Resonant Frequencies or Eigenfrequencies:
in general, of the auditory canal, and of specific musical instruments.

We have already presented the three basic characteristics of sound (pitch, loudness and quality). When analyzing sound quality we have stated that it is mainly characterized by the harmonic content and also by vibrato/tremolo and attack-decay. Finally, when analyzing harmonic content we were obliged to talk about resonant frequencies.

Resonant frequencies will be the subject of the present chapter. This discussion is split into three subchapters:

(a) resonant frequencies in general,
(b) resonant frequencies of the auditory canal, and
(c) resonant frequencies of specific musical instruments

(a) Resonant Frequencies in general.
Any object has - in general - its resonant frequencies or eigenfrequencies, and:

(1) It is easy to make an object vibrate at its resonant frequencies, difficult to make it vibrate at other frequencies. This means that we must spend energy in order to make the object vibrate at a different frequency than its own resonant frequencies. Well known examples are:

- the simple pendulum i.e. a mass hanging from a cord of length ℓ, which has eigenfrequency \( f_1 = \frac{1}{2\pi} \sqrt{\frac{g}{\ell}} \), where \( g \) is the acceleration of gravity,

- the rod pendulum i.e. a rod of length ℓ hanging from its one end which has eigenfrequency \( f_1 = \frac{1}{2\pi} \sqrt{\frac{3g}{2\ell}} \).

If we just take the rod, move it from the vertical and let it go, it will vibrate at its eigenfrequency mentioned above. If we want to make it vibrate at a different frequency we must e.g. hold it with our hands and oblige it to do so, however spending our precious energy! In both the examples presented above there is only one resonant frequency.

(2) If we create a complex excitation to an object, it will select its resonant frequencies and it will vibrate at those frequencies (can be quite a few), essentially "filtering out" soon all other frequencies present in the excitation. We can, for example, hit the rod pendulum so that it can even make a circle but after a while it will start vibrate at its own well known eigenfrequency.

(3) Most vibrating objects have many resonant frequencies. As an example we give the case of a chord with both ends fixed. If the length of the chord is \( L \), then the following condition must be satisfied:

\[
L = n \frac{\lambda_n}{2}.
\]

Here \( n \in \mathbb{N}^* = \{1,2,3,4,...\} \) and \( \lambda_n \) is the corresponding wavelength of the allowed vibration mode. Now, since the velocity of propagation of the wave, \( c \), is related to \( \lambda_n \) and the corresponding frequency, \( f_n \), by \( c = \lambda_n f_n \), it follows that:

\[
f_n = n \frac{c}{2L} = n f_1.
\]

\( f_1 \) is called the fundamental frequency and all \( f_n \) are called harmonics. In this special case, \( c \) is given by the following equation:
The tension of the chord, and \( \rho = (m/L) \) is the linear mass density. Since, finally, \( f_n = n \sqrt{\frac{T}{4mL}} \),

chords of different length have different eigenfrequencies and we can change the set of eigenfrequencies of a given chord by altering its tension. It follows e.g. that 4-fold tension of a given chord multiplies its eigenfrequencies with a factor of two (if the length is not modified and if it doesn’t break!). Notice that our chord’s vibration is transverse. The points on the chord which do not move at all are called nodes and the points which vibrate with maximum amplitude are called antinodes. As depicted in the picture above, in our case of the chord, the two ends are forced to be nodes, but there are - in general - more nodes.

A second example of an object with many resonant frequencies is an one-side-closed cylinder of length \( L \). This allows standing waves with wavelength, \( \lambda_n \):

\[
L = n \left( \frac{\lambda_n}{4} \right), \quad n = \text{odd} \ (1,3,5,...)
\]

i.e. since \( c = \lambda_n f_n \) allows frequencies:

\[
f_n = n f_1, \quad f_1 = \frac{c}{4L},
\]

where \( c \) is the speed of sound. Again, \( f_1 \) is the fundamental frequency and \( f_n \) the \( n \)th harmonic. In this case, the vibration is longitudinal. The closed end of the cylinder is forced to be a node, since no longitudinal vibration is possible there. On the contrary, the open end of the cylinder is forced to be an antinode, since the cylinder meets the free air molecules. In general, again, there are more nodes and antinodes. For example the third harmonic has two nodes and two antinodes.

A third example of an object with many eigenfrequencies is an open cylinder of length \( L \). This allows standing waves with wavelength, \( \lambda_n \):

\[
L = n \left( \frac{\lambda_n}{2} \right), \quad n = 1,2,3,4,...
\]

i.e. since \( c = \lambda_n f_n \) allows frequencies:

\[
f_n = n f_1, \quad f_1 = \frac{c}{2L},
\]

where \( c \) is the speed of sound. In this case, the vibration is longitudinal and - by force - the two ends of the cylinder are antinodes, since the cylinder meets the free air molecules. As can be seen in the picture, the fundamental vibrational mode has one node and two antinodes, the second harmonic has two nodes and three antinodes etc.

Let us now turn to membranes. Suppose we have a plane membrane, homogeneously stretched by a tension \( T \), given as force per unit length. The membrane has mass per unit area \( \mu \) and the boundary is clamped. Then the speed of the vibration is given by \( c = (T/\mu)^{1/2} \). A rectangular membrane vibrates at resonant frequencies given by the equation:

\[
f_n = n \sqrt{\frac{T}{4\mu L}}.
\]
Notice that the eigenfrequencies depend on two indices i.e. m and n. We stress that with a complex excitation the membrane will vibrate at a combination of its resonant frequencies after having expelled all “alien” frequencies. For a square membrane the equation above becomes:

\[ f_{mn} = \frac{c}{2} \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}, \quad m = 1,2,3,… \quad \text{and} \quad n = 1,2,3,… \]

where \( f_r = f_{10} = f_{01} = c/(2a) \) is the fundamental frequency. The following table depicts the first resonant frequencies of a square membrane (\( a = b \)):

<table>
<thead>
<tr>
<th>m</th>
<th>n</th>
<th>m</th>
<th>n</th>
<th>( f_{mn} )</th>
<th>fundamental and overtones</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>m</td>
<td>n</td>
<td>( f_r )</td>
<td>1st harmonic</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>( \sqrt{2} f_r \approx 1.414 f_r )</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>( \sqrt{5} f_r \approx 2.236 f_r )</td>
<td>2nd harmonic</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>( 2 \sqrt{2} f_r \approx 2.828 f_r )</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>( 3 f_r )</td>
<td>3rd harmonic</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>( 10 f_r \approx 3.162 f_r )</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>( \sqrt{13} f_r \approx 3.605 f_r )</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td>4</td>
<td>0</td>
<td>( 4 f_r )</td>
<td>4th harmonic</td>
</tr>
</tbody>
</table>

On the other hand, a circular membrane (e.g. a drum and approximately the tympanic membrane) has eigenfrequencies given by the equation:

\[ f_{mn} = \frac{x_{mn}}{a} \frac{c}{(2\pi a)}, \]

where a is the radius of the membrane and \( x_{mn} \) is the n-th root of the \( J_m(kr) \) Bessel function. In this case, the radial part of the solution is a Bessel function \( J_m(kr) \) and the boundary condition is \( k_{mn}a = x_{mn} \) i.e. the membrane is supposed to be fixed at \( r = a \). We just give the first four vibrational modes:

<table>
<thead>
<tr>
<th>Fundamental, ( f_{01} ), with ( x_{01} \approx 2.4048 )</th>
<th>( \beta' ) overtone, ( f_{21} ), with ( x_{21} \approx 5.1336 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha' ) overtone, ( f_{11} ), with ( x_{11} \approx 3.8317 )</td>
<td>( \gamma' ) overtone, ( f_{02} ), with ( x_{02} \approx 5.5201 )</td>
</tr>
</tbody>
</table>

Summarizing, the lowest resonant frequency of a vibrating object is called its fundamental frequency. If an object has more than one resonant frequencies these are often called overtones. An harmonic is defined as an integer multiple of the fundamental frequency, i.e. the nth harmonic is such that \( f_n = n f_1 \), where \( f_1 \) is the fundamental frequency. In many cases, as in the above mentioned example of the vibrating chord, the overtones are simply harmonics.

(b) Resonant Frequencies of the Auditory Canal.

The auditory canal, which is roughly 3 cm long and 0.7 cm thick, is approximately an one-side-closed tube. We have already presented the frequencies of maximum sensitivity of human hearing (picture on the right). Notice that I don’t know if this figure is completely reliable. These curves can somehow be modelled supposing that the auditory canal is roughly approximated by an one-side closed tube. There is a significant dip in the range 2 kHz - 5 kHz with a peak sensitivity around 3.5 kHz - 4 kHz. The observed peak at about 3.7 kHz at body temperature is associated with the fundamental frequency of the auditory canal and corresponds to a tube length of 2.4 cm. The high sensitivity region at 2 kHz – 5 kHz is very important for the understanding of speech. There is another enhanced sensitivity region roughly above 10 kHz which may be associated with the 3rd harmonic of the auditory canal.
The ear’s sensitivity as a function of frequency can be illustrated by playing some wav files here e.g.

<table>
<thead>
<tr>
<th>文件名</th>
<th>频率</th>
</tr>
</thead>
<tbody>
<tr>
<td>50hz.wav</td>
<td>50 Hz</td>
</tr>
<tr>
<td>200hz.wav</td>
<td>200 Hz</td>
</tr>
<tr>
<td>500hz.wav</td>
<td>500 Hz</td>
</tr>
<tr>
<td>1000hz.wav</td>
<td>1 kHz</td>
</tr>
<tr>
<td>3000hz.wav</td>
<td>3 kHz</td>
</tr>
<tr>
<td>4000hz.wav</td>
<td>4 kHz</td>
</tr>
<tr>
<td>8000hz.wav</td>
<td>8 kHz</td>
</tr>
<tr>
<td>10000hz.wav</td>
<td>10 kHz</td>
</tr>
<tr>
<td>13500hz.wav</td>
<td>13.5 kHz</td>
</tr>
<tr>
<td>20000hz.wav</td>
<td>20 kHz</td>
</tr>
</tbody>
</table>

(c) Resonant Frequencies of specific Musical Instruments.

Many of the instruments of the orchestra, those utilizing strings and air columns, produce the fundamental frequency and harmonics. Vibrating strings, open cylindrical air columns, and conical air columns will vibrate at all harmonics of the fundamental. One-side-closed cylinders will vibrate with only odd harmonics of the fundamental. In all categories specified above, the overtones can be said to be harmonic.

However, sound sources such as membranes or other percussive sources may have resonant frequencies which are not integer multiples of their fundamental frequencies. They are said to have some non-harmonic overtones.

Many musical instruments use strings like e.g. the guitar, the violin, the piano etc. Percussive musical instruments are e.g. the various types of drums. Below we also give some examples of wind instruments.

References
[1] [http://hyperphysics.phy-astr.gsu.edu/hbase/waves/funhar.html#c1](http://hyperphysics.phy-astr.gsu.edu/hbase/waves/funhar.html#c1)
[3] [http://www.kettering.edu/~drussell/Demos/MembraneCircle/Circle.html](http://www.kettering.edu/~drussell/Demos/MembraneCircle/Circle.html)
Fourier Series and Fourier Transform

By the use of the famous Fourier Series, a periodic function is expressed as a sum of harmonics. In the case of non-periodic functions a generalisation of the Fourier series is used i.e. the Fourier Transform where the sum is replaced by an integral and as a result the non-periodic function is expressed as an integral where also non-harmonics contribute to the function representation.

- Fourier Series: any periodic function can be expressed as a sum of harmonic functions i.e. cos and sin (mathematical glossary) or “harmonics” (musical glossary). Mathematically this is expressed in the following:

THEOREM 1: If a continuous function $f(t)$ is periodic with period $T$, i.e. with frequency $\nu = 1/T$, then it may be approximated arbitrarily well by a “Fourier series”:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \frac{2\pi nt}{T} + b_n \sin \frac{2\pi nt}{T} \right],$$  

where the coefficients are given by:

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos \frac{2\pi nt}{T} \, dt, \quad n = 0,1,2,3,…$$  

and

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin \frac{2\pi nt}{T} \, dt. \quad n = 1,2,3,…$$

The sinusoidal functions (cosines and sinuses) which are added in Eq. (1) are called Fourier components. The coefficients $a_n$ and $b_n$ are called Fourier coefficients. Let us now characterize the various Fourier components to gain some insight. Please have in mind that a function $\cos(at)$ or $\sin(at)$ has period $T_a = 2\pi/a$.

<table>
<thead>
<tr>
<th>$a_n$</th>
<th>$\cos \frac{2\pi nt}{T}$</th>
<th>$b_n$</th>
<th>$\sin \frac{2\pi nt}{T}$</th>
<th>period, $T$</th>
<th>frequency, $\nu$</th>
<th>comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_0$</td>
<td>1</td>
<td></td>
<td></td>
<td>$\infty$</td>
<td>0</td>
<td>any constant background</td>
</tr>
<tr>
<td>$a_1$</td>
<td>$\cos \frac{2\pi T}{T}$</td>
<td>$b_1$</td>
<td>$\sin \frac{2\pi T}{T}$</td>
<td>$T_1 = T$</td>
<td>$\nu_1 = \nu$</td>
<td>fundamental or 1st harmonic</td>
</tr>
<tr>
<td>$a_2$</td>
<td>$\cos \frac{4\pi T}{T}$</td>
<td>$b_2$</td>
<td>$\sin \frac{4\pi T}{T}$</td>
<td>$T_2 = T/2$</td>
<td>$\nu_2 = 2\nu$</td>
<td>2nd harmonic</td>
</tr>
<tr>
<td>$a_3$</td>
<td>$\cos \frac{6\pi T}{T}$</td>
<td>$b_3$</td>
<td>$\sin \frac{6\pi T}{T}$</td>
<td>$T_3 = T/3$</td>
<td>$\nu_3 = 3\nu$</td>
<td>3rd harmonic</td>
</tr>
<tr>
<td>$a_4$</td>
<td>$\cos \frac{8\pi T}{T}$</td>
<td>$b_4$</td>
<td>$\sin \frac{8\pi T}{T}$</td>
<td>$T_4 = T/4$</td>
<td>$\nu_4 = 4\nu$</td>
<td>4th harmonic</td>
</tr>
</tbody>
</table>

As can be seen from the last column of this table the onomatology characterizes “fundamental” the Fourier component which has the same period and frequency as the original function. All other Fourier components are called “harmonics” and the Fourier component with frequency $\nu_n = \nu n$ is characterized as «$n$th harmonic». One can easily show that if the function is even (odd), then only the $a_n$ ($b_n$) Fourier coefficients which multiply cosines (sinuses) survive.

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1 Ουσιαστικά, το θεόρημα λέει ότι το ορίο

$$\lim_{k \to \infty} \int_{-T/2}^{T/2} (f(t) - s_k(t))^2 \, dt = 0,$$

όπου $s_k(t)$ η αντίστοιχη πεπερασμένη σειρά. Το κατά πόσο η σειρά συγκλίνει σημειακός και αναπαριστά την $f(t)$ προσδιορίζεται από τις συνθήκες Dirichlet. Αν λοιπόν (α’) η συνάρτηση είναι τιμηματικά συνεχής και μονότονη, και (β’) σε κάθε σημείο ασυνέχειας υπάρχει το ορίο από δεξιά και από αριστερά, τότε:

$$\lim_{k \to \infty} \left[ \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \frac{2\pi nt}{T} + b_n \sin \frac{2\pi nt}{T} \right] \right] = \left\{ f(t), \right\} \left( f(t_+) + f(t_-) \right)/2, \sigmaτα σημεία ασυνέχειας$$
One of the simplest examples is the function
\[ f(t) = 1 + \cos(2\pi t / T), \]
which has period T. According to Eqs. (2),
\[ a_0 = 2 \text{ and } a_1 = 1 \]
i.e. in this case we have only the constant background and the fundamental.

A second famous example is a pulse function with a period \( T = 2\pi \), i.e.
\[
\begin{cases}
-\alpha, & -\pi < t < 0 \\
\alpha, & 0 < t < \pi \\
\end{cases}
\]
and likewise for any \( t \in \mathbb{R} \). Using Eqs. (2) we obtain:
\[
f(t) = \frac{4\alpha}{\pi} \left( \sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \ldots \right)
\]
The following two figures depict how adding the sinusoidal functions with the correct amplitudes we obtain in the limit the pulse function.

Another form of Theorem 1 is
\[
f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n \sin \left( \frac{2\pi nt}{T} + \phi_n \right), \quad \text{where } A_n = \sqrt{a_n^2 + b_n^2} \text{ and } \tan \phi_n = \frac{a_n}{b_n}. \quad (3)
\]
Yet, another form of Theorem 1 is obtained using complex notation. Suppose that a function \( f(t) \) is periodic with period \( T \), i.e. with frequency \( \nu = 1/T \), then it may be approximated arbitrarily well by a “complex Fourier series”:
\[
f(t) = \sum_{n = -\infty}^{+\infty} c_n e^{i \frac{2\pi nt}{T}} \quad (4)
\]
where

\[ c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-\frac{2\pi nt}{T}} \, dt = \begin{cases} a_0, & n = 0 \\ \frac{2}{T} \left( a_n - i b_n \right), & n > 0 \\ \frac{2}{T} \left( a_n + i b_n \right), & n < 0 \end{cases} \]  

This form of Theorem 1 via Eqs. (4-5) is the basis for the extremely important Fourier transform, which is obtained by transforming \( c_n \) from a discrete variable to a continuous one as the period \( T \to \infty \).

In the case of non-periodic functions a generalisation of the Fourier series is used i.e. the Fourier Transform where the sum is replaced by an integral and as a result the non-periodic function is expressed as an integral where also non-harmonics contribute to the function representation.

Next talk for Fourier Transform:
- Παραδείγματα επιτυχούς και μη αναλύσεως Fourier.
- Fourier analysis of MEG signal (α, μ rhythm or general).
- Παραδείγματα και τα συνθήκες της άναλυσης (useful for MEG and MRI) και \( x \leftrightarrow k \) (useful for MRI).
- The recognition of different vowel sounds of the human voice is largely accomplished by analysis of the harmonic content by the inner ear.

References
[1] Bronstein – Semendjajew, Tachenbuch der Mathematik

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Harmonic content differences in vowel sounds: http://hyperphysics.phy-astr.gsu.edu/hbase/sound/vowel.html#1
Forming the vowel sounds, vocal formants, vowel formants: http://hyperphysics.phy-astr.gsu.edu/hbase/music/vowel.html#3