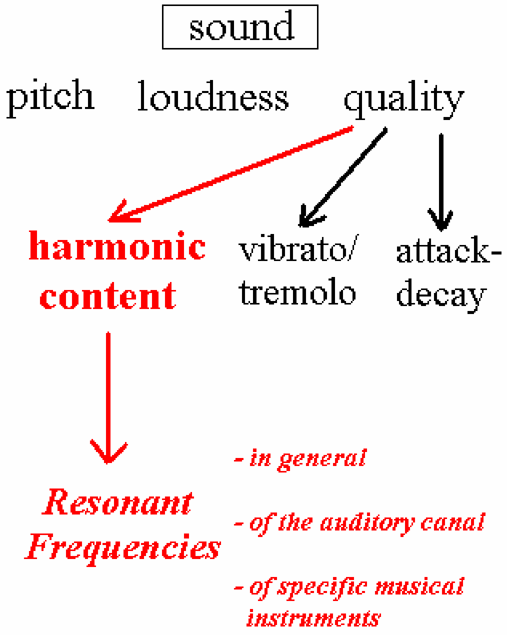
PART 3

Resonant Frequencies or Eigenfrequencies: in general, of the auditory canal, and of specific musical instruments.



We have already presented the three basic characteristics of sound (pitch, loudness and quality). When analyzing sound quality we have stated that it is mainly characterized by the harmonic content and also by vibrato/tremolo and attack-decay. Finally, when analyzing ***harmonic content*** we were obliged to talk about ***resonant frequencies***.

Resonant frequencies will be the subject of the present chapter. This discussion is split into three subchapters:

(a) resonant frequencies in general,
(b) resonant frequencies of the auditory canal, and
(c) resonant frequencies of specific musical instruments

(a) Resonant Frequencies in general.

Any object has - in general - its resonant frequencies or eigenfrequencies, and:

(1) It is ***easy*** to make an object vibrate at its resonant frequencies, ***difficult*** to make it vibrate at other frequencies. This means that we must spend energy in order to make the object vibrate at a different frequency than its own resonant frequencies. Well known examples are:

- the ***simple pendulum*** i.e. a mass hanging from a cord of length ℓ, which has eigenfrequency $f_1 = \frac{1}{2\pi}\sqrt{\frac{g}{\ell}}$, where g is the acceleration of gravity,
- the ***rod pendulum*** i.e. a rod of length ℓ hanging from its one end which has eigenfrequency $f_1 = \frac{1}{2\pi}\sqrt{\frac{3g}{2\ell}}$.

If we just take the rod, move it from the vertical and let it go, it will vibrate at its eigenfrequency mentioned above. If we want to make it vibrate at a different frequency we must e.g. hold it with our hands and oblige it to do so, however spending our precious energy! In both the examples presented above there is only one resonant frequency.

(2) If we create a complex excitation to an object, it will ***select*** its resonant frequencies and it will vibrate at those frequencies (***can be quite a few***), essentially "filtering out" soon all other frequencies present in the excitation. We can, for example, hit the rod pendulum so that it can even make a circle but after a while it will start vibrate at its own well known eigenfrequency.

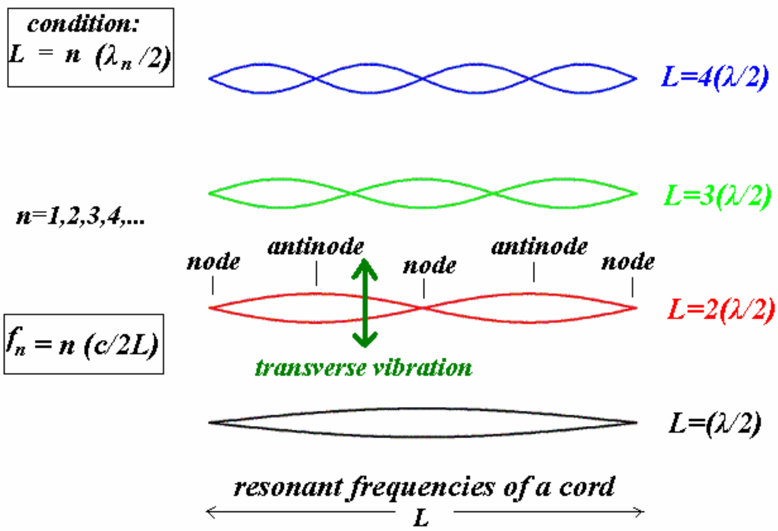
(3) Most vibrating objects have ***many*** resonant frequencies. As an example we give the case of a chord with both ends fixed. If the length of the chord is L, then the following condition must be satisfied:

$$L = n\frac{\lambda_n}{2}.$$

Here $n \in \aleph^* = \{1,2,3,4,...\}$ and λ_n is the corresponding wavelength of the allowed vibration mode. Now, since the velocity of propagation of the wave, c, is related to λ_n and the corresponding frequency, f_n, by $c = \lambda_n f_n$, it follows that:

$$f_n = n\frac{c}{2L} = n f_1.$$

f_1 is called the ***fundamental frequency***



and all f_n are called ***harmonics***. In this special case, c is given by the following equation:

$$c = \sqrt{\frac{T}{\rho}} = \sqrt{\frac{T}{(m/L)}}.$$

T is the tension of the chord, and ρ = (m/L) is the linear mass density. Since, finally,

$$f_n = n\sqrt{\frac{T}{4mL}},$$

chords of different length have different eigenfrequencies and we can change the set of eigenfrequencies of a given chord by altering its tension. It follows e.g. that 4-fold tension of a given chord multiplies its eigenfrequencies with a factor of two (if the length is not modified and if it doesn't break!). Notice that our chord's vibration is ***transverse***. The points on the chord which do not move at all are called ***nodes*** and the points which vibrate with maximum amplitude are called ***antinodes***. As depicted in the picture above, in our case of the chord, the two ends are forced to be nodes, but there are - in general - more nodes.

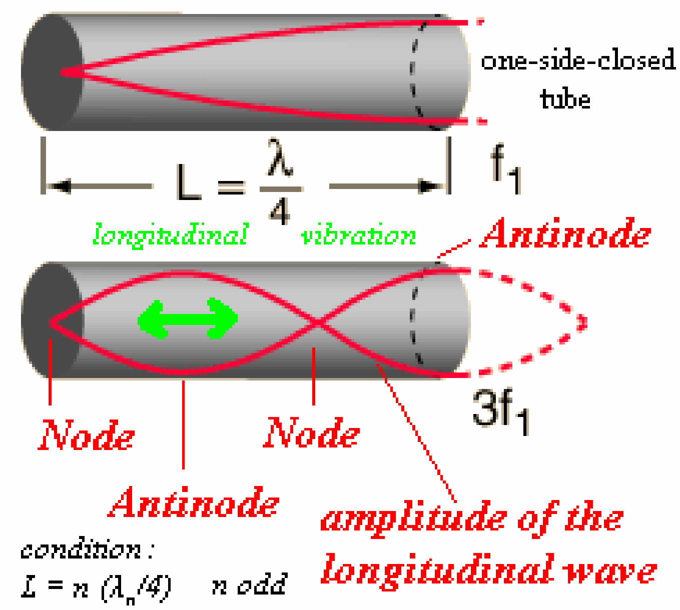
A second example of an object with many resonant frequencies is an one-side-closed cylinder of length L. This allows standing waves with wavelength, λ_n:

$$L = n\,(\lambda_n/4), \quad n = \text{odd } (1,3,5,\dots)$$

i.e. since $c = \lambda_n f_n$ allows frequencies:

$$f_n = n\, f_1, \quad f_1 = c/(4L),$$

where c is the speed of sound. Again, f_1 is the ***fundamental*** frequency and f_n the ***nth harmonic***. In this case, the vibration is ***longitudinal***. The closed end of the cylinder is forced to be a node, since no longitudinal vibration is possible there. On the contrary, the open end of the cylinder is forced to be an antinode, since the cylinder meets the free air molecules. In general, again, there are more nodes and antinodes. For example the third harmonic has two nodes and two antinodes.



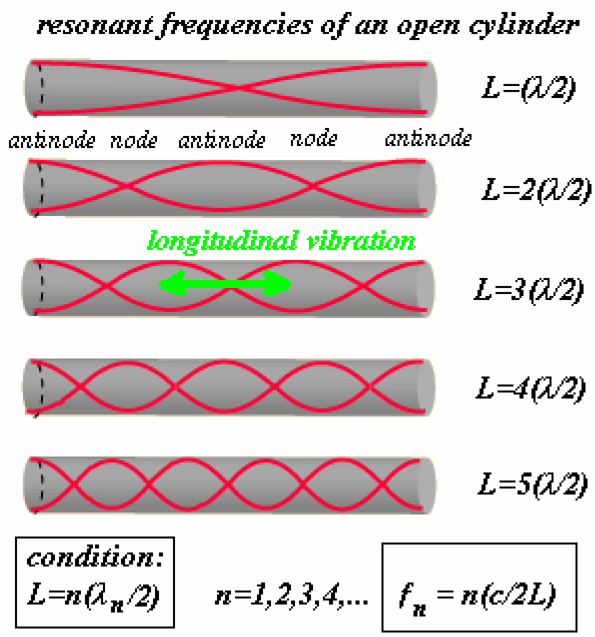
A third example of an object with many eigenfrequencies is an open cylinder of length L. This allows standing waves with wavelength, λ_n:

$$L = n\,(\lambda_n/2), \quad n = 1,2,3,4,...$$

i.e. since $c = \lambda_n f_n$ allows frequencies:

$$f_n = n\, f_1, \quad f_1 = c/(2L),$$

where c is the speed of sound. In this case, the vibration is ***longitudinal*** and - by force - the two ends of the cylinder are antinodes, since the cylinder meets the free air molecules. As can be seen in the picture, the fundamental vibrational mode has one node and two antinodes, the second harmonic has two nodes and three antinodes etc



Let us now turn to membranes. Suppose we have a plane membrane, homogeneously stretched by a tension T, given as force per unit length. The membrane has mass per unit area μ and the boundary is clamped. Then the speed of the vibration is given by $c = (T/\mu)^{1/2}$. A ***rectangular membrane*** vibrates at resonant frequencies given by the equation:

$$f_{mn} = \frac{c}{2}\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}, \qquad m = 1,2,3,\ldots \quad \text{and} \quad n = 1,2,3,\ldots$$

Notice that the eigenfrequencies depend on ***two indices*** i.e. m and n. We stress that with a *complex excitation* the membrane will vibrate at a combination of its resonant frequencies after having expelled all "alien" frequencies. For a square membrane the equation above becomes:

$$f_{mn} = \frac{c}{2a}\sqrt{m^2 + n^2} = f_f\sqrt{m^2 + n^2}, \qquad m = 1,2,3,\ldots \quad \text{and} \quad n = 1,2,3,\ldots$$

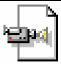
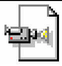
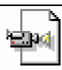
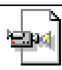
where $f_f = f_{10} = f_{01} = c/(2a)$ is the fundamental frequency. The following table depicts the first resonant frequencies of a square membrane (a = b):

m	n	m	n	f_{mn}	fundamental and overtones
0	0			rest	
0	1	1	0	f_f	1st harmonic
0	2	2	0	$2\ f_f$	2nd harmonic
1	1			$\sqrt{2}\ f_f \approx 1.414\ f_f$	
2	1	1	2	$\sqrt{5}\ f_f \approx 2.236\ f_f$	
2	2			$2\sqrt{2}\ f_f \approx 2.828\ f_f$	
0	3	3	0	$3\ f_f$	3rd harmonic
3	1	1	3	$\sqrt{10}\ f_f \approx 3.162\ f_f$	
2	3	3	2	$\sqrt{13}\ f_f \approx 3.605\ f_f$	
0	4	4	0	$4\ f_f$	4th harmonic

On the other hand, a ***circular membrane*** (e.g. a ***drum*** and ***approximately the tympanic membrane***) has eigenfrequencies given by the equation:

$$f_{mn} = x_{mn}\, c/(2\pi a),$$

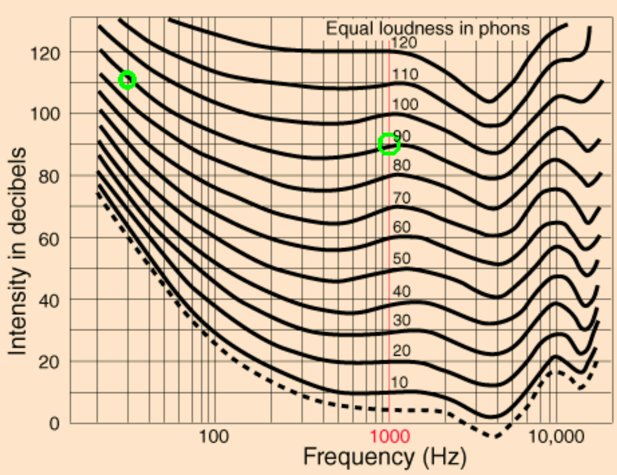
where a is the radius of the membrane and x_{mn} is the n-th root of the $J_m(kr)$ Bessel function. In this case, the radial part of the solution is a Bessel function $J_m(kr)$ and the boundary condition is $k_{mn}\ a = x_{mn}$ i.e. the membrane is supposed to be fixed at r = a. We just give the first four vibrational modes:

fundamental, f_{01}, with $x_{01} \approx 2.4048$ 	β′ overtone, f_{21}, with $x_{21} \approx 5.1336$ 
α′ overtone, f_{11}, with $x_{11} \approx 3.8317$ 	γ′ overtone, f_{02}, with $x_{02} \approx 5.5201$ 

Summarizing, the ***lowest resonant frequency*** of a vibrating object is called its ***fundamental frequency***. If an object has ***more than one*** resonant frequencies these are often called ***overtones***. An ***harmonic*** is defined as an ***integer multiple of the fundamental*** frequency, i.e. the ***nth*** harmonic is such that $\mathbf{f_n = n\ f_1}$, where f_1 is the fundamental frequency. In many cases, as in the above mentioned example of the vibrating chord, the overtones are simply harmonics.

(b) Resonant Frequencies of the Auditory Canal.

The ***auditory canal***, which is roughly 3 cm long and 0.7 cm thick, is approximately an one-side-closed tube. We have already presented the frequencies of maximum sensitivity of human hearing (picture on the right). Notice that I don't know if this figure is completely reliable. These curves can somehow be modelled supposing that the auditory canal is roughly approximated by an one-side closed tube. There is a significant ***dip*** in the range ***2 kHz - 5 kHz*** with a ***peak sensitivity*** around ***3.5 kHz - 4 kHz***. The observed ***peak*** at about ***3.7 kHz*** at body temperature is associated with the ***«fundamental frequency»*** of the auditory canal and «corresponds» to a tube length of 2.4 cm. The high sensitivity region at 2 kHz – 5 kHz is very important for the understanding of ***speech***. There is another enhanced sensitivity region roughly ***above 10 kHz*** which may be associated with the ***«3rd harmonic»*** of the auditory canal.



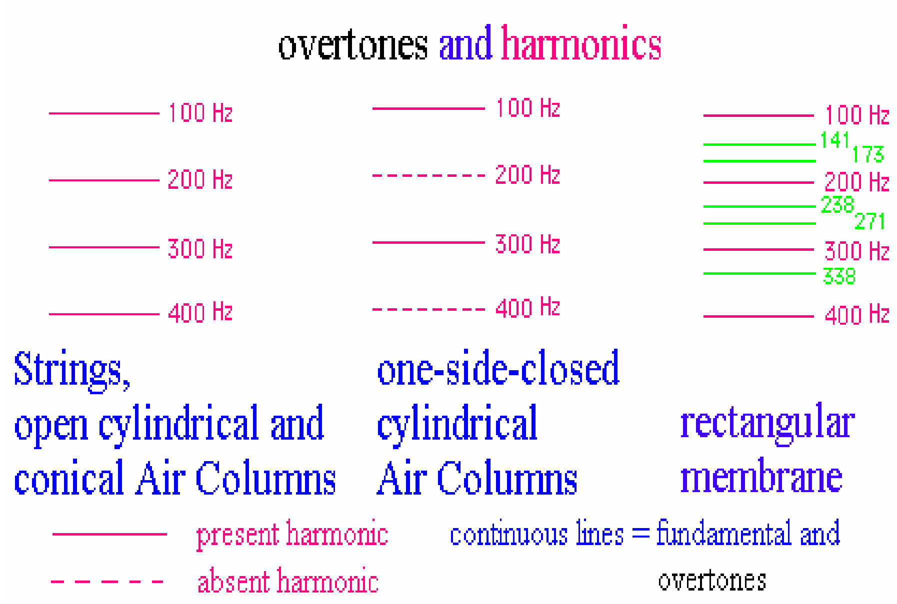
the Sensitivity of the Ear is represented by the Equal Loudness Curves

The ear's sensitivity as a function of frequency can be illustrated by playing some wav files here e.g.

50hz.wav	200hz.wav	500hz.wav	1000hz.wav	3000hz.wav
50 Hz	200 Hz	500 Hz	1 kHz	3 kHz
4000hz.wav	8000hz.wav	10000hz.wav	13500hz.wav	20000hz.wav
4 kHz	8 kHz	10 kHz	13.5 kHz	20 kHz

(c) Resonant Frequencies of specific Musical Instruments.

Many of the instruments of the orchestra, those utilizing ***strings*** and ***air columns***, produce the fundamental frequency and harmonics. Vibrating ***strings***, ***open cylindrical air columns***, and ***conical air columns*** will vibrate at all harmonics of the fundamental. ***One-side-closed cylinders*** will vibrate with only odd harmonics of the fundamental. In all categories specified above, the overtones can be said to be harmonic.



However, sound sources such as ***membranes*** or ***other percussive sources*** may have resonant frequencies which are not integer multiples of their fundamental frequencies. They are said to have some non-harmonic overtones.

Many musical instruments use ***stings*** like e.g. the ***guitar***, the ***violin***, the ***piano*** etc. ***Percussive*** musical instruments are e.g. the various types of ***drums***. Below we also give some examples of ***wind*** instruments.

Musical instruments with ***open air columns*** like the ***flute*** and the ***picolo***, produce all harmonics.

Musical instruments with ***conical air columns*** like the ***oboe*** and the ***saxophone*** produce the same resonant frequencies as an open cylinder of the same length.

flute

picolo

oboe

saxophone

Finally, the ***clarinet*** consists of a ***closed cylindrical air column*** with a bell-shaped opening at one end.

clarinet

References

[1] http://hyperphysics.phy-astr.gsu.edu/hbase/waves/funhar.html#c1
[2] Hermann Härtel and Ernesto Martin, "The vibrating-membrane problem - based on basic principles and simulations", preprint.
[3] http://www.kettering.edu/~drussell/Demos/MembraneCircle/Circle.html

Fourier Series and Fourier Transform

By the use of the famous ***Fourier Series***, a ***periodic function*** is expressed as a ***sum of harmonics***. In the case of non-periodic functions a generalisation of the Fourier series is used i.e. the ***Fourier Transform*** where the sum is replaced by an integral and as a result the ***non-periodic function*** is expressed as an ***integral where also non-harmonics contribute*** to the function representation.

● ***Fourier Series***: any ***periodic function*** can be expressed as a ***sum of harmonic functions i.e. cos and sin (mathematical glossary)*** or ***"harmonics" (musical glossary)***. Mathematically this is expressed in the following:

THEOREM 1: If a continuous function f(t) is ***periodic*** with period T, i.e. with frequency ν = 1/T, then it may be approximated arbitrarily well[1] by a "***Fourier series***":

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{2\pi nt}{T} + b_n \sin\frac{2\pi nt}{T}\right], \qquad (1)$$

where the coefficients are given by:

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\frac{2\pi nt}{T}\,dt, \qquad n = 0,1,2,3,\ldots \qquad (2a)$$

and

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\frac{2\pi nt}{T}\,dt. \qquad n = 1,2,3,\ldots \qquad (2b)$$

The sinusoidal functions (cosines and sinuses) which are added in Eq. (1) are called ***Fourier components***. The coefficients a_n and b_n are called ***Fourier coefficients***. Let us now characterize the various ***Fourier components*** to gain some insight. Please have in mind that a function cos(at) or sin(at) has period $T_a = 2\pi/a$.

a_n	$\cos\frac{2\pi nt}{T}$	b_n	$\sin\frac{2\pi nt}{T}$	period, T	frequency, ν	comment
a_0	1			∞	0	any constant background
a_1	$\cos\frac{2\pi t}{T}$	b_1	$\sin\frac{2\pi t}{T}$	$T_1 = T$	$\nu_1 = \nu$	**fundamental** or 1st harmonic
a_2	$\cos\frac{4\pi t}{T}$	b_2	$\sin\frac{4\pi t}{T}$	$T_2 = T/2$	$\nu_2 = 2\nu$	2nd harmonic
a_3	$\cos\frac{6\pi t}{T}$	b_3	$\sin\frac{6\pi t}{T}$	$T_3 = T/3$	$\nu_3 = 3\nu$	3rd harmonic
a_4	$\cos\frac{8\pi t}{T}$	b_4	$\sin\frac{8\pi t}{T}$	$T_4 = T/4$	$\nu_4 = 4\nu$	4th harmonic

As can be seen from the last column of this table the onomatology characterizes "***fundamental***" the Fourier component which has the ***same period and frequency as the original function***. All other Fourier components are called "***harmonics***" and the Fourier component with frequency ν_n = **n** ν is characterized as «**n**th harmonic». One can easily show that if the function is even (odd), then only the a_n (b_n) Fourier coefficients which multiply cosines (sinusus) survive.

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[1] Ουσιαστικώς, το θεώρημα λεει ότι το όριο

$$\lim_{k\to\infty}\int_{-T/2}^{T/2}(f(t)-s_k(t))^2\,dt = 0,$$

όπου $s_k(t)$ η αντίστοιχη πεπερασμένη σειρά. Το κατά πόσο η σειρά συγκλίνει σημειακώς και αναπαριστά την f(t) προσδιορίζεται από τις συνθήκες Dirichlet. Αν λοιπόν (α΄) η συνάρτηση είναι τμηματικά συνεχής και μονότονη, και (β΄) σε κάθε σημείο ασυνέχειας υπάρχει το όριο από δεξιά και από αριστερά, τότε:

$$\lim_{k\to\infty}\left[\frac{a_0}{2} + \sum_{n=1}^{k}\left[a_n\cos\frac{2\pi nt}{T} + b_n\sin\frac{2\pi nt}{T}\right]\right] = \begin{cases} f(t), & \textit{όπου } f \textit{ συνεχής} \\ \left(f(t_-)+f(t_+)\right)/2, & \textit{στα σημεία ασυνέχειας}\end{cases}$$

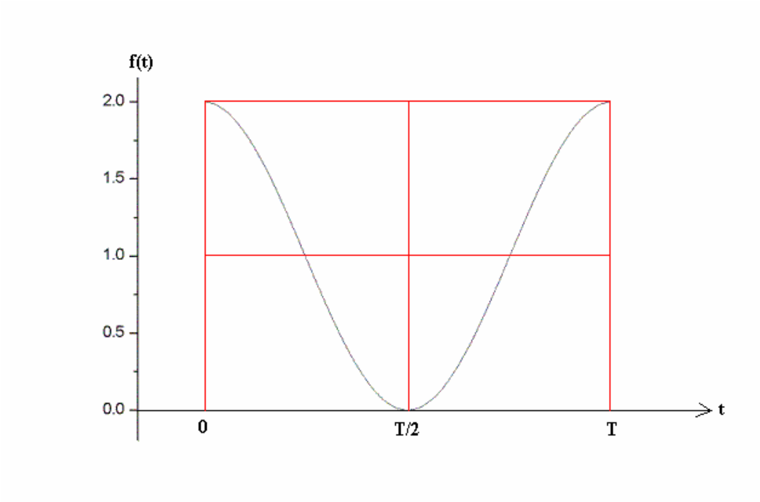
One of the simplest examples is the function

$$f(t) = 1 + \cos(2\pi t/T),$$

which has period T. According to Eqs. (2),

$$a_0 = 2 \text{ and } a_1 = 1$$

i.e. in this case we have only the constant background and the fundamental.

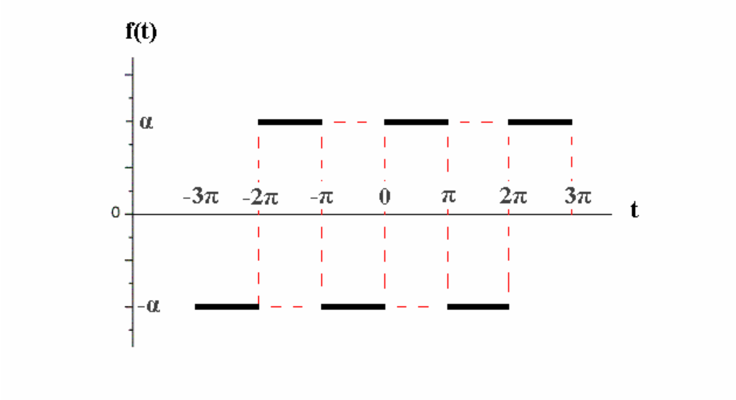


A second famous example is a pulse function with a period $T = 2\pi$, i.e.

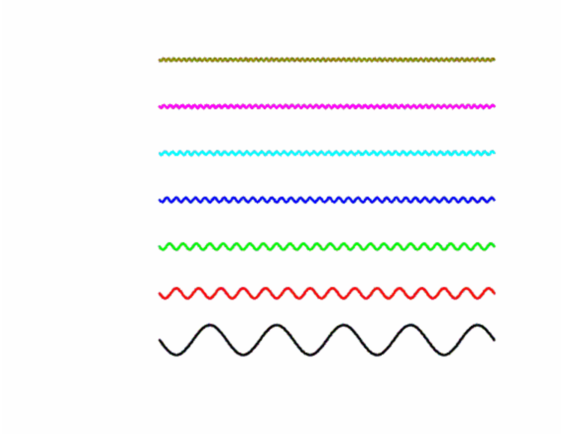
$$f(t) = \begin{cases} -\alpha, & -\pi < t < 0 \\ +\alpha, & 0 < t < \pi \end{cases}$$

and likewise for any $t \in \Re$. Using Eqs. (2) we obtain:

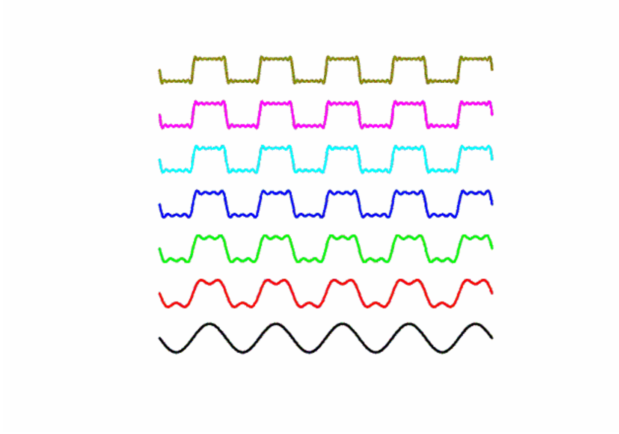
$$f(t) = \frac{4\alpha}{\pi}\left(\sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \ldots \right)$$



The following two figures depict how adding the sinusoidal functions with the correct amplitudes we obtain in the limit the pulse function..



functions we add



result after having added each function

Another form of Theorem 1 is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n \sin\left(\frac{2\pi n t}{T} + \phi_n\right), \qquad \text{where } A_n = \sqrt{a_n^2 + b_n^2} \text{ and } \tan\phi_n = \frac{a_n}{b_n}. \qquad (3)$$

Yet, another form of Theorem 1 is obtained using complex notation. Suppose that a function f(t) is ***periodic*** with period T, i.e. with frequency $\nu = 1/T$, then it may be approximated arbitrarily well by a "***complex Fourier series***":

$$f(t) = \sum_{n=-\infty}^{+\infty} c_n e^{i\frac{2\pi n t}{T}} \qquad (4)$$

where

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-i\frac{2\pi nt}{T}}dt = \begin{cases} \frac{a_0}{2}, & n=0 \\ \frac{(a_n - ib_n)}{2}, & n>0 \\ \frac{(a_{-n} + ib_{-n})}{2}, & n<0 \end{cases} \quad (5)$$

This form of Theorem 1 via Eqs. (4-5) is the basis for the extremely important ***Fourier transform***, which is obtained by transforming c_n from a discrete variable to a continuous one as the period $T \rightarrow \infty$.

● In the case of ***non-periodic*** functions a generalisation of the Fourier series is used i.e. the ***Fourier Transform*** where the sum is replaced by an integral and as a result the ***non-periodic function*** is expressed as an ***integral where also non-harmonics contribute*** to the function representation.

Next talk for Fourier Transform:

- Παραδείγματα επιτυχούς και μη αναλύσεως Fourier.
- Fourier analysis of MEG signal (α, μ rhythm or general).
- Παραδείγματα μετασχηματισμών t ↔ f (useful for MEG and MRI) και x ↔ k (useful for MRI).
- The recognition of different vowel sounds of the human voice is largely accomplished by analysis of the harmonic content by the inner ear.

References

[1] Bronstein – Semendjajew, Tachenbuch der Mathematik
[2] Fourier Series: http://mathworld.wolfram.com/FourierSeries.html, and
Fourier Transform: http://mathworld.wolfram.com/FourierTransform.html

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Next-Rest Bibliography:

Fourier analysis and synthesis: http://hyperphysics.phy-astr.gsu.edu/hbase/audio/Fourier.html#c1
Harmonic content differences in vowel sounds: http://hyperphysics.phy-astr.gsu.edu/hbase/sound/vowel.html#1
Forming the vowel sounds, vocal formants, vowel formants:
http://hyperphysics.phy-astr.gsu.edu/hbase/music/vowel.html#3