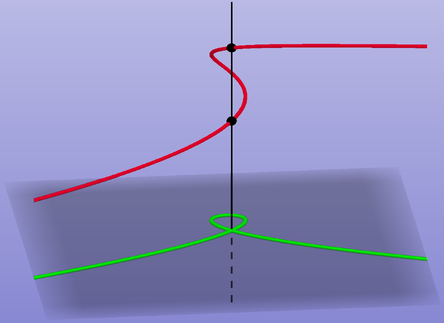
Singular Points and Blow-ups

Framework

Birational Map

A Blow-up π is a birational map but is *not* a diffeomorphism. Here, we will discuss only of blow-ups between 2-dimensional complex spaces. Such blow-ups can be used to regularize planar (complex) curves.



Definition

Let $H = \{(P, L) \in \mathbb{C}^2 \times \mathbb{P}^1(\mathbb{C}) \mid P \in L\}$
$= \{(((x, y), (w : z)) \in \mathbb{C}^2 \times \mathbb{P}^1(\mathbb{C}) \mid xz = yw\}$.
We note π the projection of H on $\mathbb{C}^2$.

Definitions

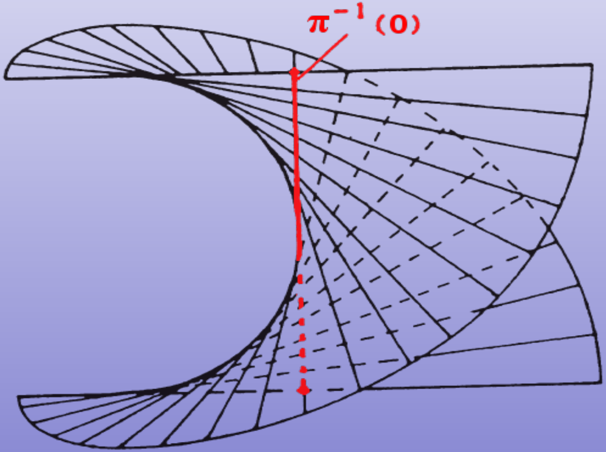
- π^{-1} is a *blow-up* of the origin of $\mathbb{C}^2$.
- π is a *blow-down* on that point.
- $\pi^{-1}(0) \simeq \mathbb{P}^1(\mathbb{C})$ is called the *exceptional divisor* of π.

Lemma

The map π is an analytic isomorphism from $H \backslash \pi^{-1}(0)$ to $\mathbb{C}^2 \backslash \{0\}$.

The space H

Figure: The map π is also called the Höpf Bundle



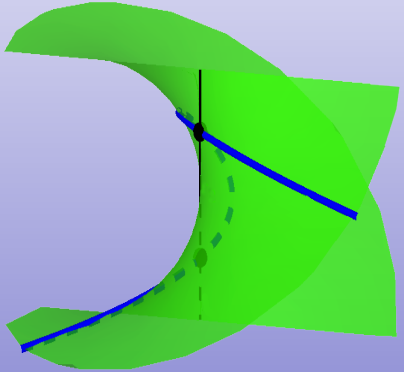
Total Preimage, Strict Preimage

Let $f \in \mathbb{C}[x, y]$.
The zero set of f will be noted Γ and we define $\hat{f} := f \circ \pi$.

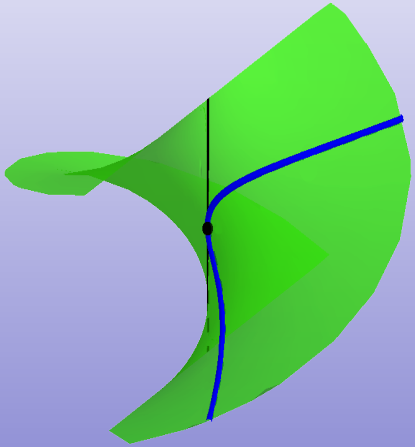
Définitions

- $\hat{f}^{-1}(0)$ is the total preimage of Γ.
  We note it $\hat{\Gamma}$.
- The closure of $\hat{f}^{-1}(0) \backslash \pi^{-1}(0)$ is the strict preimage of Γ.
  We note it Γ'.

Total Preimage, Strict Preimage



Total Preimage, Strict Preimage



Resolution of Singularity

Composing Blow-ups

A map $\pi : \Sigma \xmapsto{\pi_n} \Sigma_n \xmapsto{\pi_{n-1}} \dots \xmapsto{\pi_0} \Sigma_0$ is a blow-up composition if all the π_i's are blow-ups.

We will discuss only of blow-up compositions verifying the followings:

- i. Σ_0 is a neighborhood of $0 \in \mathbb{C}^2$.
- ii. π_0 blows-up only the origin.
- iii. π_i blows-up one or several points belonging to the exceptional divisor of π_{i-1}.

Resolution of Singularity

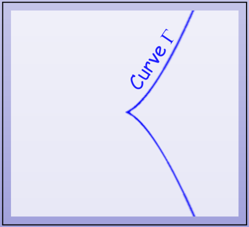
Strong Resolution of a Singularity

A *strong resolution of a singularity* is a blow-up composition verifying the followings:

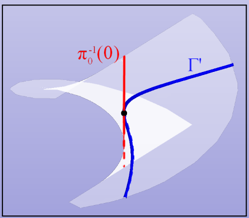
1. The strict preimage Γ' is smooth (no cusp, no self-intersection...).
2. The strict preimage intersects the exceptional divisor only transversally.
3. The strict preimage intersects the exceptional divisor only at its smooth points (ie. not at points of self-intersection of the exceptional divisor).

Resolution of Singularity

$$y^2 = x^3 \qquad\qquad y^2 = x^3 \text{ or } z^2 = x$$

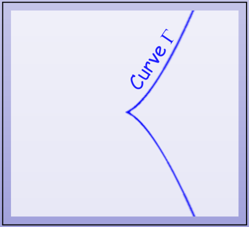


$$\pi_0^{-1} \qquad z = \frac{y}{x}$$



Resolution of Singularity

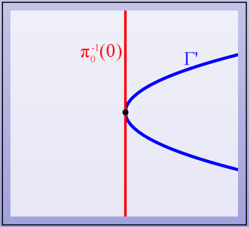
$$y^2 = x^3$$



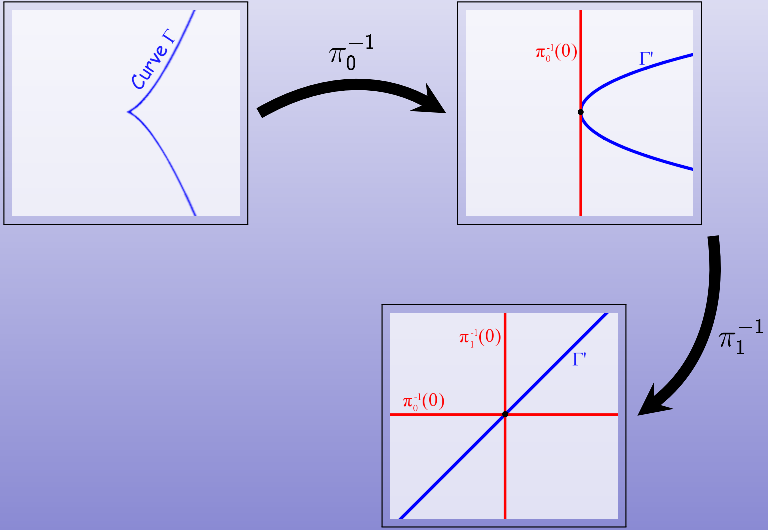
$$\xrightarrow{\pi_0^{-1}}$$

$$x = X$$
$$y = XY$$

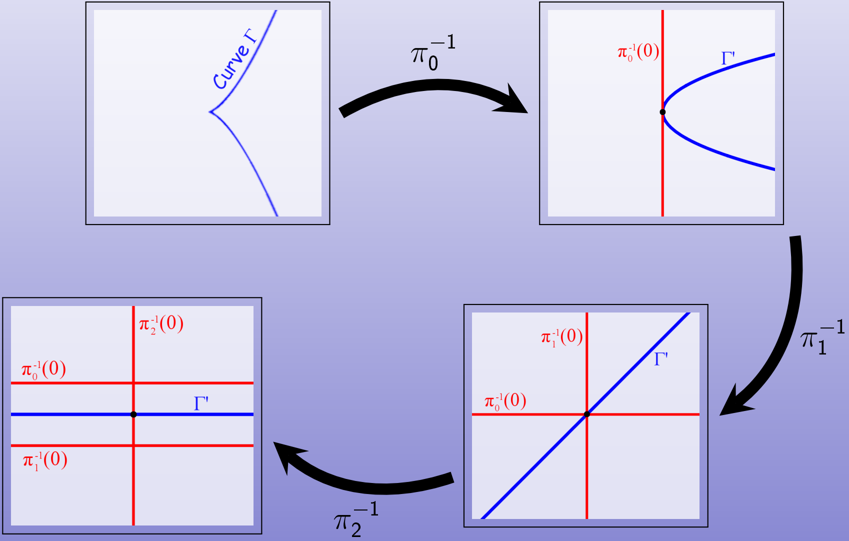
$$Y^2 = X$$



Resolution of Singularity



Resolution of Singularity



Puiseux Series

Definition

A *Puiseux Series* (at the origin) of $f \in \mathbb{C}[x, y]$ is a series $\varphi \in \mathbb{C}\left[\left[x^{\frac{1}{n}}\right]\right]$ for some n that has a limit at $x = 0$ and such that $f(x, \varphi(x)) = 0$.

Examples

$f(x, y) = y^2 - x^3$ has a unique Puiseux series: $\varphi(x) = x^{\frac{3}{2}}$

$f(x, y) = y^2 - x^3 - x^2$ has two distinct Puiseux series:

$$\begin{cases} \varphi_1(x) = x + \frac{1}{2}x^2 - \dots \\ \varphi_2(x) = -x - \frac{1}{2}x^2 + \dots \end{cases}$$

Puiseux Characteristic Pairs

Definition

The *Puiseux Characteristic Pairs* $(n_1, m_1), \ldots, (n_r, m_r)$ of the Puiseux series $\varphi(x) = \sum_{\kappa \in \mathbb{Q}} a_\kappa x^\kappa$ are defined by:

- $$\frac{n_1}{m_1} = \kappa_1 = min\{\kappa \in \mathbb{Q} \backslash \mathbb{N} \mid a_\kappa \neq 0\}$$
- $$\frac{n_i}{m_1 \ldots m_i} = \kappa_i = min\left\{\kappa \in \mathbb{Q} \mid a_\kappa \neq 0 \text{ and } \kappa \notin \frac{1}{m_1 \ldots m_{i-1}}\mathbb{N}\right\}$$

With $\gcd(n_i, m_i) = 1$ for each i.

Examples

$\varphi(x) = x^{\frac{3}{2}} + x^{\frac{7}{4}}$

$(3, 2), (7, 2)$

$\varphi(x) = x^{\frac{3}{2}} + x^{\frac{5}{3}}$

$(3, 2), (10, 3)$

Puiseux Characteristic Pairs

Property

For $i \in [\![2, r]\!]$, we have $n_{i-1} m_i < n_i$

Associated Puiseux Series

Pairs of coprime integers verifying the above property are the Puiseux pairs of some Puiseux series:

$$\begin{aligned}\varphi(x) &= x^{\frac{n_1}{m_1}} + x^{\frac{n_2}{m_1 m_2}} + \cdots + x^{\frac{n_r}{m_1 \cdots m_r}} \\ &= x^{\kappa_1} + x^{\kappa_2} + \cdots + x^{\kappa_r}\end{aligned}$$

Zariski Characteristic Pairs

Definition

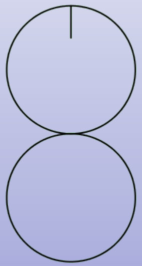
The *Zariski Characteristic Pairs* $(p_1, q_1), \ldots, (p_r, q_r)$ associated to a Puiseux series are defined by:

- $p_1 := n_1$
  $p_i := n_i - n_{i-1} m_i$
- $q_i := m_i$

Examples

$\varphi(x) = x^{\frac{3}{2}} + x^{\frac{7}{4}}$	$\varphi(x) = x^{\frac{3}{2}} + x^{\frac{7}{3}}$	
$(3,2),(7,2)$	$(3,2),(14,3)$	Puiseux Pairs
$(3,2),(1,2)$	$(3,2),(5,3)$	Zariski Pairs

Geometric Interpretation

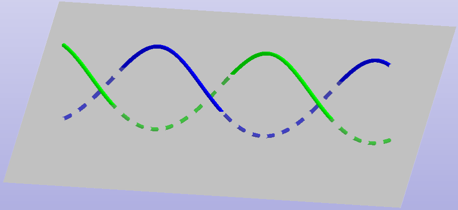


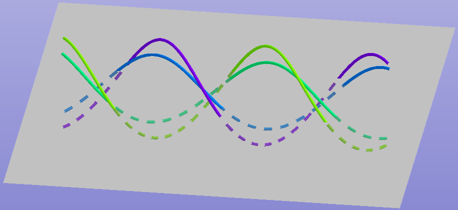
On top left, the outer circle rolls twice around itself.
On bottom right, the circle rolls only once around itself despite travelling the same distance.
The difference is due to the rotation around the inner circle.

The Zariski characteristic pairs discount that "inner rotation" (ie. the rotation of the roots due to lower-degree terms).

Geometric Interpretation

$$|x| = \epsilon \text{ in } \varphi_1(x) = x^{\frac{3}{2}} \text{ and in } \varphi_2(x) = x^{\frac{3}{2}} + x^{\frac{7}{4}}$$





Slow Approximation by Continued Fraction

For $\frac{p}{q} \in \mathbb{Q}$, we note $\frac{p}{q} = h_0 + \cfrac{1}{\ddots + \cfrac{1}{h_m}}$ its expression as a continued fraction.

Definition

The *slow approximation* of $\frac{p}{q}$ is the sequence of size $\ell(\frac{p}{q})$ defined by:

- $a_k = k$ for $1 \leqslant k \leqslant h_0$
- $a_k = h_0 + \frac{1}{k - h_0}$ for $h_0 < k \leqslant h_0 + h_1$
- $$a_k = h_0 + \cfrac{1}{\ddots + \cfrac{1}{k - (h_0 + h_1 + \cdots + h_{i-1})}}$$

  for $h_0 + \cdots + h_{i-1} < k \leqslant h_0 + \cdots + h_i$

Slow Approximation by Continued Fraction

Example

The slow approximation of $\dfrac{17}{7}$ by continued fraction is given by the following:

1 2 3 too high!

Slow Approximation by Continued Fraction

Example

The slow approximation of $\dfrac{17}{7}$ by continued fraction is given by the following:

$1 \qquad 2 \qquad 3 \qquad 2+\dfrac{1}{2} \qquad 2+\dfrac{1}{3}$ too low!

Slow Approximation by Continued Fraction

Example

The slow approximation of $\frac{17}{7}$ by continued fraction is given by the following:

$$1 \qquad 2 \qquad 3 \qquad 2+\frac{1}{2} \qquad 2+\frac{1}{3} \qquad 2+\cfrac{1}{2+\cfrac{1}{2}} \qquad 2+\cfrac{1}{2+\cfrac{1}{3}}$$

Resolution of Singularity - 1 Branch - 1 Characteristic Pair

Theorem

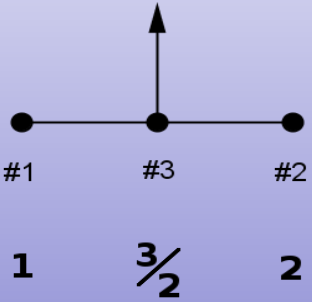
Let $f \in \mathbb{C}[x, y]$ with a singularity at the point 0 but only one branch with exactly one Zariski characteristic pair (p, q).
Then a strong resolution of the singularity is obtained by a composition of $\ell\left(\frac{p}{q}\right)$ blows-up.

Examples

$y = x^{\frac{3}{2}}$ is strongly desingularized by a composition of 3 blows-up.
$y = x^{\frac{17}{7}}$ is strongly desingularized by a composition of 7 blows-up.

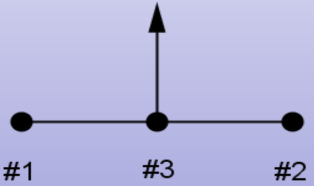
Examples

Resolution Tree of $y = x^{\frac{3}{2}}$



Examples

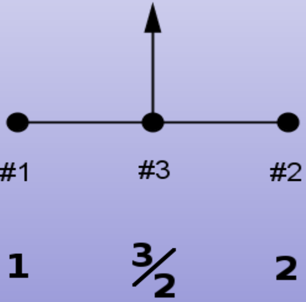
Resolution Tree of $y = x^{\frac{3}{2}}$



$y = x^{\frac{3}{2}}$ 1 $\frac{3}{2}$ 2

Examples

Resolution Tree of $y = x^{\frac{3}{2}}$



$$y = x^{\frac{3}{2}}$$

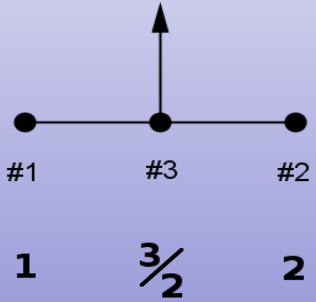
$$Y = xy$$

$$y = x^{\frac{1}{2}}$$

$$x = y^2$$

Examples

Resolution Tree of $y = x^{\frac{3}{2}}$



$y = x^{\frac{3}{2}}$

$Y = xy$

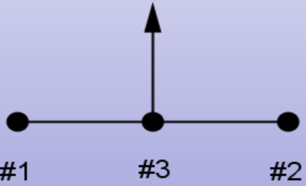
$y = x^{\frac{1}{2}}$

$x = y^2$

$x = y$

Examples

Resolution Tree of $y = x^{\frac{3}{2}}$



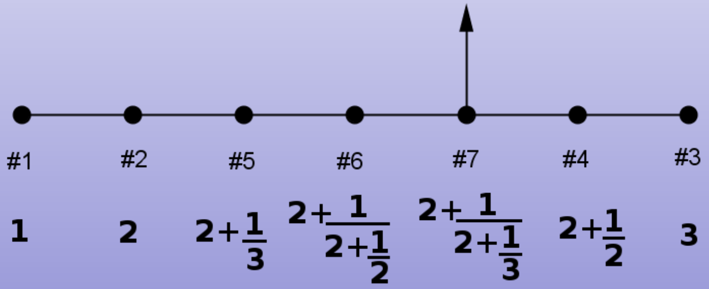
$y = x^{\frac{3}{2}}$ **1** **3/2** **2**

$Y = xy$ $y = x^{\frac{1}{2}}$ $y = 1$ $x = y$

$x = y^2$

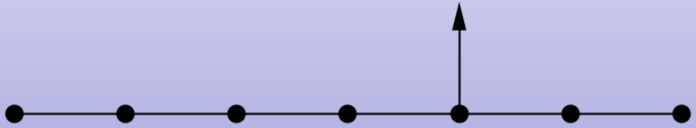
Examples

Resolution Tree of $y = x^{\frac{17}{7}}$



Examples

Resolution Tree of $y = x^{\frac{17}{7}}$



#1	#2	#5	#6	#7	#4	#3
$\mathbf{1}$	$\mathbf{2}$	$\mathbf{2+\frac{1}{3}}$	$\mathbf{2+\frac{1}{2+\frac{1}{2}}}$	$\mathbf{2+\frac{1}{2+\frac{1}{3}}}$	$\mathbf{2+\frac{1}{2}}$	$\mathbf{3}$
$y = x^{\frac{10}{7}}$	$y = x^{\frac{3}{7}}$ $x = y^{\frac{7}{3}}$	$y = x^2$	$y = x$	$y = 1$	$x = y^{\frac{1}{3}}$ $y = x^3$	$x = y^{\frac{4}{3}}$

Resolution of Singularity - 1 Branch - Several Characteristic Pairs

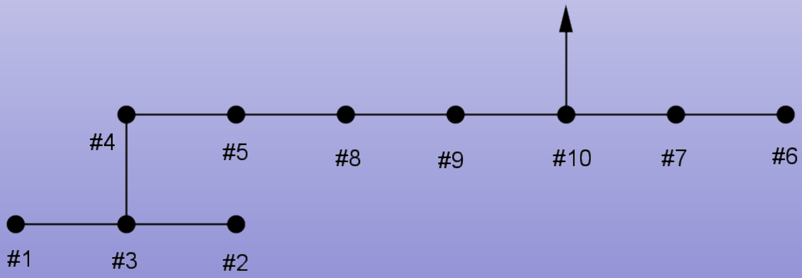
Theorem

Let $f \in \mathbb{C}[x, y]$ with a singularity at the point 0 and only one branch with r Zariski characteristic pairs $(p_1, q_1), \ldots, (p_r, q_r)$. Then the resolution tree of f is obtained by taking all the resolution trees $T\left(\frac{p_i}{q_i}\right)$ and attaching them consecutively.

A strong resolution of the singularity is thus a composition of $\ell\left(\frac{p_1}{q_1}\right) + \cdots + \ell\left(\frac{p_r}{q_r}\right)$ blows-up.

Example

Resolution Tree of $y = x^{\frac{3}{2}} + x^{\frac{38}{14}} = x^{\frac{3}{2}}(1 + x^{\frac{17}{14}})$
The Zariski characteristic pairs are $(3, 2), (17, 7)$



Introduction to the Problem

Goal

When the curve has several intersecting branches, we need both to make them smooth but also make sure that they do not intersect anymore after the blow-ups.

Remark

If B_1 and B_2 are two transversal branches, then they don't cut each other after a single blow-up.

Example

The d branches of a homogeneous polynomial of degree d are always transverse.

Coincidence Exponent

Definition

Let $\varphi \in \mathbb{C}\left[\left[x^{\frac{1}{m}}\right]\right]$ and $\varphi' \in \mathbb{C}\left[\left[x^{\frac{1}{m'}}\right]\right]$ two Puiseux series corresponding to two branches of a curve through 0.
Their *Coincidence Exponent* is defined by:

$$C(\varphi, \varphi') = \max_{\sigma, \sigma'} \{\mathrm{val}(\sigma(\varphi) - \sigma'(\varphi'))\}$$

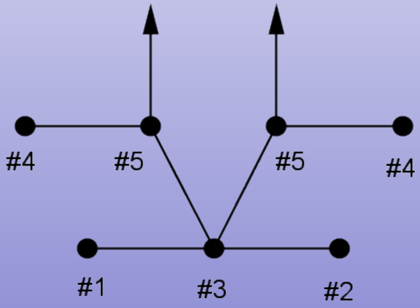
where σ (and σ') are taken among the m different choices of the m^{th} roots.

Example

$\varphi(x) = x^{\frac{5}{2}} + x^{\frac{27}{4}} + x^{10}$
$\varphi'(x) = x^{\frac{5}{2}} - x^{\frac{27}{4}} + x^{\frac{61}{6}}$
$C(\varphi, \varphi') = \mathrm{val}(x^{10} - x^{\frac{61}{6}}) = 10$

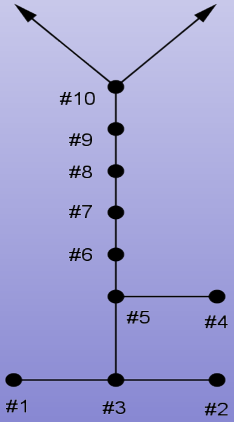
Examples

$$\varphi(x) = x^{\frac{3}{2}} + x^{\frac{7}{4}}$$
$$\varphi'(x) = x^{\frac{3}{2}} + 2x^{\frac{7}{4}}$$
$$C(\varphi, \varphi') = \frac{7}{4}$$



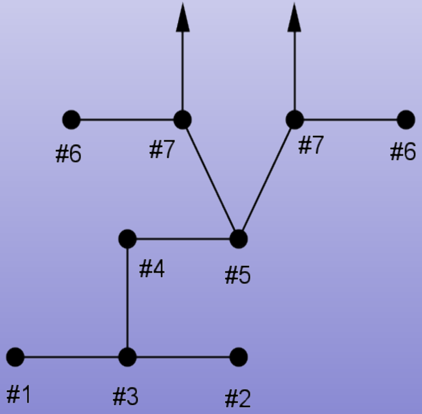
Examples

$$\varphi(x) = x^{\frac{3}{2}} + x^{\frac{7}{4}}$$
$$\varphi'(x) = x^{\frac{3}{2}} + x^{\frac{7}{4}} + x^{3}$$
$$C(\varphi, \varphi') = 3$$



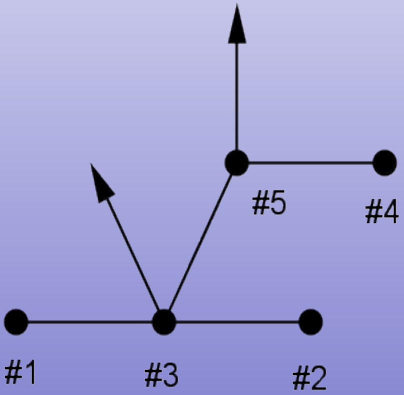
Examples

$$\varphi(x) = x^{\frac{3}{2}} + x^{\frac{11}{4}}$$
$$\varphi'(x) = x^{\frac{3}{2}} + x^{\frac{5}{2}} + x^{\frac{11}{4}}$$
$$C(\varphi, \varphi') = \frac{5}{2}$$



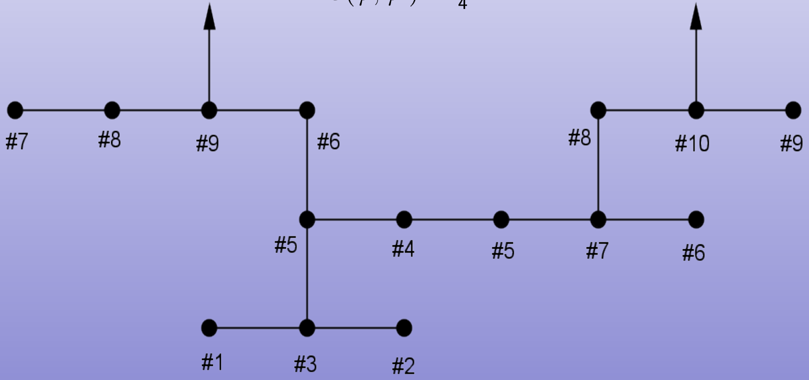
Examples

$$\varphi(x) = x^{\frac{3}{2}} + x^{\frac{7}{4}}$$
$$\varphi'(x) = x^{\frac{3}{2}}$$
$$C(\varphi, \varphi') = \frac{7}{4}$$



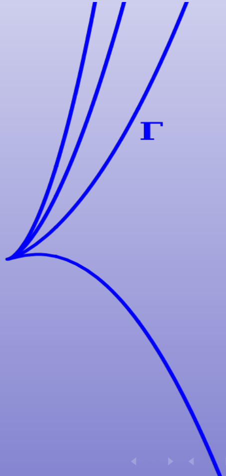
Examples

$$\varphi(x) = x^{\frac{3}{2}} + x^{\frac{7}{4}} + x^{\frac{25}{12}}$$
$$\varphi'(x) = x^{\frac{3}{2}} + x^{\frac{11}{4}} + x^{\frac{25}{8}}$$
$$C(\varphi, \varphi') = \frac{7}{4}$$



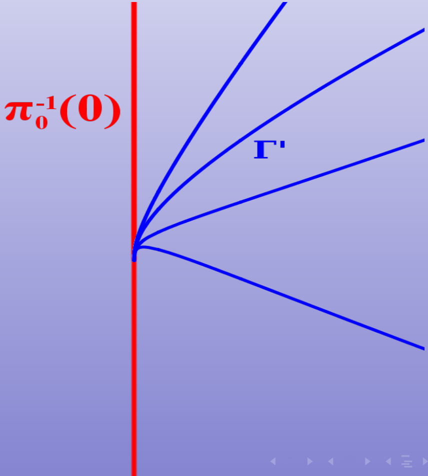
Examples

$$\varphi_1(x) = x^{\frac{4}{3}} + x^{\frac{11}{6}}$$
$$\varphi_2(x) = x^{\frac{4}{3}} + x^{\frac{11}{6}} + x^2$$



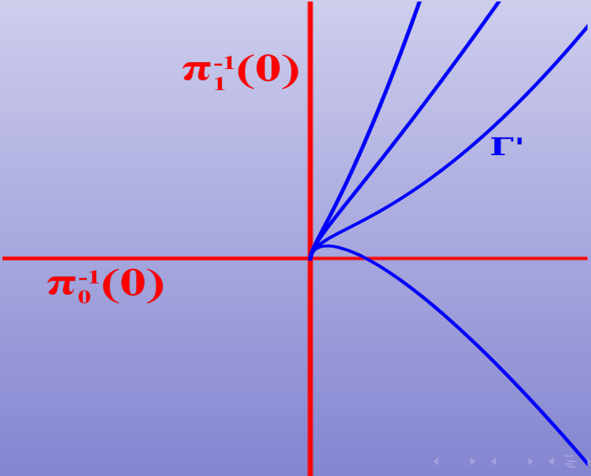
Examples

$$\varphi_1(x) = x^{\frac{1}{3}} + x^{\frac{5}{6}}$$
$$\varphi_2(x) = x^{\frac{1}{3}} + x^{\frac{5}{6}} + x$$



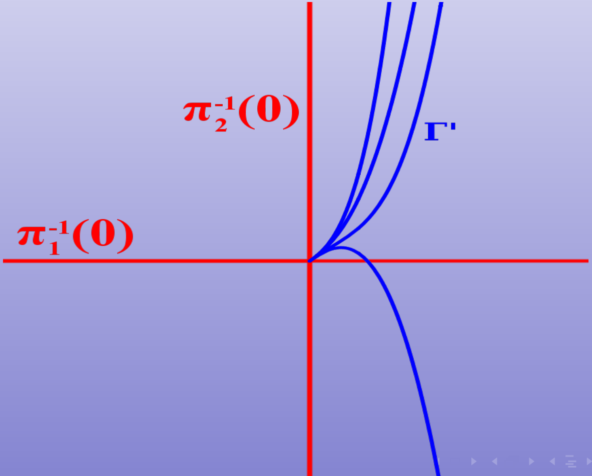
Examples

$$\varphi_1(x) = x^{\frac{1}{2}} + x^{\frac{5}{4}}$$
$$\varphi_2(x) = x^{\frac{1}{2}} + x^{\frac{5}{4}} + x^{\frac{3}{2}}$$



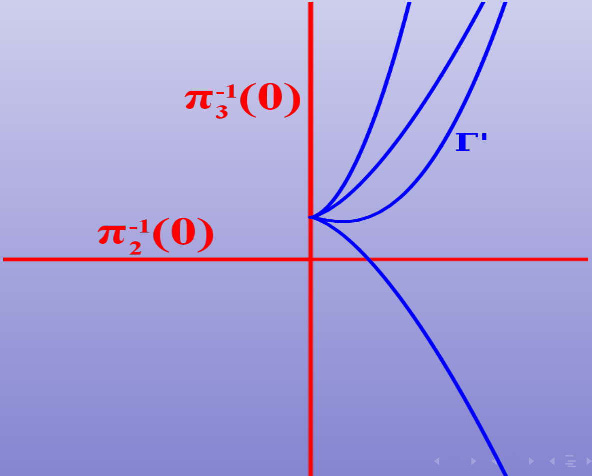
Examples

$$\varphi_1(x) = x + x^{\frac{5}{2}}$$
$$\varphi_2(x) = x + x^{\frac{5}{2}} + x^3$$



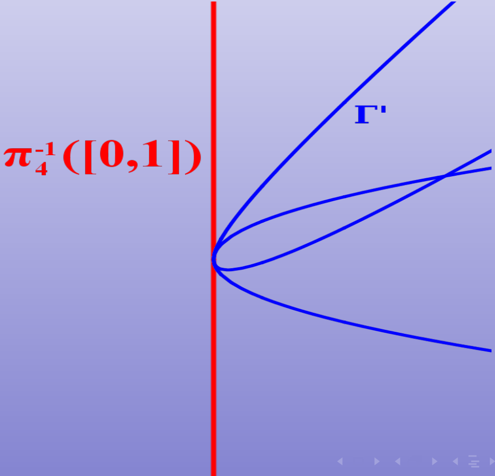
Examples

$$\varphi_1(x) = 1 + x^{\frac{3}{2}}$$
$$\varphi_2(x) = 1 + x^{\frac{3}{2}} + x^2$$



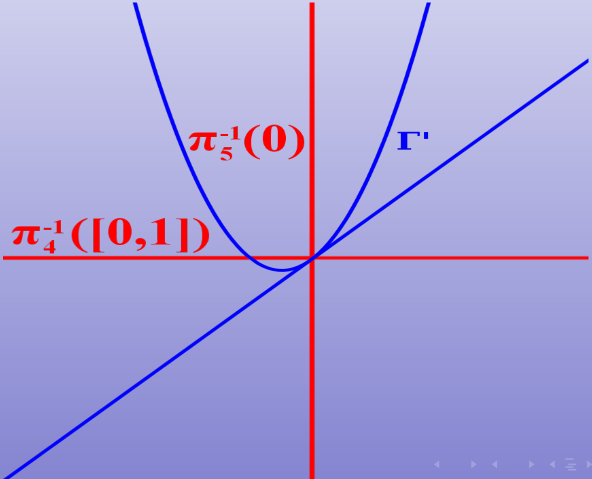
Examples

$$\varphi_1(x) = x^{\frac{1}{2}}$$
$$\varphi_2(x) = x^{\frac{1}{2}} + x$$



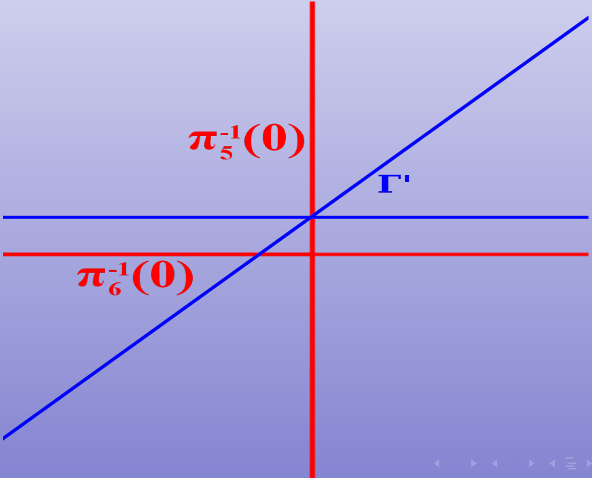
Examples

$$\varphi_1(x) = x$$
$$\varphi_2(x) = x + x^2$$



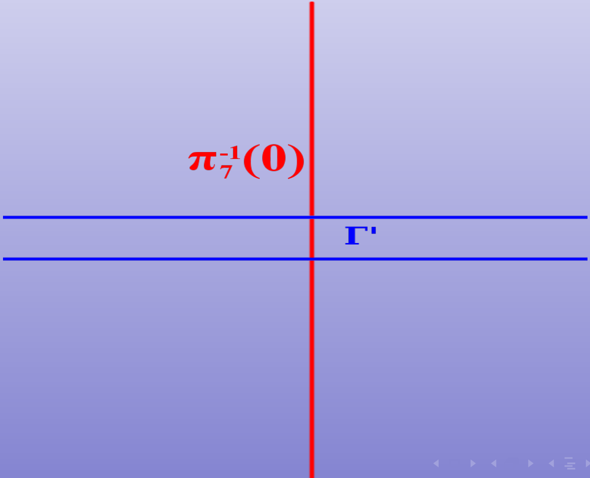
Examples

$$\varphi_1(x) = 1$$
$$\varphi_2(x) = 1 + x$$



Examples

$$\varphi_1(x) = 0$$
$$\varphi_2(x) = 1$$



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