INTEGRATING THE PHILOSOPHY OF MATHEMATICS IN TEACHER TRAINING COURSES

A Greek Case as an Example

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Abstract: The philosophy of mathematics may be assumed to provide a unifying framework that potentially supports an epistemological clarification of mathematical knowledge, as well as a critical reflection on the beliefs and values about mathematical knowledge that a teacher holds in connection with the content and the prevailing practices of mathematics teaching. Thus, the philosophy of mathematics may be considered an essential component of teachers' professional knowledge. In such a perspective, a relevant venture integrating philosophy of mathematics themes in a course of learning and teaching primary school mathematics that is offered to teachers as part of an in-service training program in Greece is presented. The rationale of the venture is outlined, selected examples are briefly presented, and issues that arise in its implementation are reported.

Key words: Teacher training, didactics, teachers' beliefs , primary school, nature of mathematics

1. BACKGROUND ORIENTATION

Arguments from three different, although interrelated, perspectives that are briefly presented in the following support directly or indirectly the thesis that the philosophy of mathematics has to be considered an indispensable component of teachers' professional knowledge. This thesis constitutes the starting point of a venture attempting to integrate selected themes from the philosophy of mathematics into teachers' training course on primary school mathematics, which is reported in this paper.

The first argument asserts the direct association of a philosophy of mathematics with fundamental features of mathematics education. Many years ago, Thom claimed that a philosophy of mathematics has powerful implications for educational practice pointing out that "In fact, whether one wishes it or not, all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics" (Thom 1973, 204). Hersh, few years later, emphasised this claim noting that

> "One's conception of what mathematics is affects one's conception of how it should be presented. One's manner of presenting it is an indication of what one believes to be most essential in it. … The issue, then, is not, What is the best way to teach? but, What is mathematics really about? …" (Hersh 1979, 33).

Generalizing and exemplifying further this position of association between a philosophy and didactics of mathematics, Steiner notes that

> "Concepts for teaching and learning mathematics—more specifically: goals and objectives (taxonomies), syllabi, textbooks, curricula, teaching methodologies, didactical principles, learning theories, mathematics education research design (models, paradigms, theories, etc.), but likewise teachers' conceptions of mathematics and mathematics teaching as well as students' perception of mathematics—carry with them or even rest upon (often in an implicit way) particular philosophical and epistemological views of mathematics" (Steiner 1987, 8).

The argument of association of a philosophy of mathematics with fundamental features of mathematics education, exemplified in the above quotations, has been principally developed on the grounds of theoretical analyses.

A second argument is that teachers' ideas, views, conceptions, or beliefs (the terms are dependent on conceptual frameworks and thus on adopted theoretical perspectives) about mathematics, its learning, and teaching, implicitly reflect, or are related to, a philosophy of mathematics. These teachers' beliefs about mathematics, in their turn, play a significant role in shaping characteristic patterns of their didactic practices, as it is widely accepted nowadays, either on the basis of empirical evidence (Thompson 1984, 1992) or on a philosophical realm (Lerman 1983, 1990; Steiner, 1987; Ernest, 1989).

Teachers' beliefs about mathematics are personally held mental constructs about the nature of mathematical knowledge, which includes, among others, beliefs about the origins of mathematical knowledge, the nature of mathematical knowledge as a discipline, the nature of mathematical problems and tasks, the relationships between mathematical knowledge and empirical reality, and, in particular, about the applicability and utility of mathematical knowledge, the nature of mathematical knowledge as a subject taught in schools (Törner 1996), as well as beliefs about oneself as a learner and user of mathematics, and more generally, beliefs about the process of learning mathematics (Ernest 1989, Pehkonen 1994).

Teachers' beliefs about mathematics reflect, or are related to, a philosophy of mathematics and, in fact, constitute a kind of *practical philosophy of mathematics*, which, as a complex, practically-oriented set of understandings, regulates and shapes to a great extent the teachers' thoughts and practices within classrooms, although subject to the constraints and contingencies of the school context. Moreover, this teachers' *practical philosophy of mathematics*, often takes precedence over knowledge, shaping the interpretation of their held knowledge and selectively admitting or rejecting new knowledge. The process of how, why, when, and under what circumstances beliefs about mathematics are adopted and defined by the individual teacher is not yet clearly and definitely described by relevant literature. In any case, their discussion falls beyond the aim of this paper.

Therefore, assuming an association of a philosophy of mathematics with mathematics education issues on the one hand and teachers' philosophical and epistemological views of mathematics on the other, then philosophy of mathematics per se has to be considered as an indispensable component of teachers' professional training in mathematics learning and teaching—a training, however, that aims at enabling teachers to develop a questioning stance towards dominant canons of mathematics education, a critical reflection of their personal didactical practices, and an increase in their professional autonomy in teaching mathematics. Or, more broadly, it needs to be a training that aims to support teachers in becoming reflective practitioners, playing an important role in the definition of the purposes and goals of their work, as well as on the means to attain them, and therefore, participating in the production of knowledge about teaching mathematics. It must be a knowledge about teaching mathematics, however, with an element of critique of the established standards.

The third argument—the assumption that a philosophy of mathematics is directly associated to a deeper understanding of mathematics as the subject matter knowledge of teaching it—seems reasonably hard to doubt.

In recent years, the importance of mathematical knowledge has been well documented in the literature, and the lack of it has been linked to less competent mathematics teaching (Rowland, Martyn, Barber and Heal, 2000,

2001) and over-reliance on commercial schemes (Millett and Johnson 1996). It has been well documented, as well, that understanding mathematics for teaching entails both knowledge *of* mathematics and knowledge *about* mathematics or, using Shulman's distinction, *substantive* and *syntactic* knowledge of mathematics (Shulman 1987, 8). Knowledge *of* mathematics includes both propositional and procedural knowledge (i.e., concepts, principles, facts, and the ways that they are organized). Ball argued that to teach mathematics effectively, teachers must have knowledge of mathematics, characterized by an explicit conceptual understanding of the principles and meanings underlying mathematical procedures and by connectedness—in contrast to compartmentalization of mathematical topics, rules, and definitions (Ball 1990). Knowledge *about* mathematics includes an understanding of the nature of mathematical knowledge and the mechanisms through which new knowledge is introduced and accepted in the community of mathematicians, as well as knowledge about proofs, rules of evidence, and structures (Schwab 1978).

In summary, the philosophy of mathematics, as well as the history and sociology of mathematics, are essential components of the domain of knowledge about mathematics. Against this background, a venture of integrating selected themes from the philosophy of mathematics into teacher training has been undertaken over the last five years as a constituent element of a course on learning and teaching primary school mathematics. This course is offered to primary school teachers by the author of this paper, as part of an in-service training program run by the Primary Education Department of the Aristotle University of Thessaloniki, Greece.

In accordance with the aims of the course, the philosophy of mathematics is conceived in a descriptive and social perspective and is considered to account for:

- Mathematical knowledge: its nature, justification, and genesis
- The objects of mathematics: their nature and origins
- The application of mathematics: its effectiveness in science, technology, and other realms
- Mathematical practice: the activities of mathematicians, both in the present and the past (Ernest 1991, 27).

In the following section, main features of the rationale of this venture are outlined, and issues that have arisen by an overall evaluation of the course are briefly reported and commented upon.

2. INTEGRATING THEMES FROM PHILOSOPHY OF MATHEMATICS IN A TEACHER TRAINING COURSE: A GREEK PROJECT

The philosophy of mathematics may be considered an essential component of teachers' professional knowledge. In such a perspective, a relevant venture integrating philosophy of mathematics themes in a course of learning and teaching primary school mathematics that is offered to teachers as part of an in-service training program in Greece is presented.

2.1 The Rationale and its Background

Any introduction of the philosophy of mathematics to teacher' training courses has to be designed on a basis framed by the proposed answers to the following core questions:

- *Content*: What topics from the philosophy of mathematics should be taught to teachers?
- *Method*: What methods are most appropriate to teach them to teachers?
- *Incorporation:* What relationships should be established between courses of philosophy of mathematics, mathematics, and didactics of mathematics offered to teachers in a training program?

Answering these questions first, and in accordance with the aims outlined above, a course was offered including topics from the philosophy of mathematics, adopting mainly an informative approach, designed and implemented on an experimental basis, and offered to teachers as a supplementary course to the main course on learning and teaching primary school mathematics. Both courses were one semester long, each taught in one-and-a-half hour lecture followed by a one-and-a-half hour discussion session. After two semesters of implementation, however, this option of distinct, although supplementary, courses was evaluated and found to have a non-significant impact on teachers' thinking. The main reason for failure was ascribed to a revealed inability of teachers to perceive any relevance between the topics of philosophy of mathematics and their immediate teaching interests.

In this account, two prerequisites of any attempt to introduce the philosophy of mathematics in a teacher training course were noticed. First, an apparent relevance of topics between the philosophy of mathematics and mathematics teaching may be a crucial factor for teachers' motivation. Second, thought-provoking questions regarding mathematics taught in primary schools could function as a catalyst in attracting teachers' interest and involvement in the philosophy of mathematics. At the same time, an empirical investigation showed that primary school teachers' prevailing beliefs about mathematics were dominated by a conception of mathematics as a fixed, predictable, absolute, certain, value-free, culture-free, and applicable body of knowledge involving a set of facts, rules, and procedures to be used in the pursuance of some external end (Chassapis 2003). In addition, teachers' poor mathematical backgrounds and a rather narrow view of mathematics as an academic discipline were also evidenced by relevant studies and ought to be considered.

Taking into account the previously summarized issues, and in addition to the main aims of teachers' training, the scheme of integrating themes from the philosophy of mathematics into the main course on learning and teaching primary school mathematics has been developed and implemented using the following rationale:

- Themes from the philosophy of mathematics have been selected and were developed along three threads stemming from the primary mathematics content:
    - Concepts (cardinal and ordinal number concepts, definitions of natural and rational numbers, questions on the nature and properties of numbers, irrational numbers, numeration systems as cultural constructs, continuity and infinity issues arisen from rational and real number concepts)
    - Processes (definition, justification and proving in mathematics, the what and the why of axiomatic systems in mathematics)
    - Applications (problem-solving and relationships between mathematical knowledge and empirical reality)
- Themes from the philosophy of mathematics are introduced and discussed using thought-provoking questions, which create dissonant situations for the teachers and thus motivate them to be actively involved in discussion, learning, and reflective thinking
- Questions concerning the nature of mathematical knowledge and practice have arisen recurrently, on many occasions, during lectures and discussions of the course—for instance, issues concerning the use of manipulatives in teaching particular mathematical concepts

The organizing concepts of this rationale are *themes* and *thought-provoking questions*. By *themes* in the philosophy of mathematics we mean collections of learning experiences that assist students in relating their learning to questions that are important and meaningful for them, as well as practice-bounded (Freeman and Sokoloff, 1996). Themes are the organizers of the philosophy of mathematics content, which is presented and discussed using questions meaningful to the teachers as starting points. They are intended to give meaning and direction to the reflection and learning process (Perfetti and Goldman, 1975). Finally, the thematic approach seems to provide an environment where knowledge can be individually and socially constructed, so it may be considered to be associated with constructivist ideas of knowing.

By *thought provoking-questions*, which create dissonant situations for the teachers and thus motivate them to be actively involved in approaching the philosophy of mathematics, we mean questions create perplexity, challenge beliefs, spot questionable issues, and potentially foster a conceptual reconstruction of mathematical knowledge and pedagogy for the teachers.

2.2 The Content and Organization of the Course

In the outline of the course on learning and teaching primary school mathematics that follows, a record is found of the attempted articulation of thematic units from the philosophy of mathematics into the content of this course. Besides the themes from the philosophy of mathematics, selected issues from the historical development of mathematical concepts are occasionally introduced to emphasize that mathematics is a constantly changing creation of human activity and not a fixed and finished, a priori existing product, which one is expected to discover. Inset examples (see Appendix at the end of this chapter) indicate the types of questions introducing the philosophy of mathematics issues, which are posed during the lectures of the course. The questions are put forward in various wordings, depending on the case at hand.

The following is an outline of the course on learning and teaching primary school mathematics, integrating themes from the philosophy of mathematics. Items in *italic* represent issues of philosophy.

UNIT 1 Mathematical concepts and their characteristics

Topics

- Fundamental features of the mathematical concepts and the initial difficulties children find in comprehending them
- The articulation and organization of the mathematical concepts in systems

Questions / Thematic unit from philosophy of mathematics
Why the organization of mathematics concepts in axiomatic systems?
Euclid: the axiomatization of geometry
Peano: the axiomatization of natural numbers
Gödel: the incompleteness theorems
What does the axiomatic organization of mathematics concepts mean for the learning of mathematics?
Further discussion issues: definition, justification, and proving in mathematics (example: Pythagorean theorem proofs)

- The graphical and symbolic representations and the linguistic expressions of the mathematical concepts
- The relationships of mathematical concepts to empirical reality

Questions / Thematic unit from philosophy of mathematics
Are mathematical concepts and truths discovered or invented?
Realism and anti-realism in mathematics

UNIT 2 Classes, sets and relations in mathematics

Topics

- Classes, sets and relations
- Equivalence and ordering relations

Questions / Thematic unit from philosophy of mathematics
Are the collections of objects used in primary mathematics classes or sets?
And how is the difference between classes and sets conceived?
The Russell's paradox and its consequences
Finite and infinite sets
Whole and part
The Cantor's paradox
The continuum hypothesis
The question of foundations of mathematics: answers, approaches and schools of thought (logicism, formalism and intuitionism)

- Logical operations
- The formation and development of logical-mathematical concepts and relations in childhood: Piagetian and socio-cultural perspectives

UNIT 3 The number concepts

Topics

- The concept of cardinal and ordinal number

  Questions / Thematic unit from philosophy of mathematics
  What does the numerocity of a class of objects mean?
  Using potency of classes of objects, which concept of cardinal and ordinal numbers is implicitly constructed in primary mathematics?
  What are numbers: objects or properties?
  What is the relation of the so constructed (and implicitly defined) concept of cardinal and ordinal numbers to counting and its outcomes?
  Number definition approaches and issues in philosophy of mathematics (Pythagoreans, Cantor, Peano, Frege)
  Do different number definitions impact any differences in teaching number concept? (Example: Cardinal and ordinal numbers in primary mathematics)

- Numeration systems and linguistic numerical expressions

  Questions / Thematic unit from philosophy of mathematics
  Why are various numeration systems invented and used over time and across cultures?
  Numeration systems as cultural constructs
  The cultural aspects of mathematical activity
  Piagetian and constructivist approaches to the formation of number concepts

- Socio-cultural approaches to the acquisition of number concepts

UNIT 4 Number operations

Topics

- Mathematical definitions and properties of number operations

  Questions / Thematic unit from philosophy of mathematics
  How is number addition and multiplication defined in mathematics?
  Do different definitions of number operations impact any differences in their teaching?
  A further discussion issue: Definition in mathematics

- Number operations in the decimal numeration system—Algorithms
- The mapping of number operations to real world situations

Questions / Thematic unit from philosophy of mathematics
What truth is expressed by a number operation and, more generally, by a mathematical statement?
Is a truth about objects, concepts, or neither?
The question of mathematical truth
The indispensability of mathematics arguments

UNIT 5 Expansions of number concept

Topics

- On the mapping of numerical concepts to real world situations: Counting, ordering and measuring
- Natural numbers and integers
- The concept of zero in natural numbers and in integers
- The fraction concept and the rational numbers
- Decimals and percentages

Questions / Thematic unit from philosophy of mathematics
What happens when ordering rational numbers?
Discreteness and continuity
Infinity and its paradoxes: From Zeno to Cantor and beyond

- The concept of irrational numbers
- The concept of real numbers

Questions / Thematic unit from philosophy of mathematics
Is any number modeled to an attribute of empirical reality and vice-versa?
Does square root of 2 exist?
The existence of mathematical entities
In what sense, if any, do mathematical entities (numbers, for instance) exist?
The nature of mathematics concepts: Mathematical entities and empirical objects
Approaches and schools of thought on the nature of mathematics and mathematical activity

UNIT 6 On methods and media for teaching mathematics in primary classroom

Topics

- Methods for teaching mathematics in primary classroom
- Tools and materials for teaching mathematics in primary classroom

Question / Thematic unit from philosophy of mathematics
Is there any, and what, meaning in the discovery methods for teaching mathematics concepts?
Are mathematics concepts embodied in manipulatives?
(Example: Multi-base Arithmetic Blocks)
The ontology of mathematics and its implications for mathematics education
A further discussion issue: What is mathematics after all?

Schematically, and in a figurative language, the attempted mode of articulation of philosophy of mathematics themes into the units of the course on learning and teaching primary school mathematics could be described as a spiraling one.

2.3 Feedback and Evaluation of the Course

The course, as outlined above, has been offered for the last two years. Each semester, after the completion of the course, the teachers that have participated are asked to write a short anonymous report evaluating the main aspects of the course, describing the most important gains they enjoyed from attending the course and spotting the main obstacles that they encountered in following the lectures and participating in the discussions of the course. All reports are being carefully and read many times, so the evaluative comments made by the teachers can be clearly elicited from their writing and the derived data recorded and analyzed.

From the analysis of the evidence collected over the two years of this course's implementation, a number of conclusions were drawn. Most of the teachers point out that their main gain from attending the course was incitement for conceptual change, or an actual conceptual reconstruction they engaged in. In many cases, it was vaguely described as a *change of mind* about mathematics. In every case, this re-conceptualization of the nature of mathematics led them to view the discipline from a perspective different from their own long-held perspective. This perspective on mathematics that is *new* for them has, among its impacts, the adoption of a critical approach towards many dominant standards of, and prevailing practices in, mathematics education. This critical approach is mostly emanated, according to all available evidence, from the possibility of questioning what is taken for granted in school mathematics—offered by their involvement in the philosophy of mathematics.

The themes from the philosophy of mathematics that are obviously appreciated by the teachers, and which seem to have the greatest effect on their thinking, are those which offer explicit opportunities for challenging their conceptions about teaching particular concepts of school mathematics (e.g., numbers) or elements of their teaching models (e.g., the use of manipulatives as embodiments of mathematical concepts or processes).

The most serious problem encountered during the implementation of the course originated from the poor mathematical backgrounds of the teachers. The lack of specific mathematical knowledge does not permit them to comprehend specific questions and ideas from the philosophy of mathematics, and delimits the extent, depth, and quality of the relevant discussions. Infinity and continuity issues are the most characteristic examples. Most of the teachers enrolled in this and other in-service courses, had already graduated from Higher Teachers' Training Colleges offering three-year courses (which was operative in Greece up to 1985, when they were replaced by university departments), so their original mathematical background is relatively poor. The outcomes of this course, as far the philosophy of mathematics is concerned, are crucially dependent on teachers' mathematical backgrounds. The repeated conclusion of many research studies, that teachers' existing knowledge and beliefs are critical in shaping what and how they learn from teacher training experiences, seems to be validated once more in our case.

A second problem, crucial for the efficacy of the course, appeared to be the lack of appropriate resources in Greek language on the philosophy of mathematics intended for primary school teachers to meet their reading requirements, which was necessary for their constructive participation in the course learning activities.

A final issue—not induced from teachers reports but from the author's experience—which must be registered in the assessment of the course, is the demand in time and work both for the preparation and for the running of each course session, placed, as a rule, on the professor.

3. CONCLUDING COMMENTS

Until recently, mathematics education has been developed in a way that is scarcely related to the philosophy of mathematics. Rationales and practical proposals in mathematics education are even now mainly informed by educational and psychological research, disregarding ideas coming from disciplines that study the nature of mathematics, as do the philosophy, history, and sociology of mathematics. Nowadays, however, this situation seems to be changing through the work of many scholars and teacher trainers.

In this paper, I have presented an attempt to integrate themes from the philosophy of mathematics in a teachers training course on learning and teaching mathematics in primary school, aimed to contribute into supporting teachers in becoming reflective practitioners. The overwhelmingly positive feedback from the majority of teachers that attended this course, was an initially surprising finding. It must be recognized, however, that no single course is a panacea for promoting aims concerning teachers' reflective thinking and acting. Teacher courses, even sequences of courses over multiple semesters, seem to make some difference, but may be insufficient to promote the kinds of changes in the conceptual understanding and ideology of teachers necessary to achieve outcomes compatible with the ideals of reflective practice. It may be that to fight against those years of learning reinforces the status quo, and that the slow accumulation and layering of reconstructive experiences over many other years will be required.

Finally, there is no doubt that a careful review of the course syllabus, together with empirical evidence or experiences from similar efforts, will likely reveal additions and deletions or a reorganization of the course content. Any such review is welcomed and necessary to establish a dialogue for optimal approaches to the integration of the philosophy of mathematics in prospective and in-service teachers training programs.

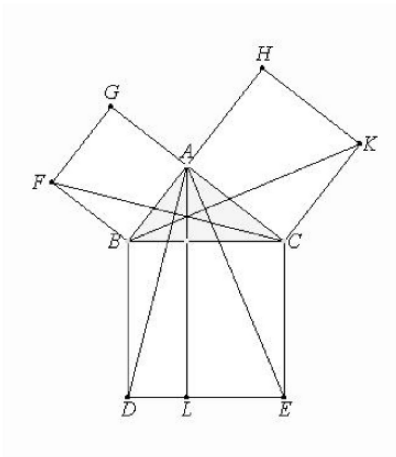
APPENDIX

Example 1: Pythagorean Theorem Proofs

In right-angled triangles, the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.

1.1: Euclid's proof based on deductive reasoning:
(Euclid's *Elements*, Book 1, Proposition 47)

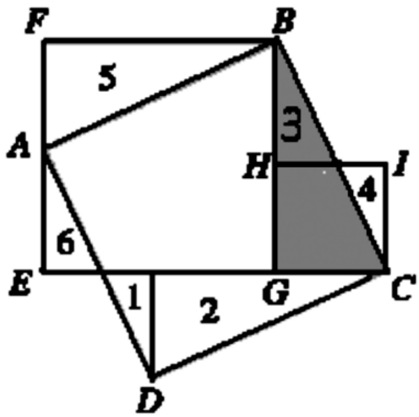
Figure 1: Euclid's Proof of the Pythagorean Theorem.



Let ABC be a right-angled triangle having the angle BAC right. The square on BC equals the sum of the squares on BA and AC. Describe the square BDEC on BC, and the squares GB and HC on BA and AC. Draw AL through A parallel to either BD or CE, and join AD and FC. Because each of the angles BAC and BAG are right, it follows that with a straight line BA, and at the point A on it, the two straight lines AC and AG (not lying on the same side) make the adjacent angles equal to two right angles. Therefore, CA is in a straight line with AG. For the same reason, BA is also in a straight line with AH. Because the angle DBC equals the angle FBA (each is right), add the angle ABC to each. The whole angle DBA equals the whole angle FBC. Because DB equals BC, and FB equals BA, the two sides AB and BD equal the two sides FB and BC, respectively, and the angle ABD equals the angle FBC. Therefore, the base AD equals the base FC, and the triangle ABD equals the triangle FBC. Now, the parallelogram BL is double the triangle ABD, for they have the same base BD and are in the same parallels BD and AL. The square GB is double the triangle FBC, for they again have the same base FB and are in the same parallels FB and GC. Therefore, the parallelogram BL also equals the square GB. Similarly, if AE and BK are joined, the parallelogram CL can also be proved equal to the square HC. Therefore, the whole square BDEC equals the sum of the two squares GB and HC. And the square BDEC is described on BC, and the squares GB and HC on BA and AC. Therefore, the square on BC equals the sum of the squares on BA and AC. In right-angled triangles, the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.
Quod erat demonstrandum (Q.E.D).

1.2: Liu Hui's proof based on the Chinese dissection process:
(Liu Hui, commentary on the Jiuzhang suanshu, third century B.C. In Swienciki, L.W. *The Ambitious Horse. Ancient Chinese mathematics problems,* Emeryville, CA: Key Curriculum Press, 2001, 58)

Figure 2: Liu Hui's Proof of the Pythagorean Theorem.



Start with square ABCD, whose side is c and area is c^2. Remove pieces 1, 2, and 3 by translation and rotate to form pieces 4, 5, and 6, respectively. The original square has now been transformed into two new squares: 1) square EFBG of side b and area b^2, 2) square GHIC of side a and area a^2.

We write Area (GHIC) + Area (EFBG) = Area (ABCD) or $a^2+b^2=c^2$.

Questions:

Which proof of the Pythagorean theorem is accepted by the discipline of mathematics, is included in mathematics textbooks, and is used in teaching mathematics?

Why?

Why are hands-on activities not accepted as justification arguments in mathematical proofs?

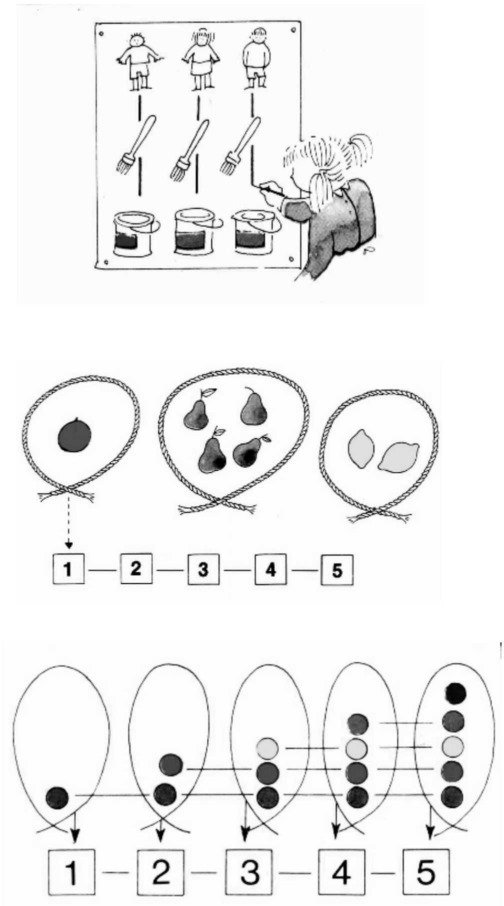
What is the aim of a proof in mathematics?

Why is deductive, and not creative, thinking mostly valued in mathematics?

Example 2: Cardinal and Ordinal Numbers in Greek Primary Mathematics

Source: *My mathematics*, 1st primary school grade, Athens: OEDB, 2003, 26, 74. (in Greek)

Figure 3: Cardinal and Ordinal Numbers in Greek Primary Mathematics.



Questions:

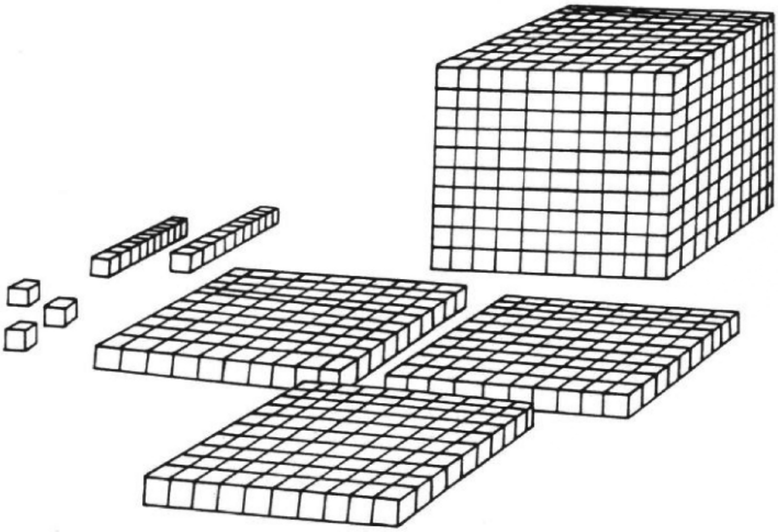
Using the potency of classes of objects, which concept of cardinal and ordinal numbers is implicitly defined in primary mathematics?

What are numbers: objects or properties?

Example 3: Multi-base Arithmetic Blocks (Base Ten)

How many blocks do you see?
These blocks stand for a certain number. What number do you think they stand for?

Figure 4: Multi-base Arithmetic Blocks (Base Ten).



Answers given by 3rd graders:

9 (counting discrete objects, disregarding differences in size and the markings on the blocks)

About 1¼ (measuring using as reference unit 1 (block/cube) + the rest make about a quarter)

923 (measuring surfaces/areas using as reference unit 1 surface = 100 therefore 6 x 100 =600 on the cube + 3x100=300 flat pieces + 20+3)

1.323 (decimal number system)

Questions:
Which interpretation do you consider as "correct"?
Why?

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