Implicitization of Varieties of High Codimension

Inspired from Chow Form Representations

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1. The Algorithm
2. The Extraneous Factor
3. The Number of Equations
4. Features & Comparisons

Chow Form

Definition

Let V be a d-dimensional irreducible variety and $H_0, \ldots, H_d$ be linear forms where $H_i(x) = u_{i0}x_0 + \cdots + u_{in}x_n$
for $i = 0, \ldots, d$.
The Chow form of V is a polynomial R_V in the variables u_{ij} such that $R_V(u_{ij}) = 0 \Leftrightarrow V \cap \{H_0 = 0, \ldots, H_d = 0\} \neq \emptyset$.

Proposition

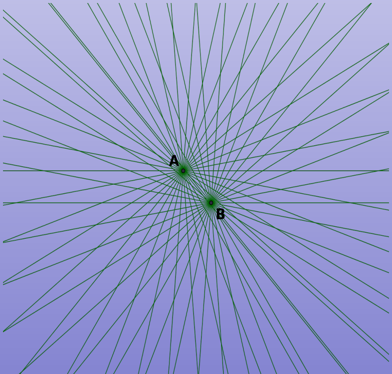
V is uniquely determined by its Chow form. More precisely, a point $\xi \in \mathbb{P}^n(\mathbb{R})$ lies in V if and only if any $(n-d-1)$-dimensional linear subspace containing ξ belongs to the Chow form (ie. the parameters defining this subspace are a root of R_V).

Chow Form

Example

$V = \text{Zeros}(y - x^2, x + y) = \{(-1, 1), (0, 0)\} = \{A, B\}$

Then the Chow Form is a polynomial in a, b, c vanishing iff A or B belongs to $ax + by + c = 0$.



$R_V(a, b, c) =$
$(ax_A + by_A + c)(ax_B + by_B + c)$

Conical Hypersurface

Curve in 3 D

For a curve $\mathscr{C}$ in $\mathbb{P}^3(\mathbb{R})$ and a point G, we have
$\mathrm{Cone}(G, \mathscr{C}) = \cup_{x \in \mathscr{C}} \mathrm{Line}(G, x)$.

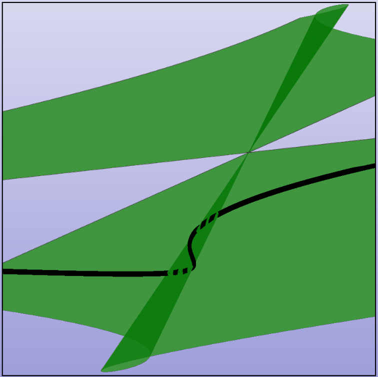
Curve in 4 D

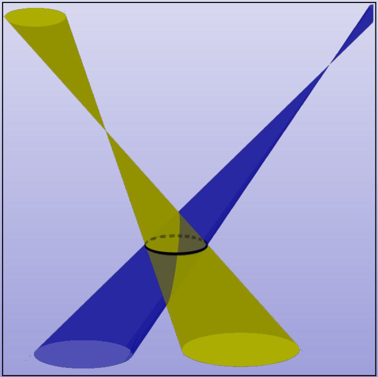
For a curve $\mathscr{C}$ in $\mathbb{P}^4(\mathbb{R})$ and two points G_1, G_2, we have
$\mathrm{Cone}(G_1, G_2, \mathscr{C}) = \cup_{x \in \mathscr{C}} \mathrm{Plane}(G_1, G_2, x)$.

General Case

For a variety V of codimension c in $\mathbb{P}^n(\mathbb{R})$ and $c-1$ points
$G_1, \ldots, G_{c-1}$, we have
$\mathrm{Cone}(G_1, \ldots, G_{c-1}, V) = \cup_{x \in V} \mathrm{Aff}(G_1, \ldots, G_{c-1}, x)$.

Examples





Resultant for Implicitization

Given $p : \mathbb{P}^d(\mathbb{R}) \to \mathbb{P}^n(\mathbb{R})$ a parameterization of V, we choose $c-1$ *generic* points $G_1, \ldots, G_{c-1}$.

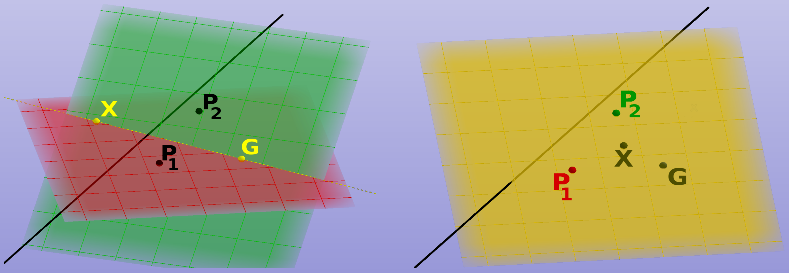
Implicitization

Let $H_0(\xi, X), \ldots, H_d(\xi, X)$ be the equations in X of random hyperplanes containing all the points G_i and ξ.
Then $\mathrm{Res}_t(H_0(p(t), X), \ldots, H_d(p(t), X))$ is a hypersurface that can be reduced to two components : $\mathrm{Cone}(G, V)$ and an extraneous hypersuface E of degree d.

The Extraneous Factor

The extraneous plane

$$\left.\begin{array}{l} H_0 = \mathrm{Aff}(G, X, P_0) \\ H_1 = \mathrm{Aff}(G, X, P_1) \end{array}\right\} \text{The extraneous plane is } \mathrm{Aff}(G, P_0, P_1).$$



The resultant vanishes when $H_0 \cap H_1$ intersects the curve.
When $X \in \mathrm{Aff}(G, P_0, P_1)$, $H_0 = H_1$ and intersects the curve anyway.

The Extraneous Factor

The extraneous hypersurface

In general, the extraneous factor is an hypersurface E.
Its equation is given by the following formula :
$\odot_{i=0}^{d}(\xi \wedge G_1 \dots G_{c-1} \wedge P_{i,1} \dots P_{i,d}) = 0$ where $P_{i,j}$ are points used to define the random hyperplanes H_i.

Degree

Note : Since the degree of the resultant is quite high compared to the degrees of $\mathrm{Cone}(G, V)$ and of the extraneous factor, they appear with some power.

$$\underbrace{\text{Resultant}}_{\text{degree } \leqslant \delta^d + d\delta^d} = \underbrace{\mathrm{Cone}(G, V)^q}_{\text{degree } \leqslant \delta^d \times q} \times \underbrace{E^p}_{\text{degree } \leqslant d \times p}$$

Number of Surfaces Needed

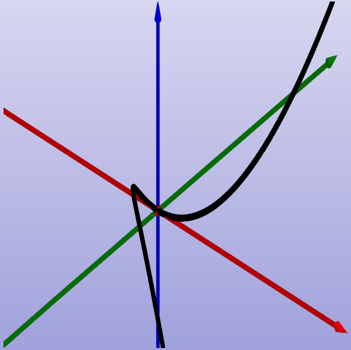
Computing one resultant gives one implicit equation for V, but we need more...

Theorem

If the variety is a space curve, noted $\mathscr{C}$, then for any choice of 3-tuples of points $\mathscr{G}_1, \mathscr{G}_2, \mathscr{G}_3$ that are not aligned and not on $\mathscr{C}$, the surfaces $\mathrm{Cone}(\mathscr{G}_1, \mathscr{C}), \mathrm{Cone}(\mathscr{G}_2, \mathscr{C}), \mathrm{Cone}(\mathscr{G}_3, \mathscr{C})$ uniquely determine the curve $\mathscr{C}$.

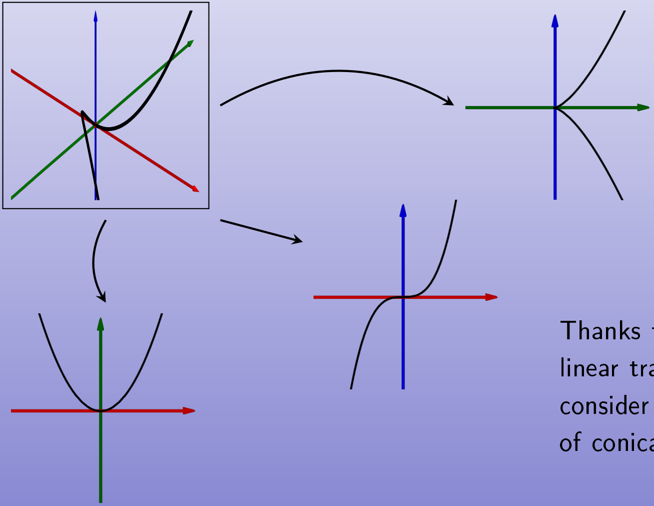
3 equations are enough for describing $\mathscr{C}$.

Sketch of Proof



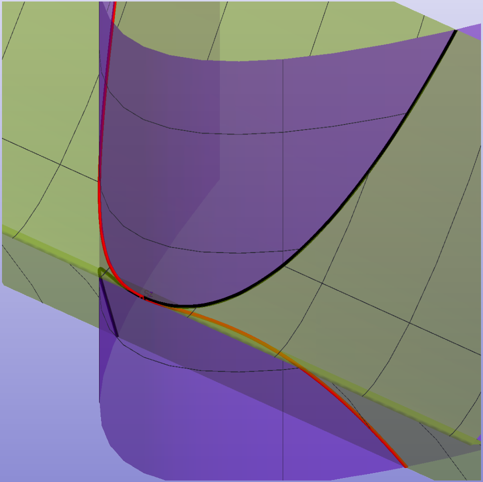
Considering a space curve, the algorithm provides conical surfaces containing the curve.

Sketch of Proof



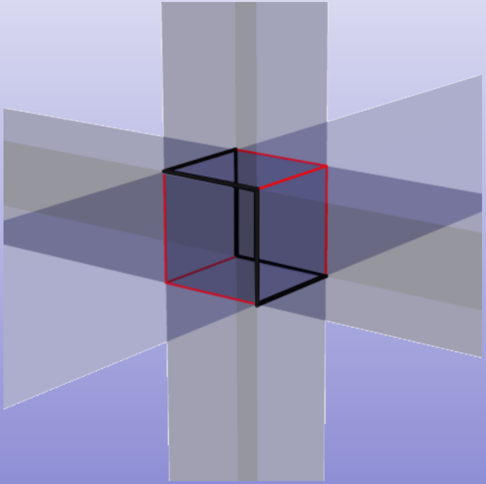
Thanks to a projective linear transformation, we consider cylinders instead of conical surfaces.

Sketch of Proof



Generically, two of these cylinders intersect along $\mathscr{C}$ and other curves. With a third cylinder using a linearly independant orientation, the extra curves are ruled out.

Non-rational Counter-example



Curves in $\mathbb{P}^n(\mathbb{R})$

Although the method works for any variety of codimension $c > 1$, it runs better for curves.

Simpler Terminating Condition

n equations are sufficient for describing a curve in $\mathbb{P}^n(\mathbb{R})$.
We don't have a terminating condition yet for arbitrary codimension.

Simpler Resultant Computation

We use the univariate Sylvester resultant instead of the multivariate sparse resultant.

Simpler Extraneous Factor

Extraneous hyperplane $= \mathrm{Aff}(G_1, \ldots, G_{n-2}, P_0, P_1)$.

Features

Pros :

- A fixed number of equations,
- Few equations,
- Possibility of having both a matrix representation (fast) and implicit equations (slower).

Cons :

- Equations are not of optimal degree,
- Matrix representation far from optimal.

Also :

- The method works better for high codimension,
- Equations have a geometric meaning.

Maple's implementation

We implemented the algorithm on Maple and Sage.

	Chow Form	Maple's `EliminationIdeal`	Matrix Representation
Runtime (representation)	Medium	Fast	Fast
Runtime (usage)	Immediate	Immediate	May be slow
Number of outputs	Low and fixed	Medium and moderatly varying	One matrix of full rank but varying size
Degree of outputs	Same as input	Either same as input or a bit lower	All matrix entries are linear
Retrieve point parameters	✗	✗	✓
Randomized	✓*	✗	✓*
Geometric Feature	Conical surfaces	✗	✗

(*) Depends on the implementation