Implicitizing Rational Curves by the Method of Moving Quadrics

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$\approx$ Algebraic Variety $\not\approx$

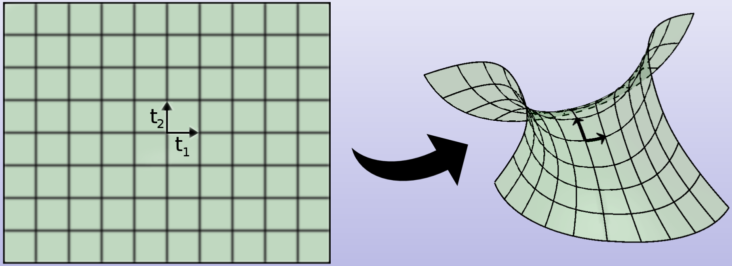
A variety is defined by algebraic (polynomial or rational) equations.
Framework : V a variety of dimension d in $\mathbb{P}^n$.

Usual Representations

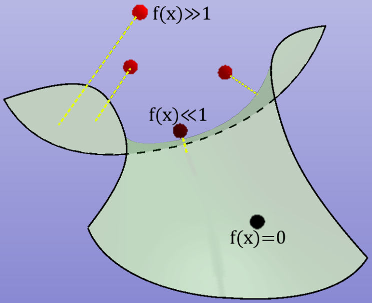
Parametric : V is drawn as
$V = \{(p_0(t) : \ldots : p_n(t)) \mid t = (t_0 : \ldots : t_d) \in \mathbb{P}^d\}$,
Implicit : V is caracterized as
$V = \{x = (x_0 : \ldots : x_n) \in \mathbb{P}^n \mid f_0(x) = f_1(x) = \cdots = 0\}$.

Typically, V is a rational space curve ($d = 1$, $n = 3$).

Parametric



Implicit



≈ Why using Implicit Representations ? ≈

Parametric Representations

- Sample and display are simple (very useful in CAGD)
- Can be local (eg. Bézier curve/patch)
- Cannot be used for any variety (not closed under intersection)

Implicit Representations

- Intersection and membership are simple ; distance is simpler
- Allow raytracing technics
- In CAGD, curves describe object borders

≊ Implicit Polynomial ≴

Numerical Unstability

Floating-point arithmetic makes the evaluation of high degree polynomials unstable : rounding errors propagate and explode.

Example

$$f(x) := (x^3 - 1)(x - 10)^{15}$$

$$f\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 0$$

$$f(-0.5 + 0.866025403784440i) \approx 1.652551896306318 + 8.724965314413668i$$

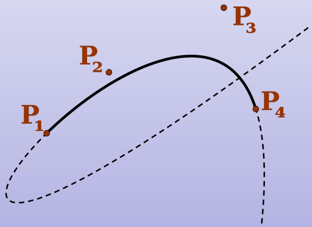
$$\widetilde{f}(-0.5 + 0.866025403784440i) = 4.19138410839463 + 1.46574416565220i$$

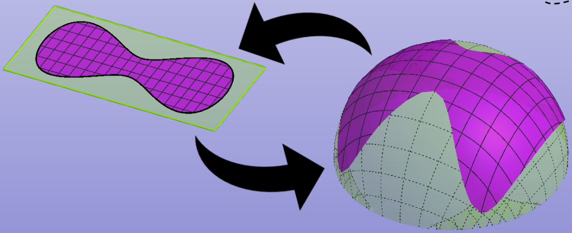
Implicit Representation

Implicit representations are not local.

Examples :

- Bézier curves self-intersecting inside their control polygon
- Surface trimming





Solution : solving the inversion problem (given a point P on V, find t such that $p(t) = P$)

≂ Matrix Representation ≴

MRep

Our Matrix Representation (or *MRep*) of V are matrices $M(x_0, \ldots, x_n)$ of generic rank r such that $x \in V \iff \mathrm{Rank}(M(x)) < r$.

Features

MRep are a kind of implicit representation :

- $P \in \mathbb{P}^n$, «$P \in V$» is easy to check,
- Intersections are much easier than for parametric,
- Can solve «Find t such that $P = p(t)$ for a given $P \in V$».

In 1995, D. Cox, T. Sederberg and F. Chen used μ-bases to build this kind of matrix.

$\approx$ The Bézout Matrix $\backsimeq$

Framework : p_0, p_1, p_2 are polynomials of degree δ in t

Sylvester is Linear

The Sylvester matrix in t of $(p_0(t)x - p_1(t)w, p_0(t)y - p_2(t)w)$ is a matrix :

- of size $2\delta \times 2\delta$
- its entries are linear in w, x, y
- its rank drops on the planar curve $(p_0(t) : p_1(t) : p_2(t))_t$

Bézout is Quadratic

The *Bézoutian* (or *Bézout Matrix*) is :

- of size $\delta \times \delta$
- its entries are quadratic in w, x, y
- its rank drops on the curve

≈ Hybrid-Bézout ≈

It is possible to trade lines and columns of the Sylvester matrix with the ones of the Bézout matrix.

Example with $(t, 1+t+\frac{t^2}{2}, 1+2t+t^2)$

$$\text{Sylvester} = \begin{pmatrix} -\frac{w}{2} & 0 & -w & 0 \\ x-w & -\frac{w}{2} & y-2w & -w \\ -w & x-w & -w & y-2w \\ 0 & -w & 0 & -w \end{pmatrix}$$

$$\text{Hybrid} = \begin{pmatrix} -\frac{w}{2} & -w & 0 \\ x-w & y-2w & -\frac{w^2}{2} \\ -w & -w & -wx-w^2+wy \end{pmatrix}$$

$$\text{Bezout} = \begin{pmatrix} -\frac{w^2}{2} & -\frac{wy}{2}+wx \\ -wx-w^2+wy & -\frac{w^2}{2} \end{pmatrix}$$

Syzygies

Let $p = \left(\frac{p_1}{p_0}, \ldots, \frac{p_n}{p_0}\right)$ a parameterization and
$I := < p_0, \ldots, p_n >$

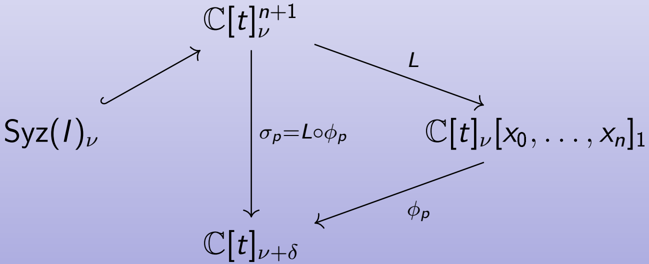
The Space of Syzygies

$$\mathrm{Syz}(I) := \left\{ h = (h_0, \ldots, h_n) \mid \sum_{i=0}^{n} h_i\, p_i = 0 \right\}$$

Syz(I) and Syz(I^2) are modules.

$$\mathrm{Syz}(I^2) := \left\{ h = (h_{00}, \ldots, h_{nn}) \mid \sum_{0 \le i \le j \le n} h_{ij}\, p_i\, p_j = 0 \right\}$$

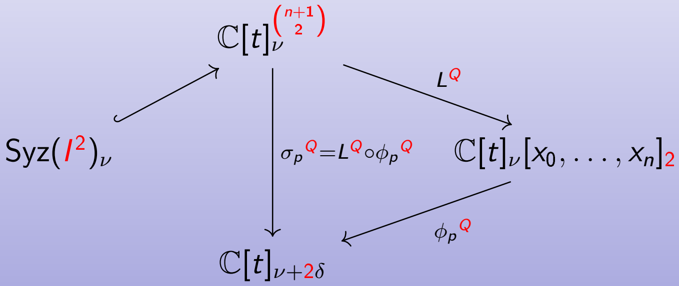
≈ The different spaces involved ≈



With :

- $L(h_0, \ldots, h_n) = \sum_i h_i x_i$
- $\phi_p(\sum_i h_i x_i) = \sum_i h_i p_i$
- $\mathrm{Syz}(I)_\nu = \mathrm{Ker}(\sigma_p)$

≈ The different spaces involved

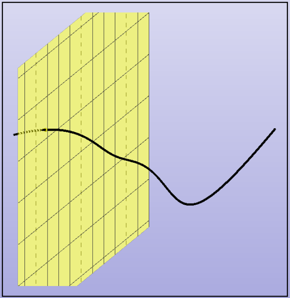


With :

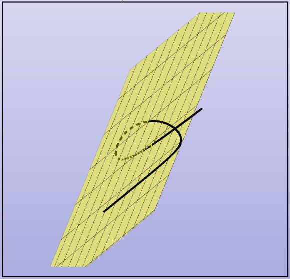
- $L^Q(h_{00}, \ldots, h_{nn}) = \sum_{ij} h_{ij}\, x_i\, x_j$
- ${\phi_p}^Q(\sum_{ij} h_{ij}\, x_i\, x_j) = \sum_{ij} h_{ij}\, p_i\, p_j$
- $\mathrm{Syz}(I^2)_\nu = \mathrm{Ker}({\sigma_p}^Q)$

≈ Aka. Moving Planes ≈

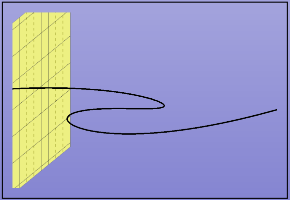
$X = t$



$tX + Z = 1$



$X = t(t-1)(t+1)$



Click on a figure to see the animation of the moving plane

≂ μ-basis ≂

Koszul Syzygies

Obvious syzygies are :

$$\mathbf{h} = (p_1, -p_0, 0, \dots, 0) \qquad p_0 p_1 - p_1 p_0 = 0$$

$$\text{or } \mathbf{h} = (0, p_2, -p_1, 0, \dots, 0) \qquad p_1 p_2 - p_2 p_1 = 0$$

$$\dots \text{or } \mathbf{h} = (-p_n, 0, \dots, 0, p_0) \qquad p_n p_0 - p_0 p_n = 0$$

They form a basis of $\mathrm{Syz}(I)_{\geq \delta}$

A μ-basis $(\mathbf{h_1}, \dots, \mathbf{h_n})$ of $\mathrm{Syz}(I)$ is a basis of minimal degree

Theorem [*Cox, Sederberg, Chen 1995*]

A μ-basis $(\mathbf{h_1}, \dots, \mathbf{h_n})$ always exists and satisfies :

$$\sum_{i=1}^{n} \deg(\mathbf{h_i}) = \sum_{i=1}^{n} \mu_i = \delta$$

≈ Algorithm MRep ≉

Input : $p = (p_0 : \ldots : p_n)$ and $\nu > 0$

1. Compute basis of $\mathrm{Syz}(I)_\nu$.
2. Write this basis as $\begin{pmatrix} S_0 \\ \vdots \\ S_n \end{pmatrix}$: rows indexed by $\mathbb{C}[t]_\nu^{n+1}$, columns indexed by basis elements.
3. Let $\mathbb{M}_\nu = \sum_i S_i x_i$.

Theorem [*LB, Thang Luu Ba 2010*]

$\mathbb{M}_\nu$ is a matrix of size $\dim(\mathbb{C}[t]_\nu) \times \dim(\mathrm{Syz}(I)_\nu)$ with linear entries in $x_0, \ldots, x_n$.
For $\nu \geq d-1$, it is a Matrix Representation of V. More accuratly, $\nu \geq \mu_n + \mu_{n-1} - 1$ where $\mu_1 \leq \cdots \leq \mu_n$ are degrees of μ-basis.

$\precsim$ Shifting Syzygies $\nsimeq$

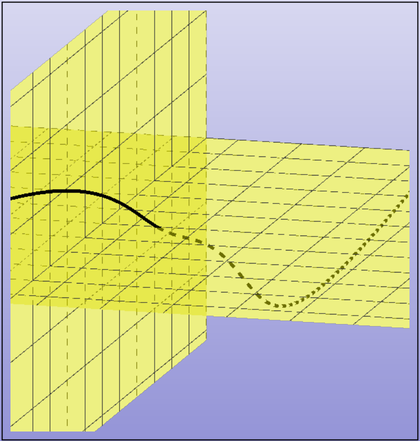
Syzygy Shifts

Linear relations $\sum_i h_i\, p_i = 0$ can be shifted to quadratic relations of the same degree :

$$\text{For } k = 0, \ldots, n, \sum_i h_i\, p_i\, p_k = 0$$

This can be used to compute $\mathrm{Syz}(I^2)_\nu$ modulo $\mathrm{Syz}(I)_\nu$ with linear algebra.

≈ Shifted Syzygy (Geometrical Meaning) ≉



Click on the figure to see the animation of the moving quadric

≈ Algorithm QMRep ≉

Input : $p = (p_0 : \ldots : p_n)$ and $\nu \geq 0$

1. Compute $\mathbb{M}_\nu$ (and $\mathrm{Syz}(I)_\nu$ in the process).
2. Compute basis of $\mathrm{Syz}(I^2)_\nu$ modulo $\mathrm{Syz}(I)_\nu$.
3. Define $\mathbb{Q}_\nu$ similar to linear MRep construction.
4. Concatenate $\mathbb{MQ}_\nu = \begin{pmatrix} \mathbb{M}_\nu & \mathbb{Q}_\nu \end{pmatrix}$.

Theorem [*LB, FY, CL 2019*]

$\mathbb{MQ}_\nu$ is a matrix with linear or quadratic entries in $x_0, \ldots, x_n$.
For $\nu \geq \mu_n - 1$, it is a Matrix Representation of V.

Moving Quadric Surfaces

Click on a figure to see the animation of the moving quadric

≈ Examples ≈

Twisted Cubic

The twisted cubic is $p = (s^3 : s^2t : st^2 : t^3)$.

Its MRep (with $\nu = 1$) is $\begin{pmatrix} -y & x & -w \\ z & -y & x \end{pmatrix}$

QMRep is the Implicit Equations

Its QMRep (with $\nu = 0$) is $\begin{pmatrix} y^2 - x\,z & w\,z - y\,x & x^2 - w\,y \end{pmatrix}$

Examples

Degree 7 Curve : MRep

A generic degree 7 curve has μ-basis of degrees $(2, 2, 3)$.

A MRep (with $\nu = 4$) is

$$\begin{pmatrix} -x-z & -z & 0 & -z & 0 & 0 & 0 & y+4z \\ \frac{11}{4}y-15z & \frac{5}{4}y+z & \frac{1}{8}y-\frac{1}{2}z & x+\frac{3}{4}y & \frac{1}{4}y & 0 & -\frac{1}{8}y-\frac{1}{2}z & -4y+16z \\ w-\frac{23}{4}y+12z & -\frac{3}{2}y+3z & \frac{3}{4}y+3z & -\frac{9}{4}y+10z & x-\frac{1}{4}y+4z & \frac{1}{4}y & \frac{1}{2}y-2z & 9y-32z \\ 5y+14z & w+\frac{5}{4}y+2z & -\frac{3}{8}y-z & \frac{5}{2}y+6z & 2z & x-\frac{1}{4}y+4z & -\frac{7}{8}y+4z & -10y-24z \\ -8z & -2z & w-z & -4z & 0 & 2z & x+y+7z & 16z \end{pmatrix}$$

Degree 7 Curve : QMRep

Its QMRep (with $\nu = 1$) is

$$\begin{pmatrix} -3x_1^2+\ldots+18x_3^2 & 32x_1^2+\ldots+48x_3^2 & 4x_1^2+\ldots-16x_3^2 & -x_1^2+\ldots-9x_3^2 \\ x_0x_1+\ldots-130x_3^2 & 100x_1^2+\ldots+2040x_3^2 & x_0x_2+\ldots-8x_3^2 & -5x_1^2+\ldots-100x_3^2 \end{pmatrix}$$

Theorem says we can use $\nu = 2$ for a 3×7 QMRep.

Comparison with Sylvester and Bézout

Using the same planar curve $\left(t : 1 + t + \frac{t^2}{2} : 1 + 2t + t^2\right)$:

$$\text{Sylvester} = \begin{pmatrix} -\frac{w}{2} & 0 & -w & 0 \\ x - w & -\frac{w}{2} & y - 2w & -w \\ -w & x - w & -w & y - 2w \\ 0 & -w & 0 & -w \end{pmatrix}$$

$$\text{Bezout} = \begin{pmatrix} -\frac{w^2}{2} & -\frac{wy}{2} + wx \\ -wx - w^2 + wy & -\frac{w^2}{2} \end{pmatrix}$$

$$\mathbb{M}_1 = \begin{pmatrix} w & 2x - 2y \\ y - 2x & w + 4x - 2y \end{pmatrix}$$

$$\mathbb{MQ}_0 = \left(w^2 - 6yx - 2wy + 2y^2 + 4x^2 + 4wx\right)$$

Comparison Table (Matrix Sizes)

Degree (μ-basis degrees)	MRep size	QMRep size
4 $(1,1,2)$	3×5	2×4
5 $(1,2,2)$	4×7	2×5
6 $(2,2,2)$	4×6	2×6
6 $(1,1,4)$	5×9	4×8
7 $(2,2,3)$	5×8	3×5 (2×4)
7 $(1,3,3)$	6×11	3×7
7 $(1,1,5)$	6×11	5×10
10 $(3,3,4)$	7×11	4×10 (3×7)
15 $(5,5,5)$	10×18	5×15

Notes : $\sum_i \mu_i = \delta$

Evenly distributed μ-bases are the generic case.

≂ Comparison Table (Runtime) ≂

Degree d and degrees $(\mu_i)_i$	$\mathbb{M}_{\mu_n+\mu_{n-1}-1}$	$\mathbb{MQ}_{\mu_n-1}$	GCD easy input	GCD difficult input
5 $(2,3)$	54ms	22ms	18ms	19ms
10 $(5,5)$	230ms	62ms	23ms	33137ms
10 $(1,9)$	230ms	121ms	29ms	6826ms
5 $(1,2,2)$	61ms	22ms	24ms	27ms
9 $(3,3,3)$	125ms	59ms	28ms	29247ms
9 $(1,4,4)$	267ms	78ms	31ms	11301ms
9 $(1,1,7)$	256ms	171ms	35ms	9922ms
15 $(5,5,5)$	377ms	167ms	35ms	32ms
15 $(1,7,7)$	929ms	199ms	37ms	34800ms
15 $(1,1,13)$	894ms	534ms	48ms	53ms

Average time over 100 random point ownership checks

Notes : GCD difficult input are rational points belonging to the curve with a high number of digits. They are not especially difficult for MRep methods.

All the computations here are exact (the GCD method may not be used with numerical datas).

Sylvester Forms

Quadratic Relations from Linear Syzygies

Some quadratic relations come from resultants of the μ-basis :

$$\deg(\mathbf{h_k}) = \mu_k, \quad \mathbf{h_k} = \sum_{i=0}^{n} \left(a_{i,0} s^{\mu_k - 1} \cdot \sigma + \sum_{j=1}^{\mu_k} a_{i,j} s^{\mu_k - j} t^{j-1} \cdot \tau \right) x_i$$

$$\deg(\mathbf{h_{k'}}) = \mu_{k'}, \quad \mathbf{h_{k'}} = \sum_{i=0}^{n} \left(b_{i,0} s^{\mu_{k'} - 1} \cdot \sigma + \sum_{j=1}^{\mu_{k'}} b_{i,j} s^{\mu_{k'} - j} t^{j-1} \cdot \tau \right) x_i$$

$$\mathrm{Res}_{\sigma,\tau}(\mathbf{h_k}, \mathbf{h_{k'}}) = \begin{vmatrix} s^{\mu_k - 1} \times \boxed{\text{linear in } x_0, \ldots, x_n} & s^{\mu_{k'} - 1} \times \boxed{\text{linear in } x_0, \ldots, x_n} \\ \boxed{\begin{array}{l} \text{degree } \mu_{k'} - 1 \text{ in } s, t, \\ \text{linear in } x_0, \ldots, x_n \end{array}} & \boxed{\begin{array}{l} \text{degree } \mu_k - 1 \text{ in } s, t, \\ \text{linear in } x_0, \ldots, x_n \end{array}} \end{vmatrix}$$

Resultant produces a quadratic relation of degree $\mu_k + \mu_{k'} - 2$. Because of factorization, it also produces syzygies of I^2 down to degree $\max(\mu_k, \mu_{k'}) - 1$.

≂ Sylvester Forms ≂

Example : $\mu_1 = \mu_2 = 2$

$\mathbf{h_1} = a_0 s\sigma + (a_1 s + a_2 t)\tau$
$\mathbf{h_2} = b_0 s\sigma + (b_1 s + b_2 t)\tau$

$$\text{Res} = s \cdot \begin{vmatrix} a_0 & a_1 s + a_2 t \\ b_0 & b_1 s + b_2 t \end{vmatrix} = s[s(a_0 b_1 - a_1 b_0) + t(a_0 b_2 - a_2 b_0)]$$

But also :

$\mathbf{h_1} = (a_0 s + a_1 t)\sigma + a_2 t\tau$
$\mathbf{h_2} = (b_0 s + b_1 t)\sigma + b_2 t\tau$
$\text{Res} = t[t(a_1 b_2 - a_2 b_1) + s(a_0 b_2 - a_2 b_0)]$

Total : 1 element of $\mathrm{Syz}(I^2)$ of degree $\mu_1 + \mu_2 - 2 = 2$
and 2 elements of $\mathrm{Syz}(I^2)$ of degree $\mu_1 + \mu_2 - 3 = 1$

≈ Computing $\mathrm{Syz}(I^2)$ with Sylvester Forms ≁

Combinatorial Formula

For each couple of degrees $\mu_k \leq \mu_{k'}$ of the μ-basis, there are :

- 1 element of $\mathrm{Syz}(I^2)_{\mu_{k'}+\mu_k-2}$
- 2 elements of $\mathrm{Syz}(I^2)_{\mu_{k'}+\mu_k-3}$

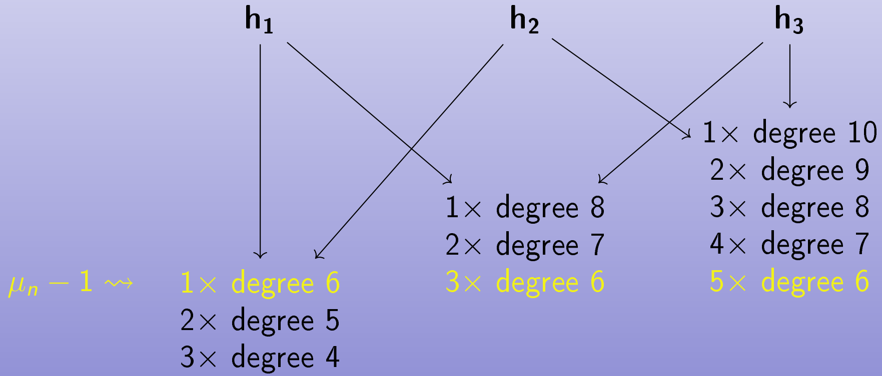
  $\vdots$
- μ_k elements of $\mathrm{Syz}(I^2)_{\mu_{k'}-1}$

Example : $\mu = (3, 5, 7)$

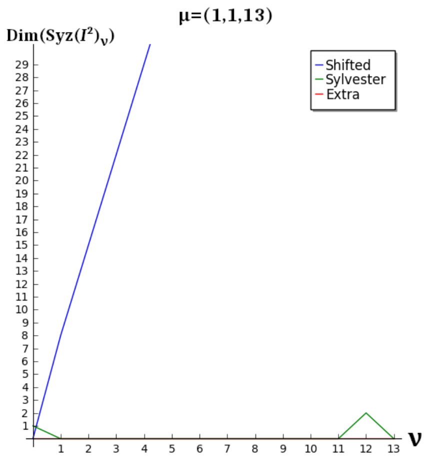
In degree $\nu = \mu_3 - 1 = 6$, $\dim(\mathrm{Syz}(I^2)_6) = 3 + 5 + 1 = 9$

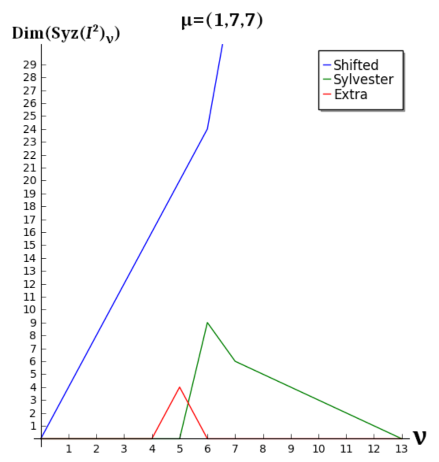
Computing Syz(I^2) with Sylvester Forms

μ-basis : $\mathbf{h_1}, \mathbf{h_2}, \mathbf{h_3}$ of degrees $\mathbf{3}, \mathbf{5}, \mathbf{7}$:

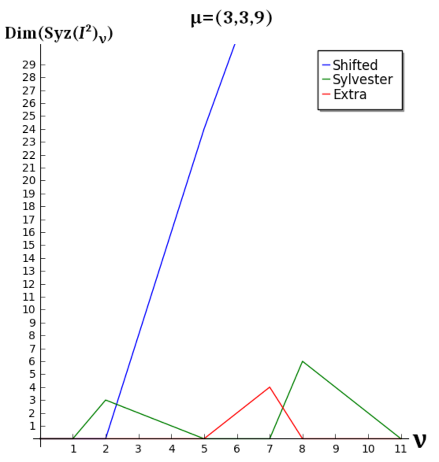


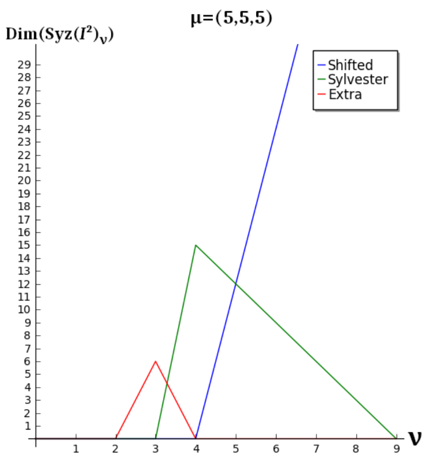
≈ Space of Quadratic Relations ≈





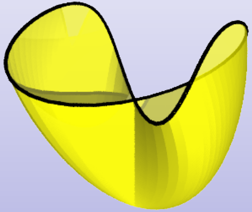
Space of Quadratic Relations





Self-Intersection

Application : border of a sliced cross-cap



$$\mathbb{MQ}_2 = \begin{pmatrix} 1-\frac{5}{3}z & y & 0 & -\frac{8}{3}xz-\frac{32}{9}z^2 & -xy-\frac{4}{3}yz & xy+\frac{4}{3}yz & x+\frac{4}{3}z-\frac{5}{3}xz-\frac{20}{9}z^2 \\ y & \frac{8}{3}z & 0 & xy-y^2+4yz & x-y+4z-\frac{5}{3}xz+\frac{5}{3}yz-\frac{20}{3}z^2 & -y^2-\frac{4}{3}yz & -y-\frac{4}{3}z+\frac{5}{3}yz+\frac{20}{9}z^2 \\ -1+\frac{5}{3}z & -y & -y^2+\frac{8}{3}z-\frac{40}{9}z^2 & -y^2-\frac{4}{3}yz & -y-\frac{4}{3}z+\frac{5}{3}yz+\frac{20}{9}z^2 & -y^2-\frac{32}{9}z^2 & -y-y^2+\frac{8}{3}z+\frac{1}{3}yz-\frac{40}{9}z^2 \end{pmatrix}$$

$(x, y, z) = (0, 0, 0)$ is a self-intersection

$\mathrm{Rank}(\mathbb{MQ}_2(0, 0, 0)) = 1$

$\mathrm{Ker}(\mathbb{MQ}_2(0, 0, 0)^T) = \{(1, 1, 1), (1, -1, 1)\} \rightsquigarrow$ the two pre-images of $(0, 0, 0)$ are $t = 1$ and $t = -1$

Curve Intersection

Two parameterized curves : $\mathcal{C}_1$ (deg. 7) and $\mathcal{C}_2$ (deg. 4).
$\mathbb{MQ}_2$ of $\mathcal{C}_1$ is of size $(3,7)$ and can be computed numerically.

$$\mathbb{MQ}_2(p_2(t)) \approx \begin{pmatrix} 48.618t^4\text{-}78.594t^3\text{-}310.76t^2+560.59t\text{-}228.16 & & \ddots \\ \ddots & \ddots & \ddots \\ \cdots & 255.76t^8\text{-}1097.4t^7+1144.4t^6+1095.2t^5\text{-}3526.1t^4+4008.7t^3\text{-}2886.4t^2+1292.7t\text{-}253.06 & \end{pmatrix}$$

$(x,y,z) \approx (-0.126, 2.743, 3.1833)$ is an intersection

Using SVD, we check that the rank drops at $t = 0.731$
$p_1(0.731) \approx p_2(0.731) \approx (-0.126, 2.743, 3.1833)$

≂ Distance Approximation ≄

Trick : $\det(\mathbb{MQ}_\nu.\mathbb{MQ}_\nu^T)$ gives a single implicit equation of $V \cap \mathbb{R}^n$
The following is a figure of
$\det(\mathbb{MQ}_\nu.\mathbb{MQ}_\nu^T) = 0$ (black) and
$\det(\mathbb{MQ}_\nu.\mathbb{MQ}_\nu^T) = \epsilon$ (yellow)