falmer Press).

Lie, S., Kjaernsli, M. and Brekke, G. (1997) *Hva i all verden skjer i realfagene? [What On Earth Is Happening In The Sciences?]* (Oslo: University Of Oslo).

Mellin-Olsen, S. (1987) *The Politics of Mathematics Education* (Dordrecht: Kluwer).

Mellin-Olsen, S. (1991) *Hvordan tenker lærere om matematikkundervisning? [How do teachers think about the teaching of mathematics?]* (Landås, Bergen: Bergen Universcity College.)

Morgan, C (1998) *Writing Mathematically. The Discourse of Investigation* (London: Falmer Press).

Morgan, C. (2002) *What does social semiotics have to offer mathematics education research?* (Paper presented at PME26 at the University of East Anglia, July 21–26.2002. (http://www.math.uncc.edu/~sae/, visited 18.02.2004).

Ongstad, S. (1994) *På sporet av fag- og undervisningssjangre? [On the track of subject- and didaktikk genre?]* (HiO report).

Ongstad, S. (1997) *Sjanger, posisjonering og oppgaveideologier. Et teoretisk-empirisk bidrag til et tverrfaglig semiotisk og didaktisk sjangerbegrep [Genre, positioning, and task ideologies: A theoretical-empirical contribution to an interdisciplinary, semiotic, and didaktic genre concept]* (Unpublished doctoral dissertation, NTHU: Trondheim).

Ongstad, S. (1999) *Vad er posisjonsanalys? [What is analysis of positioning?]*. In Säfström, C. A. and Østman, L. (eds) (Texanalys. Studentlitteratur: Lund).

Ongstad, S. (2002a) *From Utterances to Genres in Mathematics Education (and vice versa) – a dynamic triadic (socio-)semiotics exemplified*. (Paper presented at PME26 at the University of East Anglia, July 21–26.2002. (http://www.math.uncc.edu/~sae/, visited 18.02.2004).

Ongstad, S. (2002b) Positioning Early Research on Writing in Norway, *Written Communication*, Vol. 19, 3, pp. 345–381.

Ongstad, S. and Braathe, H. J. (2001) *Analysis of Positioning(s) used on Studies of Textbooks in the Didactics of Mathematics* (Paper presented at IAMTE conference in Amsterdam July 2001).

Pimm, D. (1987) *Speaking Mathematically: Communication in Mathematics Classrooms* (London: Routledge and Kegan Paul).

Pimm, D. (1995) *Symbols and Meanings in Mathematics Classroom* (London: Routledge).

Ricoeur, P. (1981) *Hermeneutics and the Human Sciences* (Cambridge: Cambridge University Press).

Rowland, T. (2000) *The Pragmatics of Mathematics Education*. Vagueness in Mathematics Discourse (London: Falmer Press).

Smidt, J. (1993) *Ny skriveforskning i Norge. En oversikt [New research on writing in Norway: An overview]* (Oslo, Norway: PUF, The Norwegian Research Council).

Smidt, J. (1999) 'All the world is a stage' – discourse roles and students positioning in the great dialog. In Dyste, O. (ed.) *A dialogical perspective and Bakhtin* (Proceedings PLF report 2. University of Bergen).

Smidt, J. (2002) Double Histories in Multivocal Classrooms, *Written Communication*, Vol. 19, 3, pp. 414–443.

Walkerdine. V. (1998) *Counting Girls Out. Girls and Mathematics*. (New Edition, London: Falmer Press).

Contact

Hans Jørgen Braathe
Oslo University College Faculty of Education
Email: HansJorgen.Braathe@lu.hio.no

The Mediation of Cultural Tools in the Learning and Teaching of Mathematics: A Selective Outline of Research Issues

Dimitris Chassapis, *Aristotle University Thessaloniki*

Abstract

A central claim of the socio-cultural approaches to human cognition which are adopting a Vygotskian perspective is that children's mental functioning and development can be accounted for in terms of their engagement in culturally organised practices in which cultural tools play a crucial role. Such tools, mediating human practical and mental functioning, have been developed in a culture over extended periods of time and have become an integral part of human activity, functioning as 'carriers' of socio-cultural patterns and knowledge. By acting as mediators, cultural tools structure human practical activity and bring into play differentiated mental processes which in turn regulate and qualitatively transform that practical activity. Mediatory means, thinking processes, and human practical activities become functionally intertwined in their development, shaping each other in a dialectical interdependence. In this account, issues concerning the impact of material and symbolic mediatory means upon mathematical concepts formation and skills development, will be presented and relevant research topics will be put forward for discussion.

Introduction

This paper presents a selective outline of fundamental aspects of a research field, which have emerged in light of a socio-cultural

approach to human cognition: the mediation of cultural tools in the learning and teaching of mathematics. The outline presented is a selective one as far as it pinpoints issues and puts forward questions arising from the particular research interests of the author.

In the first part of the paper, they are briefly presented as key concepts of a socio-cultural approach to human cognition in a Vygotskian perspective, which are closely related to, and elucidate, crucial aspects of the role played by cultural tools in human acting and thinking. While in the second part, cultural tools and their functioning are further commented on. The third and the fourth parts of the paper are respectively focused on research topics which have arisen in the field under discussion. The guiding principle of the arguments followed is that human thinking and learning develop through, and are shaped by, the activities in which people participate; activities that are social in nature and have historically developed tools, structures, and settings. Thus, people in their everyday actions and activities inherit and embody the socio-historical residue of their predecessors, so that context-independent cognition is non-existent. Mathematics learning and teaching are socio-culturally mediated activities and, from this standpoint, they cannot be understood by focusing on the individual student in isolation nor by considering the individual learner only in face to face interactions with social agents, such as teachers or parents. Mathematics learning and teaching need to be located, and therefore researched, in more encompassing cultural, historical and political contexts and practices.

Cultural tools as mediators of social to individual human cognition

The cornerstone of socio-cultural approaches to human cognition developed along a Vygotskian line of thought is the assumption that a prerequisite for understanding the unique characteristics of human cognition is the consideration of two factors. The first one is the 'historical character of human behaviour and learning' (Kozulin, 1990, p. 81), which means that individuals use not only physical experience, but also historical and cultural experience to make sense of their world. And the second factor is 'the social nature of human experience' (ibid, p. 81), which means that individual experiences are only a part of the pool of experiences that are available in the particular society. Hence, from a socio-cultural perspective, what is unique in human cognition emanates from the need and ability of human beings to mediate their actions through cultural means that are transmitted, in the course of history, from generation to generation (Vygotsky, 1978).

Analysing the use of cultural means in human acting and thinking, Vygotsky describes two types of mental functions, lower and higher. The lower mental functions are genetically inherited, and allow only for a direct, instinctive and impulsive response to the environment. The higher mental functions, on the other hand, are 'by definition, culturally mediated; they involve not a 'direct' action on the world, but 'indirect' action' (Cole and Wertsch, n.d). Vygotsky argued that higher mental functions are not merely an extension of lower mental functions, rather, they are 'a function of socially meaningful activity' (Kozulin, 1990, p. 113). So, according to the theory known as the genetic law of cultural development, 'every function in the child's cultural development appears twice: first, on the social level, and later, on the individual level; first between people (interpsychological), and then inside the child (intrapsychological)…. All the higher functions originate as actual relations between human individuals' (Vygotsky, 1978 p. 57). In other words, 'any higher mental function was external because it was social at some point before becoming an internal, truly mental function' (Vygotsky, 1981, p. 162).

Thus, what people learn in the course of their mental development is crucially dependent upon the higher mental processes that are available in their given cultural environment. This premise constitutes, in my view, a starting-point for fruitful research endeavours concerning issues of mathematics learning. For instance, it may give reasons for investigating and elucidating differences across cultures that use different conceptual structures, symbol systems, computational tools or problem-solving approaches, differences across groups of subjects within the same culture or even differences within subjects across different situations (see for a relevant discussion Nunes, 1992a).

Besides their social character, however, the human higher mental functions are invariably mediated processes (Vygotsky, 1987, p. 126). The source of mediation can be of different types: material tools, systems of symbols, conceptual structures or behaviours of other human beings. As stated by Vygotsky (1981, p. 137) mediated means included 'language; various systems for counting; mnemonic techniques; algebraic symbol systems; works of art; writing; schemes, diagrams, maps, and mechanical drawings; all sorts of conventional signs'.

In every case, the mediation of the human higher mental functions is a socio-cultural mediation; a fact which led Vygotsky to consider mediation as 'the central fact about our psychology' (Nicholl, 1998). For example, when we write we do it by using a pen, a pencil, or a computer (material tools), but, besides this material mediation, when writing we also use a symbolic tool, language (the tool of tools, according to Luria, 1929). Most, if not all, of our actions –physical and mental – are mediated by some type of cultural tool. Hence, it is the ability to function at the level of mediated higher mental processes with the tools which are historically and culturally transmitted, that distinguishes human beings from other animals. In this account, Zichenko (1985) has argued that a tool-mediated action must be considered as the appropriate primary unit of analysis for a Vygotskian account of human mental functioning, because a goal-directed, tool-mediated action necessarily encompasses and co-ordinates in a unit several interrelated intra-psychological as well as inter-psychological functions.

On the above grounds the following statements may be put forward being of particular interest here:

- every human action is done through the use of cultural tools that are social, mediate human actions and transform and restructure human 'nature' in the process. For example, having a language changes the way in which we perceive reality since words, and more generally categories, used between people in discursive practices to account for our reality expand our consciousness. They also limit it as we can only fully experience and understand those aspects of reality which we have as words for tools for thinking about them (Säljö, 1996).

- cultural tools we use do not simply facilitate our mental functions but, at the same time, they shape and transform them.
- all psychological functions begin and to a large extent remain culturally, historically and institutionally situated and are context specific (Cole and Wertsch, n.d.).
- the mind is distributed, it is not exclusively in our heads but also in the cultural tools that mediate our actions. Higher mental functions include the biological individual, the cultural mediatory tools and the socio-cultural context (Bateson, 1972, pp. 318–320).
- and finally, the rationalist principle 'from thought to action' is actually reformulated to the principle 'from action to thought'.

In summary, the issue of mediation as brought up by Vygotsky provides 'the link or bridge between concrete actions carried out by individuals and groups, on the one hand, and cultural, institutional, and historical settings, on the other' (Wertsch, del Río and Alvarez, 1995, p. 21) or to put it simply between individual and social. Hence, it plays an important role in the socio-cultural perspectives of human cognition.

In a Vygotskian perspective higher mental functions are, as mentioned, always social, mediated activities which become individual. The process through which the linking of these two levels is accomplished or through which 'the internal reconstruction of an external operation' (Vygotsky, 1978, p. 56) takes place, is called internalisation. The concept of internalisation is in many aspects a controversial one under the influence of the dichotomising thinking of 'the inner' and 'the outer' in cognitive psychology, a thinking derived from the ontological stance often referred to as Cartesian Dualism. Cartesian Dualism, as an ontological stance, conceives the world of material bodies and the world of the spiritual self as being of radical different essence and absolutely independent; and the concept of internalisation seems to be founded on a similar dualism presupposing that the social and psychological planes are separate or that a boundary exists between individual and context (for a discussion see Rogoff, 1990 and Matusov, 1998).

However, in the perspective outlined in this paper, the concept of internalisation is conceived as 'simultaneously a social and an individual process' (John-Steiner and Mahn, 1996, p. 10), in a view which is not presupposing a strict distinction between mind and environment. In such a view, the internalisation process is first and foremost seen as a confirmation of the thesis that human thinking and learning are, principally and to a large extent, social processes. In any case, the internalisation process cannot be fully understood without reference to a reciprocal externalisation process, since '(internal) mental processes manifest themselves in external actions performed by a person, so they can be verified and corrected, if necessary' (Kaptelinin, 1996, p. 109).

In this process of internalisation the role played by cultural tools is crucial. According to Vygotsky, the 'social' is composed of cultural tools, which being a constitutive part of the context, shape individual psychological processes. Thus, every human action (including in thought) is done through the use of cultural tools. Cultural tools mediate both social and individual acting and thinking. Once more, the social has supremacy over the individual, since if the particular culture in which an individual is brought up does not possess certain social tools the individual will be unable to utilise them or the higher mental functions which they facilitate.

Thus, the social character of human higher mental functions and their mediation by cultural tools cannot be separated. They are mutually constitutive and the study of one necessarily involves the study of the other in a relational conception.

Summing up, a Vygotskian approach to human cognition is grounded on four main theoretical constructs and their specific interrelationships: higher mental functions, mediation, cultural tools and internalisation. Higher mental functions are what allow human beings to distance themselves from the world and act instrumentally upon it. Because they are socially meaningful processes – thus external to any individual – higher mental functions are always mediated. This mediation is done by tools, which have been developed in a culture and have become an integral part of human activity. The higher mental functions when mastered by individuals are internalised and through this process the social becomes available as an individual tool. The process of internalisation involves not only the transformation of the object internalised but, at the same time, the change of the nature of the individual by the properties of the object as well as by the process of internalisation itself (Kozulin, 1990, p. 115).

The mediation of cultural tools in thinking

Tools, material, symbolic, conceptual or behavioural, as well as their use have been developed in a culture over extended periods of time and have become an integral part of human activity, functioning as 'carriers' of socio-cultural patterns and knowledge. By acting as mediators, they structure human practical activity and bring into play differentiated mental processes which in turn regulate and qualitatively transform that practical activity.

In Vygotsky's (1981, pp. 139–140) words, 'the inclusion of a tool in the process of behaviour (a) introduces several new functions connected with the use of the given tool and with its control; (b) abolishes and makes unnecessary several natural processes, whose work is accomplished by the tool; and (c) alters the course and individual features (the intensity, duration, sequence, etc.) of all the mental processes that enter into the composition of the instrumental act, replacing some functions with others, (i.e., it re-creates and reorganises the whole structure of behaviour just as a technical tool re-creates the whole structure of labour operations)'.

Mediatory means, thinking processes, and human practical activities become functionally intertwined in their development, shaping each other in a dialectical interdependence.

For example, using an abacus to carry out a numerical operation of addition generates different mental processes from using a piece of paper and a pencil. The principles, which rule the process of number addition, when using an abacus and a piece of paper and a pencil are, of course, the same. Both tools, based on place-value numeration, have historically developed in interdependence. A number addition using both tools is carried out by decomposing numbers into parts coinciding with the decimal system and successively adding

up these parts. However, the functional organisation of carrying out a number addition varies considerably in the two cases. When using an abacus, this process of number addition is directly imposed by the material structure of the tool; in contrast, the process is merely an artificial rule that has to be respected when using a piece of paper and a pencil. In other words, using different mediatory means structures the task of carrying out a number addition in qualitatively different ways, thus invoking different mental processes for its accomplishment, and different consequences on the formation and development of the related concepts (Chassapis, 1999).

In this context, two important distinctions require a further clarification. The first concerns the distinction between material tools on the one hand and conceptual, as well as symbolic, tools on the other; and the second distinction refers to the double characteristic of tools as both mediatory means and objects of human activity themselves.

Regarding the first distinction between material tools and conceptual and symbolic tools, (which may also be referred to as signs), it should be pointed out that despite their overall similarity, these two kinds of cultural tools cannot be considered isomorphic with respect to the functions they perform as mediatory means of human acting and thinking. As Vygotsky (1978, p. 55) put it, in his own terminology 'a most essential difference between sign and tool, and the basis for the real divergence of the two lines (of mediated activity), is the different ways that they orient human behaviour. The tool's function is to serve as the conductor of human influence on the object of activity; it is externally oriented; it must lead to changes in objects. It is a means by which human external activity is aimed at mastering, and triumphing over, nature. The sign, on the other hand, changes nothing in the object of a psychological operation. It is a means of internal activity aimed at mastering oneself; the sign is internally oriented. These activities are so different from each other that the nature of the means they use cannot be the same in both cases.'

In other words, although this is an oversimplification, the use of material tools primarily affects the objects of human activity, whereas the use of conceptual or symbolic tools primarily affects their users. However, the function of material tools and the function of conceptual and symbolic tools, no matter how antithetical they may appear, are dialectically interdependent, and in many human activities their boundaries are so mobile as to become indiscernible.

Bruner (1962, p. vii), expressing this fact, wrote that: 'man, if you will, is shaped by the tools and instruments that he comes to use, and neither the mind nor the hand alone can amount to much . . . . And if neither hand nor intellect alone prevails, the tools and aids that do are the developing streams of internalised language and conceptual thought that sometimes run parallel and sometimes merge, each affecting the other.'

This functional dualism is more profound, like the two sides of the same coin, in human practical activities mediated by material tools that rely heavily on non-linguistic sign systems, such as, for instance, diagrams, maps, charts, drawings, or tables. In such cases one side or the other is more apparent depending on the characteristics of the tools used, the nature of the activity, and the conditions of its development (Chassapis, 1999).

The second distinction refers to the double character of tools as mediatory means of human acting and thinking on the one hand, and on the other hand as objects of human activity themselves. . We use conceptual tools, such as a Pythagorean theorem, or material tools, such as a ruler, for carrying out a measurement operation, in which case the numeration system and the ruler are not objects of our activity. In situations of normal use the using of tools is done through operations, and as a rule is not conscious to the user. However, we use the same tools as objects of our inquiry if problems concerning their use or their effectiveness arise in the course of a measurement operation. But, even when difficulties with the use of tools arise, people attempt, and in many cases succeed, in developing 'new' operations that 'work around' the particular problems, so that related activities can be accomplished. Hence, tools as mediators, their use in operations, and tools as objects themselves, may be considered, to a greater or lesser extent, as mutually shaped both by the socio-cultural level of their development and by the individual level of their appropriation. Tools, both individually as well as in their use, are seen as historically crystallised knowledge, both conceptual and

procedural, which reflect the state of praxis up until the time that they are developed in a culture, which praxis in turn is shaped by the tools used, and so on. If we accept such a perspective, then the study of tools-in-use in specific activities needs to be a prerequisite of any related research endeavour. For instance, in mathematics education we need to study a proof method and its use both as applied by a mathematician and by a student in properly selected problem situations.

The mediating role of tools in learning and using mathematics

Mathematics as a discipline and as a school subject incorporates a multitude of tools in various forms. As outlined in this paper, they could be thought of as tools of generic concepts and theorems (e.g., Pythagorean theorem), methods (e.g., proof methods, algorithms), symbolic systems (e.g., decimal numeration system) or material resources (e.g., compass, ruler, calculators). In school mathematics these tools are shaped by mathematics as a discipline and its place in the particular culture, as well as by a specific school context in their acquisition by students and use in problem-solving activities. Students participate in mathematics learning activity through appropriating the use of mathematical tools proper to that activity. A Pythagorean theorem, for example, can be seen as a tool to be used in problem-solving situations requiring the correspondence between the mathematical fields of geometry and arithmetic, or as an important result that can be reached through a process of investigation.

A significant number of empirical studies have attempted to understand the mediating role played by cultural tools in learning and using mathematics. A review suggests that four aspects are particularly highlighted (Abreu 2000):

(a) the material structure and/or the logical organisation of diverse mathematical tools;
(b) the influence of specific tools on constraining or empowering the accomplishment of a mathematical task;
(c) the constraints on the use of certain tools that that can be imposed by specific social practices;

(d) the use of 'old' tools in new contexts.

A short discussion of each of these issues, illustrated by selected examples, follows.

(a) The material structure and/or the logical organisation of diverse tools

The understanding of logical organisation and/or the material structure of mathematical tools is a requirement implied by the thesis that the logic of mathematical thinking is not to be found in the mind, but in the ways specific mathematical tools have been invented and socio-culturally organised. The characteristics of specific mathematical tools, the logical organisation and/or the material structure of these characteristics, the relationships between different mathematical tools and the influences of the organisation of specific tools in relation to the strategies a person uses to think, and solve a mathematical problem, are among the most common topics in this research area.

Example: The use of various instruments to draw circles

Chassapis (1999) investigated the material structure of various instruments used to draw circles, and their influence on the formation of formally defined mathematical concepts of a circle. He noted that the compass, in contrast to the circle-drawing tracers or templates, induces by its material structure and its functional use the generative features of the formal mathematical concepts of the circle, that is, the centre and the radius. Consequently, the use of a compass in circle-drawing structures this operation in a radically different fashion from circle tracers and templates, bringing into play differentiated thinking processes, which positively influence the formation of analytical concepts by children that are analogous to the formally defined mathematical concepts of the circle.

Example: Numeration systems

Saxe and Posner (1983) classified the numeration systems of Papua New Guinea in terms of the means used to signify numerical relations (physical or verbal representations) and the fundamental organisational structure of the system (base or non-base structure). They classified about 200 different counting systems into four cat-

egories: 1) spatial representation and no base structure; 2) spatial representation and base structure; 3) non-spatial representation and no base structure; 4) non-spatial representation and base structure. Depending on the numeration system which is accessed [used], and used as a mediator, the counting operation can be a difficult or a simple task, especially the counting to thousands (Nunes and Bryant, 1996). For users of body part tally systems counting up to thousands can be a difficult task (Bishop, 1988; Lancy, 1978). These systems rely on spatial representation of a number of body parts which can vary from 12 to 68, which do not have a base and which are finite (Nunes and Bryant, 1996; Saxe and Posner, 1983). In addition, the linguistic characteristics of numeration systems are found to strongly influence a person's mastery of understanding of place value and base-10 structure, and counting, as well as calculation skills (Miura, 1987)

Example: Strategies associated with oral and written arithmetic

Nunes (1992a; 1992b) analysed the principles underlying the use of oral and written strategies in accomplishing various arithmetical tasks and concluded that in general terms the same logical properties are involved. For example the use of the strategy of decomposition to solve addition and subtraction problems in oral arithmetic relies on the property of the associativity of addition and subtraction, which is the same property involved in written arithmetic. However, she also noted differences that might affect the functional organisation of a person's activity for accomplishing particular arithmetical tasks. For example, calculations made in oral arithmetic follow the order in which people speak (from large to small numbers), whilst in written arithmetic they go in the opposite direction.

Summing up, the analysis of specific types of tools for representing mathematical ideas aims to provide an insight into the influence of the logical organisation and/or the material structure of mathematical tools on a person's thinking. This kind of study elucidates advantages and disadvantages of particular tools in empowering mathematical thinking required in specific contexts and how some tools, while deploying the same logic as others, might be easier to grasp (Nunes and Bryant, 1996).

(b) The influence of specific tools on the constraint, or empowering, of the accomplishment of a mathematical task

Material resources for carrying out physical or conceptual operations embody in their physical structure a certain logical organisation. A ruler, for instance, as an instrument to measure length embodies a standard unit, the iteration of the unit without gaps and the subdivision of the unit into smaller units. Clocks and calendars involve units to measure time. Calculators contain information on numbers and operations with numbers.
It may be assumed that someone's potential for accomplishing a mathematical task can be affected by the tools that are available to them as well as by the socio-culturally established uses of these tools.

Example: The use of a calendar

Säljö (1996a) has provided evidence that using a tool, such as a calendar, can transform a difficult task into an easy one. Counting the days included between two given dates, e.g. from March 24 until June 18, became an easy task when using a calendar, in contrast to calculations using paper and pencil. The error was produced by subtracting 24 (th of March) from 31 (st of March). In a straight subtraction 31–24 is seven, whereas the result of using a calendar to enumerate the days between 24th and 31st is eight. The use of a calendar that embodies the cardinal structure of number helped the children to overcome the difficulty by counting the actual days.

Example: Length measurement tasks

Nunes, Light and Mason (1993) investigated the extent to which the accomplishments of length measurement tasks by children were affected by the tools they used. The use of a conventional ruler marked in centimetres and half-centimetres, i.e. a tool that embodied a specific mathematical structure, and the use of pieces of string, i.e.. a non-conventional measurement tool, were contrasted and the different strategies applied in the two conditions were analysed. They concluded that the children performed significantly better where the conventional ruler was used and that in the non-conventional condition the children's difficulties might be linked to the need to re-invent basic logical properties, such as iteration and subdivision, which are embedded in the conventional tool.

In summary, these and similar studies contrasting the use of various tools to accomplish the same mathematical task or comparing the use of the same tool to accomplish diverse mathematical tasks, suggest that performance in a task can be related to the cultural tools available to support an individual's activity.

(c) The constraints on the use of certain tools that can be imposed by specific social practices

Strategies for solving problems may vary according to the social practice within which the task is located. The context of school mathematics imposes on problems a framework which is, in many instances, quite different from the framework imposed on the same problems by everyday life. Thus, it seems that the choice and use of tools for solving a problem is linked to the specific social context in which the problem is posed and solved. The distinction between 'school mathematics' and 'street mathematics' or 'realistic mathematics' is to a large extent based on such a premise (Nunes, Schliemann, and Carraher 1993).

Example: Different strategies to solve postage problems

Säljö and Wyndhamn (1993) found that students utilised different strategies to find the postage cost of a letter according to the context in which the task was presented. In the context of a school mathematics lesson most of the students engaged in some type of calculation, while in a social studies lesson most of the students found the solution by reading the table of the Swedish postal charges.

Several studies contrasting school mathematics contexts to non-institutional social contexts seem to conclude that the choice of different types of tools and the use of different strategies for solving problems derives from the meaning that a task assumes in a particular context; in connection with the definition of the tools and knowledge that is valued in this context.

(d) The use of 'old' tools in new contexts

Empirical studies have shown that, in many cases and under particular conditions, individuals approach unfamiliar situations occurring in one social context using tools that they have acquired in another context, and that situated strategies are constructed through progressive specialisation within a practice (Scribner, 1984). Although available research findings do not, in my view, allow for widely accepted conclusions, it does seem that the use of specific tools in particular contexts is selective, depending on the social value attached to tools and their uses in specific social practices. Viewed in this way, issues concerning the 'transfer' of mathematical knowledge and skills may be considered from an entirely different approach from the traditional cognitive one (e.g. Evans, 1999).

A few remarks on research methodology

Besides the issues concerning the various aspects of the mediatory means and processes in the formation and use of mathematical concepts, the methodology adopted or developed constitutes an essential component of any relevant research endeavour. As Vygotsky put it (1978, p. 65), 'the search for methods becomes one of the most important problems of the entire enterprise of understanding the uniquely human forms of psychological activity. In this case, the method is simultaneously prerequisite and product, the tool and the result of the study '.

A review of studies investigating the role of tools in the learning and teaching of mathematics points to the following characteristics as more or less common methodological constituents, although still requiring analysis in individual cases:

- the basic unit of analysis is goal-directed, since a meaningful activity is considered to encompass and co-ordinate in a unit several interrelated intra-psychological as well as inter-psychological functions.
- activities are investigated *in situ*, e.g. in the environment where they 'naturally' occur since they are socially and culturally determined.

- emphasis is put on the cultural, institutional and social settings in which these activities occur.
- the research time frame is long enough to encompass the objects of the activity and their possible transformations.
- a varied set of data-collection techniques is used, including interviews, observations andvideo as well as historical materials.

Additionally, the following two methodological aspects enrich any study investigating the impact of cultural tools and their use, on the learning and teaching of mathematics.

The first indicates that the study of this activity should not be static, The basic unit of research should be a properly selected goal-directed activity: a dynamic process in a state of continuous change and development. This implies a corresponding methodological stance aimed at including in the investigation any new, emerging, element of the activity, the actors and the tools used. For instance, in investigating how a mathematical tool is used in a problem-solving task, its use over time allowing for the usage to develop and possibly to differentiate has to be considered.

The second methodological approach concerns the investigation of the activity in relation to other activities directly interrelated, dependent on or influenced by it.

For example, in investigating learning in a mathematics classroom the activity of the teacher overlaps that of the students; the interaction of the two activities, one dependent on the other, must be analysed. The two activities are distinct, but directly interrelated. On the one hand the object of the course is viewed differently by teachers and students, and therefore their perspectives and expectations differ. We may assume that for students the aim is meeting examination requirements and their participation in the lesson is an instrument for doing so, whereas for teachers, the teaching of mathematics is the object, and the resources available for teaching the class are the activity's instruments.

Without any doubt, many other methodological issues, such as those concerning the social construction of research data, may be highlighted in any study of the role played by cultural tools in the learning and teaching of mathematics. However, a detailed review of such issues is far beyond the aim of this paper, which has sought only to introduce, although selectively, some essential aspects of this research..

References

Abreu, G. De (2000), Relationships between macro and micro socio-cultural contexts: implications for the study of interactions in the mathematics classroom, *Educational Studies in Mathematics* 41**,** pp. 1–29.

Bateson, G. (1972) *Steps to an ecology of mind: Collected essays in anthropology, psychiatry, evolution and epistemology* (San Francisco: Chandler Publishing Co).

Bishop, A. (1988) *Mathematical Enculturation: A Cultural Perspective on Mathematics Education* (Dordrecht: Kluwer).

Bruner, J. (1962) Introduction. In L. S. Vygotsky, *Thought and language* (Cambridge,Ma.: MIT Press).

Chassapis, D. (1999) The mediation of tools in the development of formal mathematical concepts: The compass and the circle as an example, *Educational Studies In Mathematics*, 37, pp. 275–293.

Cole, M. and Wertsch, J. W. (n.d) Beyond the individual-social antimony in discussions of Piaget and Vygotsky. Vygotsky Centennial Project: Massey University's Virtual Faculty, http://www.massey.ac.nz/~ALock/virtual/colevyg.htm (2003, July 2).

Evans, J. (1999) Building bridges: Reflections on the problem of transfer of learning in mathematics, *Educational Studies in Mathematics*, 39, pp. 23–44.

John-Steiner, V. and Mahn, H. (1996) Sociocultural approaches to learning and development: A Vygotskian framework, *Educational Psychologist*, 31, pp. 191–206.

Kozulin, A. (1990) *Vygotsky's psychology: A biography of ideas* (New York: Harvester Wheatsheaf).

Lancy, D.F. (1978) The indigenous mathematics project, *Papua New Guinea Journal of Education*, 14 (special issue).

Luria, A.R. (1929) The problem of the cultural development of the child, *Journal of Genetic Psychology*, 35, 493–506.

Matusov, E. (1998) When Solo activity is not privileged: Participation and internalization models of development, *Human Development*, 41, 326–349.

Miura, I.T. (1987) Mathematics achievement as a function of language, *Journal of Educational Psychology*, 79, pp. 79–82.

Nicholl, T. (1998, 6 May) Vygotsky. Vygotsky Centennial Project: Massey University's Virtual Faculty, http://www.massey.ac.nz/~ALock/virtual/trishvyg.htm (2003, July 2).

Nunes, T. (1992a) Cognitive invariants and cultural variation in mathematical concepts, *International Journal of Behavioral Development* 15(4), pp. 433–453.

Nunes, T. (1992b) Ethnomathematics and everyday cognition, in D.A. Grouws (ed.), *Handbook of Research on Mathematics Teaching and Learning* (New York: MacMillan), pp. 557–574.

Nunes, T. and Bryant, P. (1996) *Children doing mathematics* (Oxford: Blackwell).

Nunes, T., Light, P. and Mason, J. (1993) Tools for thought: the measurement of length and area', *Learning and Instruction* 3, pp. 39–54.

Nunes, T., Schliemann, A. and Carraher, D. (1993) *Street mathematics and school mathematics* (Cambridge: Cambridge University Press).

Rogoff, B. (1990) *Apprenticeship in Thinking: Cognitive Development in Social Contexts* (New York: Oxford University Press).

Rogoff, B. and Lave, J. (eds) (1984) *Everyday Cognition: Its Development in Social Context* (Cambridge, Ma.: Harvard University Press).

Säljö, R. (1996) Mental and physical artifacts in cognitive practices, in P. Reimann and H. Spada (eds), *Learning in Human and Machines* (Oxford: Pergamon), pp. 83–96.

Säljö, R. and Wyndhamn, J. (1993) Solving everyday problems in the formal setting. An empirical study of the school as context for thought, in S. Chaiklin and J. Lave (eds), *Understanding Practice: Perspectives on Activity and Context* (Cambridge, Cambridge University Press), pp. 327–342.

Saxe, G.B. and Posner, J. (1983) The development of numerical cognition: cross-cultural perspectives, in H.P. Ginsburg (ed.), *The Development of Mathematical Thinking* (London, Academic Press), pp. 291–317.

Scribner, S. (1984) Cognitive studies of work, *The Quarterly Newsletter of the Laboratory of Human Cognition* Vol. 6 (San Diego: University of California), pp. 1–50.

Vygotsky, L.S. (1978) *Mind in Society. The Development of Higher Psychological Processes* (Cambridge,Ma.: Harvard University Press).

Vygotsky, L. S. (1981) The instrumental method in psychology (originally written in 1930) in J. V. Wertsch (ed. and trans.), *The concept of activity in soviet psychology* (Armonk, N.Y.: M. E. Sharpe), pp.134–143.

Vygotsky, L. S. (1987) Thinking and speech, in R.W. Rieber and A.S. Carton (eds), *The Collected Works of L. S. Vygotsky, Volume I: Problems of General Psychology* (New York: Plenum), pp. 38–285.

Wertsch, J. V. (1985) *Vygotsky and the social formation of mind* (Cambridge, Ma.: Harvard University Press).

Wertsch, J. V. (1991) *Voices of the mind: A sociocultural approach to mediated action* (Cambridge, Ma.: Harvard University Press).

Wertsch, J. V., Del Rio, P. and Alvarez, A. (1995) Sociocultural studies: history, action and mediation, in Wertsch, J. V., Del Rio, P. and Alvarez, A (eds), *Sociocultural Studies of Mind* (New York: Cambridge University Press), pp. 1–34.

Contact

Assoc. Prof. Dr. Dimitris Chassapis
Aristotle University of Thessaloniki, School of Education
Email: dchassap@eled.auth.gr

Παιδαγωγικό Τμήμα Δημοτικής Εκπαίδευσης
Αριστοτέλειο Πανεπιστήμιο Θεσσαλονίκης

Developing a Mathematical Telos in Communities of Learners Using Dynamic Geometry

John Gardiner (1943–2004), *Sheffield Hallam University*

Abstract

A study with 11–14 year old pupils in the UK considered mathematical meaning-making using dynamic geometry on the TI92 hand-held computer in four schools. Classroom observations from audio-recording and field notes are reported and related to short and long term factors identified by previous authors. Short-term factors, many concerned particularly with the use of dynamic geometry, are the importance of explanation in social proof, transparency of resources and technology, sense/meaning making, spontaneous/scientific concepts and the tension between proof and construction when using dynamic geometry. These immediate factors operate within a classroom environment which depends on a teaching and learning atmosphere generated over time by the teacher as guardian of the classroom community. In this area socio-mathematical norms, argumentation, and the idea of a whole class ZPD have been identified as relevant factors. Winbourne and Watson (1998) advance six aspects of the development of local communities of mathematical practice, and speak of an alignment of mathematical meaning-making which they call a telos. The long and short-term factors identified in this study can provide ways in which the reflective classroom teacher can analyse, and promote, local communities of practice, both in the use of dynamic geometry and other mathematical activity.

Outline of the project

This research was carried out in four state secondary schools in the north of England, with pupils in the 11–14 age group, between 1997 and 2000. In most cases the work was done with groups of about 15–25 children, but in one case smaller groups of four to six children were used in an attempt to examine detailed learning processes. Overall the children involved were taken from across the attainment range, with some children from lower attainment sets. The project used Cabri II as available on the TI92. This is a small handheld computer with a 3cm x 6cm screen. An overhead projector could be used to show and talk about screens generated. Classroom dialogue from pupils and teachers was recorded on audio-tape and transcribed. Field notes were used to record the screens generated. Screens and dialogues with and between pupils were also recorded using scribble pads of sheets of paper covering desks. The diagrams which children were asked to work with in classrooms were designed to be easily produced on this machine and the limited nature of the hardware (at least compared to Cabri on a PC) did give the advantage that simple screens were used which led quickly to mathematical meaning-making. Classroom observations were related to factors identified by previous authors. Short-term factors included the distinction between sense and meaning-making (after Schultz, 1994, and treated more fully in this paper). In the course of the project other short-term factors have been identified. Mathematical (in this case geometrical) proof can be seen as a social exercise, as being concerned with convincing others. Ideas of conviction and proof based on the work of deVilliers (1990, 1991) and the use of proof and explanation as a social exercise have been influential. Vygotskian ideas of spontaneous and scientific concepts (Vygotsky, 1978) can be applied to the immediate meaning-making of individuals within the practice of a learning community, and the gradual emergence of geometrical ideas and their acceptance by the community of learners working with dynamic geometry has been observed during the project. Longer-term elements were efforts to promote the development of a classroom climate where a whole-class ZPD is generated, following ideas discussed by Hedegaard (1990). Cobb and Yackel (1996), Yackel (1998) and Yackel and Cobb (1996), present ideas on argumentation and socio-mathematical norms. They argue the importance of the social norms developed as a routine in classrooms, of turn-taking, listening to others, not interrupting etc, and propose the promotion in classrooms of socio-mathematical norms. Within a classroom community where dynamic geometry is being used these might include the promotion of, and respect for, generalisations and the development of what the community can accept as common knowledge. This approach parallels that of Voigt (1995) who looks at such development in terms of the moving forward of the 'taken as shared', the expansion of the body of knowledge which the community as a whole accepts. A fuller treatment of these and other factors is available elsewhere (Gardiner and Hudson, 1998, Gardiner, Hudson and Povey, 1999, Gardiner, Hudson and Povey, 2000). In analysing classroom incidents this paper concentrates on the area of sense/meaning-making (Schultz 1994).

Winbourne and Watson (1998) suggest elements which are present when there is an alignment of mathematical meaning-making, a *telos*. It is proposed that the factors identified, short and long term, can lead to the development of a local community of practice in mathematics classrooms, which can be examined according to Winbourne and Watson's criteria. The aim is to provide an analysis of ways in which various factors might contribute to an alignment of the activity of the classroom, so that all those present can see themselves as participants in the learning community.

Background to the Project

There was an attempt from the start to make the work relevant to the classroom use of dynamic geometry.

Berger (1998) makes some relevant general points in a discussion paper on the use of graphical calculators, which can equally be applied to the use of other handheld devices such as the TI 92 in whole class circumstances. After referring to the preference for the availability of hand-held machines in the classroom (as opposed to access to software in a computer suite) she continues

> Unfortunately, despite the potential and actual importance of this tool, there is not much literature dedicated to explaining or understanding how the graphic calculator,

> specifically, functions in relation to the learner. In fact, much of the literature relating to the graphic calculator is anecdotal or describes evaluative studies which fail to distinguish adequately the role of the tool from that of the instructional process … I wish to suggest that a Vygotskian approach to learning, with its emphasis on mediated activity within a particular socio-historical context, is appropriate to address the relationship between the mathematical learner and the different sign systems (multiple representations) afforded by the graphic calculator. (Berger, 1998 p. 13)

and,

> For internalisation to take place, it is not sufficient that a student is merely exposed to a new technology; rather he/she needs to engage thoughtfully with the technology … In order to interact in such a mindful way, he/she has to use the technology actively and consciously in a socially or educationally significant way. (Berger, 1998 p. 19)

She argues for dedicated research related specifically to the use of graphic calculators.

> … the learning experience is sufficiently different from (that) in a computer environment that it warrants its own dedicated research and interpretation. (Berger, 1998 p. 14)

There is, then, a need to address issues of socio-cultural aspects of learning in relation to the use of handheld calculators such as the TI92. It is hoped that the present study goes some way to answering this need, by looking at the socio-dynamics behind the use of hand-held calculators in general and in the area of dynamic geometry in particular

Technological influences

The hardware used, the TI92 hand-held computer had some influence on the course of the project. It offers most of the functions and facilities of the package available for larger machines. However, complicated screens are relatively slow to deal with. This disadvantage is balanced by the fact that the machines can be kept on the corner of a desk in the classroom, and do not dominate the physical or mental landscape of the pupils. Indeed there was a conscious attempt to make screens simple. If diagrams are easily and quickly drawn by students (and easily redrawn if they make a mistake), they can use them as an element in their classroom discussions. The hand-held nature of the TI92 means that ideas can be followed privately if pupils wish. Simple diagrams are more likely to provide a transparent window to mathematical meaning-making. (Lave and Wenger, 1991)

One machine had a port to connect to a tablet which sits on an overhead projector. This meant that teacher and pupils could demonstrate ideas to the whole class (all later machines have this feature, allowing all members of the class to present findings more easily).

Methodologies for the classroom

This study has been into the way technology, specifically dynamic geometry on the TI 92, can be used with 11–14 year old pupils. There has been emphasis all along on investigating classroom use of this technology. Hudson (2003) has pointed out the need to address the complexity of classroom processes and Watson (1998) emphasises the wide variety of different processes which are taking place in the classroom. Activity within the classroom takes place in many different areas and in ways which are variously important to the people in it. The intended learning programme will be only a subset of these activities. Cultural and societal aspects of the school and society as a whole surround the work which takes place in the classroom and may help or hinder meaning-making in the particular subject area. Hudson (2003) draws attention to ideas of Didaktik, in particular that of critical-constructive Didaktik. He suggests that this tradition recognises that the complex processes at work when teaching and learning take place in the classroom are best dealt with by a range of research approaches, which also take into account the wider societal context within which meaning-making is happening. This eclectic approach is intended to 'support pedagogical practice' and 'need(s) to be based on a combination of methods and methodologies'.

> The tradition of critical-constructive Didaktik offers a distinctive approach to educational research, which

> addresses the complexity of the processes of teaching and learning in the methodologies and methods adopted, whilst maintaining attention to considerations of meaning making within a wider societal context. (Hudson, 2003)

Hudson, drawing on Klafki (1998), identifies three method groups/ methodologies.

Historical-hermeneutical methods
intended to use scientific method to analyse and deconstruct meaningful phenomena and to relate the didactical process, seen as involving all social aspects, to the wider picture of society and culture.

Empirical methods
which are seen to be necessary when contemporary issues (for instance, in the context of this study, whole class teaching), decoded didactically by a historico-hermeneutic approach, are studied in the classroom context.

Methods of social analysis and ideology critique
no pedagogical or didactic province is seen to be outside society. Society directs the direction of educational developments by means of curricula and syllabi, and by assessment procedures. Broader aspects of the school society play their part, such as setting and other organisational arrangements, the attitudes of teachers and other students. The development of social analysis methods must be tempered by a parallel critique of the ideologies behind the type of education on offer.

The background to this approach is the acceptance that method groups/ methodologies will be found useful in particular areas of a study, but a point will be reached where the preconditions of a particular approach will mean that its usefulness declines and that a different approach will be needed to make further advances.

This eclectic approach has led to reference to a number of methodologies in this project.

The view of activity as both a method of research and the object of research, seeing Vygotsky as a methodologist/psychologist, points to a continual dialectic between method and substance, the 'tool-and-result' methodology referred to by Newman and Holzman (1993).

Eisenhart (1988) has advocated absorption into the research methods of mathematics education of the approaches used in educational anthropology. Bassey (1995) has presented the idea of the study of singularities. A further influence has been the views put forward by Brown and Dowling (1998) on the importance to classroom practitioners of both using and doing research.
The methodology adopted was to use an on-going dialogue with the literature and through this to develop a phased approach, with different areas of the literature used as the main, but not the only, framework in different phases.

This methodology is echoed in the idea of 'tool-and-result', introduced by Vygotsky:
'…the method is simultaneously prerequisite and product, the tool-and-result of the study' (Vygotsky 1978, p. 65) and discussed by Newman and Holzman (1993), who point to a distinction between tools such as hammers and screwdrivers (tool for result), and dies and jigs (tool-and-result). Hammers and screwdrivers are bought and used as needed, dies and jigs are tools designed and refined by the worker, the toolmaker. Vygotskian methodology is a 'tool-and-result'. Like the jig, it is bound up in its result. Newman and Holzman say

> The toolmaker's tool is different in a most important way. While purposeful, it is not categorically distinguishable from the result achieved by its use. Explicitly created for the purpose of helping to make a specific product, it has no reified prefabricated social identity independent of that activity. Indeed, empirically speaking, such tools are typically no more recognizable as tools than the product (often a quasi-tool or small part of a larger product) itself is recognizable as product. They are inseparable. It is the productive activity which defines both - the tool and the product (the result).

(Newman and Holzman 1993 p 38)

There were cycles of research as outlined in Table 1, the first using as a starting point and means of location, previous classroom experience and a background in the literature. Subsequent cycles each drew on the previous phase, but were also influenced by further reading of the literature, dialogue with colleagues and researcher introspection. There was also a formal dialogue with the research community, using refereed publications. In the analysis and interpretation of the data gathered in a particular phase there was generally one area of the background literature which was the principal lens used. However ideas from the literature were carried forward to other areas and data from previous phases was often re-examined using the lens of later work.

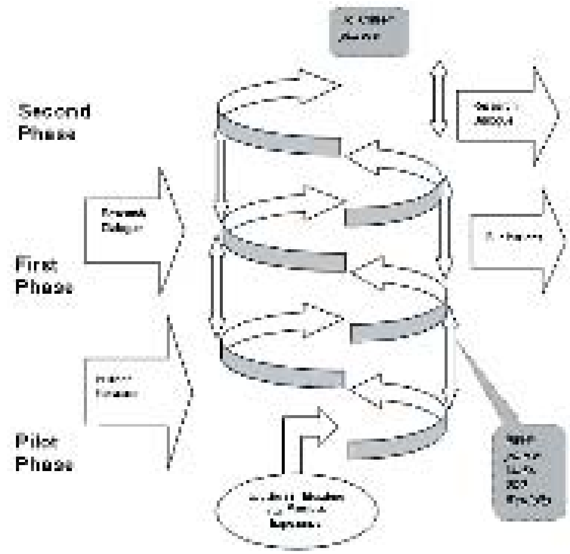


Figure 1: The pattern of research progress

Figure 1 illustrates the pattern of progress through the project. After locating the project in current literature and in my previous experience in the classroom, the main path of the research followed successive cycles as shown. Subsidiary features of the progress of the work such as continuing inputs from the literature and from dialogues with research colleagues are shown. Also indicated are inter-phase links and revisits, together with publications produced during the progress of the project.

Phase	Extent of Involvement	Content/ Direction
Pilot	Two classes of 25–30 pupils, two 80 minute sessions each	Preliminary classroom work
Phase 1	One class of 28 pupils, four sessions 80 minutes	Classroom exercises in geometry
Phase 2	Higher attaining pupils were invited to attend in lunchtime sessions, of about 40 minutes. A total of eight pupils attended for one lunchtime a week, over a period of seven weeks. There was intermittent contact with this school over the course of the project.	Meaning-making in smaller groups
Phase 3	Two classes of 25–30 pupils were involved. One was of high attainment, one a lower set. Each was seen four times, for 75 minutes each.	Classroom work, classroom dynamics
Phase 4	The sessions took place at a lunchtime mathematics club. Attendance was variable, usually about 18 and included children aged between 11 and 15, of all levels of attainment	Sense and meaning-making

Table 1: Cycles of research; progress of the project

Theoretical Background

Previous work (Gardiner and Hudson, 1998, Gardiner, Hudson and Povey, 1999, Gardiner, Hudson and Povey, 2000) has identified as relevant the areas of explanation and proof, the importance of construction and dragging in conviction within a dynamic geometry environment, and the use of public argument fostered by the teacher. This paper concentrates on the area of sense/consensus in pupils' meaning-making.

Schultz (1994) draws a distinction between sense and meaning. He sees sense as unique to an individual, with meaning implying a negotiated consensus. If a pupil interprets consensual meaning, that interpretation can be true or false of itself. The development of sense depends only on the individual who makes it. In the way these terms are used here, and applying them to dynamic geometry environments, we want the spontaneous sense perceptions of pupils to be transformed into agreed mathematical consensus, probably expressed in more or less formal mathematical language. Mason (1991) is dealing with 'sense' when he speaks of 'a sense of mustness' and acknowledges that pupils may see no need of justification except to assert that they can see that what they are saying is true.

Children have to be given the opportunity to gain a sense of geometrical truth from their exposure to the resource and that sense has then to be made into agreed meaning. This meaning-making is a reason for attempts at the formal justification of the geometrical 'sense' which has been made. In the same work, Mason (1991) suggests that teachers can use pupils' discovery of geometrical facts, and 'their gradual appreciation of the fact that there are facts' to promote learning.

> For me, the real importance of geometry is as a domain in which the fact that there are necessary and inescapable facts can be experienced, developed, manipulated to produce new facts, and, for those who wish, organised into a deductive scheme.
> (Mason, 1991 p. 76)

In more recent work Mason (2004) has debated the importance of using electronic screens to promote creativity in mathematics 'to augment the most powerful screen of all, our mental imagery'. He discusses the way in which dynamic geometry allows pupils to see that any diagram holds generality and to examine invariants in the light of what happens on dragging.

The stimulus of sense-making might be any resource in the classroom; the language which is used there, the technology or other materials.

Meira (1995) discusses the importance of representations in sense-making activity. He sees it as fundamental that representations or diagrams are seen as ‘cultural artifacts, the meanings of which are negotiated and recreated by learners in activity’. He argues that an activity consists of the actions carried out by agents in a specific social setting, involving prior conceptions, interactions with others and the use or production of conventions and artefacts. Sense-making activity will be bound up in the development of representations.

Making sense and making consensual meaning are bound up in the dialectical activity of the practice of the classroom. Consensual meaning can be seen as agreed and objective. It will eventually be expressed socially in the formal language of geometry, ‘perpendicular’, ‘mid point’, ‘congruent’ and so on. It will probably lead to further sense-making. It might be consolidated by formal argument or proof.

Sense-making and consensus-making, then, are bound together in an interactive process, the process which is known more generally as ‘meaning-making’. Social intervention by the teacher and others in this dialectic will be at, or towards, the consensus end of this dialectic. Some teachers will be better than others at moving to influence pupils’, or a pupil’s, sense-making. Some teachers will operate more towards the consensus extreme, using formal argumentation. Referent, consensual meaning is the outward sign of classroom mathematical activity, but it is bound up with sense-making, and sense-making is a delicate process, easily disrupted and much more difficult to promote. Suitably transparent resources, including dynamic geometry, can help the teacher to promote sense-making and its progress towards consensual meaning. In the classroom episodes described here, individual pupils’ sense-making is identified and the way this progresses to consensual meaning, accepted by the practice, is noted.

Class Room Episodes

Episode 1: Sense and meaning

One pattern followed was for the class to generate and discuss a simple dynamic image, and to record the result in exercise books as a diagram after the dynamic image had been appreciated. Concentration on the dynamic image first allows pupils to see diagrams as generalisations of geometric properties, rather than as fixed representations of a particular situation. The hand-held nature of the TI92 is particularly suitable for pair discussion and, indeed, as noted previously, for consigning to a corner of the desk when work on paper is preferred.

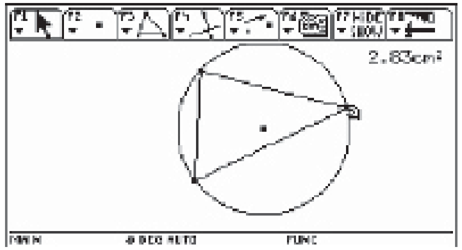


Figure 2: Simple dynamic images

The class was a lower attainment year 9 (13–14 years old) group from school C. I was concerned to present material which was appropriately transparent to these pupils who had not used the TI92 before. The screen used could be generated by these students, helped by worksheet description and overhead projector demonstration, by only a few key-presses. However it led rapidly to an opportunity for spontaneous meaning-making. The class was asked to draw a circle and a triangle with its vertices on the circle, then to measure the area of the triangle. The pupils were able to produce the diagram in Figure 2, with the area of the triangle recorded on the screen. They were then asked to investigate the effect of dragging one of the vertices, and to look for the maximum area of their triangle. This led to opportunities for class meaning-making about, among others, perpendicular bisectors, isosceles triangles and symmetry. In jointly exploring the same screen in this way, but each on their own machine, a *telos*, a bringing together of mathematical meaning-making, is created. Students are aligned in the domain provided by the technology. They are operating in a local community of practice.

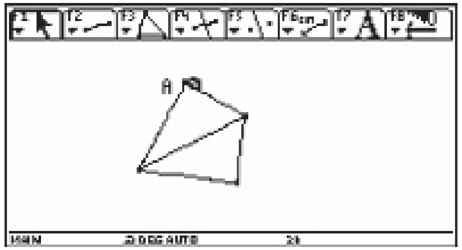
The classroom interactions between teacher/researcher (JG) and the pupils, Anne, Belle and Charlotte were audio recorded and transcribed. The following dialogue ensued.

1	JG	What area have you got?	
2	Class	General response	There was no restriction on the original diagram, a wide range of areas was possible.
3	JG	Why do we all get different answers?	
4	A	Because we all used different circles	Use of 'we' suggests the possibility of a local community of practice.
5	B	And different points	
6	JG	Look at mine while I move the point. Tell me when it will be greatest. What can we all say about our diagrams?	'What can we all say?' used to promote the local community of practice and the generation of a common direction to learning, *telos*.
7	B	It's across from the centre	Later discussion showed that Belle could demonstrate to the class the idea of the co-linearity of the mid point of one side of the triangle, the centre of the circle and the other vertex. Charlotte pointed out that the triangle was isosceles

At line 7 Belle expresses the sense she has made of the mathematics, but as noted, later discussion enabled her to be instrumental in

developing a consensus of more formally expressed meaning which was recognised by the class.

Episode 2



This diagram is made by reflecting a triangle in one of its sides. As A moves, what quadrilaterals can we get? How many kites? When do we get a rhombus? How many rhombuses? How many squares? Any other shapes?

Figure 3: Reflecting a triangle

This recording of involvement in the activity outlined in Figure 3 was made of the response at one table, of two pupils Barry and Craig, aged 12–13 from a lower attainment mathematics set. The teacher-researcher is JG. I was directing questions to the whole class of about 20. They had used the TI92 for about 40 minutes on a previous occasion.

1	JG	I asked you to move this point about. What shape do you all get?	*Developing a community of practice*
2	General response	A kite	
3	JG	I asked you how you could get a rhombus. Look at the screen and tell me when it is a rhombus A rhombus has four sides equal, not just the two pairs	*Using the OHP tablet to display the image to the class*
4	Class	Stop	
5	JG	How should I move it so it is always a rhombus?	
6	B	Turn it around or bring it in to the middle – not right in the middle though	*Sense of perpendicular bisector generated, transparent language*
7	JG	What line will it be?	
8	B	A straight line	
9	C	Symmetry	*Sense of perpendicular bisector*
10	B	Down that line, that line like that	*Waving hand in air*
11	JG	What's that angle called?	*Prompt to language of consensual meaning*
12	B	What's it called? What's it called? A right angle	*Pupil making consensus*
13	JG	So what's another name for that?	
14	B	Ninety	
15	JG	Yes ninety degrees	

16	JG	There's another word for that	*Wanting the response ‚Perpendicular bisector'*
17	B	Put a line down the middle	*Returning to transparent language,* *but making sense*

Barry and Craig have made sense of the geometrical properties in the problem. Their sense-making language is transparent (line 6 and lines 9, 10) to others in the group (it was accepted and unquestioned). Lines 11–17 use more formal language. Meaning, in the sense of socially agreed consensus, is being made.

Episode 3: Intervention by the teacher when sense is made

In the next example of classroom observation, the 'Hide and Show' function was demonstrated to the class (Figure 4) and they were asked to choose diagrams from a sheet of examples and to construct them. To do this they needed to decide what construction lines had been hidden. Figure 5 shows the work sheet and there follows a transcription of conversation between two pupils and their teacher. The pupils, Andrew and Ben, were aged 12–13, again from a low attainment group. They were working in a group with their teacher, TT and in this case decided to try the first example.

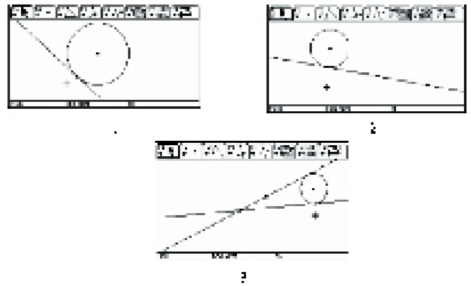


Figure 4: Hide and Show function

Figure 5: Work sheet examples

1. The line moves round the circle, always touching it.
2. The 'ball' moves down the hill.
3. The circle always touches the two lines, no matter where they move.

18	A	I want to know how to do that one, me	*Line moves round circle, example 1*
19	TT	And how are you going to do that?	
20	A	I know where the line is what he's hiding	
21	A	It's on that big circle there	
22	TT	Go on then	
23	A	Ninety degrees	*Sense of importance of perpendicular radius*
24	B	No he's hiding the line, the line what goes across from its end	*Sense making is happening*
25	TT	The line moves around the circle	
26	B	So it's at a 45 degree, no, 90 degree angle to the line, the line in the circle	*Moving to consensual meaning, with more formal language*
27	TT	Right, go for it, try to draw it. OK escape, clear all your pictures. What do you need first?	*TT has realised that sense has been made.*
28	B	Circle sir, I've done it	*They went on to draw the diagram*

In this example we see how language is used as a transparent resource in the activity of these students. In the extract their teacher, TT, was aware enough to see that sufficient sense had been made to allow progress. The activity of the pupils was then sufficiently engaged by their use of the technology as a transparent resource, so that progress could be made. They were creating cultural artefacts, both at the level of the screen and at a more fundamental level, which were used in a social setting to make first sense, then meaning.

Jones (1997) has pointed out the importance of the role of intervention by the teacher in the use of technology such as this, and I suggest that the analysis above provides a possible structure to the processes behind such intervention. The teacher needs to be sympathetic to the process of sense-making, which is sensitive and fragile. The intervention of the teacher in the episode described above is just enough to allow the pupils to move from sense towards some agreed meaning, but still allows them to use the language which they find transparent enough to allow them to make progress. In classroom incidents of this kind, teachers are aware of the need to place their interventions judiciously in the dialectic between sense-making and consensual meaning-making, using just enough formalising language to move forward pupils' meaning-making. They are using themselves and their language as well as the technology as transparent resources, being careful that the window which opens onto meaning-making is able to direct, but not obscure, the gaze of pupils.

Discussion

Winbourne and Watson (1998) have advanced factors which they propose may be present when local communities of mathematical practice are active. The long and short-term factors identified in this project as acting when dynamic geometry is used in classrooms can be related to these factors.

Criteria for local communities of mathematical practice

1. Pupils see themselves as functioning mathematically within the lesson

There are two parallel aspects to this statement which make this a powerful way of looking at local communities of practice; the idea of pupils 'seeing themselves' and analysis of what might constitute 'functioning mathematically'. Pupils' own examination of the learning community in which they find themselves, and their ability to do this, is a valuable part of the practice of the community. Indeed it can be seen as a responsibility of both teachers and learners to critically examine the learning which is going on in the community. The facility of pupils to do this is fostered by the teacher as guardian

of the local community of practice, while at the same time she critically examines her own practice.
An awareness of what constitutes 'functioning mathematically' is also fostered in their pupils by teachers. Geometrical proofs, rigorous constructions, arguing and reasoning are all made available within the medium of dynamic geometry. It is part of the thinking behind the material that mathematics is readily available, that the technology is used to help pupils to function mathematically, rather than as an end in itself. To this end, screens were kept simple, so that the class could draw a simple diagram and then use it to move forward mathematically.

Pupils who critically examine their own and others learning, and see themselves as functioning mathematically then, is the ideal here. The teacher acts as guardian of the practice and develops over time a classroom climate which promotes these values.

2. Within the lesson there is public recognition of competence

Pupils need the opportunity to demonstrate their mathematical competence, to demonstrate their progress to fuller participation in the mathematical community of practice. Pupils were able to demonstrate their ideas to the whole class using the overhead projector version of the TI 92. Pupils could collectively recognise their progress to mathematical meaning-making using the technology, demonstrating to themselves and others their mastery of new ideas and technology. Printouts of screens were made available to teachers for display.

Pupils also need the opportunity to participate in a wider classroom community and to be recognised by their peers and the teacher as socially competent in the community.

3. Learners see themselves as working together towards the achievement of a common understanding

The idea of a *telos*, of alignment in meaning-making of individual students for a period of time, is advanced by Lave (1996) and taken up by Winbourne and Watson. Hedegaard's (1990) idea of a whole class zone of proximal development is similar, and Cobb and Yackel (1994) describe the way classes can establish shared knowledge by argumentation. They describe the way in which argumentation can

be used to move forward the accepted knowledge of the class, the 'taken as shared.' In the examples of classroom activity presented above, the way in which pupils are drawn into the practice and the promotion of a shared practice by the use of class discussion to take forward the 'taken as shared' promote this approach. This aspect is reinforced by the use of the first person plural in referring to the community.

4. There are shared ways of behaving, language, habits, values and tool-use

Cobb and Yackel refer to the development of norms, social and socio-mathematical, in the classroom. These social norms are developed within and outside the classroom and school, and will be developed over time and often outside the influence of formal education. Watson (1998) has emphasised the need to consider, alongside any mathematical learning, the social learning which is happening in the classroom, about the positioning of individuals in the practice, their relationships with the teacher and their peers. The values of the school and of the society in which it operates will be central in setting social norms. Socio-mathematical norms will also be set over time, but perhaps be more dependent on the work of the mathematics teachers which pupils come into contact with. Within the material used with classes public discussion and argumentation, supported by the use of dragging in dynamic geometry is seen as furthering these aspects of the community of practice. The use of a common tool in the technology also brings individuals into the practice.

5. The shape of the lesson is dependent upon the active participation of the students

Participation in the community of practice is central to the ideas of Lave and Wenger (1991) and Lave (1996). In the communities they investigated they noted the way such participation led to individuals accepting and being accepted to their place in the learning community. The material used in classrooms in this study fostered participation. The work used in classrooms in the project was designed to be easily available to all pupils after a few key-presses and all students had their own machine so that they could generate individual images. Pupils then participated in drawing collective conclusions from their individual work in group and class discussions.

6. Learners and teachers see themselves as engaged in the same activity

This last criterion proposed by Winbourne and Watson might seem problematic if teachers see themselves as involved in the transfer of objective knowledge to their pupils. However, if we go back to an apprenticeship model of learning, with pupils aware of their role as becoming mathematicians, it is not difficult to see the practice of the classroom community, pupils and teacher, as that of critically examining the progress of learning and collectively moving forward the 'taken as shared'. If this is seen as an objective of the practice, pupils, individually and in groups, can be encouraged to see their contributions to argumentation and discussion as a valid part of the peripheral participation which is contributing to the learning which is taking place.

Model of Influences

In drawing together the threads of this discussion we can look at the way the various analyses offered by the literature can be brought together with the technology to provide a model which may be useful to teacher-practitioners and others in the classroom. This was a concern of this study from the start. In her work advocating an ethnographic approach to mathematics education research Eisenhart (1988) makes a plea that research should be relevant to the classroom. She asks education researchers to have something to say about 'what to do on Monday morning'. With this in mind it is possible to advocate a model of the influences identified here and how they interact with the technology.

In the short term we can look at ways in which the teacher, the technology and the subject content interact with the class pupils. In this area we look at the ideas of spontaneous and scientific concepts, sense and meaning-making, and, in the specific area of geometry, intuition, conviction and proof, explanation and social proof, and construction and proof. These are concerned with meaning-making at the level of the group and the individual, are involved with the specific subject content being presented. They operate at a specific moment in time but are enabled by the over-arching learning climate generated in the classroom society. Local communities of practice are

used here and elsewhere to look at the way in which the teacher and pupils develop and respond to more transitory learning alignments in the practice.

However the local community of practice has a longer-term development. It is constituted by the culture of school and society and affects the overall classroom climate. Both these interpretations are important when considering how various elements of theory can be used to analyse the learning community of the classroom.

The involvement of the teacher in both the strategic and tactical areas, long and short term, defined above is seen as paramount. Technology in the classroom may change the role of the teacher, but does not diminish it. However the consideration of technology in relation to the interrelationships between teacher, student and subject content increases the level of complexity considerably. Building on Hudson (2002) and also Hudson (this volume), Figure 6 is intended to illustrate the interrelation of these findings.

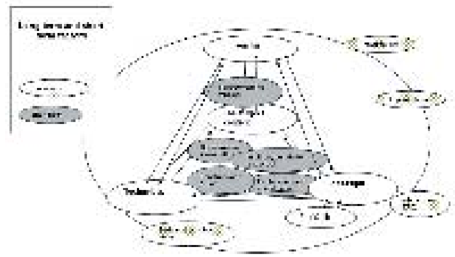


Figure 6: Factors in the development of a local community of practice

This represents a tetrahedron, with vertices composed of teacher, subject content, technology and class pupils. These four elements are seen to act on each other within the meaning-making activity of the classroom community. This activity is seen to involve the intentional engagement of the members of the community, teacher and pupils, using social interaction to further meaning-making. The criteria developed for analysing local communities of mathematical practice by Winbourne and Watson (1998) are available to provide further insight into the complex interplay of general factors which operate in the mathematics classrooms and in particular the way in which dynamic geometry can be incorporated into mathematical meaning-making. As discussed above, Winbourne and Watson suggest six factors which might affect the establishment of a local community of mathematical practice:

- pupils see themselves as functioning mathematically within the lesson;
- within the lesson there is public recognition of competence;
- learners see themselves as working together towards the achievement of a common understanding;
- there are shared ways of behaving, language, habits, values and tool-use;
- the shape of the lesson is dependent upon the active participation of the students;
- Learners and teachers see themselves as engaged in the same activity. (Winbourne and Watson, 1998, p. 183)

These ideas have been taken forward here and combined with other analysis methods. They are advanced as a selection of possible tools for the analysis of the complex interaction which is taking place in mathematics classrooms where dynamic geometry technology is being used, with the hope that involved and thoughtful practitioners will find applications outside this immediate subject content.

The complexity of the way in which the local community of practice is affected in long and short-term ways is indicated by the nature of this diagram, which can only be said to address some of the issues. School classrooms are complicated social units and mathematics learning may be only one of many things happening there. Watson reminds us that we can look at classrooms as:

social communities in which all sorts of things are being learnt (how to behave in a way that is valued by the teacher, how to be accepted by one's peers, what writing implements are fashionable….) which are not the focus of the teaching. To describe what goes on in a

classroom fully one must consider all the actions, thoughts, feelings and environmental aspects within it. (Watson 1998 p. 2)

In further applying the ideas used here, seeking to look in more detail at the way they may be applied to the classroom, it is as well to remember that we are seeking to focus on the detailed activity of a community, and a wider view would look at the influence of many socio-cultural factors. With this proviso we can look in more detail at the areas where some of the methods of analysis used may be more relevant than others.

One face of the tetrahedron indicated by Figure 6 is that composed of the relationships between Teacher, Subject Content and Class/Pupils. Adler (private communication) has remarked on the significance of teaching and learning styles, even in disadvantaged schools in South Africa, and, dealing with UK schools, the Hay McBer report (2000) identified a variable which it defined as 'classroom climate', which led to high expectations and an atmosphere in which they could be met. The ways in which argumentation, socio-mathematical norms and the development over time of a whole class zone of proximal development can lead to a classroom where individuals feel secure, are identified here. The development of such a climate, where in the words of Winbourne and Watson (1998) pupils see themselves as 'becoming mathematicians', is a long term process, but the sources referred to offer ways of developing such a classroom community. In the immediate meaning-making within the community of practice, Winbourne and Watson also offer ideas which will be more closely related to the subject content in question. Here they suggest a principle of generating a *telos*, which they describe as a momentary alignment of the meaning-making of the class. Winbourne and Watson's principles can be combined with other authors' (see introduction) to offer teachers ideas on how dynamic geometry can be used in the classroom.

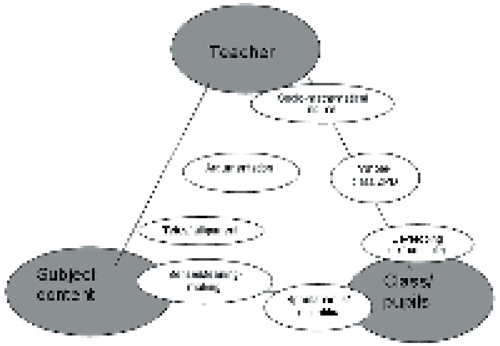


Figure 7: The relationships between teacher, subject content and class/pupils

Following these ideas we can place the elements indicated in Figure 7 as shown, with some very much in the area of interaction between teacher and class pupils, independent of subject content, with others involved in the meaning-making of class and pupils about the immediate subject content, but with others operating in the interplay between all three. These placements are arbitrary, and cannot be considered as the only elements affecting the classroom and what is going on in it at any particular time. The importance of these and other elements will depend on emphases and strategies emanating from the teachers, some within their control and many other factors over which they have little or no influence. The six criteria advanced by Winbourne and Watson (1998) are particularly directed at the long and short-tem development of a local community of mathematical practice, and it is possible to combine them into the picture already developed. Winbourne and Watson's criteria can be applied to the diagram above (Figure 7) indicating interaction between teacher, class/pupils and subject content. Figure 7 might be amended thus, with many of the factors observed more easily in the area of interaction between class/pupils and teacher.

Developing a Mathematical Telos in Communities of Learners Using Dynamic Geometry

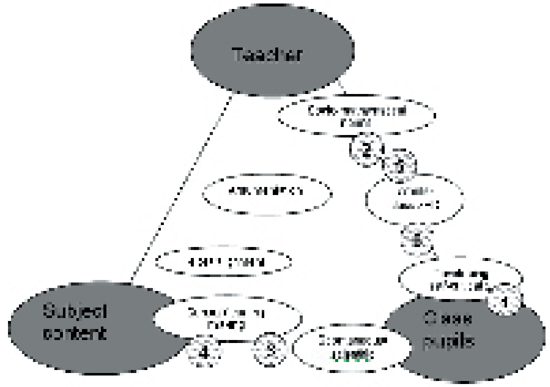


Figure 8: Incorporating Winbourne and Watson's criteria

In this paper the 'subject content' is geometry as available in a dynamic geometry environment, and factors specific to geometry which operate in this area have been identified. In particular the importance of construction as a precursor and possible replacement of proof and ideas of explanation as the driving influence in proof have been identified (Gardiner, Hudson and Povey, 2000). Other factors, referred to in Figures 7 and 8 and in the introduction to this paper are dealt with more fully elsewhere (Gardiner and Hudson, 1998, Gardiner, Hudson and Povey, 1999).

These might be incorporated as shown, but again it must be emphasised that placement of these factors is arbitrary, and the intention of this project was to identify some of the influences on the local community of practice. Their identification and influence in specific circumstances within particular classrooms when using dynamic geometry, and, perhaps more important, the interplay between them is seen as highly significant. However it is not intended to present this as anything more than an insight into a dynamic situation which, while relevant to general questions about the development of a local community of practice, may have little to say about the relative importance of the various influences in other classrooms, or indeed in the same classroom at a different time.

Whilst many of the factors identified as important are independent of the technology used, some specific points involving the use of such resources, and especially a dynamic geometry environment have been identified. They too can be discussed alongside the criteria developed by Winbourne and Watson (Figure 9).

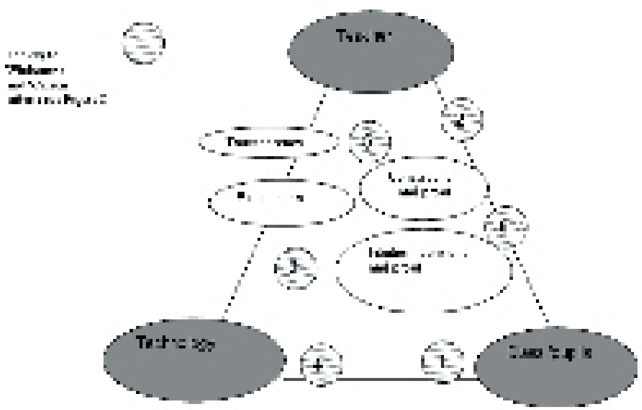


Figure 9: Relationships between teacher, technology and class/pupils

In the interaction between class/pupils and technology pupils will see themselves as acting mathematically (4) and will be engaging in shared tool-use (1). Explanation will be seen as a movement to a common understanding (3). Teachers will ensure that the use of the technology is transparent. Participation (5) using the technology available to the pupils and discussed in the class is enhanced, and public recognition of competence (2) and the acceptance of the involvement of teacher and pupils (6) is part of the learning climate already established. Again it is relevant to emphasise that these are perceived importances in a particular teaching context, and are

suggestions only of a snapshot of the relative importance of these factors within a dynamic.
This study has attempted to inform the classroom use of dynamic geometry. As such the learning of individuals has not been treated of itself, but related to the learning of the community. I have tried to say something about the way in which the sense and meaning making of the individuals in the class can be brought together by the teacher to advance the learning of the practice made up of the classroom as a whole. Individual pupils' participation in this practice will be determined by many factors. However it is the contention here that, accepting the importance of a list of criteria such as that developed by Winbourne and Watson (1998), the various factors identified, both short term and long term, can be used by the teacher/researcher to analyse and inform the development of a local community of practice.

Conclusion: the teacher as guardian of the practice

The teacher is placed initially at the vertex of the tetrahedron representing immediate learning, but this is done with some caution. Reflective teachers will regard the tetrahedron as regular and remember it can rest on any face as base. At different times the pupils, the content or the technology may be the driving influence in the classroom, but it falls to the teacher to direct this dynamic, to maintain the flow of meaning-making activity along the dialectics defined by the edges of the tetrahedron. One of Eisenhart's (1988) elements of ethnographic research is researcher introspection. It is researcher introspection as evidenced by reflective practice which drives the dialectical relationships represented by the edges of the tetrahedron. The reflective teacher is using researcher introspection constantly to analyse, and react to, what is taking place in the dialectical interrelationships represented by the edges of the face defined by pupils, content and technology/resources and is, by this introspection, addressing the complexity represented by the tetrahedron indicated in Figure 6. As ways of analysing the local community of practice constituted by those in the classroom this paper offers the idea of sense-meaning-making. Other factors are spontaneous/scientific concepts, explanation as social proof, intuition and conviction and construction as proof as ways of analysing the immediate meaning-making when dynamic geometry is used in classrooms. In another content area some of these may be relevant, some not. However that introspective researcher, the reflective classroom teacher, will have other contributions to make to the analysis of the meaning-making activity in this and other content areas, guided not only by readings in the research literature but also by past experience. If teacher and class together work in a classroom where socio-mathematical norms are established, where transparent resources are used, where argumentation is used as a learning tool, the teacher may find that some criteria advanced for the promotion of local communities of practice are met.

References

Bassey, M. (1995) *Creating Education Through Research: A Global Perspective of Educational Research for the 21st Century.* (Kirklington Press in conjunction with the British Educational Research Association).

Berger, M. (1998) Graphic Calculators: an Interpretive Framework, *For the Learning of Mathematics*, 18, 2, pp. 13–20.

Brown, A. and Dowling, P. (1998) *Doing Research/Reading Research: A mode of interrogation for education.* (London: Falmer Press).

Cobb, P. and Yackel, E. (1996) Constructivist, emergent and sociocultural perspectives in the context of developmental research. *Educational Psychologist* 31, 3–4, pp. 175–90.

De Villiers, M. (1990) The role and function of proof in mathematics. *Pythagoras*, 24, pp. 17–24.

De Villiers, M. (1991) Pupils' needs for conviction and explanation within the context of geometry. *Pythagoras,* 26, pp. 18–27.

Eisenhart, M.A. (1988) The ethnographic research tradition and mathematics education research. *Journal for Research in Mathematics Education*, 19, 2, pp. 99–114.

Gardiner, J. and Hudson, B. (1998) The evolution of pupils' ideas of construction and proof using hand-held dynamic geometry technology, in Olivier, A and Newstead, K. (eds), *Conference Proceedings of PME 22: International Group for the Psychology of Mathematics Education Conference*, 2, 337–344 (University of Stellenbosch: South Africa, July 1998).

Gardiner, J., Hudson, B. and Povey, H. (1999) What can we all say? Dynamic geometry in a whole class zone of proximal development, *Conference Proceedings of PME 23: International Group for the Psychology of Mathematics Education Conference*, 3, 1–8. (Technion: Israel Institute of Technology, July 1999)

Gardiner, J., Hudson, B. and Povey, H. (2000) Aspects of Construction and Proof in a Hand-Held Dynamic Geometry Environment, in T. Rowland and C. Morgan (eds), *Research in Mathematics Education Volume 2: Papers of the British Society for Research into Learning Mathematics* (pp.125–136). (London: British Society for Research into Learning Mathematics).

Hay, McBer *Report into Effective Teaching,* http://www.teachernet.gov.uk/docbank viewed 5/10/02.

Hedegaard, M. (1990) The zone of proximal development as basis for instruction, in L. C. Moll (ed.), *Vygotsky and Education* (pp. 349–371) (Cambridge: Cambridge University Press.)

Hudson, B. (2002) Holding complexity and searching for meaning – teaching as reflective practices. *Journal of Curriculum Studies*, 34, 1, 43–57.

Hudson, B. (2003) Approaching educational research from the tradition of critical-constructive Didaktik. *Pedagogy, Culture and Society* II, 2, 173–187.

Hudson, B. and Enser, K. (2005) (eds) *Researching the Teaching and Learning of Mathematics 1*: Proceedings of the First MATHED Intensive Programme, Institute of Comparative Education (IVE), Pädaogogische Akademie des Bundes in Oberösterreich, Linz, Austria, 2–12 July 2003. (Linz: Universitätsverlag Rudolf Trauner).

Jones, K. (1997) Children learning to specify geometrical relationships using a dynamic geometry package. *International Group for the Psychology of Mathematics Education Conference 21*, 3, pp. 121–128.

Klafki, W. (1998) Characteristics of critical-constructive Didaktik, in Gundem, B. and Hopmann, S. (eds) *Didaktik and/or Curriculum: An International Dialogue* (New York: Peter Lang) pp. 307–330.

Lave, J. (1996) Teaching, as Learning, in Practice. *Mind, Culture and Activity*, 3, 3, pp. 149–164.

Lave, J. and Wenger, E. (1991) *Situated Learning: Legitimate Peripheral Participation* (Cambridge: Cambridge University Press)

Mason, J. (2004) Mediating mathematical thinking with e-screens, in S. Johnston-Wilder (ed.) *Teaching Secondary Mathematics with ICT* (Berkshire: Open University Press).

Mason, J. (1991) Questions about geometry, in D. Pimm and E. Love (eds) *Teaching and Learning School Mathematics* (pp. 77–90) (London: Hodder and Stoughton).

Meira, L. (1995) The Microevolution of Mathematical Representations in Children's Activity. *Cognition and Instruction* 13(2), pp. 269–313.

Newman, F. and Holzman, L. (1993) *Lev Vygotsky, Revolutionary Scientist* (London: Routledge).

Vygotsky, L. S.(1978) *Mind in Society*, (M. Cole, V. John-Steiner, S. Scribner and E. Souberman, eds) (Cambridge, Ma: Harvard University Press).

Schultz, E. (1994) The Hermeneutical Aspects of Activity Theory. *Activity Theory*, 15/16 pp. 13–16.

Watson, A. (1998), *Situated Cognition and the Learning of Mathematics* (Oxford: University of Oxford Centre for Mathematics Education Research).

Winbourne, P. and Watson, A. (1998) Participating in Learning Mathematics Through Shared Local Practices, in Olivier, A. and Newstead, K. (eds), *Conference Proceedings of PME 22: International Group for the Psychology of Mathematics Education Conference*, 4, 177–184 (University of Stellenbosch, South Africa, July 1998).

Yackel, E. (1998) A Study of Argumentation in a Second Grade Classroom, in Olivier, A. and Newstead, K. (eds), *Conference Proceedings of PME 22: International Group for the Psychology of Mathematics Education Conference*, 4, pp. 209–216 (University of Stellenbosch, South Africa, July 1998).

Yackel, E. and Cobb, P. (1996) Sociomathematical norms, argumentation and autonomy in mathematics, *Journal of Research in Mathematics Education*, 27, 4, pp. 458–477.

Contact

C/o Professor Dr. Brian Hudson
Sheffield Hallam University, Division of Education and Humanities
Email: B.G.Hudson@shu.ac.uk <mailto:B.G.Hudson

That the concept ‚school algebra' includes

Jan Herman, *Charles University, Prague*

Abstract

In this paper I will concentrate on different approaches to algebra, and give a description of the main components of algebra as compared with the results of an Erasmus Intensive Program workshop: 2003 in Linz. The paper also puts forward three new sortings of the main components of algebra which arose from the workshop.

Introduction

Many experiments in the field of algebraic and pre-algebraic didactics have been done, but it is still not clear what the words 'school algebra' include. This paper is aimed at clarifying this question.

Different approaches to algebra

Algebra as generalised arithmetic:

In my opinion the most frequent approach to algebra is presenting it as generalised arithmetic. This stems from the history of mathematics where arithmetic was a base of building mathematics.
From a didactic point of view it is natural to build children's mathematical knowledge in parallel to the historical development of

mathematics. In the development of mathematics a lot of experience is needed for every break-through. In the same way children first obtain experience with numbers: the use of numbers, the use of operations with numbers, and eventually using a structure of numbers. These experiences are the basis for their understanding of algebra.

Describing algebra as generalised arithmetic is usually based on skills, which have to be mastered by a pupil, or on subject-matter, regarded as a part of algebra.

Hejny's delimitation of algebra, based on its subject-matter, is established upon:

- objects: equations, equalities, inequalities, writing with the use of letters and sets of numerical terms
- evidence: well organised sets of numerical relationships
- activities: generalisation and use of terms and
- symbolical language: tables and letters.

Slavit (1999) based his approach to algebra on skills. He views algebra as a cognitive process and its actions as associated with abstracting computation to more structural realms. This process can be manifested by the manipulation of algebraic objects and symbols.

Algebra and reification*:*

Sfard's (1991) theory of reification. takes a different point of view of algebra as generalised arithmetic. Sfard (1991) defined three stages in concept development: interiorisation, condensation, and reification. Interiorisation is the stage where the learner performs operations on lower level mathematical objects.

In the condensation stage, a complicated process is condensed into a form that becomes easier to use and think about, and the learner becomes able to combine processes, make comparisons and generalise.

Reification is termed the stage where the learner can conceive of the mathematical concept as a complete object with characteristics of its own. Concepts that have been reified can be thought of in relationship to the categories to which they belong and the characteristics of these categories may be compared to others.

From this point of view arithmetical experiences are in the interiorisation stage, algebra grows up from them in the condensation stage and the reification stage corresponds to a pure algebraic concept without connection to the real world.

Algebra as a basis of arithmetic:

Hewitt (1998) says: 'Arithmetic is impossible without algebra.' Hewitt believes arithmetic is aimed at getting answers and cannot be carried out without an algebraic structure, which indicates how to carry out a calculation. Algebra (concerned with awareness of awareness) shifts attention from the answers themselves to what must be done to find the answers.

Algebra as a language:

Algebra can be approached as a language. Malara and Navarra (2003) suggest that the natural language learning process is analogous to the learning of algebraic language.

'When a child is learning his/her natural language, he grasps meanings and rules because they accompany and support him step by step. He grasps them ingenuously, through trial, error and imitation, until he reaches school age, when he learns to read and reflect upon grammatical aspects and language syntax.' (Malara and Navarra (2003).

Algebra is connected to linguistic difficulties like organising speech, co-ordinating phrases, describing objects and situations, providing definitions, recognition, following a reasoning process and arguing through the solution of a problem. In other words, algebra should be seen as a language that not only concerns the description of reality but also amplifies its comprehension..

Algebra as a part of the curriculum

Curricula are structured according to subject-matter, but every curriculum has a theoretical background . Those backgrounds differ in their dependence on culture, didactic tradition and many other factors, and determine the main direction of algebra in that culture more than the immediate subject-matter.

Sutherland's (2000) comparative study of algebra curricula was directed at these differences.
She reports that the algebra component of the Australian (Victoria) curriculum is organised around the ideas of '**expressing generality**', '**equations and inequalities**' and '**function**'. The components of the Hungarian algebra curriculum refer only to '**work with systems of equations**' and there is a separate theme called '**relations, functions and sequences**'. The algebra component of the Canadian (British Colombia) curriculum is organised around the themes '**Pattern and Relations**' and '**Variables and Equations**'. In the curricula for Israel and 'French' Canada (Quebec) there is more emphasis on algebra as '**the study of systems of equations and inequalities**' and '**the treatment of functions**', than in other countries studied. France, Hungary, Israel and Italy place more of an emphasis on algebra as a study of '**systems of equations**' than do other countries. There are similarities between the algebra curricula of the Anglo-Saxon speaking countries (England, Australia and English -speaking Canada), and in particular this relates to an emphasis on algebra as a means of '**expressing generality and patterns**'. Italy, Hungary, France and Hong Kong introduce algebra within the context of problem situations by '**traditional word problems**' and there tends to be more emphasis on symbolic manipulation. Canada, Australia and England use more '**realistic modeling situations**' and there is less emphasis on symbolic manipulation.

Main components of algebra

Using of symbols:

The use of symbols is the most characteristic feature of algebra which everybody imagines on first hearing the word 'algebra'. Knowledge of the use of symbols is connected with having a special sense for symbols. Arcavi (1994) calls it Symbol sense. Symbol sense includes (among other aspects):

- The feeling for when to invoke symbols and also for when to abandon them.
- An appreciation that an ad hoc symbolic expression can be created for a desired purpose, and that we can engineer it.
- The choice of symbols.
- The feeling for the power of symbols: how and when symbols can and should be used in order to display relationships, generalisations and proofs which otherwise are hidden and invisible.
- An ability to manipulate and to read symbolic expressions as two complimentary aspects in solving algebraic problems.
- The awareness that one can successfully engineer symbolic relationships which express verbal or graphical given information needed to make progress in a problem, and the ability to engineer those expressions.
- The ability to select a possible symbolical representation for a problem, and, if necessary, having the courage, first, to recognise and heed one's dissatisfaction with that choice, and second, the resourcefulness to search for a better one as replacement.
- The realisation of the constant need to check the symbol meanings while solving a problem, and to compare and contrast those meanings with our own intuitions or with the expected outcome of that problem.
- Sensing the different roles symbols can play in different contexts.

Personal codes:

A personal code is a private way of expressing the process of understanding. It might begin when someone does not know the standard symbolic expression or does not know a suitable way of using the standard symbolic expression.

Examples of personal codes which appear in Filloy, Rojano's (1989) experiment are:

„Ax+B=Cx+D (A+B-D)+C
B+Ax (A+B)x."

It is hard to say if personal codes are good for pupils or not, because they are individual, but they are often the first symbolical expressions to be employed.

Unknown:

Replacing an unknown is the most common case of using symbols. In some countries the letter is used as a symbol for what is being counted from the first school years. So, in talking about algebra, we must decide which type of the letter used is an unknown and which is not. I suggest that a letter is usually called an unknown, if it represents a quantity which is required and is not known, but which it is possible to find and if it is an active part of a relationship.
The concept of the unknown is important for equations. Linchevski and Livneh (1999) say that indeed students did not seem to be able to spontaneously solve equations with the same letter on both sides of an equality by algebraic methods, since they were not able spontaneously to operate with, or on, the unknown.

Variable:

The concept of variable is more 'general' than unknown at this stage of education, because the unknown usually represents the only value, while the variable represents one 'unknown' element of a set. From this point of view, using a variable says more about the pupil's familiarity with the use of symbols and about the generality of pupils' thinking than using an unknown. Novotna and Kubinova (2001) followed this up in their research into the influence of symbolic algebraic descriptions in word problem assignments on grasping processes and on solving strategies. They found out that some pupils were able to use a variable intuitively before they learned algebra. These pupils’ levels of operating were close to generality. Gooson-Espy (1998) describes one possible way of developing meaning for the concept of variable. This concept is based on the arithmetic process by substitution of different values into the variable.

Terms:

In my view the main role of terms for the didactics of algebra is practising the work with literal symbols. It corresponds to Lincevski and Herscovics’s (1996) way of introducing terms to pupils. They start with grouping numerical terms, grouping terms involving the unknown, grouping terms in the unknown in the presence of one numerical term; they then continue by cancellation of identical numerical terms, cancelling terms in the unknown and finish with terms by restrictions on the type of equation and decomposition into a difference.

Equality, equation, inequality[2]

If we want to talk about equations in an algebraic context we have to ask: which sorts of equations are algebra and which are not. Filloy and Rojano (1989) also tried to solve this problem and they classified them in the following way: ‘In arithmetical terms, the left side of the equation corresponds to a sequence of operations performed on numbers (known or unknown); the right side represents the consequence of having performed such operations. This is what we might call the “arithmetical” notion of equality. From such a notion, an equation such as Ax+B=C can be solved by merely undoing, one by one, the operations given in the left hand sequence, starting with the number C. We shall call this type of equation ‘arithmetical.’

The arithmetical notion does not apply to an equation of the form Ax+B=Cx+D; its resolution involves operations drawn from outside the domain of arithmetic – that is, ‘operations on the unknown’.

While pupils are able spontaneously and intuitively to manipulate the numerical parts of the equations, they are not able to use the same manipulations in the context of algebraic parts of the equations. (Linchevski and Livneh (1999)).

Approximation as the first step of understanding of equation:

The most important step in understanding is correctly grasping of the symbol of equality. Pupils may not understand the processes needed to solve non-arithmetical equations such as 3x+8=7x+4 because they do not accept the initial premise that these two quantities are equivalent. (Goodson-Espy (1998)) Pupils must at least understand that the expressions on both sides of the equals sign are of the same nature (Filloy and Rojano (1989)).
The natural way for pupils without algebraic experience is to solve equations of the form Ax+B=Cx by trial-and-error methods. My previous research[3] shows that it is natural for them to use approximation methods when they are faced with equations of the form Ax+B=Cx+D. Their approximation has specific rules. They separate sides of the equation, substitute the same number for x and compare their results. After a few attempts they start to use a strategy of selection of numbers which have to be substituted. I believe approximation could be the first step to understanding equations.

***Higher levels of understanding of operations**:*

When pupils start to think more deeply about operations, they start to think algebraically and gain an ‘operation sense’. Operation sense, described by Slavit (1999), includes:

• A conceptualisation of the base components of the process. This involves an ability to break down the operation into its base components.
• Familiarity with properties which the operation is able to possess. Of these, perhaps of primary importance is an awareness of the ability to reverse the operation (invertibility). Other properties, such as commutativity, associativity, and the existence of an identity, may or may not be characteristic of a given operation, but in each case they help to clarify its general nature.
• Relationships with other operations. In addition to the relationships an operation has with its inverse, the distributive property in any field provides a means of connecting two operations, such as addition and multiplication.
• Facility with the various symbol systems associated with the operation. For example, multiplication is commonly expressed using each of the following symbols: x, ., ( ).

• Familiarity with operation contexts. Experience with different contexts of the operation can provide various perspectives on which a student can develop understanding of that operation. For example, using join, compare, and part-whole situations has been shown to be useful in the development of the operation understanding of addition.
• Familiarity with operation facts. Knowledge of certain operation facts has been shown to enable more advanced approaches to a given task. For example, operation facts of addition could lead to the following invented strategy: 7+8=15 since 8=3+5, so 7+8=7+3+5=10+5=15.
• Ability to use the operation without concrete or situational referents. A student who can perform an operation on abstract numeric values or other mental objects clearly has an advanced sense of the use of that operation.
• Ability to use the operation on unknown or arbitrary inputs. Perhaps a higher level of operation sense/understandings is exhibited when the student is enacting his or her understanding of the operation on quantities that are unknown or arbitrary.
• Ability to relate the use of the operation across different mathematical objects.
• Ability to move back and forth between the above conceptions.

Generalisation:

If we accept an idea of algebra as a generalised arithmetic, it is natural that generalisation is a component of algebra. 'Acts of generalisation and abstraction give rise to formalisms that support syntactic computations that, in turn, can be examined for structures of their own, usually based in their concrete origins. These structures seem to have three purposes,

1) to enrich understanding of the systems they are abstracted from,
2) to provide intrinsically useful structures for computations, freed of the particulars that they were once tied to, and
3) to provide the base for yet higher levels of abstraction and formalisation'. (Kaput (1995))
The ability to generalise is formed from early childhood and it could be strengthened by asking questions like those of Arcavi (1995): 'Do

you observe any pattern? Can you generalise? Can you justify your generalisation?'

Aiming at a structure of solving processes and algorithms:

Aiming at a structure of solving processes and algorithms is important for formalizing general properties. It is necessary to observe and articulate how that counting is being carried out. It is a double level of awareness – awareness of awareness – which is required for an individual to be in a position to write an algebraic statement (Hewit (1998)).

The particular importance of this component of algebra can be observed, for example,in the tasks of TIMSS and TIMMS-R (2001) in the algebraic section and in Goodson-Espy's (1998) experiment, where one of the most important tasks is: 'Review the previous tasks and write formal symbolic representations for those problems that were solved using intuitive methods'. (Goodson-Espy, 1998)

Modelling of real situations:

Algebraic modelling, closely related to generalizing, is mostly used in word problem-solving processes. It includes abstraction from non-essential details and transformation into a new, more transparent, problem with the same set of solutions. During the transformation, the other components of algebra – symbols – are used for substitution of unknown quantities.

*Routine***:**

This is not popular, but meaningless and syntactic symbol-pushing is the major student activity in algebra (Arcavi (1995)).

In traditional algebraic language teaching, a child begins by studying rules; in other words, formal manipulation. Therefore, grammatical aspects, procedures and syntax come before an understanding of meanings. There is a tendency to teach algebraic syntax and to neglect algebraic semantic. (Malara andNavarra (2003))

Arcavi (1995) suggests this could be caused by the timing of the teaching of algebra, because 'algebra, as it is often taught, presents tools too soon, before the questions these tools help to answer are meaningfully understood'.(Arcavi (1995))

In my view it is not always bad and nonsensical progress to use algebra automatically without thinking about the meaning, because algebra can be used as a tool for solving problems and it is more economical and quicker to apply algebra automatically, than to think before every step of solution.

Workshop

Description of my workshop

The workshop was held at the Erasmus Intensive Program MATHED 2003 in Linz. The aim was cooperation on delimitation of the concept of school algebra. The delimitation was designed to summarise experiences of the teaching and learning of algebra in different countries and from different personal points of view of algebra. Twelve staff members and Ph.D. students from Austria, the Czech Republic, Finland, Great Britain, Greece, Latvia and Norway participated in the workshop. The 80-minute session was divided into two parts.

In the first part we studied what the concept of school algebra includes and how to sort and structure it.

In the second part we constructed appropriate tasks for testing children's algebraical abilities. This paper presents the first part: what the concept of school algebra includes. In looking for the main components of algebra, we used brainstorming methods. In re-sorting and structuring the components, we used group work with a final open discussion.

Summary of my expectations

I was expecting that: 1) the main approach to algebra would be as generalised arithmetic, because that is the traditional approach usually taught at all school stages. 2) All the main components listed in the introduction would be described, because those are the only

main components common to every country. 3) The sorting of those components would be similar to Hejny's sorting (listed above), because it arose from the approach to algebra as a generalised arithmetic and was based on subject-matter which, I believed to be familiar to participants.

Products of the workshop

A variety of components of algebra became evident; they are listed here with commentaries:

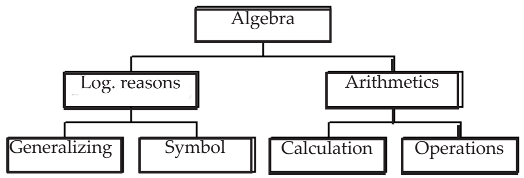
- symbols for numbers – because school algebra is about symbolising – because algebra is a generalised mathematics
- equations – are tools or skills for solving problems
- generalising – by generalising we can express patterns between two sets
- modelling of real situations – only possible if you master algebra
- relations, operations – they are connected with rules of algebra
- calculating – is related to operations
- arithmetic – as a base of algebra, especially inverse operations
- identifying patterns – as a base for generalising
- logical reasoning – the most important ability for algebra
- transcription to symbols – is connected to modeling
- variable x fixed values
- unknown number

3 types of sorting emerged.

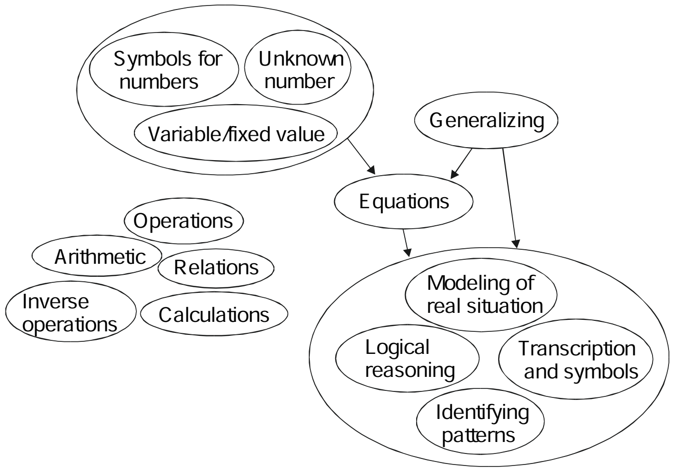
Sorting 1

mathematical thinking process	*skills*
identifying symbolising generalising reasoning conjecturing modelling	solving equations using operations

Sorting 2



Sorting 3



Comparison of products of workshop with expectations

Components

As I expected symbols, equations, generalisation and modelling appeared as components of algebra.

Operations and relations also appeared, but it was not clear what sort of use of them was considered.

Terms did not appear. I believe that participants supposed that those terms were included in equations and modelling. I expected that routine would be described as a component of algebra but it was not. I believe discussion of routine and the undirected use of algorithms in mathematical education today is not popular and could explain why routine did not appear.

The special component of algebra, which had to be the most important of all the components, was logical reasoning. I partly agree with this. It is true that if you want to use algebra correctly, you have to be able to reason logically, but logical reasoning is a part of every mathematical domain and not only the mathematical. It is a part of physics, chemistry, mother languages and many other subjects. Therefore I do not believe that it is necessary to describe logical reasoning as a component of algebra.

Sorting

Sorting 1 of components, which sorted them on mathematical thinking processes and skills was excellent. It added one new component of algebra – conjecturing. I think conjecture is close to the trial and error method and the approximation method.

The sorting is oriented on activities which the pupil has to know. This dynamic conception of algebra evokes Hewitt's (1998) conception of algebra as a basis of arithmetic.
The second, Sorting 2, was more general and was probably based on the conception of algebra as a generalised arithmetic. It is not clear

where modelling of real life is in the sorting, and which relations connect it with algebra. There is no mention of subsumed equations. I think this sorting is too theoretical for use in a didactics of algebra.

The third, Sorting 3, was set-like and I think it is similar to Hejny's sorting. Operations, inverse operations, relations, calculations are placed next to arithmetic. It means that their level could be higher than arithmetical. They are placed together, without relation to the rest of the objects. This could hint to a concept of algebra as generalised arithmetic.

The set of equations is connected to the set of symbols and the set of generalising. Generalising is connected with the set which we could call abilities and skills, but generalising is not a part of that set. This indicates the special importance which was attached to generalising.

Conclusions

The workshop was fruitful. Three sortings of components of algebra arose from it, surprisingly not solely based on the concept of algebra as a generalised arithmetic. A new component of algebra – logical reasoning – strongly emphasised by participants, appeared, but routine, an important component of algebra, did not.
These sortings could be valuable for the didactics of mathematics and for the investigative didactics of algebra, because they give another view of the concept of school algebra.

References

Arcavi, A. (1994) Symbol sense: Informal sense-making in formal mathematics, *For the Learning of Mathematics* 14 (3), 24–35.
Arcavi, A. (1995) Teaching and Learning Algebra, *Journal of Mathematical Behavior* 14, 145–162.
Filloy, E. and Rojano, T. (1989 Solving equations, *For the Learning of Mathematics* 9, 19–25.
Goodson-Espy, T. (1998) The roles of reification and reflective abstraction in the development of abstract thought, *Educational Studies in Mathematics* 36, 219–245.

Hejny, M. (unpublished) *Protoalgebra*. (Results of research carried out with support of the GAÈR project 406/93/0974).
Hewitt, D. (1998) Approaching arithmetic algebraically, *Mathematical Teacher* 163, 19–29.
Kaput, J.J. (1995) A research base supporting long term algebra reform? In Slavit, D. 1999, The role of operation sense in transitions from arithmetic to algebraic thought, *Educational Studies in Mathematics* 37, 251–274.
Linchevski, L. and Herscovics, N. (1996) Crossing the cognitive gap between arithmetic and algebra, *Educational Studies in Mathematics* 30, 39–65.
Linchevski, L. and Livneh, D. (1999) Structure sense, *Educational Studies in Mathematics* 40, 173–196.
Malara, N.A. and Navarra, G. (2003) *Arithmetic pathways towards favouring pre-algebraic thinking*, (: Bologna: Pitagora Editrice).
Novotna, J. and Kubinova, M. (2001), The influence of symbolic algebraic descriptions in word problem assignments on grasping processes and on solving strategies, *Proceedings of the 12th ICMI Study Conference The Future of the Teaching and Learning of Algebra*. (Melbourne: University of Melbourne), 494–500.
Sfard, A. (1991) On the dual nature of mathematical conceptions, in Goodson-Espy, T. 1998, The roles of reification and reflective abstraction in the development of abstract thought, *Educational Studies in Mathematics* 36, 219–245.
Slavit, D. (1999) The role of operation sense in transitions from arithmetic to algebraic thought, *Educational Studies in Mathematics* 37, 251–274.
Sutherland, R. (2000) *A Comparative Study of Algebra Curricula* (Bristol: University of Bristol).
TIMSS (Repeat 1999: 2001), *Ulohy z matematiky a prirodnich ved pro zaky 8. Rocniku* (Prague: Tauris).

Contact

Jan Herman
Charles University of Prague, Faculty of Education
Email: hermos@c-box.cz <mailto:hermos@c-box.cz>

Footnotes

[1] The paper is based on the concept of early school algebra which is taught from 5th to 9th grade (10–15 year old pupils). The word algebra is used in this sense. It is widely accepted that functions are a component of algebra, but formation of the concept of function starts later than in those cases of the components of algebra listed below. Therefore functions are not described in this paper.
[2] Pupils understanding of the concept of linear inequality was studied by Goodson-Espy (1998).
[3] The research reported here is a part of my Diploma Thesis written at the Faculty of Education, Charles University, Department of Mathematics and Mathematical Education.

Teaching-Studying-Learning Mathematics: Approaches to research for holding complexity

Brian Hudson, *Sheffield Hallam University and Umeå University*

Abstract

This paper explores differences between traditions in relation to teaching and learning. Taking the Anglo/American curriculum tradition as my starting point, I seek to highlight the ways in which the Central and Northern European tradition of Didaktik has offered a new dimension and fresh insights to the notion of reflective practice. In particular the paper highlights the way in which the tradition of Didaktik offers tools for recognising and holding the complexity associated with teaching-studying-learning processes. Furthermore it explores the potential of these tools for the analysis and evaluation of the teaching-studying-learning of mathematics by seeking to make connections with other frameworks for analysis in the field..

Introduction

The paper draws on my involvement in the development of a European module on *Didaktik* that was supported by the European Commission under the SOCRATES action programme between 1998 and 2002. The aim of this module was to develop resources for teacher education/educational sciences students in Europe using open and distance learning methodologies and new technologies. My contribution to this project was principally in

the field of mathematics education and the use of information and communications technologies (Hudson, Ongstad and Pepin, 2002). Further background details to the project and to the issues raised in this paper are discussed in Hudson (2002).

Background

In seeking to address the differences between different traditions in relation to teaching and learning, it is first of all necessary to acknowledge that terms such as 'curriculum' and 'Didaktik' are strongly culture-bound. Furthermore, it is necessary to recognise that the comparison of meaning across linguistic boundaries is fraught with a variety of difficulties, as is highlighted by Kansanen (1995a, b). In my development of the ideas contained in this paper the direction given by Seel (1999) has been influential. He argues that Didaktik may be conceived as the human or social science whose subject is the planned (institutionalised and organised) support for learning to acquire *Bildung*. This is an elusive concept to capture in English and has variously been translated as 'formation', 'education' and 'erudition'. The latter derives from the Latin *eruditio* as used by Comenius and is the suggested translation by Hopmann and Künzli (1992). However Westbury (2000a, b) suggests that 'formation' is the best English translation to capture the German sense of the term. This is based on the connotations of the verb *bilden* (to form or to shape) and has close associations with the notion of religious or spiritual formation when applied to the preparation of a member of the religious clergy. In its turn, *Bildung* can be seen to be a state of being that can be characterised by a cluster of attributes described by terms such as 'educated', 'knowledgeable', 'learned', 'literary', 'philosophical', 'scholarly', and 'wise'. *Allgemeinbildung* is a wider development of this concept and refers to a general competence for a productive coping with life with regard to co-existence and survival, including life in society and basic communicative and technical abilities and values (Friehs and Seel 2000).

In his discussion of the nature of Didaktik, Seel (1999) emphasises that human beings are born into a culture and a cultural environment, including a social system. The acquisition of, and the ability to deal with, cultural objects may be conceived as a major part of the process of acquiring *Bildung*. This emphasis on the social context and societal goals is a distinctive characteristic of the tradition of Didaktik. Thus Klafki (1995, 1998) highlights three main elements of contemporary *Bildung*. The first is *self-determination*, by which is meant that every member of society is to be enabled to make independent, responsible decisions about her or his individual relationships and interpretations of an interpersonal, vocational, ethical or religious nature; second, *co-determination*, which refers to both the rights and responsibilities of each member of society to contribute, together with others, to the cultural, economic, social and political development of the community; third, *solidarity*, which means that the individual rights to self-determination and opportunities for co-determination can only be represented and justified if it associated with action to help others. This implies not only the recognition of equal rights but also active support for those whose opportunities for self-determination and co-determination are limited or non-existent for reasons of, e.g. social conditions, lack of privilege, political restrictions, or oppression. In considering teaching and learning in school from such a perspective, they are seen to be a complex nexus of interaction, social learning and content-related acquisition of knowledge and abilities. Such a perspective mirrors recent and current debates in mathematics education in relation to social and cultural aspects of teaching and learning, e.g. the socio-cultural psychology of Vygotsky (1962) and the related fields of activity theory, e.g. Mellin-Olsen 1987, and social practice theory, e.g. Lave and Wenger 1991.

Exploring differences between Didaktik and curriculum

In Hudson (2002) I have highlighted fully what I see as the most significant differences between the different traditions of Didaktik and curriculum. I then use this as the basis to outline what the study of Didaktik has offered to the development my own thinking and practice. In this paper I will draw on those ideas selectively to focus attention on the aspect of recognising and holding complexity. At the outset I think that it is worth noting, as others have pointed out, that Didaktik is a tradition of thinking and studying teaching and learning which is virtually unknown in the English-speaking world (Kansanen 1995 b, 1999, Hopmann and Riquarts 2000, Westbury

Teaching-studying-learning Mathematics: Approaches for holding complexity

2000a). On the question of its relevance and potential, I follow Westbury (2000a) in emphasising that Didaktik provides some ways of thinking that highlight some very important, and universal, educational questions that are not well-articulated in the Anglo-American curriculum tradition.

In considering these differences, Westbury (2000a) points out that in the US curriculum tradition, the dominant idea has been *organisational*. This involves an emphasis on the tasks associated with the building of systems of schools. These systems have a curriculum-as-manual as a central part of their overall organisational framework. This curriculum contains the templates for coverage and methods that are seen as guiding, directing, or controlling the routine classroom work of a school, or of an entire school system. Such an approach results in a view of the role of the teacher as an employee of the school system, who is concerned with implementing the system's curricula in a relatively mechanical fashion.

On the other hand, within the German tradition the state curriculum, i.e. *Lehrplan*, has not been seen as something, which could or should explicitly direct a teacher's work. Although a *Lehrplan* does outline prescribed content for teaching, this is seen to be an authoritative selection from cultural traditions that can only become educative as it is interpreted and given life by teachers. Within this tradition there is an emphasis on teachers' professional autonomy and on their freedom to teach without control by a curriculum in the US sense. This is also emphasised by Seel (1999) who highlights the tension between the notion of *relative pedagogical autonomy* for schools and teachers in German-speaking countries with the more narrowly focussed Anglo-American concept of *teaching theory*. Seel also draws attention to Shulman's (1987) critique of a lesson-related instructional theory, which is seen as being too limited for a research basis for professional practice.

Within the Anglo-American tradition the social and cultural world is seen as an 'objective' structure (Reid 1998) and the task of curriculum is to present this structure to students, on the assumption that culture and society can be reduced simply to facts to be learned. In contrast Reid noted that within the tradition of Didaktik the social and cultural world is 'subjectified': it is seen that there are things to be learned, but students are to be encouraged to find their own path. As Künzli (2000) indicates, the 'Didaktiker' does not begin by asking how a student learns or what a student should be able to do or know. Rather he or she looks first at a prospective object of learning in terms of *Bildung*, to ask what it can and should signify to the student, and how students themselves can experience this significance.

In drawing on this tradition as a conceptual framework for research on teaching, Gudmundsdottir *et al.* (2000) highlight Klafki's (1995, 2000) five questions and compare them to Shulman's (1987) model of pedagogical reasoning and action. Shulman's model offers a framework for analysing teachers' knowledge that distinguishes between different categories of knowledge:

- knowledge of subject matter;
- pedagogical content knowledge;
- knowledge of other content;
- knowledge of the curriculum;
- knowledge of learners and their characteristics;
- knowledge of educational aims (purposes and values and their philosophical and historical backgrounds);
- knowledge of educational context (character of school communities and cultures); and
- general pedagogical knowledge (broad principles and strategies of classroom management and organisation).

Gudmundsdottir *et al.* (2000) offer each model as an example of practical theoretical constructs and highlight four characteristics common to both: the primacy of practice; the grounding of the concepts and theoretical models in practice; a basic notion of historicity that takes into account past, present and future; and, finally, a strong hermeneutic and interpretative stance on research and scholarship
Klafki''s five questions are based on the earlier thinking of Hermann Nohl and Erich Weniger who developed the idea, in contrast to the objectivism of previous thinkers, that a double-relativity constitutes the very essence of the contents of education: the value of any content can only be ascertained with reference to the individual learner and with a particular human, historical situation in mind, with its

attendant past and anticipated future. Klafki's questions are based on the view that any preparation for teaching is not a technical, but rather an interpretative issue, i.e. an issue to be considered in the light of a pedagogical situation. Thus he asks:

1. What wider or general sense or reality do these contents exemplify and open up for the learner? What basic phenomenon or fundamental principle, what law, criterion, problem, method, technique or attitude can be grasped by dealing with this content as an 'example'?

2. What significance does the content in question or the experience, knowledge, ability or skill to be acquired through this topic already possess in the minds of the children in my class? What significance should it have from a pedagogical point of view?

3. What constitutes the topic's significance for the children's future?

4. How is the content structured? (which has been placed into a specifically pedagogical perspective by questions 1, 2 and 3?)

5. What are the special cases, phenomena, situations, experiments, persons, elements of aesthetic experience, and so forth, in terms of which the structure of the content in question can become interesting, stimulating, approachable, conceivable, or vivid for children of the stage of development of this class?

Didaktik reflection and teaching

As I have already noted, Westbury (2000a) observes that in the USA there is vision of a strong and overt formal control over teachers as employees of the school system. In this context professionalism is a contested aspiration for teachers, and this is reflected in the language around US teacher education: teachers are 'trained' and 'certified' to teach the curriculum and then 're-trained' and 'in-serviced'. In contrast, within the German system teachers are 'licensed' as self-determining *professionals* who work within a larger institutional framework that directs, but does not control, the details of their work. As in the case of lawyers and engineers, the work of teachers is based on an expectation of autonomy of practice and a system of self-discipline and peer review rather than of external control. The notion of *Bildungsideale*, as the central social value of teaching as a profession, is of central importance. In such a context, the ideas of *curriculum change* and of *school reform* take on very different meanings than they do in the contemporary UK and USA: teachers must make their own independent judgement that new ways are preferable to the old ways – in the light of *their* sense of the central social value associated with *Bildung*. Keitel and Hopmann (1995) note that *Didaktik reflection* should be seen as the way of linking the intentions of the teacher with the state curriculum, i.e. the *Lehrplan*. As such Klafki's model reflects a tradition that is deeply rooted in the history of German education.

Recognising and holding complexity

Engagement with the tradition of Didaktik has given me a fresh perspective on the recognition and the complexity of what in the English-speaking world is referred to as 'learning'. Differences between cultures, and also the way in which language can operate, either mask or highlight this complexity. For example in Russian there is only one word, *Obuchenie*, for teaching/learning, and there is no sharp conceptual distinction made between the terms. This idea has a parallel in German with in the word *Unterrichtfach*, which Kansanen (1995a) suggests is best translated as *teaching/studying/learning*. This phenomenon is reflected in Swedish by the word *Undervisning* and in Finnish by *Opetus*. A parallel way of thinking would seem to underlie the work of Lave (1996) with her emphasis on 'teaching *as learning* in practice'. Through recognising the complexity of the process of 'learning', particular attention is given to the *studying* aspect of this process, i.e. those key functions that need to be fulfilled in order to achieve the goal/end point of the process, which might be interpreted as *a state of learning*.

A key tool for the analysis of the complex relations between teacher, student and content in the teaching-studying-learning process is the Didaktik triangle (see Figures 1 and 2). As Kansanen and Meri (1999) emphasise, the Didaktik triangle should be treated as a whole, although this is almost impossible to do in practice. They point out that the most common approach is to take the *pedagogical*

relation between the teacher and the student(s) as a starting point. As Kansanen and Meri (1999) also note, the pedagogical relation between the teacher and the student is taken as the significant starting point in *Geisteswissenschaftliche*, i.e. human science, pedagogy.

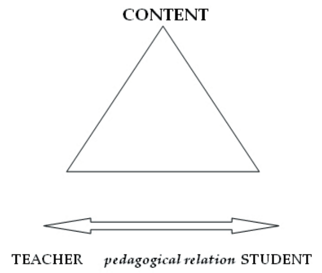


Figure 1: Pedagogical relation in the didaktik triangle

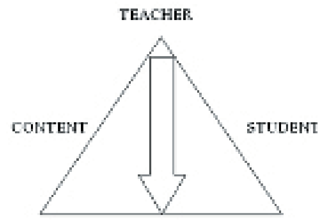


Figure 2: The didaktik relation in the Didaktik triangle

However the relationship between the teacher and the content must also be considered, and then the teacher's 'competence' is brought into focus. They also emphasise that teaching in itself does not necessarily imply learning and, therefore, the preferred term for the activities of students is 'studying'. It is through studying that the instructional process can be observed, while the invisible part of this relation may be learning. The teacher's key task is guiding this relation.

In other words, within Didaktik the didaktik relation is a relation to another relation, and concentration on this *set* of relationships is the core of a teacher's professionalism. And in view of the complexity of this set of relations as it manifests itself in any situation, it is difficult to think that the didaktik relation could be organised universally, or according to some technical rules. Consequently teachers' own practical theories and pedagogical thinking are seen to be of vital importance.

Approaches to analysis and evaluation

In approaching the evaluation of teaching-studying-learning situations it is suggested in Hudson *et al.* (2002) that action research combined with the use of the framework provided by Didaktik offers a possible approach for assisting the process of systematic reflection on experience. Furthermore by reflecting systematically on experience at an **individual** level the opportunity is provided to link theory with practice at the **social** level. These techniques and methods might be considered as tools for illuminating the *interplay* between reflection and experience at the individual level (Klafki, 2000), which in turn can help inform the *interactive relationship* between theory with practice at the social level. The interplay between reflection and experience and the interactive relationship between theory and practice are represented in Figure 3 below:

Figure 3: The interplay between **reflection** and **experience** and the interactive relationship between **theory** and **practice**

A key question in relation to reflection is 'Reflection on what?' One technique for helping to identify what aspects to focus on is to return to the *Didaktik* triangle – see Figure 4. Attention will be focused on different aspects of the triangle according to what has been chosen to reflect on. For example if you are concerned with developing your ability to 'notice' your on-the-spot decisions in the moment you will focus on the upper vertex of the triangle. If your concern is for promoting whole class interactive discussion then your focus will be on the relationship between teacher and student, i.e. the right-hand side of the triangle. This approach is consistent with that of the discipline *of noticing* (Davis, 1990 and Mason, 1994 and 2002) which aims to provide a systematic means, or discipline, whereby teachers may become more aware of their classroom practice, and thereby develop alternative responses to incidents and events that occur in their day-to-day teaching. This is also consistent with the view of the teacher as being at the heart of the teaching-studying-learning situation within the tradition of *Didaktik*. Furthermore it stands in sharp contrast with the managerialist framework for curriculum development and specification in the tradition of the Anglo-

American curriculum theory (Westbury, 2000a). The approach is concerned with change but not about research changing teachers – rather with teachers changing themselves through researching their own practice. As such it is consistent with action research approaches and can be seen to focus attention on the role of the teacher within the *Didaktik* triangle. This approach towards the evaluation of teaching-studying-learning situations is discussed further in Hudson *et al.* (2002).

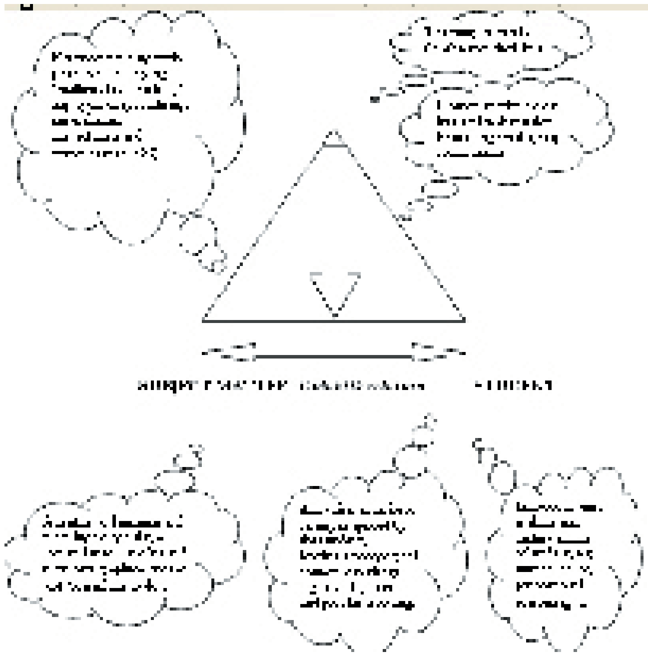


Figure 4: The Didaktik triangle as an evaluation tool

However not only is the Didaktik triangle an evaluation tool but also it can be used as a research tool in the process of data analysis and interpretation. This approach was prompted in part by Gudmundsdottir *et al.* (2000) who note that whilst Didaktik analysis

is not research based, when Kalfki's five questions are applied as a research instrument, a teacher's understanding of the 'organic power' embedded in the content itself can be explored and an understanding of how the content's 'germinative forces' and 'productive drives' can be exposed (p. 321).

Discussion

For an example of the application of some of these ideas, the chapter by Gardiner (2004) in this volume provides an example. In considering the roles of technology in relation to the complex interrelationships between teacher, students and subject matter, researchers are confronted with an even greater degree of complexity than that faced in traditional settings. In his study, Gardiner aimed at holding the complexity of the teaching-studying-learning situation rather than retreating from the challenge e.g. into a 'dyadic' mode of thinking, as discussed by Ongstad or the 'Cartesian' dualism as highlighted by Chassapis, both in this volume. A central aim of this study was to illuminate the role of the teacher in response to the challenge of promoting the development of a mathematical 'telos' in communities of learners using dynamic geometry.

References

Davis, J. (1990) The role of the participant observer in the discipline of noticing, in Seeger, F. and Steinbring, H. (1990), *Proceedings of the Fourth Conference on the Systematic Co-operation between Theory and Practice in Mathematics Education: Overcoming the Broadcast Metaphor*, Institute für Didaktik der Mathematik der Universitet Bielefeld, Bielefeld.

Friehs, B. and Seel, H. (2000) *The concept of 'Bildung' and its impact on German pedagogy*. Paper presented to meeting of Socrates European Module Didaktik at Charles University, Prague (Department of Education, Karl-Franzens University, Graz, Austria).

Gardiner, J. (2004) Developing a mathematical telos in communities of learners using dynamic geometry, in B. Hudson and K. Enser (eds) *Researching the teaching and learning of mathematics: Proceedings of MATHED 2003 Intensive Programme*, Pädagogische Hochschule Linz, Austria.

Gudmundsdottir, S., Reinertsen, A. and Nordtømme, N. (2000) Klafki's Didaktik analysis as a conceptual framework for research on teaching, in I. Westbury, S. Hopmann and K. Riquarts (eds), *Teaching as a Reflective Practice: The German Didaktik Tradition* (Mahwah, NJ: Lawrence Erlbaum Associates), 319–334.

Hopmann, S. and Künzli, R. (1992) Didaktik–Renaissance. *Bildung and Erziehung*, 45 (2), 117–135.

Hopmann, S and Riquarts, K. (2000) Starting a dialogue: a beginning conversation between the Didaktik and curriculum traditions, in I. Westbury, S. Hopmann and K. Riquarts (eds), *Teaching as a Reflective Practice: The German Didaktik Tradition* (Mahwah, NJ: Lawrence Erlbaum Associates), 3–11.

Hudson, B., Ongstad, S. and Pepin, B. (2002) *Didaktik der Mathematik (Teaching-Learning Mathematics)* http://ive.pa-linz.ac.at/emdid/latest/frontpage.html (visited 23 Aug 2003).

Hudson, B. (2002) Holding complexity and searching for meaning – teaching as reflective practice, *Journal of Curriculum Studies*, 34, 1, 43–57.

Kansanen, P. (ed.) (1995 a) *Discussion on Some Educational Issues VI* Research Report No. 145 (Helsinki, Finland: Department of Teacher Education, University of Helsinki).

Kansanen, P. (1995 b) The Deutsche Didaktik. *Journal of Curriculum Studies*, 27 (4), 347–352.

Kansanen, P. (1999) The Deutsche Didaktik and the American research on teaching, in B. Hudson, F. Buchberger, P. Kansanen, and H. Seel (eds), *Didaktik/Fachdidaktik as Science(-s) of the Teaching Profession?* TNTEE Publications, 2 (1), 21–36. http://tntee.umu.se/publications/publications.html (visited 23 Aug 2003).

Kansanen, P. and Meri, M. (1999) The didactic relation in the teaching-studying-learning process, in B. Hudson, F. Buchberger, P. Kansanen, and H. Seel (eds), *Didaktik/Fachdidaktik as Science(-s) of the Teaching Profession?* TNTEE Publications, 2 (1), 107–116. http://tntee.umu.se/publications/publications.html (visited 23 Aug 2003).

Keitel, C. and Hopmann, S. (1995) Editorial. *Journal of Curriculum Studies*, 27 (1), 1–2.

Klafki, W. (1995) Didactic analysis as the core of preparation for instruction (Didaktische Analyse als Kern der Unterrichtsvorbereitung). *Journal of Curriculum Studies*, 27 (1), 13–30.

Klafki, W. (1998) Characteristics of critical-constructive Didaktik, in B. B. Gundem and S. Hopmann (eds), *Didaktik and/or Curriculum: An International Dialogue* (New York: Peter Lang), 307–330.

Klafki, W. (2000) Didaktik analysis as the core of preparation of instruction, in I. Westbury, S. Hopmann and K. Riquarts (eds), *Teaching as a Reflective Practice: The German Didaktik Tradition* (Mahwah, NJ: Lawrence Erlbaum Associates), 197–206.

Lave, J. and Wenger, E. (1991) *Situated Learning: Legitimate Peripheral Participation* (Cambridge: Cambridge University Press).

Lave, J. (1996) Teaching, as learning, in practice, *Mind, Culture, and Activity*, 3 (3), 149–164.

Mason, J. (1994) Researching from the inside in mathematics education – locating the I–You relationship, in J. P. da Ponte and J. F. Matos (eds), *Proceedings of Eighteenth Conference for the Psychology of Mathematics Education*, 29 July–3 August 1994, University of Lisbon, Portugal, 1, 176–193.

Mason, J. (2002) *Researching Your Own Practice: The Discipline of Noticing*, London and New York: RoutledgeFalmer.

Mellin-Olsen, S. (1987) *The Politics of Mathematics Education* (Dordrecht, The Netherlands: Reidel).

Reid, W. A. (1998) Systems and structures or myths and fables? a cross-cultural perspective on curriculum content, in B. B. Gundem and S. Hopmann (eds), *Didaktik and/or Curriculum: An International Dialogue* (New York: Peter Lang), 11–27.

Shulman, L. S. (1987) Knowledge and teaching: foundations of the new reform. *Harvard Educational Review*, 5 (1), 1–22.

Seel, H. (1999): 'Allgemeine Didaktik' ('General Didactics') and 'Fachdidaktik' ('Subject Didactics'), in B. Hudson, F. Buchberger, P. Kansanen, and H. Seel (eds), *Didaktik/Fachdidaktik as Science(-s) of the Teaching Profession?* TNTEE Publications, Umea, Sweden, 2, 1, 85–93.
http://tntee.umu.se/publications/publications.html (visited 23 Aug 2003).

Tyler, R. (1949) *Basic Principles of Curriculum and Instruction* (Chicago: University of Chicago Press).

Vygotsky, L. S. (1962) *Thought and Language* (Cambridge, MA: MIT Press).

Westbury, I. (2000 a) Teaching as a reflective practice: what might Didaktik teach curriculum? In I. Westbury, S. Hopmann and K.

Riquarts (eds), *Teaching as a Reflective Practice: The German Didaktik Tradition* (Mahwah, NJ: Erlbaum), 15–40.

Westbury, I. (2000 b) Introduction, in Peter Menck, *Looking into Classrooms: Papers on Didactics* (Norwood, NJ: Ablex), xi–xx.

Contact

Division of Education and Humanities, Faculty of Development and Society, Sheffield Hallam University
Email: B.G.Hudson@shu.ac.uk

Department of Interactive Media and Learning, Faculty of Teacher Education, Umeå University

TRIADIC POSITIONING(S) AND/IN MATHEMATICS EDUCATION

Sigmund Ongstad, *University College, Oslo*

Abstract

The paper presents a triadic framework for the study of communication in education. Utterance is seen as the basic unit for analysis and the triadic view implies that the form, the content and the use of the utterance are seen as mutual and reciprocal and hence inevitably triadic. Any main element of communication such as an utterance, a text, a genre, a discourse or a context is principally seen as triadic within which communicators will have to communicate, both as utterers and interpreters. A model is presented, and an approach developed on the basis of the framework, analysis of positionings, is outlined briefly. Validity is concentrated on in particular.

The last part of the paper deals with two examples from the field of mathematics education, one presents different excerpts from a textbook in mathematics education and the other deals with a research project focusing on investigative teaching in mathematics. This 'empiric' part of the paper *is deliberately not finalized*, since the idea is not to *prove* anything, but to illustrate and to discuss how triadic thinking, that is a communicational approach, and hence analysis of positioning, *can* be used to reframe and interpret texts and utterances *systemically*.

The framework is open, and can be seen as some kind of simplified 'Wissenschafts-theorie' or general theory of science, based on a communicational view of research and education. Hence most methodologies can in principle be fitted in or related to the framework. One main idea is that the overall framework will be far more extensive than any particular approach and can therefore function as a kind of re-contextualization, creating at least a minimum of compatibility with other approaches. The paper also touches upon the way in which a triadic approach allows for a more differentiated understanding of validity.

A triadic model

According to Bühler (1934), Habermas (1984), Bakhtin (1986), Halliday (1978, 1994), (Martin 1997) and Ongstad (2002a and 2002b) signs, utterances, texts, discourses, genres, and contexts, in short, any main aspect or level of communication, could be seen as basically triadic. To utter (and hence to interpret) is a dynamic and never-ending shifting balance of form, content and use. These syntactic, semantic, and pragmatic blurred processes are not only regulated by internal mechanisms in any sign system, but also by a crucial dynamic between said and unsaid (utterance/text and genre/context).

Hence two main *processes* are involved, a 'horizontal' (A) blending of form, content and function, studied as dynamics (positionings) of structure, reference and action while uttering, and a 'vertical' (B) where mentally stored elements, or 'meaning potential' in Halliday's terms, work as active resources in theme-rheme processes (Halliday 1994). These are intricate dynamics of given (theme) and new (rheme), as utterers or interpreters unfold a text. For instance in the headline *Bush to Iran* 'Bush' can be seen as known/given/theme and 'to Iran' as unknown/new/rheme.

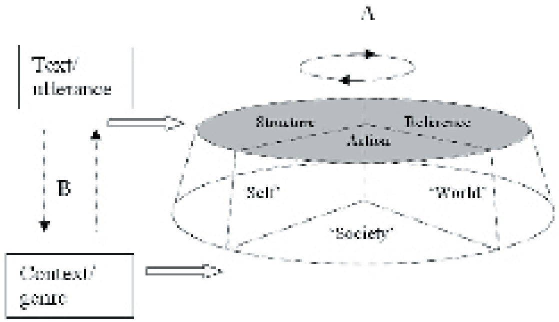


Figure 1: The relationship between the three major aspects on the concrete level of utterance/text (the grey 'surface') and their respectively corresponding three major aspects at the immanent level of context/genre, constituting the 'lifeworld' of any communicator (the 'invisible' lower part).

Further utterances are on the one hand being generated partly by the use of the already stored resources (ie, 'upwards' processes). On the other hand utterances become a potential for future meaning (ie, 'downwards' processes). In other words a contextual genre system will function as an advanced, constantly accumulating, meaning potential for communication by offering partly open, partly closed *pre-balances* (expectations) of form, content, and use. Blanks for instance, will focus on form, definitions content and commands acting. Other prototypical genres can serve as examples of onesidedness, even if in each the utterance will express, refer and address in parallel. However thousands of other genres are found in the discursive landscape *in between*. In addition the genre system of a culture is constantly shifting; more slowly in a traditional society and faster in an expanding culture. Although context is not necessarily the same as genre, genres are inevitably part of context (Erickson and Schultz, 1981; Duranti and Goodwin, 1992; Martin 1997). This implies that the perception of genre is broad and open and that it can not be seen as a defined, precise, restricted category (Freadman, 1994; Ongstad, 2002b). Genre is not a formal text-type, but rather a method of communicating.

Aesthetics-epistemology-ethics

While communicating, we consciously and unconsciously *evaluate*. Firstly, we relate emotionally through personal characterisations or experiences varying from negative, through neutral to positive; such as 'bad' and 'nice': hence creating a 'subject-related' *aesthetics*. This attitude may symptomatically be present in the expressivity of the utterance (Bakhtin, 1986). In linguistics as a discipline this subjective aspect is left out, but for learners to understand it is crucial and inevitable.

Secondly, contents and references can be considered by, or evaluated from, a true or a false regime, thus establishing an 'object-related' *epistemology.* Epistemology positions utterances from the perspective of knowledge. This aspect is dominant in the hard science, which is necessary to achieve valid research . Finally acts can be judged as being right or wrong in some respect (or effective, just, functional and the like) thus founding a 'norm-related' *ethics*. Ethics has recently become an important issue in studies of communication (Bordum, 2001; Habermas, 1998; Bauman, 1995). Hard sciences tend to leave out this aspect (cf. the 'Oppenheimer-syndrome'), and have therefore generally failed to take *responsibility* for how their research/knowledge is used or how a whole academic discipline may *function* as a tool or a means in the cultural domain. In other words, ethics is understood in the broadest possible sense.

However even if presented separately here, these and all other triadic aspects will always occur blurred or together: they are *reciprocal* or mutual within a systemic framework. Hence as a reader of this very paragraph you may for instance consider, consciously or unconsciously, the quality/style of writing (the expressivity) and how it affects your emotions,. For example, how true and valid you think the epistemological claims are as parts and as a whole, and of what help these thoughts are, separately and as a whole. Focusing on the triad in this way actually brings us back to a historical starting point in practical communication. In the ancient discipline of rhetoric, speakers would consider (respectively) the *pathos*, *logos* and *ethos* of their utterances to achieve wholeness and an optimal balance.

Validities

All the three main aspects of an utterance: the form/structure, the content/ reference, and the use/act, as well as the subjective evaluation described above, point to validity as a challenge. Depending on the positioning of the research, and accordingly its object, one faces different expectations of validity. Regarding the dilemma of choice Habermas holds:

> The validity-theoretical interpretation of Bühler's functional scheme offers itself as a way out of the difficulties of speech-act theory because it does justice to all the three aspects of a *speaker* coming to an understanding with *another person* about *something*. It incorporates within itself the truth contained in the use-theory of meaning and at the same time overcomes the types of one-sidedness specific to intentionalistic and formal semantics .... A validity-theoretic interpretation of Bühler's functional scheme further leads to the assumption that with a speech act 'MP,' S takes up relations *simultaneously* to something in the objective world, to something in the subjective world and to something in the social world (Habermas 1998:73,76).

The proposed system needs to be further related to a larger framework or a more general phenomenology. Habermas (1984) holds a triadic view on communication and validity that will have paradigmatic implications. He sees utterances as related to three integrated 'lifeworlds': to the *objective* worlds, about which, in principle true statements can be made; to the *social* worlds, which consist of all interpersonal relations; and to the *subjective* worlds which are experiences to which each communicator has privileged access (Habermas 1984:100). All these aspects and their relations will inevitably be triadic:

> Thus, to the different structural components of the lifeworld (culture, society, personality) there correspond reproduction processes (cultural reproduction, social integration, socialization) based on different aspects of communicative action (understanding, coordination, sociation which are rooted in the structural components of speech acts

> (propositional, illocutionary, expressive). These structural correspondences permit communicative action to perform its different functions and serve as a suitable medium for the symbolic reproduction of the lifeworld (Habermas 1984: xxv/McCarthy, translator's introduction).

While uttering, communicators, continuously and more or less (un-)consciously, will evaluate all three aspects in three different, but always interrelated ways. These intertwined evaluations function as *discursive validation* (Habermas 1984; Bakhtin 1986). Therefore validity connected to aesthetics will be related to the emotional 'nice-ugly' axis, and in Habermas' terms appear as truthfulness in relation to subjectivity. Further validity connected to epistemology will be related to evaluation along the content axis from 'true to false', which will appear, in Habermas' terms, as truth in relation to objectivity. Finally validity connected to ethics will be related to evaluation along the axis of action as 'good' towards 'bad', and appear, in Habermas' terms, as correctness in relation to normativity (Habermas, 1984; Ongstad 1999b).

However the shifting balances between aesthetics, epistemology, and ethics will be involved in any communication and meta-communication. Nevertheless, when there is a tendency to isolate one aspect too much – to overdo the dominant – research runs the risk of developing blind ideologies, which may be labelled by opponents and critics in negative or pejorative ways, such as subjectivism, formalism, essentialism, constructivism, functionalism, activism and the like. To reduce the risk of such blindness, research should originate from a broad semiotic, communicational perspective, which will help to contextualize the focus.

Summary

Figure 2 sumarises some main concepts: (1) Form, content and use are the basic constituents of both utterance and genre. (2) Emotionality is related to form: we search for essence in content and efficiency in acts. (3). These when evaluated, consciously or not, establish aesthetics, epistemology and ethics as separate, but at the same time related, fields. (4) These can be connected to a further division of a person's *lifeworld* in three major dimensions: aesthetics related to self, epistemology to world and ethics to society.again, all mutually related and with no clear-cut division between them as they interact. (5) Validity in these three intertwined fields and dimensions is respectively related to subjectivity, objectivity and normativity, but with no clear discursive borders. Again, all aspects are systemically related to each other, although positioning or discursive *focusing* will put each aspect discursively and thus *mentally* in the forefront.

1. content
2. essence
3. epistemology
4. world
5. objectivity

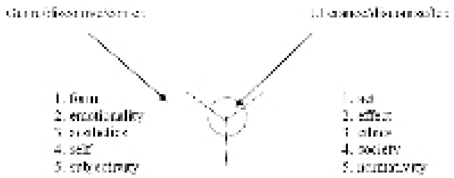


Figure 2: The internal relationship between aspects of utterance/genre (1), the core of main aspects when experienced, considered and measured as an isolated phenomenon (2), fields of judgement (3), relation to lifeworld (4) and basis for validation (5).

The challenge of further differentiation

I will briefly refer here to five different models based on different kinds of triadic thinking. My point is just to indicate that a methodological move from a general theoretical level 'down' to a more specific one, will have to face the problem or the challenge of understanding/investigating how triadic communicational thinking will *differentiate each triadic aspect* from the general to the specific. This question/issue can not be answered objectively. What we can see is that different focusing creates different triads. However these triads are still inherited through the communicational nature of the utterance. The different figures/positionings here are, in consecutive or-

der, 'grammar' (Halliday), 'discourse' (Hernadi), 'learning theories' (Illeris), 'didaktik' (Künzli) and 'pedagogical psychology' (Rørvik). It is not my intention to consider which differentiated triads are valid or invalid. The crucial point is that any research will normally have to relate to a communicational framework of some kind: implicitly or explicitly, triadic or not. The following five are explicit and triadic. However apart from Halliday (and to some extent Hernadi) none of them seems conscious of how, or even *that*, their triads relate to communication or *utterance* as triadic. I will not go into details of each figure. My only point is that these approaches are triadic and that in different ways, and with different concepts and points of departure, each searches a further differentiation of triadic aspects.

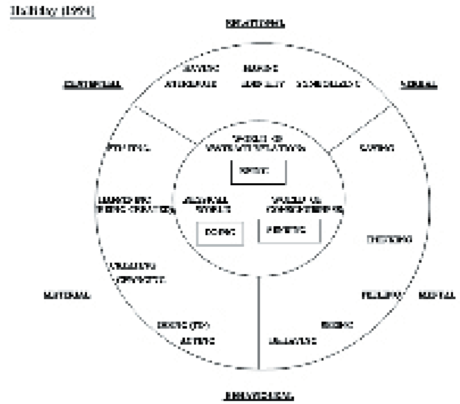


Figure 3: Halliday's systemic understanding of the relationship between six *aspects* of communication (concepts outside the outer circle) and communicational *processes* (-ing-forms) (between the circles) and basic, existential relationships to the 'world' (in the inner circle).

It should be mentioned that the basic triads Halliday and his followers work from in their textual analyses are, at the level of language, *mode*, *field* and *tenor* and, for so-called metafunctions, *textual*, *ideational* and *interpersonal* (Halliday, 1994; Martin 1997).

While one may find triadic perceptions of signs, texts, utterances and communication in the works of Aristotle, Dewey (1916), Habermas (1984), Bauman (1995) and many others, fewer have related the question of triads to the level of genre/context (Halliday 1978 and 1994, Bakhtin 1986, and Martin 1997). Using Martin (1997) we can configure the relationship between the two levels and between triadic aspects in the following way:

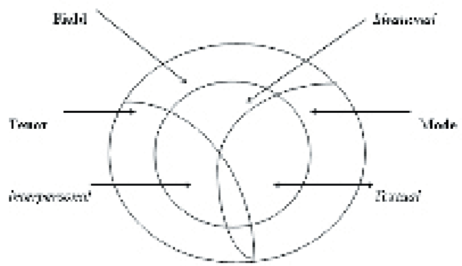


Figure 4: Functional diversification of *language* (the small circle consisting of the textual, interpersonal and ideational aspects) and *social context* (the bigger circle consisting of mode, tenor and field, respectively) after Martin (1997: 5).

Thus according to Hallidayians (or Systemic Functional Linguistics, SFL), language has three metafunctions: ideational, interpersonal and textual, and their function and relationship should be understood as follows:

> Ideational resources are concerned with representation, interpersonal resources with interaction and textual resources with information flow. In SFL this intrinsic functional organization is projected on to context, redounding with the variables of field, tenor and mode – where field focuses on institutional practices, tenor on social relations and mode on channel (Martin 1997: 4).

I will not discuss their coined concepts here, but just underline the main idea. According to this, language is a (triadic) realization of social context and vice versa, social context comprises patterns of language patterns. In other words the main aspects in utterances (texts) and genres (contexts) correspond systemically. This standpoint seems to match Bakhtin's triadic understanding of utterance (and hence even of genre?).

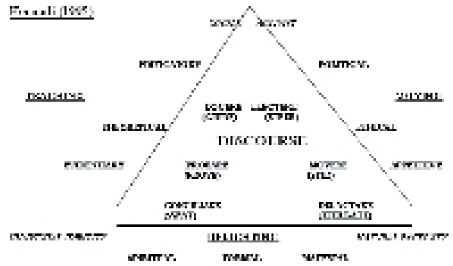
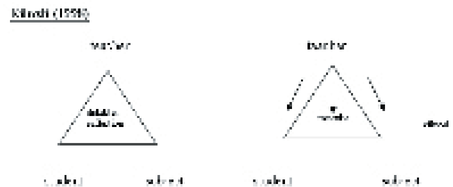


Figure 5: Hernadi's model for three aims of discourse: to move, to delight and to teach.



Figures 6a and 6b: Didaktik and the didaktic triangle and the teacher's attention in the didaktic triangle. The arrows in the didaktic triangle to the right illustrate three *dimensions* of teaching. The *doctrinaire* implies that teachers must know the subject they teach. The *maieutic* means,that teachers, like midwives, in a Socratian way, can help the learner to think about the subject. Therefore teachers need knowledge of the student's knowledge, skills, needs, interests and abilities. Finally teachers need to see *ethical* consequences of their interrelationship with the subject

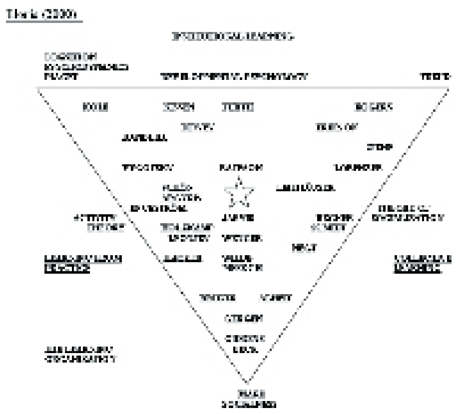


Figure 7: Positions in learning theory seen as a field of tensions, according to Illeris (2000: 190). (My translations from Danish)

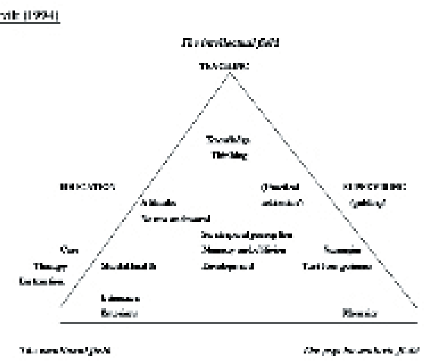


Figure 8: Basic elements in pedagogical psychology and a model for the field of pedagogical psychology. This 'manipulated' figure is a simplified combination of two figures in Rørvik (1994), a textbook for student teachers. Not all elements mentioned are included in the figure. Placed *within* the triangle are (most of) the specific chapters in the book; the extrinsic notions are activities or labels used in the

field. The main idea in Rørvik's overview is that students should get a picture of *where* they are during the study relative to the specific content of the field of pedagogical psychology. I underline that I do not claim, by referring to his triangles, that I necessarily share Rørvik's view or that it is fully compatible with other models presented. I simply indicate the tendency to create overviews; to *position* the field.

Positioning

What I have presented so far is *a theoretical framework*. A more specific point of departure is required to operationalize this framework. (Ongstad, 1999c). The concept coined for this purpose is *positioning(s)*. In the following I will outline the idea of *positioning as simultaneity* between aesthetics, epistemology, and ethics, or between processes of expressing, referring and addressing, to use Bakhtin's notions (Bakhtin, 1986). This concept will allow for *possible* meanings *within* and *relative to* the above framework when interpreting communication from an outside position. The concept of positioning is found, among others, in the work of Harré and van Langenhove (eds) (1997). My own use of the concept of positioning has however developed independently from their understanding. However, the basic configuration seems similar: positioning is seen as a *discursive* communicational phenomenon. My own conceptualization is related to triadic perceptions in the works of Bühler, Bakhtin, Habermas, and Halliday, even if none of those scholars have used the term positioning.

Positioning is in itself an empty and relational concept and can only make sense and have specific meaning combined with a given *focus*. The very act of focusing has some important basic logical implications: Firstly a focus creates a *figure* and hence simultaneously a contextual *ground*, for instance a particular text and its context. Secondly the figure can only be focused from a semiotic, discursive, communicational *position*, which normally will be tacit about itself and thus ideological in its nature (Bakhtin 1986). An ideology is something we think *from* rather than *on* (Ricoeur, 1981). Figure, background and position are logically and inevitably interrelated. Accordingly positioning(s) of someone's positioning(s) can hardly be seen as 'objective', but as *relative*. The validity of positionings of positionings depends on what and how one focuses, discursively.

One challenge in applying theory to practice is how a system is connected to the particular, to specificity, to the individual, to the unique, to the (always) new. In other words, how does the general macro relate to the specific micro. Such particularities have been a major challenge for many approaches, such as general sociology, analytic Marxism, the idea of habitus (Bourdieu 1989), grammar-based approaches (Halliday 1994), genre approaches (Martin, 1997) and even for a theory of communicational actions (Habermas 1984). They are all working from, or concerned with, the general, and seem to get into trouble the closer they get to the unique and specific.

This is not just a theoretical problem. It is one of the main challenges for any pedagogical practice leaning on theory. Dilemmas from the Australian genre debate serve as examples: should one prioritize the text or the genre, the student or the students, individuality or collectiveness, the specific or the general (Reid, 1987)? The ideal answer should be *both*. However which single model can handle such complexity? Positioning, then, is a concept that *tries* to help to bridge such gaps, by refusing to make definite oppositions such as subjectivity versus objectivity, objectivity versus normativity or normativity versus subjectivity, and by accepting the close and dynamic relationship between utterance and genre, text and context. In addition it accepts and utilises not only the relative nature of language and communication but even the paradoxical relationship between them.

Further positioning is systemic, which means that no part of the conceptual network should be given priority before an object is focused or a position is given, that is, an aspect is given *dominance*. This principle obstructs the idea that any of the main aspects or relations can have 'universal' priority over other aspects. It all depends on the purpose. It also means that the system should be seen as open-ended, relational and non-hierarchical, that is, as *a systemic network and framework*. Further it implies that all constitutive concepts and elements are mutually defined and that no category or concept within the conceptual network is valid in itself, but only in relation to the

communicational semiotic framework as a whole (Halliday 1994, Habermas 1984).

Accordingly an utterance can, in the broadest sense, be seen as a dynamic encounter of self, world, and society, of each person's lifeworld, and research must balance the major elements: aesthetics, epistemology and ethics. Utterance then has to be understood broadly. (Foucault even saw the pyramids as utterances.) Hence positioning will happen *within* the gravity of triadic aspects in communication, sometimes falling into typicalities such as very expressive or mainly cognitive or clearly social; or combinations such as socio-cognitive, psycho-analytic or socio-constructive. Usually such stereotypes are simplifications.

All communicational elements then can be seen as triads and as dynamics of utterance and genre. As meaning, potential *genres* allow utterances to say more than is obvious on the surface. Communicators, even including silent *observers* of communication, are positioned *within* the systemic framework of double triads of utterances and genres. The framework is not a method, but analyses of positioning(s) are. Positioning applies to communication as an object, as well as to any communicational aspect or element in the research process at a specific or a general level. These aspects cannot be seen as *categories* but as systemically related elements. The kind of validity we attach to them will depend on how the genre in question will balance (or foreground) aesthetics, epistemology or ethics.

Summing up, positioning(s) should be seen as a process (and a product) in which sign-users locate themselves, the world and others, semiotically in utterances. The processes and the products are framed by the impact of embodied meaning resources, by genres (or discourses). Communicators have to move between structuring form, expressing the uttering self, referring to the world and addressing others. Hence in the processes a self, a world and a society are established as integrated, reciprocal both for utterer and interpreter. Positioned as products (most) of the utterances first occur as visible, physical, concrete cultural artefacts. In one sense they are all 'dead' since they just exist as structured materiality. To (re-)create *meaning*, the processes have to be (re-)established and at least partly reconfirmed. This can only happen through the initiation of the immanent resource system (the genre system). Hence as long as there are active genre systems, there is potential for maintaining meaning related to these physical forms that are able to communicate. When genres are dead (or lost) or unknown to the receiver the forms degenerate simply to materiality, and communication fades out.

Positioned as processes, expressing-referring-addressing are in principle immanent even if the expressing of form can be sensed. In other words, the study of positioning is an interpretative activity. As a process, positioning brings in elements for uttering from the embodied resource system in balanced theme-rheme processes. In other words, a certain mixture of old/given and new, based on syntactic-semantic-pragmatic 'rules' or conditions for the sign-system and the genre(s) in question. In principle there is no unique positioning opportunity for the observer, the researcher. The same general framework is also valid for the spectator. To consider, to look at, to categorize, to interpret is just a question of using particular genres that help to form the utterances from these positions.

Positioning emotionality in a textbook in mathematics education

Background and hypothesis

A significant pattern of mathematics in educational settings is competition (Braathe and Ongstad, 2001). From primary school to university the system produces 'winners'. Most teacher educators of mathematics and writers of mathematics textbooks are likely to belong to those who have not only survived the system, but have probably developed mostly positive attitudes to, and a personal interest in, t the discipline. On the one hand, this may create engagement for both writers and readers. On the other, this positive attitude may contribute to a certain 'blindness' vis-à-vis problems which student teachers with different experiences may find.

If we consider utterances as a discursive 'place' where attitudes and engagement will surface, a textbook in mathematics, for instance,

could be regarded as displaying *symptoms* of certain attitudes. Normally 'pure' mathematics will have no place for emotions. The focus is almost entirely on the topic. However in teacher education, the content and the communication of mathematics now seems to have moved from the mathematics alone towards a more didaktic/educational attitude to mathematics. In this approach there are probably fewer restrictions to the expression of emotionality within mathematical texts.

In the amalgamation of the two fields, mathematics and the didaktics of mathematics, writers' positive attitudes may surface and compete in the text with knowledge of mathematics. This overlap is likely to occur where new understanding is added to old. It is likely to appear at the discursive borderline where new mathematical elements are embedded in the text together with given/known knowledge.

Emotionality in mathematics and mathematics education

In Norway, mathematics textbooks for student teachers are classified by the authors in three main categories, as 'mathematics' (for teachers) (Breteig and Venheim, 1999), as (combined) mathematics and didaktik of mathematics (Nygaard, Hundeland and Pettersen, 1999) or as didaktik of mathematics (Germ. 'fach-didaktik') for the teaching of mathematics (Solvang, 1992, Herbjørnsen 1998, Johnsen Høines 2001). What I find interesting is the in between-category, the explicit target to cover both aspects, mathematics and didaktik, in one book. In other words, the book could be seen as an (extensive) utterance where a dyadic, language-like mathematics and a triadic, communicative didaktik are mixed. I will focus on a few passages that can be related to the *affective* domain in one particular book. The title of the textbook, *Aha*, uses the Norwegian interjection 'aha' to allude to the heuristic feeling of sudden discovery (like Archimedes' famous Eureka exclamation) (Nygaard, Hundeland and Pettersen, 1999). I underline that these few utterances are in no way representative of the book as a whole. My intention is to read the text extracts as *symptoms* of a tension of disciplinary positionings, where emotionality plays a role.

In the preface the writers point towards some characteristics of the book. They underline that they want to change student teachers' conceptions of what mathematics is or could be, and that teachers should be able to *mediate* ('formidle') mathematics. Their explicit intention is to initiate and to train the ability of (*didaktic*) *reflection*. A further focus is their conscious use of the word *you*, as they view the text as explicitly directed to the student teacher(s). The organization of the chapters shows that the two main aspects of mathematics: mathematics and didaktik, are balanced in the following way: In all except chapter 11, mathematics plays the main role. However in most chapters there is at least one didaktic aspect: such as what is knowledge of mathematics, children's learning of numerical representation, algorithms in school, probability calculation as a topic in the curriculum, didaktic analysis of the concept of 'function', etc.

Example 1

In chapter 11 some Norwegian mathematicians are described, partly from the point of view of their biographies, in a narrative story-telling form, and partly from the point of view of their research. The writers describe how a particular theorem about prime numbers was explained by the Norwegian mathematician Atle Selberg. (The example is difficult to translate into English for reasons that soon become obvious.)

For this achievement [bragd] Selberg got Field's medal [the highest award in mathematics]. Atle Selberg is considered as Norway's greatest/biggest [største] contemporary mathematician.
If your teacher boasts about that he is the greatest/biggest [største] mathematician, then he has to mean 'great/big' [stor] differently from how we use the word here. Perhaps your teacher means that he is the fattest mathematician when he says he is great/big [(stor)] (Nygaard, Hundeland and Pettersen, 1999:596)!

Example 2

One chapter concerns the history of equations, starting with first grade equations and ending with the impossibility of solving fifth grade equations with an algebraic formula for the solution (Norwegian =

'løsningsformel'). The historical 'race' in the Renaissance between Italian theorists to be the first to solve third grade equations (and to keep the secret) is described. (Nygaard, Hundeland and Pettersen, 1999: 410–411). Moving from third to fourth grade equations the following lines occur:

You learned to solve second grade equations in high school, now it will probably be fun [blir det nok gøy] to learn to solve the third grade equation. The fourth grade equation we are afraid you must wait to learn [må du dessverre vente med å lære] until you choose a more advanced course in mathematics (Nygaard, Hundeland and Pettersen, 1999: 411).

Finally we will turn to the presentation of the history of the fifth grade equation (Nygaard, Hundeland and Pettersen, 1999:412–413).

Example 3

$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$

For hundreds of years mathematicians tried to solve the fifth grade equation above. In the 1820s the young Norwegian mathematician Niels H. Abel believed he had solved it, and none of the professors he showed it to could find anything wrong with the solution. Nevertheless, not long after, he found that he was wrong. He now decided to turn his 'project' upside down so to speak, and on the contrary to try to prove that it was unsolvable by the use of a formula for solution ['løsningsformel'], which he eventually achieved in 1824.

What Abel did was to find which characteristics such a formula had to have, in order to prove that irrespective of how you use the formula for the solution, it will not have the appropriate quality [egenskap]. Smart, don't you agree?

It further becomes clear that for $n > 4$, there is no general algebraic formula for the solution of the nth-grade equation. Hence with this result from the 19th century we must just forget becoming famous for finding an algebraic formula for the solution of, for instance, the general eighth-grade equation (Nygaard, Hundeland and Pettersen, 1999:413).

Example 4. Samples from chapters 5.2 and 5.3

P. 185: Next is 3/7, <u>then</u>. [Så var det 3/7, da.]

P. 186: Now it will be <u>exciting</u> to see what happens when we 'drive' the algorithm ofdivision (...) [Nå blir det spennende å se hva som skjer når vi kjører divisjonsalgoritmen (...)]

P. 187: <u>Elegant, or what?</u> [Elegant, hva?]

P. 191: With this word we can accumulate the observations in this paragraph to a <u>nice</u>, useful and important result: (...) [Med dette ordet kan vi slå sammen observasjonene i dette avsnittet til et fint, nyttig og sentralt resultat: (...)]

P. 192: <u>Fortunately</u> there is a faster, and just as reliable method to find the answer [(...)Heldigvis finnes det en raskere, og like sikker metode for å finne svaret (...)]

P. 197: Why, is this so, <u>then</u>? [Hvorfor er nå dette slik da?]

P. 198: (...) <u>and in multiplication we are now experts</u>. [(...) og multiplikasjon er vi jo blitt eksperter på nå.] (Nygaard, Hundeland and Pettersen, 1999: 183–201, my underlinings in all examples.)

Analysis/comments/exemplifications

Relating the discussion earlier about positioning and the section on validity, to the examples above, we can say that it has to be considered whether the emotionality indicated covers what these writers wanted to express. We also have to consider whether there actually

(objectively) exists something like the 'category' we have 'found' and should test whether our readers will accept our speech act as good or bad (adequate) in the given context (pragmatic validity). Above all we should signal which of these aspects are most interesting and relevant in this context, is it the *quality* of the 'category' or is achieving absolute *accuracy* (producing a truth) or is it the basis from which one would like to *act* that is most important?

The focus for writers in this field is generally to present specific topics of mathematics for student teachers. Thus the book is an utterance which prioritises a *description* of mathematical knowledge. For the authors there are several layers of *contexts* around it. The book is written in a certain *genre*, but partly and deliberately the utterers, although rarely break some genre expectations, for instance by adding some *affectives* where they did not have to. We, as observers however, can choose to focus on an aspect, *a figure*, which in my case is *emotionality*. The rest of the text will accordingly become background or context (or co-text). It is my research interest, that is my *platform* or my *position*, which establishes this particular *focus*. I am making a static 'category' of the emotional, dynamic aspect in this textbook.

I interpret the tendency to emotionality, although scarce and sometimes just vague, to what I have called *ideology*. One important ideology for most subject teachers, even in mathematics, is that their subject is *fun*, and that this fun is taken for granted. Hence they are mostly *engaged*. This engagement is partly related to *sense* : They have invested in their professional knowledge, and this knowledge has become part of a more coherent understanding, which may help in motivating their teaching and it is often *symptomatically* visible in their discourse, their utterances.

A problem with this *enthusiasm* is that it may sometimes communicate itself differently depending on students' attitudes to mathematics. Let us imagine that there are student teachers who did not understand the explanations, before the writers exclaimed 'Smart' and 'Elegant'. They may not have been very inspired by the level of *intensity* in the presentation. This is not a critique of the writing. The point is that the *affective domain* easily gets under undermines *aesthetic evaluation*, and therefore interpreters of such writing or such utterances cannot easily judge its value and function as utterances. *Emotional discourse* may be met with *subjective* and/or normative reactions.

The main *genre* of the text is a traditional textbook in mathematics blurred with more open genre expectation for the didaktik of mathematics. The *epistemological* aspects, the knowledge elements of mathematics, as well as the *ethical* aspects, the sense of doing and learning more mathematics, are seen as *positive* and inevitable. An implicit *ideology* is that mathematics in the educational system is a matter of course and *worthwhile* learning. This attitude creates and supports a predominantly positive and *enthusiastic* attitude in the writing of the book (as utterance). As utterance it mostly follows genre expectations in mathematics. However in the more didaktic parts the *enthusiasm* (aesthetics) shines through the varnish of *neutral* epistemology and ethics. Mathematics is *fun* (example 2), it is acceptable to make fun/jokes (even if not taken from mathematics) (example 1) and a certain proof is *smart* (example 3). In addition whole chapters, for instance 5.2 and 5.3 are 'driven' or fuelled by this engagement (*then, exciting, elegant (or what), nice, fortunately, now, then*).

The presentation of *new* elements added to *the given* is generally presented neutrally, however now and then quite *positively*, trying to show the epistemological progress as exciting. The silent ideology of this presentation is that learning in any case is not associated with negative problems, for instance the constant threat of not understanding, of being slow to learn. On the contrary there are some *elements of admiration for advanced thinking*. There is a 'race' encapsulated in mathematics as a discipline for understanding and accumulating new knowledge fast. This race is not evident in mathematics as a research field. Its ideology even finds its way into mathematical education, where smartness and brightness and deep understanding is adored and cultivated within the lines of mathematical utterances (Braathe and Ongstad, 2001).

To recap, these examples are rare and not at all representative. After all the book contains more than 700 pages of plain mathematics. The general impression is a quite traditional presentation, common to most mathematics textbooks. This genre is characterised by almost

no emotional exclamation. Hence in general these focused, positive elements are few and it could be argued that it would be wrong to generalise from them. On the other hand it could be held that it is not necessary to can count the number of positive, neutral and negative words and then summarise. We should look at the *intensity* and at the strategic placing of the words and we should consider the cultural tradition of using emotions.

Finally it is important to be *self-critical*. My positioning(s) focuses emotionality and aesthetics and this will surely blind me to other aspects of the intention of these textbook authors, never mind how other readers may react. These 'fragmented' glimpses must be gathered together in a fuller didaktic analysis. Considering the emotional self-positioning(s) of textbook authors can only be one aspect of a more advanced understanding.

A positioning of Barbara Jaworski's 'Teaching Triad'

In further differentiation of a triadic framework it is easy to lose sight of the implication of triadic forces, and it is hence tempting to see utterances, focused figures, and signs as structural, that is as dyadic *objects*. The reduction from triads to dyads is in reality implicitly a *de-contextualisation*, that help the researcher or the analyst to obtain an object that can be researched according to essentialistic research methodologies. Sometimes such a simplification may be not only valid and fruitful, but even necessary.

One illustration is from Jaworski (1994). One of her points of departure is *the teaching triad*. That is from her research position, her 'figure', her hypothesised object. Another is (socio-)constructivism, which, relative to the triad, puts stress on the student as learner. The concept of 'the teaching triad' is crucial in her study in *Investigating Mathematics Teaching*. It is described as the relationship between three aspects: *management of learning* (ML), *sensitivity of students* (SS) and *mathematical challenge* (MC). Her approach is to follow each of these three aspects separately when studying three different teachers. Part of the strategy is to validate the relevance of the teaching triad. Thus in three different chapters she focuses on *Clare: Origins of the Teaching Triad; Mike: Episodes and the Teaching Triad*; and *Ben: Af-*

firming the Teaching Triad. In the beginning of the chapter about Ben, Jaworski explains that:

> (...) I decided to offer my language of the triad. I simply said to him that I had found the three 'headings' (ML, SS, MC) useful in describing teaching which I had seen in other classes, and wondered if those headings might mean anything to Ben in terms of his own teaching.
> His immediate response was, 'I feel that management of learning *is* my job as a teacher. As a teacher, that's my role in the classroom – as opposed to managing knowledge' (Jaworski, 1994:144).

Ben also produced a piece of paper on which he had made some notes under each heading. Jaworski finds that Ben's perception is compatible with hers (or vice versa):

> Management of learning: (as opposed to management of knowledge?) *I like to be a manager of learning.* (...) *My role: organiser of activity or questions, chairperson, devil's advocate, challenger, listener, learner, making students aware of other students*. (...)*I am not a judge*.
> Sensitivity: – feelings – threat (need for success) (...) *(everyone should be able to start the activity, success breeds success* (...) *choosing the activity level of difficulty chosen by the student – not today!* (...) *to the needs of 30 students – what a challenge.* (...) *sensitivity to students, my role.*
> Mathematical challenge: (...) *-everywhere ... from the teacher good from the students – and it takes off!* (...) *But how do we get there?* (Data item 9.1: Notes written by Ben (November 1988)) (Jaworski, 1994:144–145.)

Jaworski found, through their discussions, that Ben's view of the teaching triad could be modelled as in Figure 9. She argues that it was hard to avoid the impact of the triad, in approaching Ben's teaching, and hence she decided to analyse the teaching from this perspective.

Sensitivity of students. MC= Mathematical challenge. The figure should be understood as two circles/ellipses/aspects partly overlapping each other both being included in a larger, more general circle/ellipsis/aspect.

Ben is a teacher who wants to give priority to *investigative teaching* (Jaworski's main topic). Jaworski observed, among others, lessons where he was working on 'vectors', and Ben is, partly self-critically, classifying these lessons as, in his terms, 'didactic', and other than his normal teaching. 'Didactic' to him means 'giving knowledge out', which implies to him 'a *transmission* view of teaching' (Jaworski, 1994:158). In discussions and in her design and development of her research project Jaworski pushed this tension, the 'didactic/constructivist' tension, which she saw as fundamental. I skip how she follows this hypothesis analytically and methodologically and go straight to one of her conclusions:

> I perceived that the term 'didactic' was used when Ben felt that information had to be conveyed which he could not approach through exploration or questioning. (...) Perhaps for him, didactic was associated with exposition, and giving definitions, and he saw that giving definitions, although inevitable, seemed to involve a process of conveyance rather than encouraging active construction. This is indicative of the didactic-constructivist tension. (...) It was possible to see much of Ben's teaching as investigative, and as deriving from a constructivist philosophy, but he seemed to have trouble with the intersubjectivity of knowledge (Jaworski, 1994: 169).

Jaworski finds that Ben is grappling with the *didactic tension*, a term coined by Mason, and indirectly inspired by Brousseau's notion of *the didactic contract*. This tension can be explained as a dilemma: The *more* explicit and detailed teachers are about what they wish from students, the more likely students are to take *form* for substance/content, that is to take the easy way. The *less* explicit teachers are about what they expect, the less likely students are to notice what is really going on and to see the point (Mason, 1988:33; Jaworski, 1994: 180).

Jaworski further follows up the intersubjectivity problem in her theoretical discussion:

> Epistemologically, the status of intersubjective knowledge is a problem. It can certainly not be viewed as objective, although language patterns often seem to equate it with objective forms of knowledge. However, as Chapter 2 argued, the status of knowledge within a constructivist paradigm is itself problematic. Status seems less important than the value of the concept, which is to provide a bridge between individual construction and some consensus in mathematical understanding within a community. Development of knowledge can be seen as a collective process of individual construction within a community rich in socio-cultural influences, the purpose of which is to reach a high degree of intersubjectivity. This has powerful implications for classroom approaches. Their purpose must be to achieve this intersubjectivity (Jaworski, 1994:212).

Finally, regarding *validation* of her own research, Jaworski holds:

> I have offered ideas to the wider research community and learned from interactions and responses. The validity of the research rests with this intersubjectivity as well as on attempts by the researcher to critique interpretations, recognize limitations and make the basis of judgements theoretically and contextually clear (Jaworski, 1994:213).

Analysis, suggestions and comments

In principle Jaworski seems to have a triadic view, however not explicitly communicational. A 'problem' with her teaching triad, though, is that it does not seem to be explicitly *reciprocal*, which means that not all three aspects are seen as inevitable. It is rather a question of three aspects put together (a dyad plus a monad). This becomes clear when she does not comment on Ben making a dyad of the triad by including sensitivity of students and mathematical challenge *within* management of learning. Nevertheless her own analyses of Clare's and of Mike's teaching followed each of the three aspects separately. When she comes to Ben, this pattern is not followed though. If her three aspects are interpreted sympathetically, it could be said that the triad actually consists of the main aspects of the classical didaktic triangle: teacher, student and subject.

One clear tendency is not to consider content or subject or discipline as an independent acting force, in spite of the fact that she speaks of the challenge from mathematics. At least in the first part of the book mathematics is perceived of as being generated by the students and the teacher. I think the uncertainties are related to *the positioning effect of constructivism*, giving priority to learning (positioning learning as the *dominant*). What we can see at work is that *teaching* has lost its status as an *independent aspect* in the triad, at least compared to the didaktic triangle. Or, in other words, teaching is primarily seen as a balance or as a dilemmatic relationship between student and school subject. Constructivism's current 'victory' in written curricula in many European countries may serve as an illustration of an effect of this positioning. My point here is not to criticise such a development as wrong in its own terms, but just to point to what accordingly is communicatively *figured* (and hence dominant) and what is pushed into the *background* as tacit context. Disciplinary triangles of the kind presented in the discussion , can actually help learners/student teachers to position the approach in question.

However, towards the end of her book, it becomes clear that Jaworski finds the (lowered) status of mathematics as a subject in its own terms as problematic. Giving priority to the individual learner and the way in which this learner will construct his/her knowledge of mathematics as a discipline is *backgrounded*. And as she, as a true constructivist, hesitates to call the content of mathematics a discipline for *knowledge*, her alternative focus is finally to give priority to *intersubjective* knowledge. This positioning actually makes students' progress in *mathematics* the most important goal. Ironically, this is even the goal for directions that are sceptical about radical and social constructivism.

Finally, some thoughts on the terms/concepts didactics versus didaktik in Jaworski's book: According to German, French and Nordic perceptions, *didaktik* is not a pejorative term in the way didactic/didactics generally is in English. In Jaworski's writing there is a mixture of both terms without making the difference an issue. Thus for Ben, 'didactic' is the negative counterpart to the positive 'investigative'. Jaworski admits that since contact between herself and Ben was based on the fact that she was researching, that is describing, not judging, investigative teaching as an approach, she probably (in my words), positioned Ben to 'justify' this kind of teaching, and that he was consequently rather negative about his own so-called 'didactic teaching'. Explaining the 'didactic tension' (based on ideas of Brousseau and Mason) changes the picture, at least for readers with a background in continental didaktik, but perhaps not for Anglo-Saxon readers. This positioning can therefore, at the end of the day, be *ambiguous*: one aspect functioning as *descriptive* and epistemological and the other as *expressive* or perhaps even as a *normative* characterization.

Conclusion

Positioning can be used more openly and tentatively as I have mainly done here. In other studies I have used positioning analysis more narrowly, following word for word, utterance for utterance, in a text or from a classroom (Hudson, Ongstad and Pepin, 2002). As positioning is a relative concept it can be used both on the macro and the micro level. Anything can in principle be positioned. Once the focus is decided, it is crucial to locate where in the triadic, semiotic, communicational lifeworld this positioning takes place, and hence which validities different choices may imply. For mathematics education it is a significant tension that the 'mother discipline', mathematics, is developed by the axiomatic change of the triad to a dyad, and that

education, on the other hand, is significantly communicational (or triadic). To explain this paradox is a major challenge (Ongstad, 2002a and 2002c, 2004).

Important concepts in positioning theory

figure	a focused object
ground	context as result of focusing
context	any leftover by the process of focusing a text or an utterance as meaning
position	communicational/logical/discursive/theoretical/ ideological 'site' for focusing
positioning	the process of creating a figure and hence a ground from a position
positionings	the product(s) of the process of positioning
closed object	a semiotic monad or a dyad (mathematics needs closed objects, and hence mathematics uses means to achieve that, such as axioms and definitions)
open object	a semiotic triad (mathematics education is by definition open and needs tools to stabilise a figure to get a researchable object)
axiom	a-taken-for-granted premiss which will be not validated
definition	a meta-language proposition made to achieve a certain conceptual stability
monad	a closed, whole (indivisible) phenomenon
dyad	a whole constituted by two necessary aspects
triad	a whole constituted by three reciprocal/mutual aspects
sign	something that stands for something (else) in some respect
dynamic sign	something that stands for something (else) for someone in some respect
utterance	a triadic, dynamic, open sign, a figure that is related to its genre-like context,
that has	a sufficiently clear (or intended or possible) beginning and end
genres	sorts or kinds of communication

a genre a particular sort or kind of communication

communication an utterance (someone utters by something, something else to someone)
(this perception is accordingly different from the view that someone has to
be *reached* or has to respond before we can speak of communication)

dyadic positioning the making of an object/figure by means of structural-semantic principles

triadic positioning the paradoxical approach to a communicational phenomenon as triadic
(consists of triadic sets). (Implies the hermeneutic circle.)

ideology the unspoken communicational value of context or genre in relation to
a specific text or an utterance

Overview of some communicational triads

Utterance, genre, language as	form	content	use
Utterance, genre, language as	structure	reference	act
Processes	structuring	referring	addressing
Process as focused characteristics	expressivity	referentiality	addressivity
Fields of language and semiotics	syntax	semantics	pragmatics
Grammatically metaphorized as	adjective	noun	verb
Seen as pronoun	I	it	you
Contexts of the body	self	world	society
Domain of reality (Habermas)	inner nature	outer nature	society
Halliday´s language contexts	mode	field	tenor
Contexts of semiotic space	forum (form)	universe (of meaning)	arena (acting)

Triads used by different theorists

Bühler (1934) the language sign	expression	representation	appeal
Language sign (in German)	(Ausdruck)	(Darstellung)	(Appell)
Aspect, focused by receiver	symptom	symbol	signal
Halliday (1978) metafunctions/	textual	ideational	interpersonal
context/register	mode	field	tenor
Bakhtin (1986) constituents (of utterance)	expressivity	referentiality	addressivity
Bauman (1995) processes	aesthetic	cognitive	moral
crucial elements	emotional intensity	knowledge	responsibility

Habermas (1984)

Aspect of lifeworld	inner nature	outer nature	society

Form of appearance for reality relations		subjectivity	objectivity normativity
Speech acts	expressive	propositional	illocutionary
General functions	experiences	matters	relations
Mode	expressive	cognitive	interactive
Speech act	representative	constative	regulative
Theme	utterer´s intention	propositional content	interpersonal relation
Validity claim	truthfulness/ veracity	truth	fairness, usefulness
Components of the lifeworld	personality	culture	society
Reproduction processes	socialization	cultural reproduction	social integration
Communicative action	sociation	understanding	coordination

Hernadi (1995)

Major elements	self	nature	society
Other major elements	existence	culture	experience
Rhetorical aims	delighting	teaching	moving
Discourse involves	individual identity	natural facticity	social activity
Psychological capacities	feeling	knowing	willing
Evaluative criteria	beauty	truth	justice

Traditional

'Traditional' concepts	emotional	cognitive	social
'Traditional' concepts	existential	informative	addressive
'Traditional' concepts	beauty	truth	goodness
Fields by evaluation	aesthetics	epistemology	ethics
Evaluations (ranging from)	ugly – nice	false – true	wrong – right
Societal domains	art	science	politics
Outcome from the domains	artwork	research	decision/ regulation
Traditional triads	feelings	thought	will
Alliterations	heart	head	hand
	beauty	brain	brawn
	identity	idea	interrelation

Aspects positioned as:

State of the body (as a noun)	being	thinking	doing
Bodily processes (as a verb)	being	thinking	doing
Effect in/of the sign process	impulse	category	process

Positioning(s) of professional fields, approaches, and isms when dominant

Aspects of language/ semiotics	syntax	semantics	pragmatics
General approaches	aesthetics	epistemology	ethics
Professions and disciplines (examples)	psychology	science	sociology
Different (neg.) characterizations	subjectivism	objectivism	activism
Different (neg.) characterizations	expressivism	positivism	functionalism
Different (neg.) characterizations	formalism	essentialism	pragmatism

References

Bakhtin, M. (1986) *Speech Genres and Other Late Essays* (Austin: University of Texas Press).

Bauman, Z. (1995/1993) *Postmodern Etik*. [Postmodern Ethics] (Gothenburg: Daidalos).

Bordum, A. (2001) *Diskursetik og den positive selvreference*. [Ethics of discourse and the positive self reference] (Copenhagen: Samfundslitteratur).

Bourdieu, P. (1989) *Outline of a Theory of Practice* (Cambridge, UK: Cambridge University Press).

Breteig, T. and Venheim, R. (1999) *Matematikk for lærere*. [Mathematics for Teachers] (Oslo: TANO).

Braathe, H.J. and S. Ongstad (2001) Egalitarianism meets ideologies of mathematical education – instances from Norwegian curricula and classrooms. *ZDM*, Vol. 33 (5).

Bühler, K. (1934/1965) *Sprachtheorie* (Stuttgart: Fischer).

Bühler. K. (1990/1934) *Theory of Language. The Representiall Function of Language* (Trans. D. F. Goodwin.) (Amsterdam: Benjamin).

Dewey, J. (1916) *Democracy and Education* (New York: The Macmillan Company).

Duranti, A. and Goodwin, C (eds.) (1992) *Rethinking context. Language as an interactive phenomenon* (Cambridge, UK: Cambridge University Press).

Erickson, F. and Schultz., J. (1981) When is a context? Some issues and methods in the analysis of social competence, in J. Green and C. Wallat (eds.), *Ethnography and language in educational settings* (Norwood, NJ: Ablex).

Herbjørnsen, O. (1998) *Rom, form og tall. Matematikkdidaktikk for barnetrinnet*. [Space, form and number. Didaktik of mathematics for primary school.] (Oslo: Tano-Aschehoug).

Hernadi, P. (1995) *Cultural Transactions. Nature, Self, Society* (London: Cornell).

Habermas, J. (1981) *Theorie des kommunikativen Handelns*. (Vol. 2.) (Frankfurt am Main: Suhrkamp).

Habermas, J. (1984) *The Theory of Communicative Action*. (Vol 1.) (trans. T. McCarthy) (Boston: Beacon)

Habermas, J. (1998) *Postmetaphysical Thinking*: *Philosophical Essays* (London: Polity Press).

Halliday, MAK (1978) *Language as social semiotic (*London: Arnold).

Halliday, MAK (1994) *An Introduction to Functional Grammar. Second Edition* (London: Arnold).

Harré, R. and L. van Langenhove (1997) *Positioning theory* (Oxford, UK: Blackwell).

Hudson, B., Ongstad, S. and Pepin, B. (2002) *Didaktik der Mathematik: teaching-learning mathematics* (Electronic resource book in didaktik/fach-didaktik.) http://ive.pa-linz.ac.at/emdid/latest/default.html.

Illeris, K. (2000) *Læring – aktuell læringsteori i spenningsfeltet mellom Piaget, Freud og Marx*. [Learning – current theories of learning in the field of tension between Piaget, Freud and Marx.] (Roskilde Universitetsforlag).

Johnsen Høines, M. (2001) *Begynneropplæringen. Fagdidaktikk for barnetrinnets matematikkundervisning*. [Education for beginners. Fachdidaktik for the teaching of mathematics in primary school] (Bergen: Caspar forlag).
Jaworski, B. (1994) *Investigating Mathematics Teaching* (London: Falmer).
Künzli, R. (1998) The Common Frame and the Places of Didaktik, in Gundem, B. and S. Hopmann (eds.) *Didaktik and/or Curriculum. An International Dialogue* (New York: Peter Lang).
Langenhove, L. van and R. Harré (1993) *Positioning and Autobiography: Telling Your Life, i*n N. Coupland and J. Nussbaum (eds) *Discourse and Life-span Development (*London: Sage).
Martin, J. (1997) Analysing genre: functional parameters, in Christie, F. and J. Martin (eds) *Genre and Institutions* (London: Cassell).
Nygaard, O. et al (1999) *Aha. Matematikk og matematikkdidaktikk* [Aha. Mathematics and didaktik of mathematics] (Kristiansand: Høyskoleforlaget).
Ongstad, S. (1999b) Self-positioning(s) and students' task reflexivity – a semiotic macro concept exemplified. *Journal of Structural Learning & Intelligent Systems*, 14, 125–152.
Ongstad, S. (1999c) Vad är positioneringsanalys? 'Självpositionering' i en (post)modern skola som exempel [What is positioning analysis? 'Self positioning' in a (post)modern school as an example], in C. A. Säfström and L. Östman (eds): *Textanalys. En introduktion til syftesrelaterade analyser* [Text analysis. An introduction to purpose related analyses] (pp.148–176) (Lund: Studentlitteratur).
Ongstad, S. (2001) *Positioning aesthetics, epistemology and ethics in didaktik of subjects*. Paper at the conference of the European Educational Research Association, Lille, France.
Ongstad, S. (2002a) Genres – from static, closed, extrinsic, verbal dyads to dynamic, open, intrinsic, semiotic triads, in R. Coe, L. Lingard and T. Teslenko (eds), *The Rhetoric and Ideology of Genre: Strategies for Stability and Change* (pp. 297–320) (Cresskill, NJ: Hampton Press).
Ongstad, S. (2002b) Positioning Early Research on Writing in Norway. *Written Communication*. 19/3.
Ongstad, S. (2002c) From Utterances to Genres in Mathematics Education (and vice versa) – a dynamic triadic (socio-)semiotics exemplified. Paper at PME26, Norwich, July. (See PME26's website, Discussion Group, Semiotics: http://www.math.uncc.edu/~sae/)

Ongstad, S. (2004) *Språk, kommunikasjon og didaktikk.* [Language, communication and didaktik](Bergen: Fagbokforlaget og Landslaget for norskundervisning).
Ricoeur, P. (1981) *Hermeneutics and the Human Sciences* (Cambridge: Cambridge University Press).
Rørvik, H. (1994) *Læring og utvikling* [Learning and development] (Oslo: Universitetsforlaget).
Solvang, R. (1992) *Matematikkdidaktikk* [Didaktik of mathematics] (Oslo: NKI-forlaget).

Contact

Prof. Dr. Sigmund Ongstad
Oslo University College, Faculty of Education
Email: Sigmund.Ongstad@lu.hio.no

Public and personal inerpretation of a poing, a straight line and their relation: A comparison of phylogenesis and ontogenesis

Magdalena Prokopová, *Charles University Prague*

Abstract

The purpose of the paper is to consider possible tools for verifying whether there are parallels between the historical development of the public interpretation of a point and a straight line and their relation and possible cognitive developments examined in today's generation of pupils. The structure consists of an historical analysis and an abstraction of the main points gleaned from that analysis, placed in a rough hierarchy of development. This is followed by the exploration of students' personal interpretations concerning the notion of a point and a straight line and their relation which, by considering the ages of the students, can also be put in a rough hierarchy of development. The two resulting hierarchies are compared and conclusions drawn.

Introduction

The phenomenon of infinity has been with us since the time of ancient mathematics. It has always awakened wonder and apprehension. We can focus on infinity from different points of view. Here we will be concerned with the phenomena of geometry, their relationships and their interpretations, as they relate to the phenomenon of infinity.

A point and a straight line are fundamental objects of the Euclidian geometry which is taught at basic and secondary schools. Philosophers have meditated on the nature of a point and a straight line from long before Euclid's time (from the 6th century BC). But it was Euclid (about 325–265 BC) who delimited the concept of a point and a straight line (and others) by means of a definition in the First Book of his Elements (*Stocheia*). The phylogenesis of a point and its relation to a straight line is marked out by names such as Viète, Kepler, Leibniz, Newton, Bolzano and Cantor.

The research project attempts to describe phenomena connected with students' understanding of the notion of a point and its relation to a straight line and to compare their possible cognitive development with the historical development outlined above. If there are a number of similarities we can use our understanding of phylogenesis to understand better the way in which contemporary students build the concept of infinity.

Theoretical Background

Phenomena in geometry are everything that we can see in the world of geometry. We can understand phenomena as multifarious **relations**. Or each person can understand them individually; he or she may interpret these phenomena and their relations differently. However a general interpretation of them arises from public consensus. So we can define the following two sorts of interpretation:

- **public interpretation**
- **personal interpretation**

Public interpretation: indicates an interpretation of a phenomenon or a relation between phenomena that is accepted in the framework of some model of a mathematical theory. For example, a straight line is an arbitrary extendable *eutheia* in the framework of the plane model of Euclid's geometry and a straight line is a set of points (fulfilling other conditions) in the framework of set conception. Public interpretation is not attributable to a real person, although it usually has its author or authors. It is usually written in a official text. It is presented to pupils in textbooks or by teachers. It is generally accepted, it is given by higher authority and so it cannot conflict with an interpretation of other phenomenon or related phenomena in the framework of the mathematical model in which we are working.

Personal interpretation: means an interpretation accepted by one particular person and usually assigned just to him or her. It is formed under the influence of public interpretation (through textbooks or teachers) and personal interpretations of other people (teachers, classmates, etc.). It cannot be specified since even the individual cannot realise it in its totality, although he or she works with it incessantly. The interpretation of one phenomenon or the relation between some phenomena may contradict the interpretation of some other phenomenon or relation. This can occur because the person does not realise which mathematical model he or is using. If we assume the need for harmony and absence of contradiction, the realisation of contradiction in a person's mind can motivate him or her towards a better understanding of these phenomena or relations of phenomena.

Infinity has many different **manifestations**. For example, when we consider a straight line, we may focus on its length or its width or its number of points. The phenomenon of infinity is present in all cases, but these are not the same manifestations. Rodrigo de Arriaga (1592–1667) distinguished between the following manifestations of the phenomenon of infinity [Vopěnka 2001]:

- **infinity as far as size is concerned**
- **infinity as far as the number of elements is concerned**
- **infinity as far as intensity is concerned**

We meet the first and second manifestations in geometry. De Arriaga reflected the third form in connection with qualities of God, for example, God's infinite love or wisdom etc. The size of infinity can be expressed as:

- **infinitely large**
- **infinitely small**

We speak about infinite length, when we consider the length of a straight line. Conversely, when we think about its width, we say that a straight line has no width. These explanations do not contradict. Each needs different approaches and each has different obstacles to understanding them.

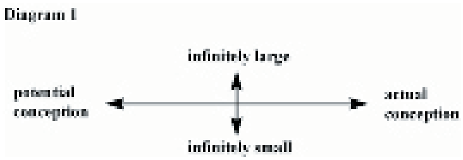
When we consider a straight line as a set of points, we say that there are infinitely many points. In explanation we speak of an infinite process. Infinity in relation to the number of elements also manifests itself in two different forms.

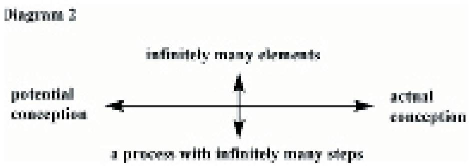
- **infinitely many elements**
- **a process with infinitely many steps**

A **density** of points is very important when we think about a number of points of a straight line (or a segment). We can describe it with the help of the concept of a distance of points. A density is a proportion between a number of points and a length in common conception. But we are focusing only on two qualitatively different cases – **discrete** and **continuous** spacing of points on an object. It depends just on the distance between two adjacent points, if we interpret it as finite or infinitely small or zero.

We use two conceptions – processional and conceptual, when we solve some problems [Hejný, Stehlíková 1999]. We consider potential and actual conception in connection with the phenomenon of infinity. If we consider an infinite length of a straight line from a potential point of view, we mean the possibility of arbitrarily extending the straight line. So we think first of a segment which can be of arbitrary length but never finished. Our approach to a straight line is therefore processional because we are the extenders of the segment. On the other hand, the completeness pertains only to an actual conception. Then we consider the straight line finished because it does not have the beginning and the end, and we are unable to extend a straight line because it is already infinite So we see a straight line as a conceptual model.

The following two diagrams illustrate different manifestations of the phenomenon of infinity in relation to geometrical objects and their possible interpretations.





We have chosen two basic geometrical objects – a point and a straight line – for investigation of students' understanding of infinity. These objects are very familiar, as students start to work with them very early and use them relatively often. On the other hand, the phenomenon of infinity impacts differently on these two objects..

Before we try to compare public and personal interpretations of a point and a straight line with the help of these two diagrams, it is necessary to be sure which world of geometry we are in. Modern geometry works in the classical world of geometry. However ancient geometrical theory was only based in the natural world of geometry from ancient times until the development of set theory. Vopěnka explains the main differences between these two approaches [Vopěnka 1996].

When we reflect on geometrical objects or constructions etc., we think in the natural world of geometry. This is very closely linked with our (not only sensory) competencies. A point is so small that it is the smallest object we can perceive. For example, we see a point like a dot on a piece of paper. But if we use a magnifying glass we do not see the dot like a point. The point has moved with our competencies, it has moved with our horizon. Geometrical objects are situated on a

horizon of our natural world of geometry and they are very strongly connected. We can simplify this by saying that the farther away the horizon , the smaller the point. But modern mathematics discovered a new world of geometry, which does not depend on an observer. In classical geometry all geometrical objects disappear behind the horizon. There is 'absolute smallness' of a geometrical point or 'absolute straightness' of a straight line.

Historical Development of the Concept

Constructional problems were the main concern of Greek geometry in the 6th and 5th centuries BC. In this period attempts can be identified to create geometrical structures and to formulate opinions about what is and what is not constructible. The concepts of a point and a straight line are fundamental to the idea of construction and the historical evidence suggests that the focus of reasoning gradually moved from concrete problems towards ideal objects.

Pythagoras (580 – 500 BC) defines, via Proclus, a point as a monad that has a position. A point is thus a basic unit, which has a placement. Pythagoras' school understands space as a sum total of points. [Struik 1963, p. 42]

Parmenides (from Elea, about 530 – 470 BC) considers how the world is recognizable. He compares the sensory and intellectual cognition and asks, for example: What is a point? A dot on a paper? Or is the dot only a model of the point? [Hejný 1985, pp. 50–51]

Zeno (from Elea, 490 – 430 BC) masterfully gives a true picture of the conflict between sensory perception and intellectual abstraction and idealization in his paradoxes.

Democritus (470 – 360 BC) developed his mathematical atomism,[1] possibly as a response to Zeno's paradoxes. This theory posits the existence of an invisible unit, which has a non-zero size and cannot be infinitely divided into geometrical objects. For Democritus, all mathematical objects have direct material models; an ideal mathematical point and a straight line do not exist. [Drábek 1998, p.175]

Aristotle (from Stageira, 384 – 322 BC) views and compares a unit and a point as monads, the former having no position while the latter has position. [Hejný 1985, p. 68]

Eudoxos (408 – 390 BC), based on Plato's (427/8 – 347 BC) philosophy of ideas, formulates his concept of continuous geometry, which culminates in Euclid's work.

Euclid (about 325 – 265 BC) defines a point in the following way: 'A point is that which has no part'. (*Semeion estin hu meros uthen.*) He explains a point in accordance with Plato's conceptions of the world of ideas. So the idea of a part is not present in the idea of a point. A placement only exists in the idea of a concrete point. Euclid also claims: 'Borders of a line are points'. (*Grammes de perata semeia.*) Euclid does not highlight the continuity of a line (and other objects). He probably considers this to be evident . This explains his first postulate: 'to draw a straight line from any point to any point'. The second postulate says 'to produce a finite straight line in a straight line'. We see that Euclid understood a straight line as 'potentially infinite', 'continuous' and 'containing points'.

The period up to the Renaissance was very quiet for mathematics and science . With the Renaissance came further developments. Scientists of the 16th and 17th centuries, like **Kepler**, **Viète**, **Cavalieri** or **Torricelli**, developed methods which led to infinitesimal calculus. They came back to Democritus' atomism. Cavalieri used a theory of 'indivisible elements' (in *Geometria indivisibilius continuorum, 1635*) and a straight line formed by a movement of points. [Struik 1963, p. 96]

Descartes discovered analytic geometry (in *Géométrie, 1637*), using algebra in geometry. Abscissae were introduced only by **Leibniz**.

Set theory, founded by **Bolzano** and **Cantor**, brought a new view outlook to every geometrical object and became an ideal tool for investigating the phenomenon of infinity.

Phenomena of Phylogenesis

In this historical development of the understanding of a point and its relation to a straight line, the following phenomena can be identified:

A general characterization

- detachment from a picture (the state when a person is not looking at a picture, but has it on his or her mind)
- gradual movement from the pictorial form of representation of a point towards the abstract form, from real objects to ideal objects

A concept of a point

- determination of position
- size and position
- unit of space
- smallness
- isolation
- materiality
- indivisibility
- an endpoint of a segment
- algebraic description (abscissae)

A concept of a straight line

- straightness
- thinness
- arbitrary extension
- limitless
- infinite length
- continuum
- algebraic description
- orientation
- a set of points (elements)

Let us try to analyse the public interpretation of a straight line in Euclidean geometry by using our diagrams. Euclid presented the following definitions in his *Elements (Stocheia)*:

1. Eutheia (line) is length without width.
2. A straight line is that which is straight to its points.
3. To lead a line from an arbitrary point to an arbitrary point.
4. To produce a finite straight line in a straight line.

He definitely considered a straight line to be in the natural world of geometry. We can put these definitions in our diagrams in this way:

1. Diagram 1: infinitely small + actual conception
2. Diagram 1: infinitely small + actual conception (zero curvature
3. Diagram 1: infinitely long + potential conception
4. Diagram 2: infinitely many elements + potential conception

This research project attempts to describe phenomena connected with students' understanding of the notion of a point and its relation to a straight line and to compare possible cognitive development with the historical development outlined above.

Methodology

We prepared a list of questions to help to find out which manifestations of infinity a respondent attaches to a point and a straight line (and/or a ray, a segment). (The age of the respondents is indicated in parenthesis.)
1. 'What is a point?' *(age 9–18)*

2. Consider the following problem: We have a segment *AB*. Divide it by means of a point *C* in the ratio of 2:3. Divide it into the two parts in your mind which originated in the division by point *C*. What will the boundary points of both segments be? *(age 9–18)*

3. How many points does a straight line (or a ray or a segment) consist of? *(age 9–18)*

4. What is a straight line? *(age 9–18)* What are parallels? *(age 12–18)*

5. How many straight lines are there which cross the point A and intersect the straight line P ? How many straight lines are there which cross the point A and don't intersect the straight line P? (age 9–12)

6. How many parallels of a given straight line are there? How many straight lines are there which cross a given point? How many parallels are there between two given parallels? (age 12–18)

7. There are two rays, one is an proper subset of the other one. Which of them is longer? (age 9–18)

8. There are two rays A and B. We mark the ray B as it is on the picture. Which of these two 'lines' is longer: A or the marked part of B? (age 13–18)

9. There is a square. We cut it into two equal rectangles. We put one rectangle next to the other one so that a new rectangle is formed (see figure below). We cut it again and put one part next to the other one again. We can repeat it again and again. Which object will we have 'in the end'? (age 13 – 18)

Semi-structured interviews with 19 pupils and students from the age of 9 to 18 years were carried out. Every question, except the second one, was used in one-to-one interviews (where the experimenter spoke to one student only); the second one was used in interviews in which two or three students spoke to the experimenter. We used the same expressions in follow-up questions, which had been used earlier by the students interviewed. The experimenter asked them about the size and the shape of a point and reasons for their answers. Among the follow-up questions was: What is the shortest distance between two different points?

All the interviews were recorded and later transcribed.

Illustrations from the Interviews and Comments

From the transcriptions of the interviews, the following extracts were chosen to illustrate some aspects of the students' understanding of a point and a straight line.

Interview 1: Jakub (a boy), 12 years old
E27: So, does a point have a size?
J28: It can have some size in reality. (He points to a mark on a paper.)
E29: But I am not talking about the drawn point. I am speaking about the proper geometrical point.
J30: (3 secs) *So, the point's* (5 secs) *size is this.* (He points to a mark on a paper.)
E53: Do points B and D have a midpoint between them ?
J54: Yes, they do.
E55: But you said that there was nothing between them.
J56: Hmm. So the midpoint (3 secs) *has no size, so it can be there, as nothing.*

This boy appears to feel that there are two different worlds – the physical and the geometrical world. An object, which has zero size, cannot exist in the physical world. Yet he can consider it in the natural world of geometry. (Diagram 1: infinitely small + potential conception)

Interview 2: One year later. Jakub (boy), 13 years old
E1: What's a point?
J2: (6 secs) *So, a point is* (3 secs) *a thing which determines some position, it demarcates something.* (3 secs) *It can't demarcate anything, it is somewhere.*
J6: It doesn't have any size. It has no content. We can never draw a point, we can only indicate it.
J8: It's hard to imagine something that has no size.
E9: How do you know that a point has no size?
J10: If I pictured a point like a circle, I could divide it wherever I liked. And even then if we could not see it, it would still be possible to divide it further.
J12: It is not possible to divide a point in this way . It demarcates only one place.

We can see a marked difference in this boy's understanding of geometrical objects after one year. He can think and speak about ideal geometrical objects (and does not need a concrete point). He highlights a zero size, a position, an indivisibility and the demarcation of a point. (Diagram 1: infinitely small + actual conception, Diagram 2: a process with infinitely many steps + actual conception)

Interview 3: Marek (boy), 11 years old
E27: There are two rays, one is a subset of the other one. Which of them is longer?
M28: (5 sec) Neither.
E29: Why? (10 sec) One girl said that both of them have the same length. Do you agree with her?
M30: Yes, I do.
E31: But another girl said that the ray B is longer.
M32: No, the ray A can finish farther.

Marek understands a ray like an infinitely long line, but his conception is certainly potential (M32). He explained the incomparability of two rays with the help of their extendibility, especially his own ability to extend them. (Diagram 1: infinitely long + potential conception)

Interview 4: *Marta (girl), 16 years*
E73: How many points does a straight line have?
M74: (3 sec) Infinitely many. (2 sec) Because the points are so inflated that they transform to a continuous line, which can form only when all points are there. There are infinitely many points because the straight line is infinitely long.

Marta is able to work with abstract notions. She appreciates connections of points and a straight line like a continuous set of points in the plane model of Euclid's geometry. Her discussion of the number of points could be said to be based on potential also, but she speaks about continuity too, and it indicates her actual conception. She can use both ideas. (Diagram 1: infinitely long + actual conception, Diagram 2: infinitely many elements + actual conception)

Phenomena of Ontogenesis

By analyzing the transcriptions from the interviews, the following phenomena have been identified which characterize students' understanding of the notion of a point and a straight line:

A general characterization

• detachment from a picture (the state when a person is not looking at a picture, but has a concept in his or her mind)
• a gradual movement from the pictorial form of representation of a point towards the abstract form, from real objects to ideal objects

A concept of a point

• determination of position
• size and position
• unit of space
• smallness
• isolation
• shape
• indivisibility
• an endpoint of a segment
• algebraic description (abscissas)

A concept of a straight line

• straightness
• thinness
• arbitrary extending
• limitless
• infinite length
• continuum
• algebraic description
• orientation
• a set of points (elements)

Conclusions: The Comparison of Phylogenesis and Ontogenesis

In general, the students in the study appear to use similar concepts to those we can see in the historical development of the understanding of a point and its relation to a straight line. There are two differences, however, concerning the notion of a point. The students did not discuss the physical representation of a point, and indeed we have been unable to find any mention of the shape of a point in the historical analysis. At this stage in the development of a mathematical schema, the notion of a shape is not an exact mathematical term. As far as the ancient mathematicians were aware of it, they probably considered a point an object without a shape or as just such a shape. The emphasis laid on the physical representation of a point in the phylogenesis corresponds to the greater priority attached to the tactile sense for appreciation of the essence of phenomena compared with today. In contrast, the students in the study tended to prefer the visual sense.
Many students are able to think about a point and a straight line (and other geometrical objects) only by way of a picture. This corresponds to the first phase of the historical development of plane geometry, in which notions both of a real object and an ideal object are present. Pupils and students use both, and these ideas are sometimes contrasted with one another. Students become gradually able to 'leave' the physical world and think in the ideal geometrical world.
All the pupils and students interviewed highlight the smallness of a point. But only some of them admit its zero size. This is a natural enough opinion, as everything that we meet in everyday life has a finite size. Children do not understand a line or a segment as a 'set' which is compactly ordered. This corresponds to Democritus' geometrical atomism. They emphasize that it is 'only one place' like a unit of space.

In general, older students speak about the determination of a position via a point. This may be partly due to what they have learnt of school geometry and, in particular, analytical geometry.
Only a few students admit the indivisibility of a point. This is possible only if they stop thinking about a point as an object on paper.
Younger students (under 15 years old) often answered 'A straight line has no point'. or 'A ray has just one point'. or 'A segment has just

two points'. There are two reasons: They think only about a picture: 'two points are drawn'. Or they can understand a straight line with two marked points as a segment and two rays. This corresponds to the historical development of the relation between points and other geometrical objects. Points were understood as borders of a line or vertices of a rectangle. [Eisenmann 2002, pp. 9–10]

Students' personal interpretations of a point, a straight line (and/or a ray and/or a segment) and their relationships are often contradictory. Where students are helped to realize this they are better able to understand these concepts. It can be a motivation to engage with this field of mathematics.

There appears to be a match between the historical and possible cognitive developments of the notion of a point. We can identify similar phenomena in students' understanding of the notion of a point compared with its historical development. An appreciation of the historical development may assist in anticipating students' potential difficulties in understanding.

References

Becvárová, M. (2000) Eukleidovy Základy. Jejich vydání a překlady. (Praha: Prometheus).

Drábek, J. (1998) Démokritova atomistická metoda. Sborník Matematiky v proměnách věků I. (Praha: Prometheus).

Eisenmann, P. (2001) Point, segment, line – secondary school pupils' ideas. *8th Czech-Polish Mathematical School.* (Ústí n.L: Acta Universitatis Purkynianae).

Eisenmann, P. (2002) Propedeutika infinitesimílního počtu. (Ústí n. L: Acta Universitatis Purkynianae).

Hejný, M. (1986). In Znám, Š a kol. *Pohl'ad do dejín matematiky*. (Bratislava: SNTL).

Hejný, M a kol. (1990) Teória vyučovania matematiky 2. (Bratislava: SPN).

Hejný, M., Stehlíková, N. (1999) Číselné představy dětí. (Praha: PedF UK).

Kuřina, F. (1996) Deset pohledů na geometrii. M a (Praha: ALBRA).

Monaghan, J. (2001) Young peoples' ideas of infinity. *Educational Studies in Mathematics,* Vol. 48, 2–3, pp. 239–257 (Dordrecht, Boston, London: Kluwer Academic Publishers).

Prokopová, M. Some remarks about pupils' concept of a point. *9th Polish-Czech Mathematical School*. Krakow [in press].

Servít, F. (1907) *Eukleidovy základy (Elementa)*. (Praha: JČMF).

Struik, D. J. (1963) Dějiny matematiky. (Praha: Orbisa).

Tall, D. (2001) Natural and formal infinities. *Educational Studies in Mathematics,* Vol. 48, 2–3, pp. 199–238 (Dordrecht, Boston, London: Kluwer Academic Publishers).

Vopěnka, P. (1989) *Rozpravy s geometrií.* (Praha: Panorama).

Vopěnka, P. (2000) Úhelný kámen evropské vzdělanosti a moci. (Praha: Práh).

Contact

Magdalena Prokopová
Charles University, Prague, Faculty of Education
Email: prokopova@pf.ujep.cz

Footnotes

[1] Democritus' idea of a geometrical atom was used later in mathematical analysis by Viete and Kepler in the 16th century.