Compact and efficient implicit representations

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HELLENIC REPUBLIC
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Marie Skłodowska-Curie
Actions

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ARCADES Network

- 13 PhD students
- 6 hosting countries ; Barcelona, Glasgow, Sophia-Antipolis, Wien, Linz, Athens, Oslo
- 14 research centers and industrial partners
- 2 secondments per PhD student (in my case : SINTEF, Oslo, and RISC, Linz)
- Algebraic Geometry, Rigidity Theory, Computer Graphics, CAD-CAE. . .

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$\approx$ Parametric and Implicit $\not\approx$

Let V be a variety of dimension d in $\mathbb{C}^n$.
Its codimension is $c = n - d$.

Parametric

A (rational) parametric description of V is a rational function $p : \mathbb{C}^d \to \mathbb{C}^n$.

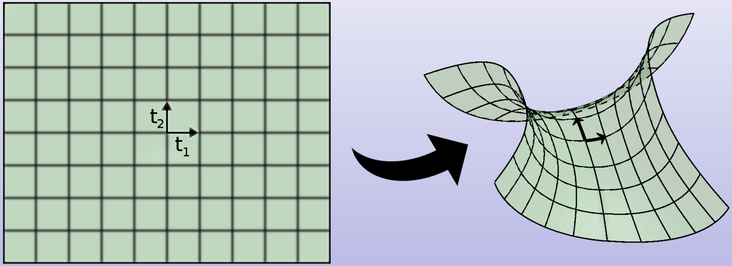
$$V = \left\{ p(t) \mid t \in \mathbb{C}^d \right\}$$

Implicit

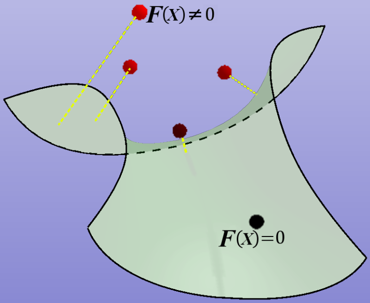
A standard implicit description of V is a set of polynomial functions $F_i : \mathbb{C}^n \to \mathbb{C}$.

$$V = \{ x \in \mathbb{C}^n \mid F_i(x) = 0, \forall i \}$$

Parametric



Implicit



Parametric and Implicit

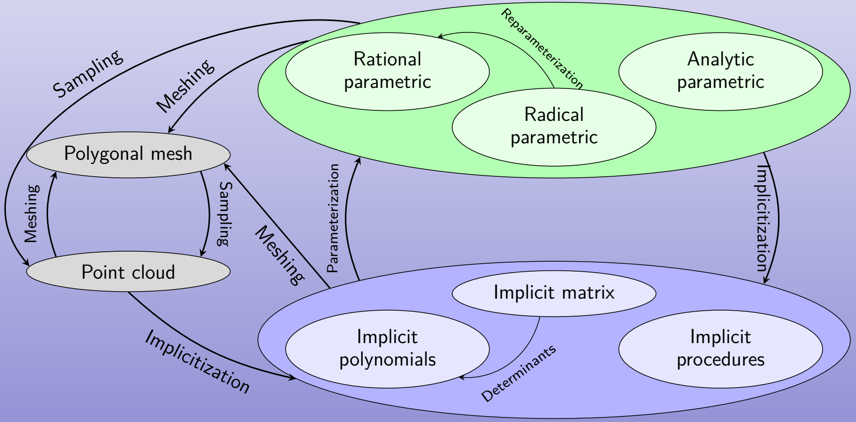
Parametric Representations

- Extensively used in CAGD
- Sample and display are simple
- Can be local (eg. Bezier curve/patch)
- Cannot be used for any variety (not closed under intersection)

Implicit Representations

- Provide geometric and algebraic informations (degree, genus, ideal...)
- Intersection and membership are simple
- Allow raytracing technics
- Describe the whole variety

≈ Representation Catalogue ≈



≈ Implicitization Toolbox ≈

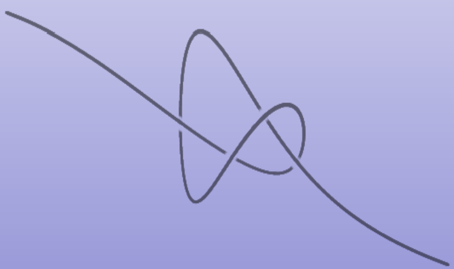
Remark : «$f_i(x,t) := x_i - p_i(t) = 0$» is an easy first step towards implicitization, where p is a parametrization.

Tools

- **Groebner bases**
  Using algebraic tools (ideals, euclidean division...), find equations equivalent to $(f_i)_i$ in $(\mathbb{C}[x,t] \setminus \mathbb{C}[x]) \coprod \mathbb{C}[x]$,
- **Elimination theory**
  Using resultants, eliminate the variable(s) t,
- **Syzygy theory**
  Using syzygies and μ-bases, build convenient implicit representations.

≂ Implicit Representation Problems ≃

Problem (I) : the number of equations required is not obvious when codim > 1.



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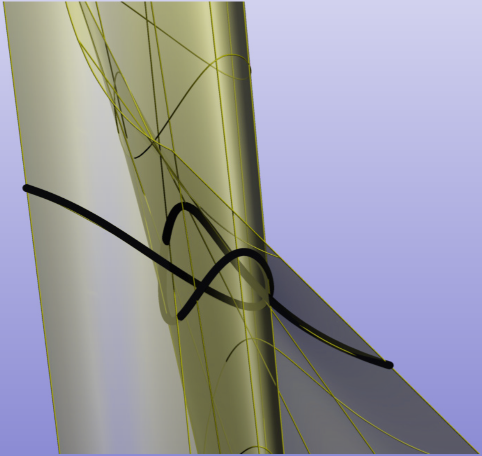

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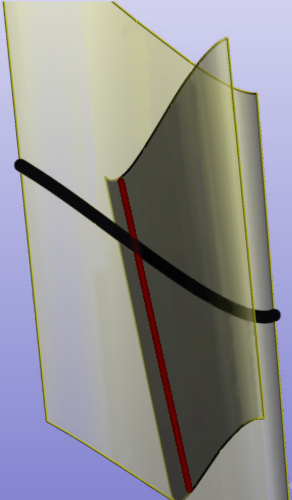
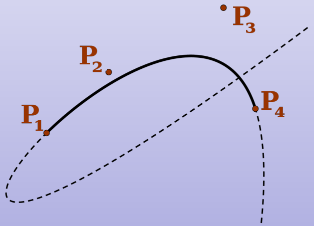


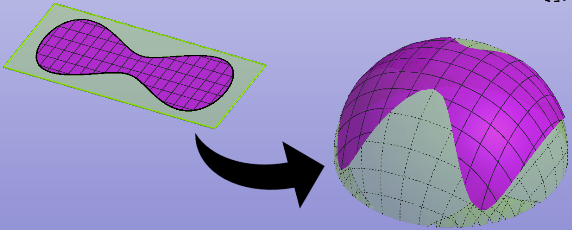
Implicit Representation Problems

Problem (II) : implicit representations are not local.

Examples :

- Bézier curves self-intersecting inside their control polygon
- Surface trimming





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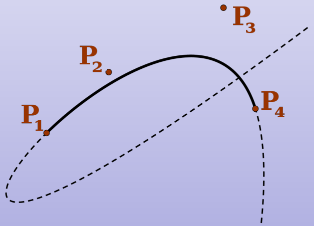

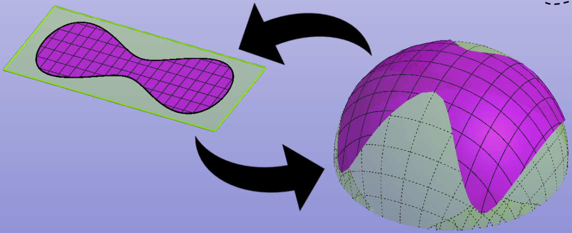

Solution : solving the inversion problem

≂ Implicit Representation Problems ≴

Problem (III) : Instability
Floating-point arithmetic makes the evaluation of high degree polynomials instable : rounding errors propagate and explode.

Example

$$P(x) := (x^3 - 1)(x - 10)^{15}$$

$$P\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 0$$

$$P(-0.5 + 0.866025403784440i) \approx 1.652551896306318 + 8.724965314413668i$$

$$\widetilde{P}(-0.5 + 0.866025403784440i) = 4.19138410839463 + 1.46574416565220i$$

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≂ Swept Volume ≴

C. Laroche. *An Implicit Representation of Swept Volumes based on Local Shapes and Movements*. arXiv, 2020. Joint work with A. Raffo

RISC Software GmbH (Hagenberg, Austria) develops tools to simulate drilling and shaping tools.

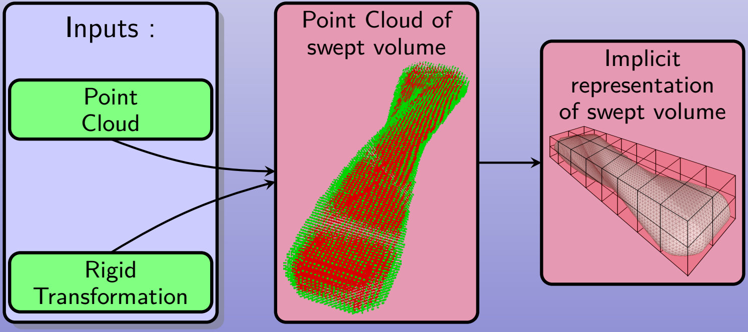
Moving tool

A *base tool* B is a bounded 3D model given by local implicit patches $(A_i, f_i)_{1 \leq i \leq N}$:

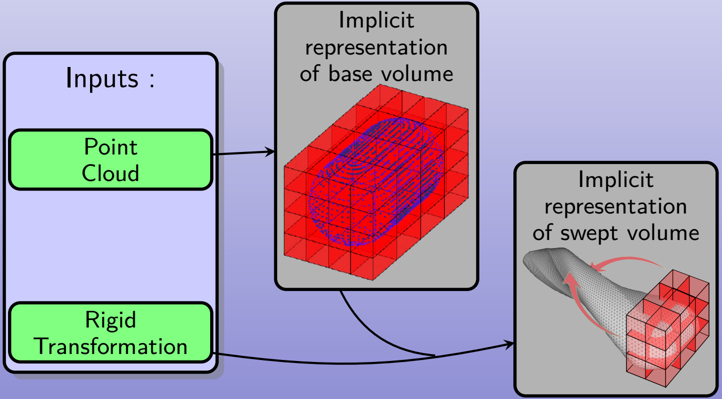
- A_i is an area (ball, cube, convex polygon...)
- f_i is a local implicit procedure : given $x \in A_i, f_i(x) \leq 0 \iff x \in (B \cap A_i)$

A *sweeping transformation* is a piecewise smooth map $I \to \mathrm{Iso}^+(\mathbb{R}^3)$

≈ Swept Volumes : previous strategy

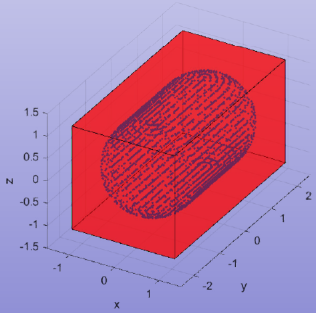


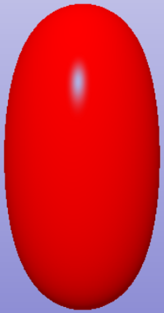
Swept Volumes : new strategy



≃ Example of local implicit patches : LR-BSplines ≄

We combine LR-BSplines and sweeping transformations to have implicit representation of swept volume.
↳ We can use it for boolean operations (intersection, difference, etc).





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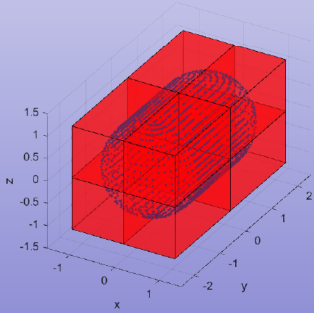

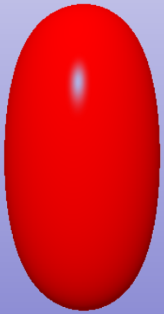

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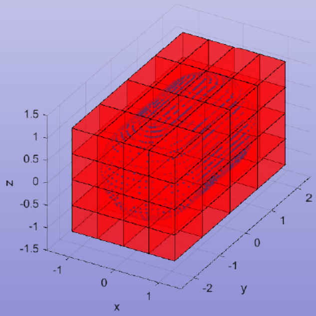
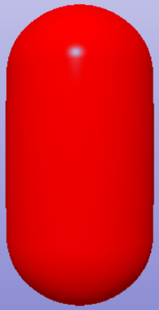
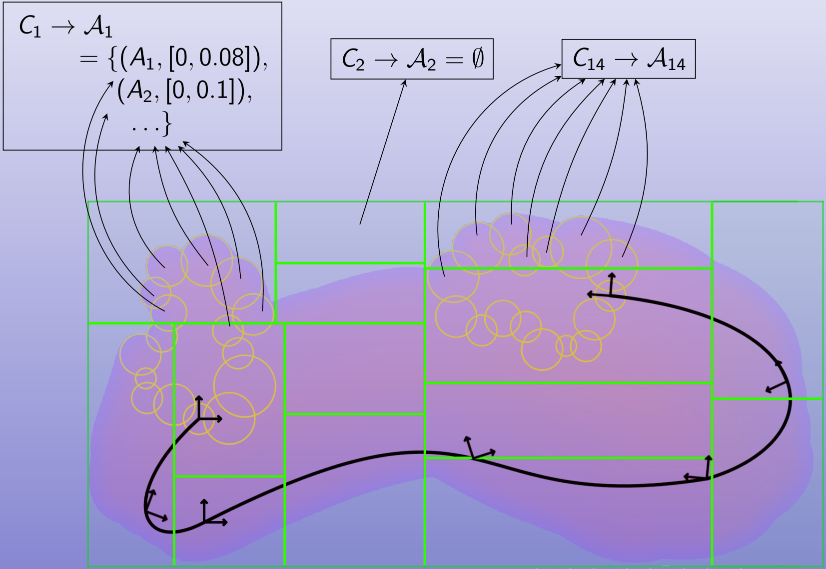




≈ Structure of our implicit representation ≉



SINTEF (Oslo, Norway) develops C++ code manipulating geometric objects (GoTools).
Let $f_0, \ldots, f_n$ be polynomials in n variables.
The coefficients of f_i are $\{c_{i,\alpha} \mid 0 \leq i \leq n \text{ and } \alpha \in \Delta_i\}$ with $\text{Support}(f_i) \subset \Delta_i \subset \mathbb{N}^n$.

Dense Resultant

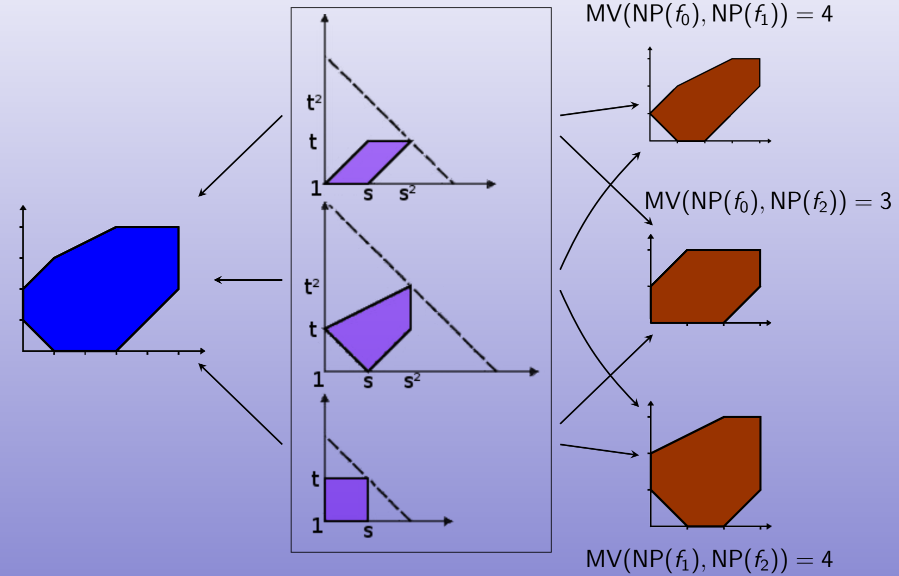
Use $\Delta_i = \{\alpha \mid |\alpha| \leq \deg(f_i)\}$
Then there is a polynomial Res_M in $\mathbb{C}[c_{i,\alpha}]$ such that :

- $\text{Res}_M(C) = 0 \iff \exists x$ such that $f_0(x) = \cdots = f_n(x) = 0$
- $\forall i, \deg_i(\text{Res}_M) = \prod_{j \neq i} \deg(f_j)$

Sparse Resultant

Use $\Delta_i =$ Newton Polytope of $\text{Support}(f_i)$. Then there is a polynomial Res_S verifying the same properties except that $\deg_i(\text{Res}_S) = \text{MixedVolume}((\Delta_j)_{j \neq i})$

Sparse Resultant



≈ Sparse Resultant Matrix ≉

Algorithm (Maple [*IZE 2000*] and C++ [*CL 2018*])

1. Compute Newton polytopes Q_i of f_i,
2. Compute mixed subdivision of Minkowski sum $Q := Q_0 + \cdots + Q_n$ (from lower hull of a generic lifting), Each cell is given by $\sigma = S_0 + \cdots + S_n$ where $S_i \subset Q_i$ and $\exists j_\sigma$ such that $\dim(S_{j_\sigma}) = 0$, Each point $p \in ((Q + \delta) \cap \mathbb{N}^n)$ belongs to a unique cell $\sigma(p)$, where δ is a small generic translation.
3. Construct

$$M := (\mathrm{coeff}(x^{p - S_{j_\sigma(p)}} f_i, x^q))_{p,q \in ((Q+\delta)\cap\mathbb{N}^n)}$$

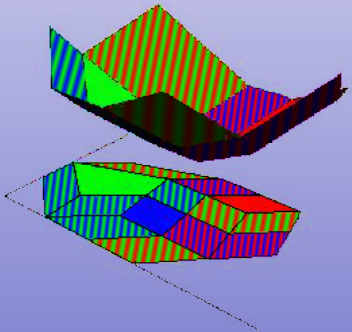
http://users.uoa.gr/~claroche/publications/SparseResultant.zip

Mixed Subdivision

$f_0 = a_{00} + a_{10}x + a_{21}x^2y + a_{11}xy$ $\quad$ $f_1 = b_{01}y + b_{22}x^2y^2 + b_{21}x^2y + b_{10}x$

$f_2 = c_{00} + c_{01}y + c_{11}xy + c_{10}x$

Each cell is $S_0 + S_1 + S_2$ where at least one is reduced to a point



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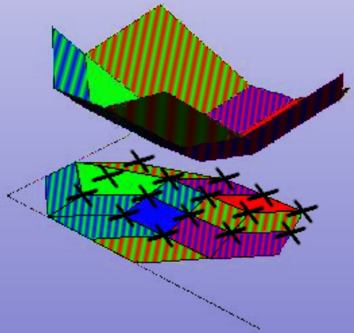
Sparse resultant matrix is indexed by integer points of Q



$$M = \begin{pmatrix} xyf_0 \\ xy^2f_0 \\ xy^2f_1 \\ \vdots \\ x^3y^2f_0 \\ x^4y^3f_2 \end{pmatrix}$$

≈ Chow Form ≉

I. Z. Emiris, C. Konaxis, C. Laroche and I. Kotsireas. *Matrix representations by means of interpolation*. ISSAC '17, pp149-156, jul 2017.
I. Z. Emiris, C. Konaxis and C. Laroche. *Implicit representations of high-codimension varieties*. CAGD, 74 :101764, oct 2019.

Definition

Let $H_0, \ldots, H_d$ be linear forms where $H_i(X) = u_{i0}X_0 + \cdots + u_{in}X_n$ for $i = 0, \ldots, d$.
The Chow form of the variety V is the *single* polynomial R_V in the variables u_{ij} such that $R_V(u_{ij}) = 0 \Leftrightarrow V \cap \{H_0 = 0, \ldots, H_d = 0\} \neq \emptyset$.

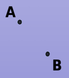
Proposition

V is uniquely determined by its Chow form. More precisely, a point $x \in \mathbb{C}^n$ lies in V if and only if any $(n - d - 1)$-dimensional linear subspace containing x belongs to the Chow form (ie. the parameters defining this subspace are a root of R_V).

Chow Form

Example

$V = \text{Zeros}(Y - X^2, X + Y) = \{(-1,1),(0,0)\} = \{A, B\}$
Then the Chow Form is a polynomial in a, b, c vanishing iff A or B belongs to $aX + bY + c = 0$.



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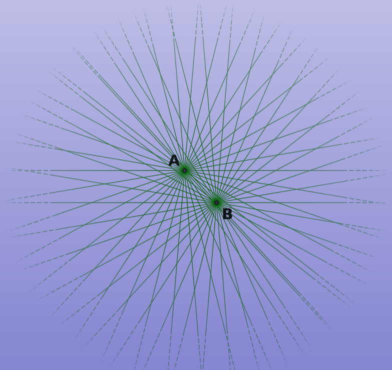

$$R_V(a,b,c) = (ax_A + by_A + c)(ax_B + by_B + c)$$

≈ Conical Hypersurface ≈

Curve in 3 D

For a curve $\mathscr{C}$ in $\mathbb{C}^3$ and a point G, we have
$\mathrm{Cone}(G, \mathscr{C}) = \cup_{x \in \mathscr{C}} \mathrm{Line}(G, x)$.

Curve in 4 D

For a curve $\mathscr{C}$ in $\mathbb{C}^4$ and two points G_1, G_2, we have
$\mathrm{Cone}(G_1, G_2, \mathscr{C}) = \cup_{x \in \mathscr{C}} \mathrm{Plane}(G_1, G_2, x)$.

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General Case

For a variety V of codimension c in $\mathbb{C}^n$ and $c-1$ points $G_1, \dots, G_{c-1}$, we have
$\mathrm{Cone}(G_1, \dots, G_{c-1}, V) = \cup_{x \in V} \mathrm{Aff}(G_1, \dots, G_{c-1}, x)$.

≿ Resultant for Implicitization ⋦

Given $p : \mathbb{C}^d \to \mathbb{C}^n$ a parameterization of V,
we choose $c-1$ *generic* points $G_1, \ldots, G_{c-1}$ and $\xi_{01}, \ldots, \xi_{dd}$.

Implicitization

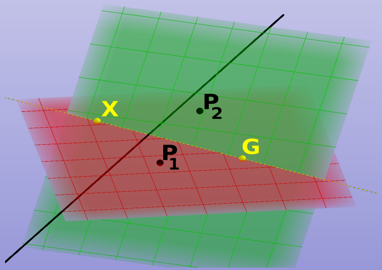
$$\mathrm{Res}_t \left(\begin{vmatrix} G_1 & \cdots & G_{c-1} & \xi_{01} & \cdots & \xi_{0d} & p(t) & X \\ 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & 1 \end{vmatrix}, \ldots, \begin{vmatrix} G_1 & \cdots & G_{c-1} & \xi_{d1} & \cdots & \xi_{dd} & p(t) & X \\ 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & 1 \end{vmatrix} \right)$$

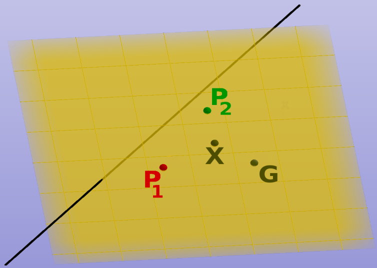
is a hypersurface that can be reduced to two components : Cone(G, V) and an extraneous hypersuface E of degree d.

≂ The Extraneous Factor ≉

The extraneous plane

$$\left.\begin{array}{l} H_0 = \mathrm{Aff}(G, \xi_0, X) \\ H_1 = \mathrm{Aff}(G, \xi_1, X) \end{array}\right\} \text{The extraneous plane is } \mathrm{Aff}(G, \xi_0, \xi_1).$$





The resultant vanishes when $H_0 \cap H_1$ intersects the curve.
When $X \in \mathrm{Aff}(G, \xi_0, \xi_1)$, $H_0 = H_1$ and intersects the curve anyway.

≃ The Extraneous Factor ≄

The extraneous hypersurface [*IZE, CK, CL 2017*]

In general, the extraneous factor is an hypersurface E of degree d. Its equation is given by the following formula :
$\bigodot_{i=0}^{d}(G_1 \dots G_{c-1} \wedge \xi_{i1} \dots \xi_{id} \wedge X) = 0$

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Degree

Note : Since the degree of the resultant is quite high compared to the degrees of Cone(G, V) and of the extraneous factor, they appear with some power.

$$\underbrace{\text{Resultant}}_{\text{degree } \leqslant \delta^d + d\delta^d} = \underbrace{\text{Cone}(G, V)^q}_{\text{degree } \leqslant \delta^d \times q} \times \underbrace{E^p}_{\text{degree } \leqslant d \times p}$$

≍ Curves in $\mathbb{C}^n$ ≴

http://users.uoa.gr/~claroche/publications/ChowFormImplicitize.zip

Although the method works for any variety of codimension $c > 1$, it runs better for curves.

Curves in $\mathbb{C}^n$

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Simpler Terminating Condition [*IZE, CK, CL 2017*]

n equations are sufficient for describing a curve in $\mathbb{C}^n$. We don't have an optimal terminating condition for arbitrary codimension.

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Simpler Resultant Computation

We use the univariate Sylvester resultant instead of the multivariate sparse resultant.

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Simpler Extraneous Factor

Extraneous hyperplane $= \mathrm{Aff}(G_1, \ldots, G_{n-2}, \xi_0, \xi_1)$.

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≈ Syzygies ≉

L. Busé, C. Laroche and F. Yıldırım. *Implicitizing rational curves by the method of moving quadrics*. CAD, 114 :101–111, sep 2019.

Let $p = \left(\frac{p_1}{p_0}, \ldots, \frac{p_n}{p_0}\right)$ a parameterization of a curve with $\deg(p_i) = \delta$ and $I := < p_0, \ldots, p_n >$.

The Space of Syzygies

$$\mathrm{Syz}(I) := \left\{ h = (h_0, \ldots, h_n) \mid \sum_{i=0}^{n} h_i \, p_i = 0 \right\}$$

$\mathrm{Syz}(I)$ is a module.

≍ Syzygies ≭

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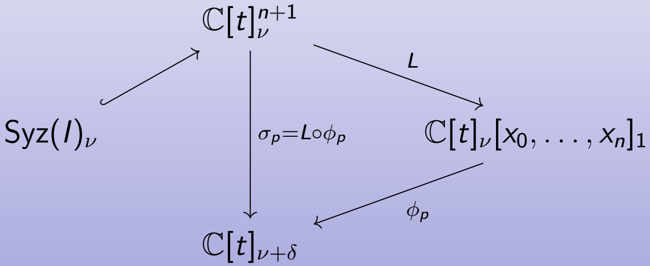
$\mathrm{Syz}(I)$ and $\mathrm{Syz}(I^2)$ are modules.

$$\mathrm{Syz}(I^2) := \left\{ h = (h_{00}, \ldots, h_{nn}) \mid \sum_{0 \le i \le j \le n} h_{ij}\, p_i\, p_j = 0 \right\}$$

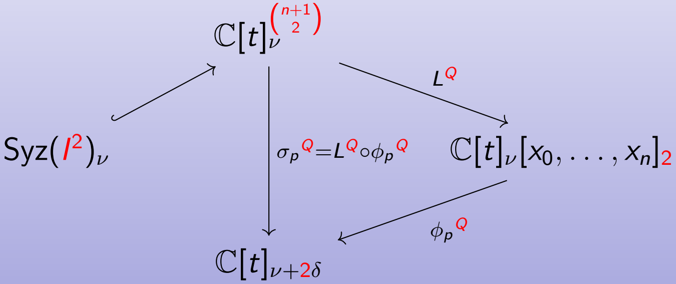
≈ The different spaces involved ≈



With :

- $L(h_0, \ldots, h_n) = \sum_i h_i \, x_i$
- $\phi_p(\sum_i h_i \, x_i) = \sum_i h_i \, p_i$
- $\mathrm{Syz}(I)_\nu = \mathrm{Ker}(\sigma_p)$

The different spaces involved

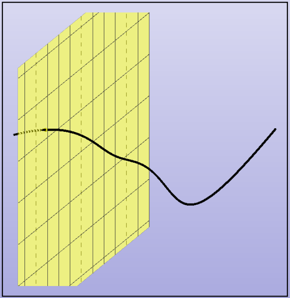


With :

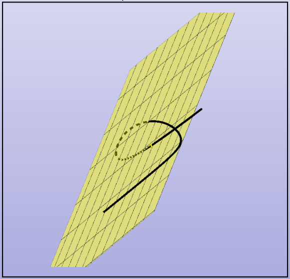
- $L^Q(h_{00},\ldots,h_{nn}) = \sum_{ij} h_{ij}\, x_i\, x_j$
- ${\phi_p}^Q(\sum_{ij} h_{ij}\, x_i\, x_j) = \sum_{ij} h_{ij}\, p_i\, p_j$
- $\mathrm{Syz}(I^2)_\nu = \mathrm{Ker}({\sigma_p}^Q)$

≈ Aka. Moving Planes following the Curve ≈

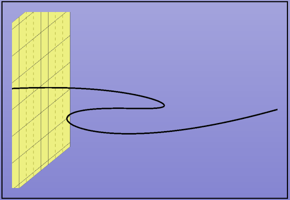
$X = t$



$tX + Z = 1$



$X = t(t-1)(t+1)$



Algorithm MRep

Input : $p = \left(\frac{p_1}{p_0}, \ldots, \frac{p_n}{p_0}\right)$ and $\nu > 0$

1. Compute basis of $\mathrm{Syz}(I)_\nu$.
2. Write this basis as $\begin{pmatrix} S_0 \\ \vdots \\ S_n \end{pmatrix}$: rows indexed by R_ν^{n+1}, columns indexed by basis elements.
3. Let $\mathbb{M}_\nu = \sum_i S_i x_i$.

Theorem

$\mathbb{M}_\nu$ is a matrix of size $\dim(R_\nu) \times \dim(\mathrm{Syz}(I)_\nu)$ whose entries are linear in $x_0, \ldots, x_n$.
For $\nu \geq d - 1$, it is a Matrix Representation of V. More accuratly, $\nu \geq \mu_n + \mu_{n-1} - 1$ where $\mu_1 \leq \cdots \leq \mu_n$ are degrees of μ-basis.

≈ Algorithm QMRep ≉

Input : $p = \left(\frac{p_1}{p_0}, \ldots, \frac{p_n}{p_0}\right)$ and $\nu \geq 0$

1. Compute $\mathbb{M}_\nu$ (and $\mathrm{Syz}(I)_\nu$ in the process).
2. Compute basis of $\mathrm{Syz}(I^2)_\nu$ modulo $\mathrm{Syz}(I)_\nu$.
3. Define $\mathbb{Q}_\nu$ similar to MRep construction.
4. Concatenate $\mathbb{MQ}_\nu = \left(\mathbb{M}_\nu \quad \mathbb{Q}_\nu\right)$.

Theorem [*LB, FY, CL 2019*]

$\mathbb{MQ}_\nu$ is a matrix whose entries are linear or quadratic in $x_1, \ldots, x_n$. For $\nu \geq \mu_n - 1$, it is a Matrix Representation of V.

(A matrix depending on $x_0, \ldots, x_n$ is a Matrix Representation of V when its rank drops on $\underline{\mathbf{x}}$ iff $\underline{\mathbf{x}} \in V$.)

≈ Examples ≈

Twisted Cubic

The twisted cubic is $p = (t, t^2, t^3)$.

Its MRep (with $\nu = 1$) is $\begin{pmatrix} x & z & y \\ 1 & y & x \end{pmatrix}$

QMRep is the Implicit Equations

Its QMRep (with $\nu = 0$) is $\begin{pmatrix} x\,y - z & z\,x - y^2 & x^2 - y \end{pmatrix}$

≈ Examples ≈

Degree 7 Curve : MRep

A generic degree 7 curve has μ-basis of degrees $(2, 2, 3)$.
A MRep (with $\nu = 4$) is

$$\begin{pmatrix} -x-z & -z & 0 & -z & 0 & 0 & 0 & y+4z \\ \frac{11}{4}y-15z & \frac{5}{4}y+z & \frac{1}{8}y-\frac{1}{2}z & x+\frac{3}{4}y & \frac{1}{4}y & 0 & -\frac{1}{8}y-\frac{1}{2}z & -4y+16z \\ 1-\frac{23}{4}y+12z & -\frac{3}{2}y+3z & \frac{3}{4}y+3z & -\frac{9}{4}y+10z & x-\frac{1}{4}y+4z & \frac{1}{4}y & \frac{1}{2}y-2z & 9y-32z \\ 5y+14z & 1+\frac{5}{4}y+2z & -\frac{3}{8}y-z & \frac{5}{2}y+6z & 2z & x-\frac{1}{4}y+4z & -\frac{7}{8}y+4z & -10y-24z \\ -8z & -2z & 1-z & -4z & 0 & 2z & x+y+7z & 16z \end{pmatrix}$$

Degree 7 Curve : QMRep

Its QMRep (with $\nu = 1$) is

$$\begin{pmatrix} -3x^2+\ldots+18z^2 & 32x^2+\ldots+48z^2 & 4x^2+\ldots-16z^2 & -x^2+\ldots-9z^2 \\ x+\ldots-130z^2 & 100x^2+\ldots+2040z^2 & y+\ldots-8z^2 & -5x^2+\ldots-100z^2 \end{pmatrix}$$

≈ Examples ≉

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Theorem says we can use $\nu = 2$ for a 3×7 QMRep.

Moving Quadrics following the Curve

≈ Sylvester Forms ≈

Quadratic Relations from Linear Syzygies

Some quadratic relations come from resultants of the μ-basis :

$$\deg(\mathbf{h_k}) = \mu_k, \quad \mathbf{h_k} = \sum_{i=0}^{n} \left(a_{i,0} s^{\mu_k - 1} \cdot \sigma + \sum_{j=1}^{\mu_k} a_{i,j} s^{\mu_k - j} t^{j-1} \cdot \tau \right) x_i$$

$$\deg(\mathbf{h_{k'}}) = \mu_{k'}, \quad \mathbf{h_{k'}} = \sum_{i=0}^{n} \left(b_{i,0} s^{\mu_{k'} - 1} \cdot \sigma + \sum_{j=1}^{\mu_{k'}} b_{i,j} s^{\mu_{k'} - j} t^{j-1} \cdot \tau \right) x_i$$

$$\mathrm{Res}_{\sigma,\tau}(\mathbf{h_k}, \mathbf{h_{k'}}) = \begin{vmatrix} s^{\mu_k - 1} \times \boxed{\textit{linear in } x_0, \dots, x_n} & s^{\mu_{k'} - 1} \times \boxed{\textit{linear in } x_0, \dots, x_n} \\ \boxed{\begin{array}{l} \textit{degree } \mu_{k'} - 1 \textit{ in } s, t, \\ \textit{linear in } x_0, \dots, x_n \end{array}} & \boxed{\begin{array}{l} \textit{degree } \mu_k - 1 \textit{ in } s, t, \\ \textit{linear in } x_0, \dots, x_n \end{array}} \end{vmatrix}$$

Resultant produces a quadratic relation of degree $\mu_k + \mu_{k'} - 2$. Because of factorization, it also produces syzygies of I^2 down to degree $\max(\mu_k, \mu_{k'}) - 1$.

≈ Sylvester Forms ≈

Example : $\mu_1 = \mu_2 = 2$

$$\mathbf{h_1} = a_0 s\sigma + (a_1 s + a_2 t)\tau$$
$$\mathbf{h_2} = b_0 s\sigma + (b_1 s + b_2 t)\tau$$
$$\text{Res} = s \cdot \begin{vmatrix} a_0 & a_1 s + a_2 t \\ b_0 & b_1 s + b_2 t \end{vmatrix} = s[s(a_0 b_1 - a_1 b_0) + t(a_0 b_2 - a_2 b_0)]$$

But also :

$$\mathbf{h_1} = (a_0 s + a_1 t)\sigma + a_2 t\tau$$
$$\mathbf{h_2} = (b_0 s + b_1 t)\sigma + b_2 t\tau$$
$$\text{Res} = t[t(a_1 b_2 - a_2 b_1) + s(a_0 b_2 - a_2 b_0)]$$

Total : 1 element of $\text{Syz}(I^2)$ of degree $\mu_1 + \mu_2 - 2 = 2$
and 2 elements of $\text{Syz}(I^2)$ of degree $\mu_1 + \mu_2 - 3 = 1$

≂ Computing $\mathrm{Syz}(I^2)$ with Sylvester Forms ≄

Combinatorial Formula

For each couple of degrees $\mu_k \leq \mu_{k'}$ of the μ-basis, there are :

- 1 element of $\mathrm{Syz}(I^2)_{\mu_{k'}+\mu_k-2}$
- 2 elements of $\mathrm{Syz}(I^2)_{\mu_{k'}+\mu_k-3}$

  $\vdots$

- μ_k elements of $\mathrm{Syz}(I^2)_{\mu_{k'}-1}$

Example : $\mu = (3, 5, 7)$

In degree $\nu = \mu_3 - 1 = 6$, $\dim(\mathrm{Syz}(I^2)_6) = 3 + 5 + 1 = 9$

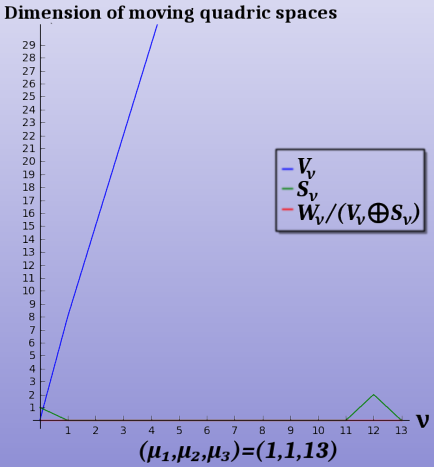
≈ Computing $\mathrm{Syz}(I^2)$ with Sylvester Forms ≈

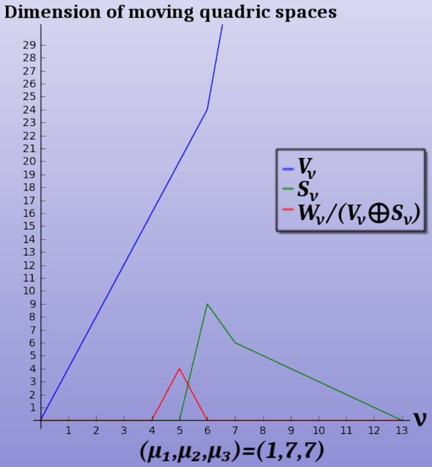
μ-basis : $\mathbf{h_1}, \mathbf{h_2}, \mathbf{h_3}$ of degrees $\mathbf{3}, \mathbf{5}, \mathbf{7}$:

$\mathbf{h_1}$ $\quad$ $\mathbf{h_2}$ $\quad$ $\mathbf{h_3}$

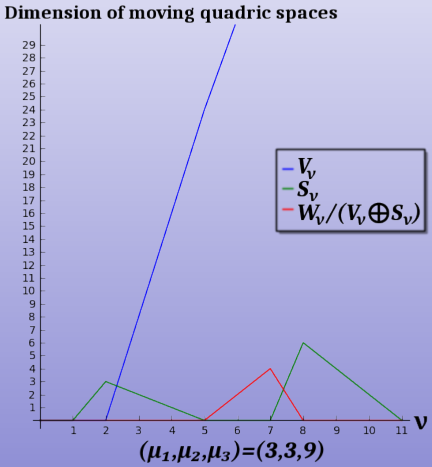
	$\mathbf{h_1}, \mathbf{h_2}$	$\mathbf{h_1}, \mathbf{h_3}$	$\mathbf{h_2}, \mathbf{h_3}$
			$1\times$ degree 10
			$2\times$ degree 9
		$1\times$ degree 8	$3\times$ degree 8
		$2\times$ degree 7	$4\times$ degree 7
$\mu_n - 1 \rightsquigarrow$	$1\times$ degree 6	$3\times$ degree 6	$5\times$ degree 6
	$2\times$ degree 5		
	$3\times$ degree 4		

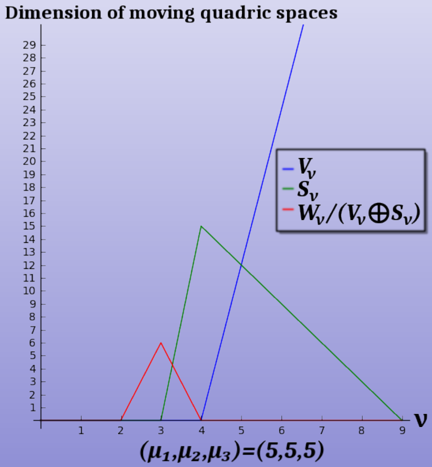
≈ Space of Quadratic Relations ≈





Space of Quadratic Relations





$\mathbb{MQ}_\nu$ is a Matrix Representation for $\nu \geqslant \mu_3 - 1$

$A := \mathbb{C}[x_0, x_1, x_2, x_3]^{\mathrm{hom}}$

$R := \mathbb{C}[s, t; x_0, x_1, x_2, x_3]^{\mathrm{hom}}$ (all the moving hypersurfaces)

$\mathfrak{m} :=< s, t >$ (localisation ideal)

$\mathbf{h_1}, \mathbf{h_2}, \mathbf{h_3} := \mu$-basis for p

$J :=< \mathbf{h_1}, \mathbf{h_2}, \mathbf{h_3} >$ (gen. by moving hyperplanes $\to p$)

$J' := \mathrm{Zeros}(\mathbf{h} \mapsto \mathbf{h}(s, t; p(s, t)))$ (moving hypersurfaces $\to p$)

J' is the saturation of J

$\mathbf{h_1}, \mathbf{h_2}, \mathbf{h_3}$ form a regular sequence in R outside $V(\mathfrak{m})$.

$$J' = (J :_R \mathfrak{m}^\infty) = \{\mathbf{h} \in R \text{ such that } \exists k \in \mathbb{N}, \mathbf{h}\mathfrak{m}^k \subset J\}$$

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$\mathbb{MQ}_\nu$ is a Matrix Representation for $\nu \geq \mu_3 - 1$

The theorem we want to prove is rewritten as :

$$\forall \nu \geq \mu_3 - 1, \ J'_\nu = (J'\langle 2\rangle)_\nu$$

where $J'\langle 2\rangle$ is generated by $J' \cap R_{\deg(x)\leq 2}$.

$\mathbb{MQ}_\nu$ is a Matrix Representation $\iff$ $J'_\nu = (J'\langle 2\rangle)_\nu$

Čech complex

The Čech complex of $B := R/J$ is given by :

$$\mathcal{C}^\bullet_{\mathfrak{m}}(B) : 0 \to B \xrightarrow{\check{c}_0} B_{x_0} \oplus B_{x_1} \oplus B_{x_2} \oplus B_{x_3} \xrightarrow{\check{c}_1} \cdots \xrightarrow{\check{c}_3} B_{x_0x_1x_2x_3} \to 0$$

Local cohomology

Except for $H^0_{\mathfrak{m}}$, the local cohomology of B with support in $\mathfrak{m}$ is the cohomology of the Čech complex :

$$\begin{aligned} H^0_{\mathfrak{m}}(B) &= \{\mathbf{h} \in B \text{ such that } \exists k \in \mathbb{N},\ \mathbf{h}\mathfrak{m}^k = 0\} \\ &= (J :_B \mathfrak{m}^\infty)/J = J'/J \\ H^i_{\mathfrak{m}}(B) &= \mathrm{Ker}(\check{c}_i)/\,\mathrm{Im}(\check{c}_{i-1}),\ \text{ for } i \geqslant 1 \\ H^2_{\mathfrak{m}}(R) &\simeq A \otimes_{\mathbb{C}} \check{S},\ \text{ where } \check{S} := \frac{1}{st}\mathbb{C}[s^{-1}, t^{-1}] \end{aligned}$$

$\precsim$ Koszul Complex $\not\precsim$

$$\mathbb{MQ}_\nu \text{ is a Matrix Representation } \iff J'_\nu = (J'\langle 2\rangle)_\nu$$

Remember that $J = < \mathbf{h_1}, \mathbf{h_2}, \mathbf{h_3} >$ is graded both w.r.t. $\mathbb{C}[s,t]$ and to $\mathbb{C}[x_0, x_1, x_2, x_3]$.
Degree shifts w.r.t. $\mathbb{C}[s,t]$: $[-]$
Degree shifts w.r.t. $\mathbb{C}[x_0, x_1, x_2, x_3]$: $\{-\}$

$$K_\bullet : 0 \to R\,[-\mu_1 - \mu_2 - \mu_3]\,\{-3\} \xrightarrow{d_3} \cdots$$

$$\cdots \xrightarrow{d_2} R[-\mu_1]\{-1\} \oplus R[-\mu_2]\{-1\} \oplus R[-\mu_3]\{-1\} \xrightarrow{d_1} R \to 0$$

We have that $H_0(K_\bullet) = R/J = B$.

Lemma

$$H_2(H^2_{\mathfrak{m}}(K_\bullet)) \xrightarrow{\sim} H^0_{\mathfrak{m}}(B) = J'/J$$

≈ Double Complex ≉

In order to prove the lemma, consider $\mathcal{C}^{\bullet}_{\mathfrak{m}}(K_{\bullet})$:

$$
\begin{array}{ccccccc}
 & & 0 & & \cdots & 0 & \\
 & & \downarrow & & & \downarrow & \\
0 \to & & R\left[-\mu_1-\mu_2-\mu_3\right]\{-3\} & \to \cdots \to & & R & \to 0 \\
 & & \downarrow & & & \downarrow & \\
\vdots & & \vdots & & \ddots & \vdots & \vdots \\
 & & \downarrow & & & \downarrow & \\
0 \to & R\left[-\mu_1-\mu_2-\mu_3\right]\{-3\}_{x_0x_1x_2x_3} & & \to \cdots \to & & R_{x_0x_1x_2x_3} & \to 0 \\
 & & \downarrow & & & \downarrow & \\
 & & 0 & & \cdots & 0 &
\end{array}
$$

≳ Spectral Sequences ≴

The spectral sequence corresponding to the column filtration of this double complex converges at the second step because $\mathbf{h_1}, \mathbf{h_2}, \mathbf{h_3}$ form a regular sequence outside $V(\mathfrak{m})$:

$$\begin{array}{cccc} H^0_{\mathfrak{m}}(H_3(K_\bullet)) & H^0_{\mathfrak{m}}(H_2(K_\bullet)) & H^0_{\mathfrak{m}}(H_1(K_\bullet)) & H^0_{\mathfrak{m}}(B) \\ 0 & 0 & 0 & H^1_{\mathfrak{m}}(B) \\ 0 & 0 & 0 & 0 \end{array}$$

The row filtration of our double complex gives another spectral sequence that also converge at the second step :

$$\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ H_3(H^2_{\mathfrak{m}}(K_\bullet)) & H_2(H^2_{\mathfrak{m}}(K_\bullet)) & H_1(H^2_{\mathfrak{m}}(K_\bullet)) & H_0(H^2_{\mathfrak{m}}(K_\bullet)) \end{array}$$

They both converge to the same limit : the homology of the total complex of $\mathcal{C}^\bullet_{\mathfrak{m}}(K_\bullet)$. Thus $H_2(H^2_{\mathfrak{m}}(K_\bullet)) \xrightarrow{\sim} H^0_{\mathfrak{m}}(B)$.

$\mathbb{MQ}_\nu$ is a Matrix Representation $\iff H^2_{\mathfrak{m}}(K_1)_\nu = 0$

$H_2(H^2_{\mathfrak{m}}(K_\bullet))$ is obtained from the sequence $H^2_{\mathfrak{m}}(K_\bullet)$:

$$H^2_{\mathfrak{m}}(K_3) \to H^2_{\mathfrak{m}}(K_2) \to H^2_{\mathfrak{m}}(K_1)$$

Using $H^2_{\mathfrak{m}}(R) \simeq A \otimes_{\mathbb{C}} \check{S}$ and the expression of K_1, we obtain :

$$(H^2_{\mathfrak{m}}(K_1))_\nu \simeq \left(\check{S}_{\nu-\mu_1} \oplus \check{S}_{\nu-\mu_2} \oplus \check{S}_{\nu-\mu_3}\right) \otimes_{\mathbb{C}} A\{-1\}$$

In particular, $\forall \nu \geqslant \mu_3 - 1$, $H^2_{\mathfrak{m}}(K_1)_\nu = 0$.

The decisive exact sequence

The following sequence of graded A-modules is exact for $\nu \geqslant \mu_3 - 1$.

$$\left(\check{S}_{\nu-\mu_1-\mu_2} \oplus \check{S}_{\nu-\mu_1-\mu_3} \oplus \check{S}_{\nu-\mu_2-\mu_3}\right) \otimes_{\mathbb{C}} A\{-2\} \to (J'/J)_\nu \to 0$$

Thus $(J'/J)_\nu = ((J'/J)\langle 2\rangle)_\nu$ and $J'_\nu = (J'\langle 2\rangle)_\nu$.

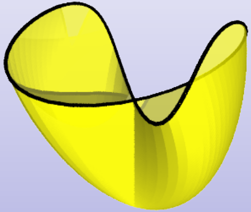
Comparison Table (Space Curves)

Degree (μ-basis degree)	MRep size	QMRep size
$4\ (1,1,2)$	3×5	2×4
$5\ (1,2,2)$	4×7	2×5
$6\ (2,2,2)$	4×6	2×6
$6\ (1,1,4)$	5×9	4×8
$7\ (2,2,3)$	5×8	$3 \times 5\ (2 \times 4)$
$7\ (1,3,3)$	6×11	3×7
$7\ (1,1,5)$	6×11	5×10
$10\ (3,3,4)$	7×11	$4 \times 10\ (3 \times 7)$
$15\ (5,5,5)$	10×18	5×15

Notes : $\sum_i \mu_i = \delta$
Evenly distributed μ-bases are the generic case.

≂ Self-Intersection ≠

Application : border of a sliced cross-cap



$$\mathbb{MQ}_2 = \begin{pmatrix} 1-\frac{5}{3}z & y & 0 & -\frac{8}{3}xz-\frac{32}{9}z^2 & -xy-\frac{4}{3}yz & xy+\frac{4}{3}yz & x+\frac{4}{3}z-\frac{5}{3}xz-\frac{20}{9}z^2 \\ y & \frac{8}{3}z & 0 & xy-y^2+4yz & x-y+4z-\frac{5}{3}xz+\frac{5}{3}yz-\frac{20}{3}z^2 & -y^2-\frac{4}{3}yz & -y-\frac{4}{3}z+\frac{5}{3}yz+\frac{20}{9}z^2 \\ -1+\frac{5}{3}z & -y & -y^2+\frac{8}{3}z-\frac{40}{9}z^2 & -y^2-\frac{4}{3}yz & -y-\frac{4}{3}z+\frac{5}{3}yz+\frac{20}{9}z^2 & -y^2-\frac{32}{9}z^2 & -y-y^2+\frac{8}{3}z+\frac{1}{3}yz-\frac{40}{9}z^2 \end{pmatrix}$$

$(x, y, z) = (0, 0, 0)$ is a self-intersection

$\mathrm{Rank}(\mathbb{MQ}_2(0, 0, 0)) = 1$

$\mathrm{Ker}(\mathbb{MQ}_2(0, 0, 0)^T) = \{(1, 1, 1), (1, -1, 1)\} \rightsquigarrow$ the two pre-images of $(0, 0, 0)$ are $t = 1$ and $t = -1$

≈ Curve Intersection ≉

Two parameterized curves : $\mathcal{C}_1$ (deg. 7) and $\mathcal{C}_2$ (deg. 4).
$\mathbb{MQ}_2$ of $\mathcal{C}_1$ is of size $(3,7)$ and can be computed numerically.

$$\mathbb{MQ}_2(p_2(t)) \approx$$

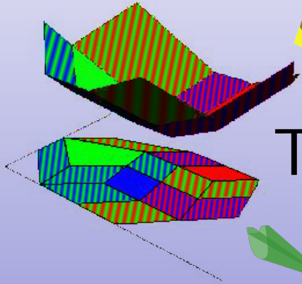
$$\begin{pmatrix} 48.618t^4\text{-}78.594t^3\text{-}310.76t^2+560.59t\text{-}228.16 & & \ddots \\ \ddots & \ddots & \ddots \\ \cdots\ 255.76t^8\text{-}1097.4t^7+1144.4t^6+1095.2t^5\text{-}3526.1t^4+4008.7t^3\text{-}2886.4t^2+1292.7t\text{-}253.06 & & \end{pmatrix}$$

$(x,y,z) \approx (-0.126, 2.743, 3.1833)$ is an intersection

Using SVD, we check that the rank drops at $t = 0.731$
$p_1(0.731) \approx p_2(0.731) \approx (-0.126, 2.743, 3.1833)$

$\approx$ Distance Approximation $\backsimeq$

Trick : $\det(\mathbb{MQ}_\nu.\mathbb{MQ}_\nu^T)$ gives a single implicit equation of $V \cap \mathbb{R}^n$
The following is a figure of
$\det(\mathbb{MQ}_\nu.\mathbb{MQ}_\nu^T) = 0$ (black) and
$\det(\mathbb{MQ}_\nu.\mathbb{MQ}_\nu^T) = \epsilon$ (yellow)



Thank you