PART 4

Fourier Series and Fourier Transform

By the use of the famous ***Fourier Series***, a ***periodic function*** is expressed as a ***sum of harmonics***. In the case of non-periodic functions a generalisation of the Fourier series is used i.e. the ***Fourier Transform*** where the sum is replaced by an integral and as a result the ***non-periodic function*** is expressed as an ***integral where also non-harmonics contribute*** to the function representation.

● ***Fourier Series***: any ***periodic function*** can be expressed as a ***sum of harmonic functions i.e. cos and sin (mathematical glossary)*** or ***"harmonics" (musical glossary)***. Mathematically this is expressed in the following:

THEOREM 1: If a continuous function f(t) is ***periodic*** with period T, i.e. with frequency $\nu = 1/T$, then it may be approximated arbitrarily well[1] by a "***Fourier series***":

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{2\pi nt}{T} + b_n \sin\frac{2\pi nt}{T}\right], \tag{1}$$

where the coefficients are given by:

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\frac{2\pi nt}{T}\,dt\,, \qquad n = 0,1,2,3,\dots \tag{2a}$$

and

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\frac{2\pi nt}{T}\,dt\,. \qquad n = 1,2,3,\dots \tag{2b}$$

The sinusoidal functions (cosines and sinuses) which are added in Eq. (1) are called ***Fourier components***. The coefficients a_n and b_n are called ***Fourier coefficients***. Let us now characterize the various ***Fourier components*** to gain some insight. Please have in mind that a function cos(at) or sin(at) has period $T_a = 2\pi/a$.

a_n	$\cos\frac{2\pi nt}{T}$	b_n	$\sin\frac{2\pi nt}{T}$	period, T	frequency, ν	comment
a_0	1			∞	0	any constant background
a_1	$\cos\frac{2\pi t}{T}$	b_1	$\sin\frac{2\pi t}{T}$	**$T_1 = T$**	**$\nu_1 = \nu$**	**fundamental** or 1st harmonic
a_2	$\cos\frac{4\pi t}{T}$	b_2	$\sin\frac{4\pi t}{T}$	$T_2 = T/2$	$\nu_2 = 2\nu$	2nd harmonic
a_3	$\cos\frac{6\pi t}{T}$	b_3	$\sin\frac{6\pi t}{T}$	$T_3 = T/3$	$\nu_3 = 3\nu$	3rd harmonic
a_4	$\cos\frac{8\pi t}{T}$	b_4	$\sin\frac{8\pi t}{T}$	$T_4 = T/4$	$\nu_4 = 4\nu$	4th harmonic

As can be seen from the last column of this table the onomatology characterizes "***fundamental***" the Fourier component which has the ***same period and frequency as the original function***. All other Fourier components are called "***harmonics***" and the Fourier component with frequency $\nu_n = n\,\nu$ is characterized as «**n**th harmonic».

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[1] Ουσιαστικώς, το θεώρημα λεει ότι το όριο

$$\lim_{k\to\infty}\int_{-T/2}^{T/2}(f(t) - s_k(t))^2\,dt = 0\,,$$

όπου $s_k(t)$ η αντίστοιχη πεπερασμένη σειρά. Το κατά πόσο η σειρά συγκλίνει σημειακώς και αναπαριστά την f(t) προσδιορίζεται από τις συνθήκες Dirichlet. Αν λοιπόν (α΄) η συνάρτηση είναι τμηματικά συνεχής και μονότονη, και (β΄) σε κάθε σημείο ασυνέχειας υπάρχει το όριο από δεξιά και από αριστερά, τότε:

$$\lim_{k\to\infty}\left[\frac{a_0}{2} + \sum_{n=1}^{k}\left[a_n\cos\frac{2\pi nt}{T} + b_n\sin\frac{2\pi nt}{T}\right]\right] = \begin{cases} f(t), & \text{όπου } f \text{ συνεχής} \\ \left(f(t_-) + f(t_+)\right)/2, & \text{στα σημεία ασυνέχειας}\end{cases}$$

One can easily show that if the function is even (odd), then only the a_n (b_n) Fourier coefficients which multiply cosines (sinusus) survive.

Example 1

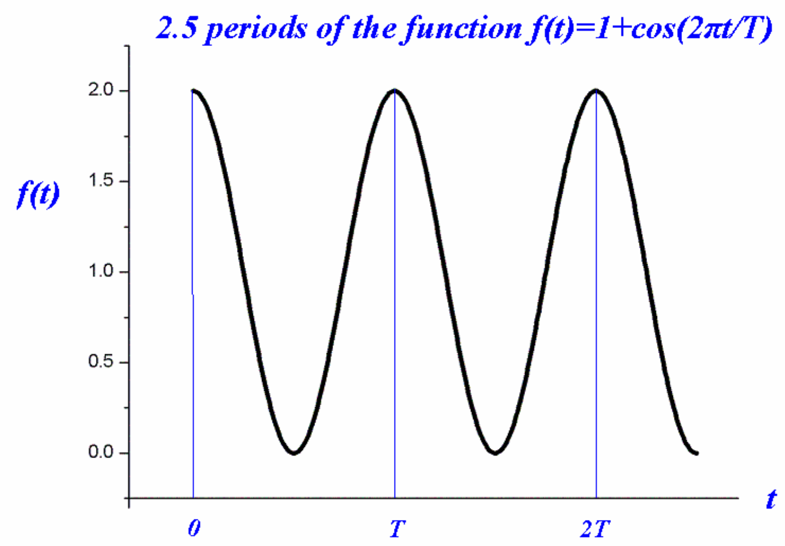
One of the simplest examples is the function

$$f(t) = 1 + \cos(2\pi t/T),$$

which has period T. According to Eqs. (2),

$$a_0 = 2 \text{ and } a_1 = 1$$

i.e. in this case we have only the constant background and the fundamental.



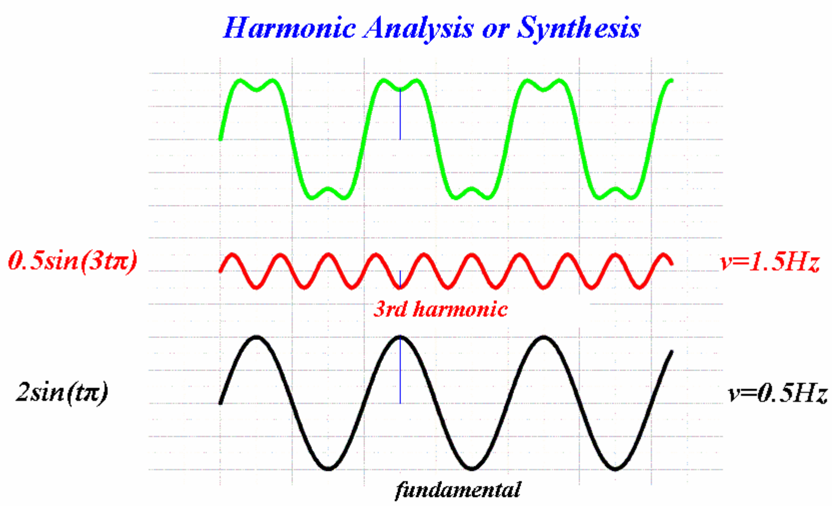
a_n	$\cos\frac{2\pi nt}{T}$	b_n	$\sin\frac{2\pi nt}{T}$	period, T	frequency, ν	comment
$a_0=2$	1			∞	0	any constant background
$a_1=1$	$\cos\frac{2\pi t}{T}$			$T_1 = T$	$\nu_1 = \nu$	**fundamental** or 1st harmonic

Example 2

The picture on the right shows an example of a composite curve (green) which is the algebric sum of the other two curves:
(a) the black one i.e. the fundamental with frequency 0.5 Hz, and
(b) the red one i.e. the 3rd harmonic with frequency 1.5 Hz.

The process of adding the black and the red curve to construct the green one is the ***harmonic synthesis***. The opposite process of finding which components are needed to construct the green curve is called ***harmonic analysis***.

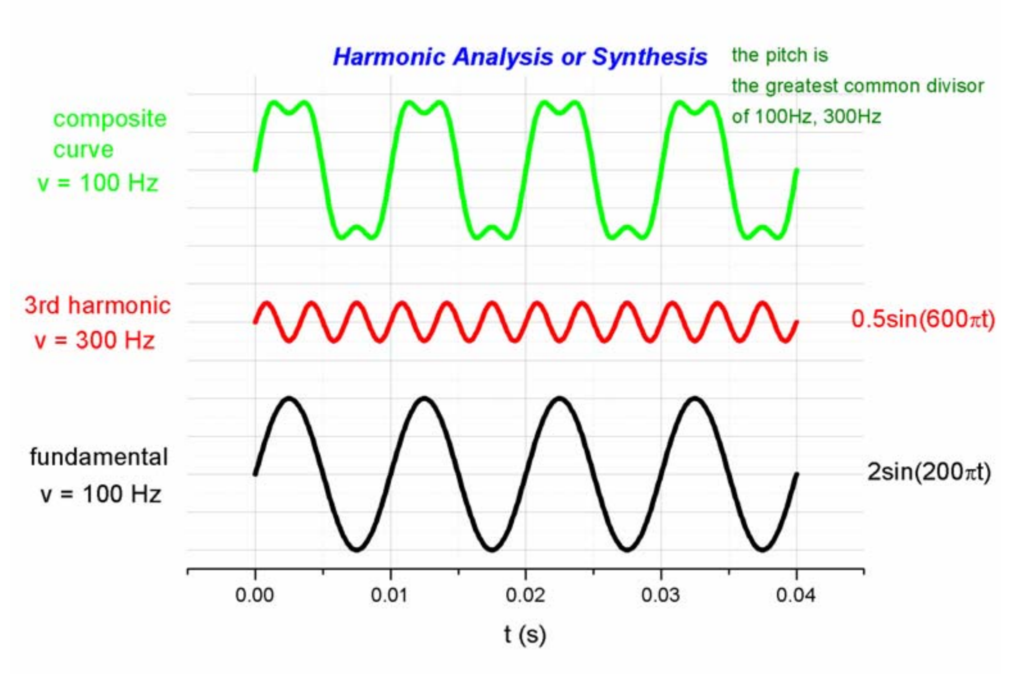
The frequency of the composite green curve is 0.5Hz i.e. the greatest common divisor (GCD) of (0.5 Hz, 1.5 Hz). In this case it coincides with the fundamental frequency.



a_n	$\cos\frac{2\pi nt}{T}$	b_n	$\sin\frac{2\pi nt}{T}$	period, T	frequency, ν	comment
		$b_1=2$	$\sin\frac{2\pi t}{T}$	$T_1 = T$	$\nu_1 = \nu$	**fundamental** or 1st harmonic
		$b_3=0.5$	$\sin\frac{6\pi t}{T}$	$T_3 = T/3$	$\nu_3 = 3\,\nu$	3rd harmonic

Almost the same example (Joos Fig.7) of a composite curve (green) which is the algebric sum of the other two curves:
(a) the black one i.e. the fundamental with frequency 100 Hz, and
(b) the red one i.e. the 3rd harmonic with frequency 300 Hz.

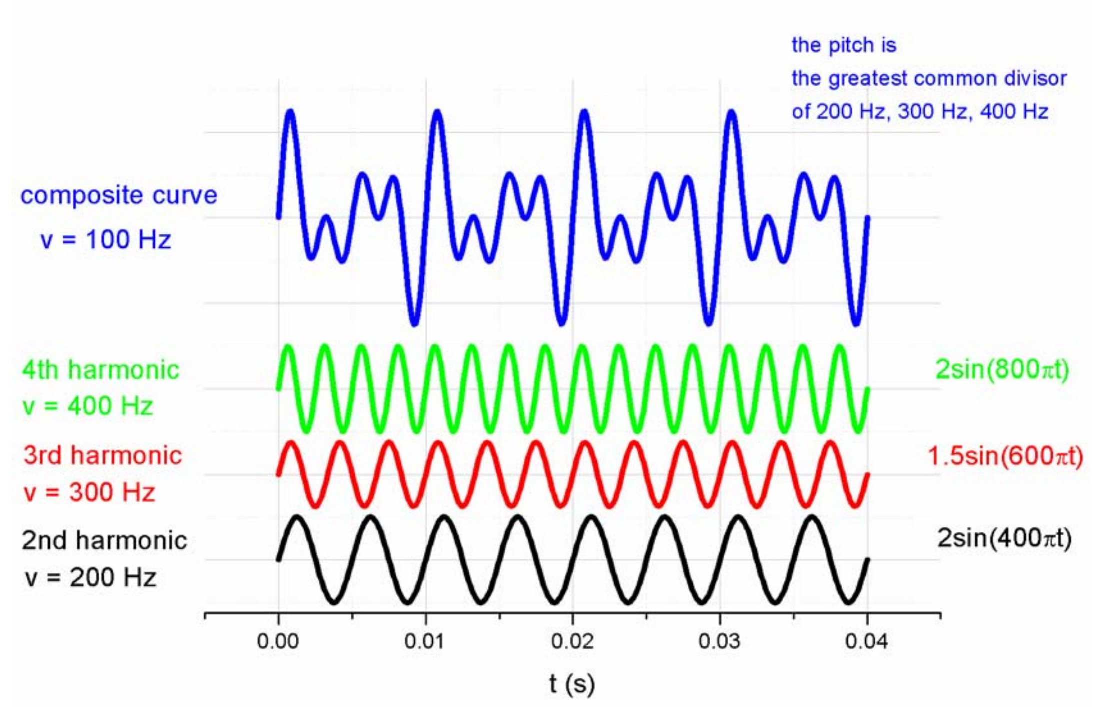
The frequency of the composite green curve (“pitch”) is 100 Hz i.e. the GCD of (100 Hz, 300 Hz). In this case it coincides with the fundamental frequency.



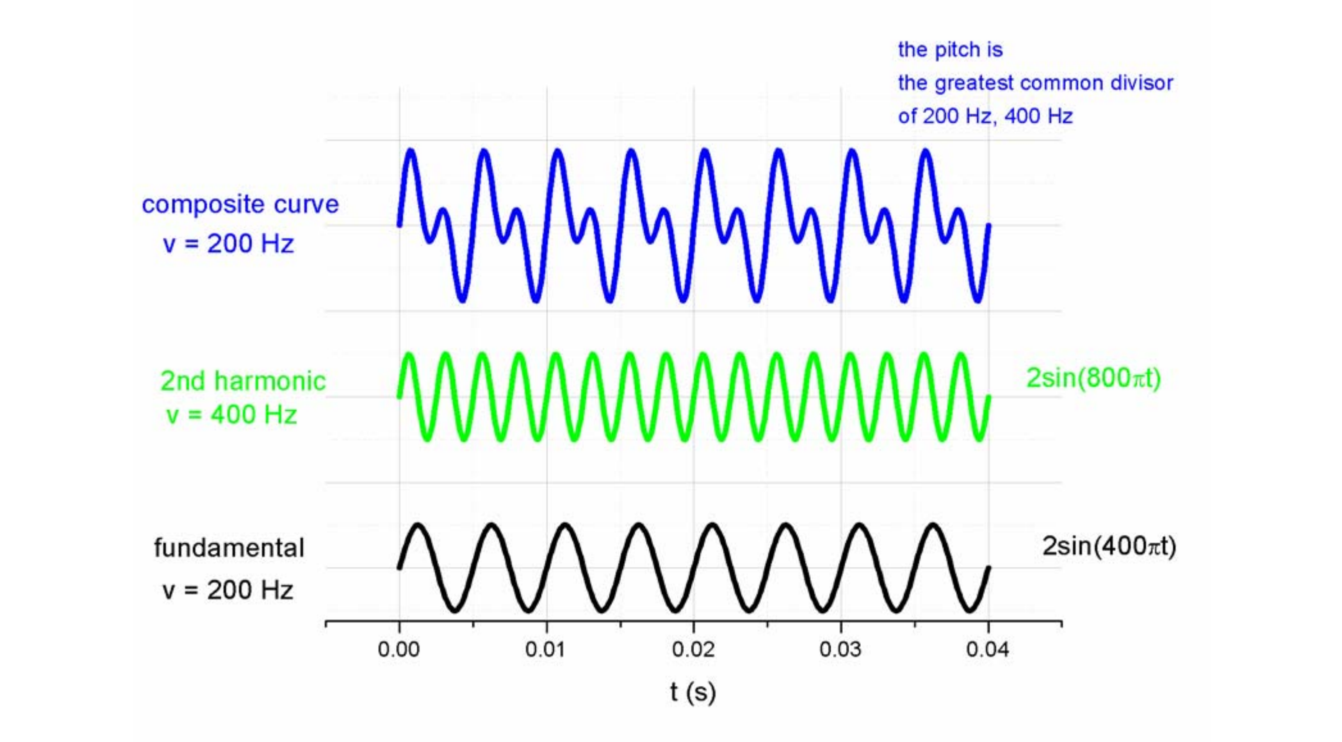
a_n	$\cos\frac{2\pi nt}{T}$	b_n	$\sin\frac{2\pi nt}{T}$	period, T	frequency, ν	comment
		$b_1=2$	$\sin(200\pi t)$	**$T_1 = 1/100$ s**	**$\nu_1 = 100$ Hz**	**fundamental** or 1st harmonic
		$b_3=0.5$	$\sin(600\pi t)$	$T_3 = 1/300$ s	$\nu_3 =$ **300 Hz**	3rd harmonic

Example 3(Joos Figs. 9-10)

A composite wave (blue) whose components do not include the fundamental frequency i.e. the frequency of the wave is ν but its components are 2ν, 3ν, 4ν (“the fundamental component ν is missing from the energy spectrum”). Specifically, the blue curve (100 Hz) is made up from the sum of the black (200 Hz), the red (300 Hz) and the green (400 Hz) curve. The frequency of the composite curve (“pitch”) is the GCD of (200 Hz, 300 Hz, 400 Hz). Although none of its components is 100 Hz, the resulting composite wave has a pitch of 100 Hz!



Now, if we remove the 300 Hz component, the resulting blue curve will have a frequency of 200 Hz i.e., again, the pitch is the GCD of the components' frequencies (200 Hz, 400 Hz). Now, the pitch of the composite curve coincides with the fundamental frequency (200Hz).



The corresponding fact of perception is that by removing the 300 Hz component we have transformed a wave of pitch 100 Hz, into a wave of pitch 200Hz! ***Πρέπει να το δοκιμάσω αυτό.***

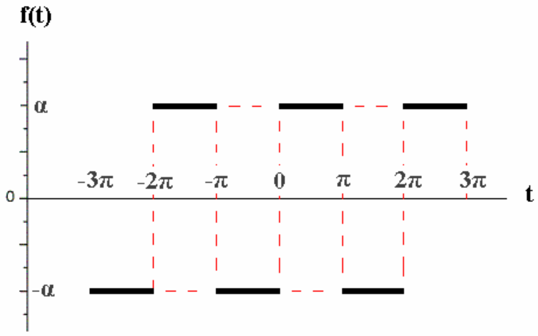
Να δοκιμάσω επίσης το παρακάτω. Listening to a composite sound wave with components 600 Hz, 800 Hz and 1000 Hz, a pitch of 200 Hz will be heard. Then, if to these a fairly strong component at 300 Hz is added, the pitch will be heard to drop one octave to 100 Hz. Thus, the pitch can be defined as the greatest common divisor (GCD) of the frequencies actually present.

Notice Joos paragraph 1.23 referring to Figs. 6,8. "It might seem that not only the sinusoid but equally well some other shape, e.g. Fig. 8A or 8B, could be taken as basic, so that an arbitrary exactly repetitive wave could than be analysed into components of that chosen basic shape, each component having the same shape as the other components though with a different period and presumably a different amplitude too. IT IS NOT POSSIBLE TO DO SO. As soon as it is decided that the components shall all have the same shape and shall be exactly repetitive, that settles it: the components will be sinusoids. The sinusoid is as basic to harmonic analysis as the integers 1,2,3 … are to ordinary arithmetic". Είναι έτσι; Δες επίσης τι ακριβώς είναι τα wavelets. (Μήνυμα Σταυρινού)

Example 3

A second famous example is a pulse function with a period $T = 2\pi$, i.e.

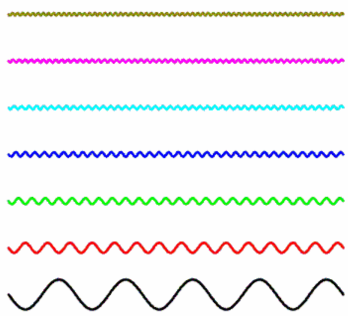
$$f(t) = \begin{cases} -\alpha, & -\pi < t < 0 \\ +\alpha, & 0 < t < \pi \end{cases}$$



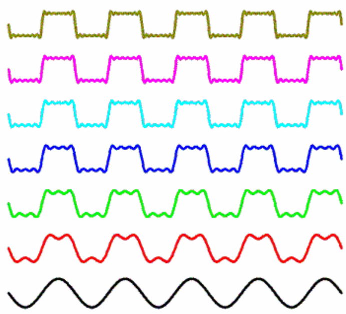
and likewise for any $t \in \Re$. Using Eqs. (2) we obtain:

$$f(t) = \frac{4\alpha}{\pi}\left(\sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \dots\right)$$

The following two figures depict how adding the sinusoidal functions with the correct amplitudes we obtain in the limit the pulse function..



functions we add



result after having added each function

Another form of Theorem 1 is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n \sin\left(\frac{2\pi nt}{T} + \phi_n\right), \qquad \text{where } A_n = \sqrt{a_n^2 + b_n^2} \text{ and } \tan\phi_n = \frac{a_n}{b_n}. \tag{3}$$

Yet, another form of Theorem 1 is obtained using complex notation. Suppose that a function f(t) is ***periodic*** with period T, i.e. with frequency $\nu = 1/T$, then it may be approximated arbitrarily well by a "***complex Fourier series***":

$$f(t) = \sum_{n=-\infty}^{+\infty} c_n e^{i\frac{2\pi nt}{T}} \tag{4}$$

where

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-i\frac{2\pi nt}{T}} dt = \begin{cases} \frac{a_0}{2}, & n = 0 \\ \frac{(a_n - ib_n)}{2}, & n > 0 \\ \frac{(a_{-n} + ib_{-n})}{2}, & n < 0 \end{cases} \tag{5}$$

This form of Theorem 1 via Eqs. (4-5) is the basis for the extremely important ***Fourier transform***, which is obtained by transforming c_n from a discrete variable to a continuous one as the period $T \rightarrow \infty$.

● In the case of ***non-periodic*** functions a generalisation of the Fourier series is used i.e. the ***Fourier Transform*** where the sum is replaced by an integral and as a result the ***non-periodic function*** is expressed as an ***integral where also non-harmonics contribute*** to the function representation.

Next talk for Fourier Transform:

- Παραδείγματα επιτυχούς και μη αναλύσεως Fourier.
- Fourier analysis of MEG signal (α, μ rhythm or general).
- Παραδείγματα μετασχηματισμών t ↔ f (useful for MEG and MRI) και x ↔ k (useful for MRI).
- The recognition of different vowel sounds of the human voice is largely accomplished by analysis of the harmonic content by the inner ear.

References

[1] Bronstein – Semendjajew, Tachenbuch der Mathematik

[2] Fourier Series: http://mathworld.wolfram.com/FourierSeries.html, and
Fourier Transform: http://mathworld.wolfram.com/FourierTransform.html

Created by C. Simserides for "Sound Properties: part3".
Comments should be addressed to csimseri@ifn-magdeburg.de

Next-Rest Bibliography:

Fourier analysis and synthesis: http://hyperphysics.phy-astr.gsu.edu/hbase/audio/Fourier.html#c1
Harmonic content differences in vowel sounds: http://hyperphysics.phy-astr.gsu.edu/hbase/sound/vowel.html#1
Forming the vowel sounds, vocal formants, vowel formants:
http://hyperphysics.phy-astr.gsu.edu/hbase/music/vowel.html#3

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http://hyperphysics.phy-astr.gsu.edu/hbase/music/vowel.html#3
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Comments should be addressed to csimseri@ifn-magdeburg.de