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On the Theory of Left/Right Almost Groups and Hypergroups with their Relevant Enumerations

Christos G. Massouros [1,*] and Naveed Yaqoob [2]

[1] Core Department, Euripus Campus, National and Kapodistrian University of Athens, GR 34400 Euboia, Greece

[2] Department of Mathematics and Statistics, Riphah International University, Sector I-14, Islamabad 44000, Pakistan; nayaqoob@ymail.com

[*] Correspondence: ChrMas@uoa.gr or Ch.Massouros@gmail.com

Abstract: This paper presents the study of algebraic structures equipped with the inverted associativity axiom. Initially, the definition of the left and the right almost-groups is introduced and afterwards, the study is focused on the more general structures, which are the left and the right almost-hypergroups and on their enumeration in the cases of order 2 and 3. The outcomes of these enumerations compared with the corresponding in the hypergroups reveal interesting results. Next, fundamental properties of the left and right almost-hypergroups are proved. Subsequently, the almost hypergroups are enriched with more axioms, like the transposition axiom and the weak commutativity. This creates new hypercompositional structures, such as the transposition left/right almost-hypergroups, the left/right almost commutative hypergroups, the join left/right almost hypergroups, etc. The algebraic properties of these new structures are analyzed and studied as well. Especially, the existence of neutral elements leads to the separation of their elements into attractive and non-attractive ones. If the existence of the neutral element is accompanied with the existence of symmetric elements as well, then the fortified transposition left/right almost-hypergroups and the transposition polysymmetrical left/right almost-hypergroups come into being.

Keywords: hypercompositional algebra; magma; left/right almost-group; left/right almost-hyper group; transposition axiom; Mathematica

1. Introduction

This paper is generally classified in the area of hypercompositional algebra. Hypercompositional algebra is the branch of abstract algebra which studies hypercompositional structures, i.e., structures equipped with one or more multi-valued operations. Multi-valued operations, also called hyperoperations or hypercompositions, are operations in which the result of the synthesis of two elements is multi-valued, rather than a single element. More precisely, a *hypercomposition* on a non-void set H is a function from $H \times H$ to the powerset $P(H)$ of H. Hypercompositional structures came into being through the notion of the *hypergroup*. The hypergroup was introduced in 1934 by Marty, in order to study problems in non-commutative algebra, such as cosets determined by non-invariant subgroups [1–3].

In [4] the *magma* is defined as an ordered pair $(H, \perp)$ where H is a non-void set and "$\perp$" is a law of synthesis, which is either a composition or a hypercomposition. Per this definition, if A and B are subsets of H, then:

$$A \perp B = \{a \perp b \in H \mid a \in A, b \in B\}, \text{ if } \perp \text{ is a composition}$$

and

$$A \perp B = \bigcup_{(a,b) \in A \times B} (a \perp b), \text{ if } \perp \text{ is a hypercomposition}$$

In particular if $A = \varnothing$ or $B = \varnothing$, then $A\perp B = \varnothing$ and vice versa. $A\perp b$ and $a\perp B$ have the same meaning as $A\perp\{b\}$ and $\{a\}\perp B$. In general, the singleton $\{a\}$ is identified with its member a. Sometimes it is convenient to use the relational notation $A \approx B$ instead of $A \cap B \neq \varnothing$. Then, as the singleton $\{a\}$ is identified with its member a, the notation $a \approx A$ or $A \approx a$ is used as a substitute for $a \in A$ or Aa. The relation $\approx$ may be considered as a weak generalization of equality, since, if A and B are singletons and $A \approx B$, then $A = B$. Thus, $a \approx b\perp c$ means either $a = b\perp c$, when the synthesis is a composition or $a \in b\perp c$, when the synthesis is a hypercomposition.

It is possible that the result of the synthesis of a pair of elements in a magma is the void set, when the law of synthesis is a hypercomposition. Then, the structure is called a *partial hypergroupoid*, otherwise, it is called a *hypergroupoid*.

Every law of synthesis in a magma induces two new laws of synthesis. If the law of synthesis is written multiplicatively, then the two induced laws are:

$$a/b = \{x \in E \mid a \approx xb\}$$

and

$$b\backslash a = \{x \in E \mid a \approx bx\}$$

Thus $x \approx a/b$ if and only if $a \approx xb$ and $x \approx b\backslash a$ if and only if $a \approx bx$. In the case of a multiplicative magma, the two induced laws are named *inverse laws* and they are called the *right division* and the *left division*, respectively [4]. It is obvious that, if the law of synthesis is commutative, then the right division and left division coincide.

A law of synthesis on a set H is called *associative* if the property,

$$(x\perp y)\perp z = x\perp(y\perp z)$$

is valid, for all elements x, y, z in H. A magma whose law of synthesis is associative, is called an *associative magma* [4].

Definition 1. *An associative magma in which the law of synthesis is a composition is called a semigroup, while it is called a semihypergroup if the law of synthesis is a hypercomposition and $ab \neq \varnothing$ for each pair of its elements.*

A law of synthesis $(x, y) \rightarrow x\perp y$ on a set H is called *reproductive* if the equality,

$$x\perp H = H\perp x = H$$

is valid for all elements x in H. A magma whose law of synthesis is reproductive is called a *reproductive magma* [4].

Definition 2. *A reproductive magma in which the law of synthesis is a composition is called a quasigroup, while it is called a quasi-hypergroup if the law of synthesis is a hypercomposition and $ab \neq \varnothing$ for each pair of its elements.*

Definition 3. [4] *An associative and reproductive magma is called a group, if the law of synthesis is a composition, while it is called a hypergroup if the law of synthesis is a hypercomposition.*

The above unified definition of the group and the hypergroup is presented in [4], where it is also proved analytically that it is equivalent to the well-known dominant definition of the group.

A composition or a hypercomposition on a non-void set H is called *left inverted associative* if:

$$(a\perp b)\perp c = (c\perp b)\perp a, \text{ for every } a, b, c \in H,$$

while it is called *right inverted associative* if

$$a \perp (b \perp c) = c \perp (b \perp a), \text{ for every } a, b, c \in H.$$

The notion of the inverted associativity was initially conceived by Kazim and Naseeruddin [5] who endowed a groupoid with the left inverted associativity, thus defining the LA-semigroup. A magma equipped with left inverted assisiativity is called a *left inverted associative magma,* while if it is equipped with right inverted assisiativity is called a *right inverted associative magma.*

Recall that if $(E, \perp)$ is a magma, then the law of synthesis:

$$(x, y) \rightarrow x \perp^{op} y = y \perp x$$

is called the *opposite* of "$\perp$". The magma $(E, \perp^{op})$ is called the *opposite magma* of $(E, \perp)$ [4].

Theorem 1. *If* $(H, \perp)$ *is a left inverted associative magma, then* $(H, \perp^{op})$ *is a right inverted associative magma.*

Proof.
$$a \perp^{op} (b \perp^{op} c) = (b \perp^{op} c) \perp a = (c \perp b) \perp a =$$
$$= (a \perp b) \perp c = c \perp^{op} (a \perp b) = c \perp^{op} (b \perp^{op} a) \quad \square$$

As it is detailed in [4] the group and the hypergroup satisfy exactly the same axioms, i.e., the *reproductive axiom* and the *associative axiom.* This led to the unified definition of the group and the hypergroup that was repeated as Definition 3 in this paper. Using the same approach, the definition of the *left/right almost-group* and the *left/right almost-hypergroup* is:

Definition 4. (FIRST DEFINITION OF THE LEFT/RIGHT ALMOST GROUP/HYPER GROUP) *A reproductive magma which satisfies the axiom of the left inverted associativity is called a left almost-group (LA-group), if the law of synthesis on the magma is a composition, while it is called a left almost-hypergroup (LA-hypergroup) if the law of synthesis is a hypercomposition. A reproductive, right inverted associative magma, is called a right almost-group (RA-group) or a right almost-hypergroup (RA-hypergroup) if the law of synthesis is a composition or a hypercomposition respectively.*

Remark 1. *Obviously if the law of synthesis is commutative, then the LA- or RA- groups and hypergroups are groups and hypergroups respectively, indeed:*

$$(a \perp b) \perp c = (c \perp b) \perp a = a \perp (c \perp b) = a \perp (b \perp c)$$

As shown in Theorem 11 in [4], when the law of synthesis is a composition, then the reproductive axiom is valid if and only if the inverse laws are compositions. Hence another definition can be given for the left/right almost-group:

Definition 5. (SECOND DEFINITION OF THE LEFT/RIGHT ALMOST-GROUP) *A magma which satisfies the axiom of the left/right inverted associativity is called a left/right almost-group (LA-group/RA-group), if the law of synthesis on the magma is a composition, and the two induced laws of synthesis are compositions as well.*

Thus, if the law of synthesis on a magma is written multiplicatively, then the magma is a left/right almost-group if and only if it satisfies the axiom of the left/right inverted associativity and both the right and the left division, a/b, $b \backslash a$ respectively, result in a single element, for every pair of elements a, b in the magma.

In a similar way, because of Theorem 14 in [4], if $a/b \neq \varnothing$ and $b \backslash a \neq \varnothing$, for all pairs of elements a, b of a magma, then the magma is reproductive. Therefore, a second definition of the left/right almost-hypergroup can be given:

Definition 6. (SECOND DEFINITION OF THE LEFT/RIGHT ALMOST-HYPERGROUP) *A magma which satisfies the axiom of the left/right inverted associativity is called a left/right almost-hypergroup (LA-hypergroup/RA-hypergroup), if the law of synthesis on the magma is a hypercomposition and the result of each one of the two inverse hypercompositions is nonvoid for all pairs of elements of the magma.*

Example 1. *This example proves the existence of non-trivial left almost-groups (Table 1) and right almost-groups (Table 2).*

Table 1. Left almost-group.

∘	1	2	3
1	1	2	3
2	3	1	2
3	2	3	1

Table 2. Right almost-group.

∘	1	2	3
1	1	3	2
2	2	1	3
3	3	2	1

Example 2. *This example presents a non-trivial left almost-hypergroup (Table 3) and a non-trivial right almost-hypergroup (Table 4).*

Table 3. Left almost-hypergroup.

∘	1	2	3
1	{1}	{1}	{1,2,3}
2	{1}	{1}	{2,3}
3	{2,3}	{2,3}	{1,2,3}

Table 4. Right almost-hypergroup.

∘	1	2	3
1	{2}	{3}	{1,3}
2	{1,2}	{1,3}	{1,2,3}
3	{1,3}	{1,2,3}	{1,2,3}

Remark 2. *It is noted that in the groups, the unified definition which uses only the reproductive axiom and the associative axiom, leads to the existence of a bilaterally neutral element and consequently to the existence of a symmetric element for each one of the group's elements, as it is proved in Theorem 2 in* [4]*. The same doesn't hold in the case of the left or right almost-groups. Indeed, if e is a neutral element in a left almost-group, then:*

$$(a \perp e) \perp c = a \perp c \text{ and } (a \perp e) \perp c = (c \perp e) \perp a = c \perp a$$

Hence the composition is commutative and therefore the left almost-group is a group.

A direct consequence of Theorem 1 is the following theorem:

Theorem 2. *If* $(H, \perp)$ *is a left almost-group or hypergroup, then* $(H, \perp^{op})$ *is a right almost-group or hypergroup respectively.*

Corollary 1. *The transpose of the Cayley table of a left almost-group or hypergroup, is the Cayley table of a right almost-group or hypergroup respectively and vice versa.*

Example 3. *The transpose of the Cayley Table 5 which describes a LA-hypergroup is Table 6, which describes a RA-hypergroup.*

Table 5. Left almost-hypergroup.

∘	1	2	3
1	{1}	{1,2}	{1,3}
2	{1,3}	{1,2,3}	{3}
3	{1,2}	{2}	{1,2,3}

Table 6. Right almost-hypergroup.

∘	1	2	3
1	{1}	{1,3}	{1,2}
2	{1,2}	{1,2,3}	{2}
3	{1,3}	{3}	{1,2,3}

Corollary 2. *The cardinal number of the left almost-groups which are defined over a set H is equal to the cardinal number of the right almost-groups defined over H.*

Corollary 3. *The cardinal number of the left almost-hypergroups which are defined over a set H is equal to the cardinal number of the right almost-hypergroups defined over H.*

Hypercompositional structures with inverted associativity were studied by many authors (e.g., Yaqoob et al. [6–10], Hila et al. [11,12], etc.).

Special notation: In the following, in addition to the typical algebraic notations, we use Krasner's notation for the complement and difference. So, we denote with A··B the set of elements that are in the set A, but not in the set B.

2. Hypercompositional Structures with Inverted Associativity

The hypergroup, was progressively enriched with additional axioms, thus creating a significant number of special hypergroups, e.g., [13–55]. Prenowitz enriched the hypergroup with an additional axiom, in order to use it in the study of geometry [56–61]. More precisely, he introduced the *transposition axiom*

$$a/b \cap c/d \neq \varnothing \text{ implies } ad \cap bc \neq \varnothing, \text{ for all } a,b,c,d \in H,$$

into a commutative hypergroup of idempotent elements, i.e., a commutative hypergroup H, all the elements of which satisfy the properties $aa = a$ and $a/a = a$. He named this new hypergroup *join space*. Prenowitz was followed by others, such as Jantosciak [62,63], Barlotti and Strambach [64], Freni [65,66], Massouros [67–71], Dramalidis [72], etc. For the sake of terminology unification, the commutative hypergroups which satisfy the transposition axiom are called *join hypergroups*. It has been proved that the join hypergroups also comprise a useful tool in the study of languages and automata [35–38,73,74]. Later on, Jantosciak generalized the above axiom in an arbitrary hypergroup as follows:

$$b\backslash a \cap c/d \neq \varnothing \text{ implies } ad \cap bc \neq \varnothing, \text{ for all } a,b,c,d \in H.$$

He named this particular hypergroup *transposition hypergroup* [33].

A *quasicanonical hypergroup* or *polygroup* [26–29] is a transposition hypergroup H containing a scalar identity, that is, there exists an element e such that $ea = ae = a$ for each a in H. A *canonical hypergroup* [17–19] is a commutative polygroup. A canonical hypergroup may also be characterized as a join hypergroup with a scalar identity [23]. The following Proposition connects the canonical hypergroups with the RA-hypergroups.

Proposition 1. *Let $(H, \cdot)$ be a canonical hypergroup and "/" the induced hypercomposition which follows from "·". Then $(H, /)$ is a right almost-hypergroup.*

Proof. In a canonical hypergroup $(H, \cdot)$ each element $x \in H$ has a unique inverse, which is denoted by x^{-1}. Moreover $e/x = x^{-1}$ and $y/x = yx^{-1}$ [4,32,34]. Thus:

$$a/(b/c) = a/\left(bc^{-1}\right) = a\left(bc^{-1}\right)^{-1} = a\left(cb^{-1}\right) = (ac)b^{-1} = (ca)b^{-1} = c\left(ab^{-1}\right) =$$
$$= c\left(ba^{-1}\right)^{-1} = c/ba^{-1} = c/(b/a)$$

Hence, the right inverted associativity is valid. Moreover, in any hypergroup holds [4,75]:

$$a/H = H/a = H$$

Thus, the reproductive axiom is valid and therefore $(H, /)$ is a RA-hypergroup. □

Subsequently, the left and right almost-hypergroups can be enriched with additional axioms. The first axiom to be used for this purpose is the transposition axiom, as it has been introduced into many hypercompositional structures and has given very interesting and useful properties (e.g., see [75,76]).

Definition 7. *If a left almost-hypergroup $(H, \cdot)$ satisfies the transposition axiom, i.e.,*

$$b\backslash a \cap c/d \neq \varnothing \text{ implies } ad \cap bc \neq \varnothing, \text{ for all } a, b, c, d \in H$$

then it will be called transposition left almost-hypergroup. If a right almost-hypergroup satisfies the transposition axiom, then it will be called transposition right almost-hypergroup.

Example 4. *This example presents a transposition left almost-hypergroup (Table 7) and a transposition right almost-hypergroup (Table 8).*

Table 7. Transposition left almost-hypergroup.

∘	1	2	3
1	{1,2,3}	{1,2,3}	{1,2,3}
2	{1,3}	{1,2}	{1,3}
3	{2,3}	{2,3}	{1}

Table 8. Transposition right almost-hypergroup.

∘	1	2	3
1	{1,2}	{1,2,3}	{1,3}
2	{1,2}	{1,3}	{1,2,3}
3	{1,2,3}	{1,2}	{1,3}

In [4] the *reverse transposition axiom* was introduced:

$$ad \cap bc \neq \varnothing \text{ implies } b\backslash a \cap c/d \neq \varnothing, \text{ for all } a, b, c, d \in H$$

Thus, the following hypercompositional structure can be defined:

Definition 8. *If a left (right) almost-hypergroup* $(H, \cdot)$ *satisfies the reverse transposition axiom, then it will be called a reverse transposition left (right) almost-hypergroup.*

Definition 9. *A hypercomposition on a non-void set H is called left inverted weakly associative if*

$$(ab)c \cap (cb)a \neq \varnothing, \text{ for every } a, b, c \in H,$$

while it is called right inverted weakly associative if

$$a(bc) \cap c(ba) \neq \varnothing, \text{ for every } a, b, c \in H.$$

Definition 10. *A quasi-hypergroup equipped with a hypercomposition which is left inverted weakly associative is called a weak left almost-hypergroup (WLA-hypergroup), while it is called a weak right almost-hypergroup (WRA-hypergroup) if the hypercomposition is right inverted weakly associative.*

Recall that a quasi-hypergroup which satisfies the weak associativity is called H_V-group [77].

Proposition 2. *A commutative WLA-hypergroup (or WRA-hypergroup) is a commutative H_V-group.*

Proof. Suppose that H is a commutative WLA-hypergroup, then:

$$(ab)c \cap (cb)a \neq \varnothing \Leftrightarrow (ab)c \cap a(cb) \neq \varnothing \Leftrightarrow (ab)c \cap a(bc) \neq \varnothing.$$

Hence H is an H_V-group. $\square$

Proposition 3. *Let* $(H, \cdot)$ *be a left almost-hypergroup. An arbitrary subset I_{ab} of H is associated to each pair of elements* $(a, b) \in H^2$ *and the following hypercomposition is introduced into H:*

$$a * b = ab \cup I_{ab}.$$

Then $(H, *)$ *is a WLA-hypergroup.*

Proof. Since xH and Hx are subsets of $x*H$ and $H*x$ respectively, it follows that the reproductive axiom holds. On the other hand:

$$(a * b) * c = (ab \cup I_{ab}) * c = (ab) * c \cup I_{ab} * c = \left[\bigcup_{r \in ab} (rc \cup I_{rc})\right] \cup \left[\bigcup_{s \in I_{ab}} (sc \cup I_{sc})\right] =$$

$$= (ab)c \cup \left[\bigcup_{r \in ab} I_{rc}\right] \cup \left[\bigcup_{s \in I_{ab}} (sc \cup I_{sc})\right]$$

and

$$(c * b) * a = (cb \cup I_{cb}) * a = (cb) * a \cup I_{cb} * a = \left[\bigcup_{r \in cb} (ra \cup I_{ra})\right] \cup \left[\bigcup_{s \in I_{cb}} (sa \cup I_{sa})\right] =$$

$$= (cb)a \cup \left[\bigcup_{r \in cb} I_{ra}\right] \cup \left[\bigcup_{s \in I_{cb}} (sa \cup I_{sa})\right]$$

Since $(ab)c = (cb)a$, it follows that $a * (b * c) \cap (a * b) * c \neq \varnothing$. $\square$

Proposition 4. *If* $\bigcap\limits_{a,b \in H} I_{ab} \neq \varnothing$, *then (H, $*$) is a transposition WLA-hypergroup.*

It is obvious that if the composition or the hypercomposition is commutative, then the inverted associativity coincides with the associativity. Thus, a commutative LA-hypergroup is simply a commutative hypergroup. However, in the hypercompositions there exists a

property that does not appear in the compositions. This is the weak commutativity. A hypercomposition on a non-void set H is called *weakly commutative* if

$$ab \cap ba \neq \varnothing, \text{ for all } a, b \in H.$$

Definition 11. *A left almost commutative hypergroup (LAC-hypergroup) is a left almost-hypergroup which satisfies the weak commutativity. A right almost commutative hypergroup (RAC-hypergroup) is a right almost-hypergroup which satisfies the weak commutativity. A LAC-hypergroup (resp. RAC-hypergroup) which satisfies the transposition axiom will be called join left almost-hypergroup (resp. reverse join left almost-hypergroup). A LAC-hypergroup (resp. RAC-hypergroup) which satisfies the reverse transposition axiom will be called reverse join left almost-hypergroup (resp. reverse join right almost-hypergroup).*

Example 5. *This example presents a join left almost-hypergroup (Table 9) and a join right almost-hypergroup (Table 10).*

Table 9. Join left almost-hypergroup.

∘	1	2	3
1	{2,3}	{1,2,3}	{1,3}
2	{1,2}	{1,3}	{1,2,3}
3	{1,3}	{2,3}	{1,2}

Table 10. Join right almost-hypergroup.

∘	1	2	3
1	{1,3}	{1,2}	{1,2,3}
2	{1,2,3}	{1,3}	{1,2}
3	{1,3}	{1,2}	{2,3}

A weak left almost-hypergroup which satisfies the weak commutativity will be named a *weak left almost commutative hypergroup* (WLAC-hypergroup). Analogous is the definition of the *weak right almost commutative hypergroup* (WRAC-hypergroup),

Proposition 5. *If* $(H, \cdot)$ *is a LAC-hypergroup, then* $(H, *)$ *is WLAC-hypergroup.*

Proposition 6. *Let* $(H, \cdot)$ *be a left almost-hypergroup and* $I_{ab} = I_{ba}$ *for all* $a, b \in H$, *then* $(H, *)$ *is a WLAC-hypergroup.*

Proposition 7.

i. *If* $(H, \cdot)$ *is a LAC-hypergroup and* $\bigcap_{a,b \in H} I_{ab} \neq \varnothing$, *then* $(H, *)$ *is a weak join left almost-hypergroup.*

ii. *If* $(H, \cdot)$ *is a LA-hypergroup,* $I_{ab} = I_{ba}$ *for all* $a, b \in H$ *and* $\bigcap_{a,b \in H} I_{ab} \neq \varnothing$, *then* $(H, *)$ *is a weak join left almost-hypergroup.*

Corollary 4. *If H is a left almost-hypergroup and w is an arbitrary element of H, then H endowed with the hypercomposition*

$$x * y = xy \cup \{x, y, w\}$$

is a weak join left almost-hypergroup.

Remark 3. *Analogous propositions to the above 3–7, hold for the right almost-hypergroups as well.*

3. Enumeration and Structure Results

The enumeration of hypercompositional structures is the subject of several papers (e.g., [78–87]). In [78] a symbolic manipulation package is developed which enumerates the hypergroups of order 3, separates them into isomorphism classes and calculates their cardinality. Following analogous techniques, in this paper, a package is developed (see Appendix A) which, when combined with the package in [78], enumerates the left almost-hypergroups and the right almost-hypergroups with 3 elements, classifies them in isomorphism classes and computes their cardinality.

For the purpose of the package, the set $H = \{1, 2, 3\}$ is used as the set with three elements. The laws of synthesis in H are defined through the Cayley Table 11:

Table 11. General form of a 3-element magma's Cayley table

$\circ$	1	2	3
1	a_{11}	a_{12}	a_{13}
2	a_{21}	a_{22}	a_{23}
3	a_{31}	a_{32}	a_{33}

where the intersection of row *i* with column *j*, i.e., a_{ij}, is the result of i∘j. As in the case of hypergroups, in the left almost-hypergroups or right almost-hypergroups, the result of the hypercomposition of any two elements is non-void (see Theorem 4 below). Thus, the cardinality of the set of all possible magmas with 3 elements which are not partial hypergroupoids, is 7^9 = 40 353 607. As it is mentioned above in the commutative case the left inverted associativity and the right inverted associativity coincide with the associativity. Among the 40 353 607 magmas only 2520 are commutative hypergroups, which are the trivial cases of left almost-hypergroups and right almost-hypergroups. Thus, the package focuses on the non-trivial cases, that is on the non-commutative magmas. The enumeration reveals that there exist 65 955 reproductive non-commutative magmas which satisfy the left inverted associativity only and obviously the same number of reproductive non-commutative magmas which satisfy the right inverted associativity only. That is, there exist 65 955 non-trivial left almost-hypergroups and 65 955 non-trivial right almost-hypergroups. Moreover, there exist 7036 reproductive magmas which satisfy both the left and the right inverted associativity i.e., there exist 7036 both left and right almost-hypergroups. Furthermore, there are 16 044 reproductive magmas that satisfy the left inverted associativity, the right inverted associativity and the associativity. This means that there exist 16 044 structures which are simultaneously left almost-hypergroups, right almost-hypergroups and hypergroups. Finally, there do not exist any reproductive magmas which satisfies both the left (or right) inverted associativity and the associativity.

The following examples present the worth mentioning cases in which a hypercompositional structure is (a) both left and right almost-hypergroup and (b) simultaneously left almost-hypergroup, right almost-hypergroup and non-commutative hypergroup.

Example 6. *The hypercompositional structure described in Cayley Table 12 is both left and right almost-hypergroup.*

Table 12. A LA- and RA-hypergroup.

∘	1	2	3
1	{1}	{1,2}	{1,2,3}
2	{1,2}	{3}	{1,3}
3	{1,2,3}	{1,2,3}	{1,2,3}

Example 7. *The hypercompositional structure described in Cayley Table 13 is simultaneously left almost-hypergroup, right almost-hypergroup and non-commutative hypergroup.*

Table 13. A non-commutative hypergroup which is also left and right almost-hypergroup.

∘	1	2	3
1	{1}	{1}	{1,2,3}
2	{1}	{1}	{2,3}
3	{1,2,3}	{1,2,3}	{1,2,3}

A magma though, with three elements, can be isomorphic to another magma, which results from a permutation of these three elements. The isomorphic structures which appear in this way, can be considered as members of the same class. These classes can be constructed and enumerated, with the use of the techniques and methods which are developed in [78]. Having done so, the following conclusions occurred:

- The 65 955 non-trivial left almost-hypergroups are partitioned in 11 067 isomorphism classes. 10 920 of them consist of 6 members, 142 have 3 members, 4 have 2 members and the last one is a one-member class. The same are valid for the 65 955 non-trivial right almost-hypergroups.
- The 7036 both left and right almost-hypergroups are partitioned in 1174 isomorphism classes. 1172 of them consist of 6 members, while the other 2 have 2 members.
- The 16 044 noncommutative structures which are simultaneously left almost-hyper groups, right almost-hypergroups and hypergroups are partitioned in 2733 isomorphism classes. 2617 of them consist of 6 members, 110 have 3 members and the last 6 have 2 members.

The above results, combined with the ones of [78] for the hypergroups, are summarized and presented in the following Table 14:

Table 14. Classification of the LA-hypergroups and the RA-hypergroups with three elements.

	Total Number	Isomorphism Classes	Classes with 1 Member (Rigid)	Classes with 2 Members	Classes with 3 Members	Classes with 4 or 5 Members	Classes with 6 Members
non-trivial left almost-hypergroups	65 955	11 067	1	4	142	0	10 920
non-trivial right almost-hypergroups	65 955	11 067	1	4	142	0	10 920
non-trivial both left and right almost-hypergroups	7036	1174	0	2	0	0	1172
simultaneously left almost-hypergroups, right almost-hypergroups and non-commutative hypergroups	16 044	2733	0	6	110	0	2617
non-commutative hypergroups (satisfying the associativity only)	4628	800	0	1	56	0	723
simultaneously left almost-hypergroups and non-commutative hypergroups	0						
simultaneously right almost-hypergroups and non-commutative hypergroups	0						
commutative hypergroups (trivial left almost-hypergroups and trivial right almost-hypergroups)	2520	466	6	3	78	0	399
hypergroups	23 192	3999	6	10	244	0	3739

In the above table we observe that there is only one class of left almost-hypergroups and only one class of right almost-hypergroups which contains a single member. The member of this class is of particular interest, since its automorphism group is of order 1. Such hypercompositional structures are called *rigid*. Additionally, observe that there are 6 rigid hypergroups all of which are commutative. A study and enumeration of these rigid hypergroups, as well as other rigid hypercompositional structures is given in [81]. The following Table 15 presents the unique rigid left almost-hypergroup, while Table 16 describes the unique rigid right almost-hypergroup with 3 elements.

Table 15. The rigid left almost-hypergroup of three elements.

∘	1	2	3
1	{1,2,3}	{2,3}	{2,3}
2	{1,3}	{1,2,3}	{1,3}
3	{1,2}	{1,2}	{1,2,3}

Table 16. The rigid right almost-hypergroup of three elements.

∘	1	2	3
1	{1,2,3}	{1,3}	{1,2}
2	{2,3}	{1,2,3}	{1,2}
3	{2,3}	{1,3}	{1,2,3}

More generally, the next theorem is valid:

Theorem 3. *A non-void set H equipped with the hypercomposition:*

$$xy = \begin{cases} H, & \text{if } x = y \\ H \cdot \cdot \{x\}, & \text{if } x \neq y \end{cases}$$

becomes a rigid left almost-hypergroup, while equipped with the hypercomposition:

$$xy = \begin{cases} H, & \text{if } x = y \\ H \cdot \cdot \{y\}, & \text{if } x \neq y \end{cases}$$

becomes a rigid right almost-hypergroup. Both of them satisfy the transposition axiom.

Remark 4. *There exist 81 hypergroupoids of two elements, 3 of which are LA-hypergroups, 3 are RA-hypergroups, 4 are commutative hypergroups and 4 are non-commutative hypergroups. The enumeration of the two-element crisp and fuzzy hypergroups can be found in [82].*

4. Algebraic Properties

In hypercompositional algebra it is dominant that a hypercomposition on a set E is a mapping of $E \times E$ to the non-void subsets of E. In [4], it is shown that this restriction is not necessary, since it can be proved that the result of the hypercomposition is non void in many hypercompositional structures. Such is the hypergroup (see [4], Theorem 12, and [75], Property 1.1), the weakly associative magma ([4], Proposition 5) and consequently the H_V-group ([76], Proposition 3.1). The next theorem shows that in the case of the left/right almost hypergroups the result of the hypercomposition is non-void as well. The proof is similar to the one in [4], Theorem 12, but not the same due to the validity of the inverted associativity in these structures, instead of the associativity.

Theorem 4. *In a left almost-hypergroup or in a right almost-hypergroup H, the result of the hypercomposition of any two elements is non-void.*

Proof. Suppose that H is a left almost-hypergroup and $cb = \varnothing$, for some c, b in H. Because of the reproductivity the equalities $H = Hc = cH$ and $H = Hb = bH$ are valid. Then, the left inverted associativity gives:

$$H = Hc = (Hb)c = (cb)H = \varnothing H = \varnothing,$$

which is absurd. Next assume that H is a right almost-hypergroup and $bc = \varnothing$, for some b, c in H. The right inverted associativity gives:

$$H = cH = c(bH) = H(bc) = H\varnothing = \varnothing,$$

which is absurd. □

Proposition 8. *A left almost-hypergroup H is a hypergroup if and only if $a(bc) = (cb)a$ holds for all a, b, c in H.*

Proof. Let H be a hypergroup. Then, the associativity $(ab)c = a(bc)$ holds for all a, b, c in H. Moreover because of the assumption, $a(bc) = (cb)a$ is valid for all a, b, c in H. Therefore $(ab)c = (cb)a$, thus H is a left almost-hypergroup. Conversely now, suppose that $a(bc) = (cb)a$ holds for all a, b, c in H. Since H is a left almost-hypergroup, the sequence of the equalities $a(bc) = (cb)a = (ab)c$ is valid. Consequently, H is a hypergroup. $\square$

Theorem 5. *In any left or right almost-hypergroup H the following are valid:*

i. $a/b \neq \varnothing$ and $b \backslash a \neq \varnothing$, for all a, b in H,

ii. $H = H/a = a/H$ and $H = a \backslash H = H \backslash a$, for all a in H,

iii. *the non-empty result of the induced hypercompositions is equivalent to the reproductive axiom.*

Proof. (i) Because of the reproduction, $Hb = H$ for all $b \in H$. Hence, for every $a \in H$ there exists $x \in H$, such that $a \in xb$. Thus, $x \in a/b$ and, therefore, $a/b \neq \varnothing$. Similarly, $b \backslash a \neq \varnothing$.

(ii) Because of Theorem 4, the result of the hypercomposition in H is always a non-empty set. Thus, for every $x \in H$ there exists $y \in H$, such that $y \in xa$, which implies that $x \in y/a$. Hence, $H \subseteq H/a$. Moreover, $H/a \subseteq H$. Therefore, $H = H/a$. Next, let $x \in H$. Since $H = xH$, there exists $y \in H$ such that $a \in xy$, which implies that $x \in a/y$. Hence, $H \subseteq a/H$. Moreover, $a/H \subseteq H$. Therefore, $H = a/H$.

(iii) Suppose that $x/a \neq \varnothing$, for all $a, x \in H$. Thus, there exists $y \in H$, such that $x \in ya$. Therefore, $x \in Ha$, for all $x \in H$, and so $H \subseteq Ha$. Next, since $Ha \subseteq H$ for all $a \in H$, it follows that $H = Ha$. Per duality, $H = aH$. Conversely now, per Theorem 4, the reproductivity implies that $a/b \neq \varnothing$ and $b \backslash a \neq \varnothing$, for all a, b in H. $\square$

Proposition 9. *In any left or right almost-hypergroup H the following are valid:*

i. $a(b/c) \cup a/(c/b) \subseteq (ab)/c$ and

ii. $(c \backslash b)a \cup (b \backslash c) \backslash a \subseteq c \backslash (ba)$

for all a, b, c in H.

Proof. (i) Let $x \in a(b/c)$. Then, $b/c \cap a \backslash x \neq \varnothing$ (1). Next, if $x \in a/(c/b)$, then $a \in x(c/b)$ or $c/b \cap x \backslash a \neq \varnothing$ (2). From both (1) and (2) it follows that $xc \cap ab \neq \varnothing$. So, there exists $z \in ab$, such that $z \in xc$ which implies that $x \in z/c$. Hence, $x \in (ab)/c$. Thus (i) is valid. Similar is the proof of (ii). $\square$

Corollary 5. *If A, B, C are non-empty subsets of any left or right almost-hypergroup H, then the following are valid:*

i. $A(B/C) \cup A/(C/B) \subseteq (AB)/C$ and

ii. $(C \backslash B)A \cup (B \backslash C) \backslash A \subseteq C \backslash (BA)$.

Proposition 10. *Let a, b, c, d be arbitrary elements of any left or right almost-hypergroup H. Then the following are valid:*

i. $(b \backslash a)(c/d) \subseteq (b \backslash ac)/d \cap b \backslash (ac/d)$,

ii. $(b \backslash a)/(d/c) \subseteq (b \backslash ac)/d$,

iii. $(c \backslash d) \backslash (a/b) \subseteq d \backslash (ac/b)$.

Proof. (i) Let $x \in (b \backslash a)(c/d)$. Then, because of Proposition 9.i, there exists $y \in b \backslash a$, such that $x \in y(c/d) \subseteq (yc)/d$. Thus $xd \cap yc \neq \varnothing$ or $xd \cap (b \backslash a)c \neq \varnothing$. Because of Proposition 9.ii, it holds that: $(b \backslash a)c \subseteq b \backslash ac$. Therefore $x \in (b \backslash ac)/d$ (1). Next, since $x \in (b \backslash a)(c/d)$, there exists $z \in c/d$, such that $x \in (b \backslash a)z$. Because of Proposition 9.ii, the inclusion relation $(b \backslash a)z \subseteq b \backslash (a z)$ holds. Thus, $bx \cap az \neq \varnothing$ or equivalently $bx \cap a(c/d) \neq \varnothing$ or, because of Proposition 9.i, $bx \cap ac/d \neq \varnothing$. Therefore, $x \in b \backslash (ac/d)$ (2). Now (1) and (2) give (i).

(ii) Suppose that $x \in (b \backslash a)/(d/c)$. Then there exists $y \in b \backslash a$ such that $x \in y/(d/c) \subseteq (yc)/d$. Thus, $xd \cap yc \neq \varnothing$ or equivalently $xd \cap (b \backslash a)c \neq \varnothing$. But according to Proposition 9.ii, the inclusion $(b \backslash a)c \subseteq b \backslash ac$ holds. Hence, $x \in (b \backslash ac)/d$.

(iii) can be proved in a similar manner. $\square$

Corollary 6. *If A, B, C, D are non-empty subsets of any left or right almost-hypergroup H, then the following are valid:*

i. $(B \backslash A)(C/D) \subseteq (B \backslash AC)/D \cap B \backslash (AC/D),$
ii. $(B \backslash A)/(D/C) \subseteq (B \backslash AC)/D,$
iii. $(C \backslash D) \backslash (A/B) \subseteq D \backslash (AC/B).$

Proposition 11. *Let a, b be elements of a left or right almost-hypergroup H, then:*

i. $b \in (a/b) \backslash a$ *and*
ii. $b \in a/(b \backslash a).$

Proof. (i) Let $x \in a/b$. Then $a \in xb$. Hence, $b \in x \backslash a$. Thus, $b \in (a/b) \backslash a$. Therefore, (i) is valid. The proof of (ii) is similar. $\square$

Corollary 7. *If A, B are non-empty subsets of any left or right almost-hypergroup H, then:*

i. $B \subseteq (A/B) \backslash A$ *and*
ii. $B \subseteq A/(B \backslash A).$

Remark 5. *The above properties are consequences of the reproductive axiom and therefore are valid in both, the left and the right almost-hypergroups as well as in the hypergroups [39].*

Proposition 12.

i.   *In any left almost-hypergroup the following property is valid:*

$$(a/b)/c = (bc) \backslash a \quad (mixed\ left\ inverted\ associativity)$$

ii.   *In any right almost-hypergroup the following property is valid:*

$$c \backslash (b \backslash a) = a/(cb) \quad (mixed\ right\ inverted\ associativity).$$

Proof. (i) Let $x \in (a/b)/c$. Then the following sequence of equivalent statements is valid:

$$x \in (a/b)/c \Leftrightarrow xc \cap a/b \neq \varnothing \Leftrightarrow a \in (xc)b \Leftrightarrow a \in (bc)x \Leftrightarrow x \in (bc) \backslash a.$$

Similar is the proof of (ii). $\square$

Corollary 8.

i.   *If A, B, C are non-empty subsets of a left almost-hypergroup H, then:*

$$(A/B)/C = (BC) \backslash A.$$

ii.   *If A, B, C are non-empty subsets of a right almost-hypergroup H, then:*

$$C \backslash (B \backslash A) = A/(CB).$$

Proposition 13.

i.   *In any left almost-hypergroup the right inverted associativity of the induced hypercompositions is valid:*

$$b \backslash (a/c) = c \backslash (a/b).$$

ii.	*In any right almost-hypergroup the left inverted associativity of the induced hypercompositions is valid:*

$$(b \backslash a)/c = (c \backslash a)/b.$$

Proof. For (i) it holds that:

$$b \backslash (a/c) = \{x \in H \mid bx \cap a/c \neq \varnothing\} = \{x \in H \mid a \in (bx)c\} =$$
$$= \{x \in H \mid a \in (bx)c\} = \{x \in H \mid a/b \cap cx \neq \varnothing\} = c \backslash (a/b).$$

Regarding (ii) it is true that:

$$(b \backslash a)/c = \{x \in H \mid b \backslash a \cap xc \neq \varnothing\} = \{x \in H \mid a \in b(xc)\} =$$
$$= \{x \in H \mid a \in c(xb)\} = \{x \in H \mid c \backslash a \cap xb \neq \varnothing\} = (c \backslash a)/b.$$

This completes the proof. $\square$

Corollary 9.

i.	*If A, B, C are non-empty subsets of a left almost-hypergroup H, then:*

$$B \backslash (A/C) = C \backslash (A/B).$$

ii.	*If A, B, C are non-empty subsets of a right almost-hypergroup H, then:*

$$(B \backslash A)/C = (C \backslash A)/B.$$

5. Identities and Symmetric Elements

Let H be a non-void set endowed with a hypercomposition. An element e of H is called *right identity*, if $x \in x \cdot e$ for all x in H. If $x \in e \cdot x$ for all x in H, then e is called *left identity*, while e is called *identity* if it is both right and left identity. An element e of H is called *right scalar identity*, if $x = x \cdot e$ for all x in H. If $x = e \cdot x$ for all x in H, then e is called *left scalar identity*, while e is called *scalar identity* if it is both right and left scalar identity [4,13,18,88]. When a left (resp. right) scalar identity exists in H, then it is unique.

Example 8. *The Cayley Tables 17 and 18 describe left/right almost-hypergroups with left/right scalar identity.*

Table 17. Left almost-hypergroup with left scalar identity.

∘	1	2	3
1	{1}	{2}	{3}
2	{3}	{1,2,3}	{1,3}
3	{2}	{1,2}	{1,2,3}

Table 18. Right almost-hypergroup with right scalar identity.

∘	1	2	3
1	{1}	{3}	{2}
2	{2}	{1,2,3}	{1,3}
3	{3}	{1,2}	{1,2,3}

If the equality $e = ee$ is valid for an identity e, then e is called *idempotent identity* [4,39,88]. e is a *right strong identity*, if $x \in x \cdot e \subseteq \{e, x\}$ for all x in H while e is a *left strong identity*, if $x \in e \cdot x \subseteq \{e, x\}$. e is a *strong identity*, if it is right and left strong [4,32,34,39,88–90]. Note that the strong identity needs not be unique [32,34,39]. Both the scalar identity and the strong identity are idempotent identities.

Theorem 6. *If e is a strong identity in a left almost-hypergroup H, then*

$$x/e = e \backslash x = x, \text{ for all } x \in H - \{e\}$$

Proof. Suppose that $y \in x/e$. Then $x \in ye \subseteq \{y, e\}$. Consequently $y = x$. □

Let e be an identity element in H and x an element in H. Then, x will be called *right e-attractive*, if $e \in e \cdot x$, while it will be called *left e-attractive* if $e \in x \cdot e$. If x is both left and right e-attractive, then it will be called *e-attractive*. When there is no likelihood of confusion, e can be omitted. See [32] for the origin of the terminology.

Proposition 14. *In a left almost-hypergroup with idempotent identity e, $e \backslash e$ is the set of right e-attractive elements of H and e/e is the set of left e-attractive elements of H.*

Proof. Suppose that x is a right attractive element of H. Then $e \in ex$. Thus $x \in e \backslash e$. Additionally, if $x \in e \backslash e$, then $e \in ex$. Hence $e \backslash e$ consists of all the right attractive elements of H. The rest follows in a similar way. □

When the identity is strong and x is an attractive element, then $e \cdot x = x \cdot e = \{e, x\}$, while, if x is non-attractive, then $e \cdot x = x \cdot e = x$ is valid. In the case of a strong identity, the non-attractive elements are called *canonical*. See [32] for the origin of the terminology.

Proposition 15. *If x is not an e-attractive element in a left almost-hypergroup with idempotent identity e, then xe consists of elements that are not e-attractive.*

Proof. Suppose that $y \in xe$ and assume that y is attractive. Then

$$ye \subseteq (xe)e = (ee)x = ex.$$

Since $e \in ey$, it follows that $e \in ex$, which is absurd. □

Proposition 16. *If x is a right (resp. left) e-attractive element in a transposition left almost-hypergroup with idempotent identity e, then all the elements of xe are right (resp. left) e-attractive.*

Proof. Let $y \in xe$, then $x \in y/e$. Moreover, $x \in e \backslash e$. Thus $e \backslash e \cap y/e \neq \varnothing$, which implies that $ee \cap ey \neq \varnothing$. Therefore, $e \in ey$. □

Proposition 17. *If x is a right (resp. left) attractive element in a transposition left almost-hypergroup with idempotent identity e, then its right (resp. left) inverses are also right (resp. left) attractive elements.*

Proof. Since $e \in ex$, it follows that $x \in e \backslash e$. Moreover, if x' is a right inverse of x, then $e \in xx'$. Therefore, $x \in e/x'$. Consequently, $e \backslash e \approx e/x'$. Transposition gives $ee \approx ex'$ and since e is idempotent, $e \in ex'$. Thus, x' is right attractive. □

Corollary 10. *If x is not a right (resp. left) attractive element in a transposition left almost-hypergroup with idempotent identity e, then its right (resp. left) inverses are also not right (resp. left) attractive elements.*

Proposition 18. *If a left almost-hypergroup (resp. right almost-hypergroup) has a right scalar identity (resp. left scalar identity), then it is a hypergroup.*

Proof. Suppose that e is a right scalar identity in a left almost-hypergroup H. Then for any two elements b, c in H we have:

$$bc = (be)c = (ce)b = cb.$$

Hence H is commutative and therefore H is a hypergroup. Similar is the proof for the right almost-hypergroups. □

Proposition 19. *If a left almost-hypergroup or a right almost-hypergroup H has a strong identity e and if it consists only of attractive elements, then $x, y \in xy$.*

Proof. $(ex)y = \{e, x\}y = \{e, y\} \cup xy$ and $(yx)e = yx \cup \{e\}$. Since $(ex)y = (yx)e$ it derives that $yx \cup \{e\} = xy \cup \{e, y\}$. Similarly, $xy \cup \{e\} = yx \cup \{e, x\}$. Therefore:

$$xy \cup \{e\} = yx \cup \{e, x\} = xy \cup \{e, y\} \cup \{e, x\} = xy \cup \{e, x, y\}.$$

Hence, $x, y \in xy$. □

Corollary 11. *If a left almost-hypergroup or a right almost-hypergroup H has a strong identity e and if it consists of attractive elements only, then*

i. $x \in x/y$ and $x \in y \backslash x$, *for all $x, y \in H$*
ii. $x/x = x \backslash x = H$, *for all $x, y \in H$*

Proposition 20. *If a left almost-hypergroup or a right almost-hypergroup H has a strong identity e and if the relation $e \in bc$ implies that $e \in cb$, for all $b, c \in H$, then H is a hypergroup.*

Proof. For any two elements in H we have:

$$cb \subseteq \{e, c\}b = (ec)b = (bc)e = \{e\} \cup bc. \text{ (i)}$$

If $e \in bc$ then $cb \subseteq bc$. Moreover,

$$bc \subseteq \{e, b\}c = (eb)c = (cb)e = \{e\} \cup cb. \text{ (ii)}$$

According to the assumption if $e \in bc$, then $e \in cb$. Hence $bc \subseteq cb$. Thus $cb = bc$. Next, if $e \notin bc$ then $e \notin cb$ and (i) implies that $cb \subseteq bc$ while (ii) implies that $bc \subseteq cb$. Thus $cb = bc$. Consequently H is commutative and therefore H is a hypergroup. □

Proposition 21. *If a left almost-hypergroup or a right almost-hypergroup H has a strong identity e and if A is the set of its attractive elements, then*

$$A/A \subseteq A \text{ and } A\backslash A \subseteq A.$$

Proof. Since A is the set of the attractive elements, $e/e = A$ and $e\backslash e = A$ is valid. Therefore:

$$A/A = (e\backslash e)/(e/e) \subseteq (e\backslash ee)/e = (e\backslash e)/e$$

But $e\backslash e = e/e$, thus

$$(e\backslash e)/e = (e/e)/e = ee\backslash e = e\backslash e = A$$

Consequently $A/A \subseteq A$. Similarly, $A\backslash A \subseteq A$. $\quad\square$

An element x' is called *right e-inverse* or *right e-symmetric* of x, if there exists a right identity $x' \neq e$ such that $e \in x \cdot x'$. The definition of the *left e-inverse* or *left e-symmetric* is analogous to the above, while x' is called *e-inverse* or *e-symmetric* of x, if it is both right and left inverse with regard to the same identity e. If e is an identity in a left almost-hypergroup H, then the set of the left inverses of $x \in H$, with regard to e, will be denoted by $S_{el}(x)$, while $S_{er}(x)$ will denote the set of the right inverses of $x \in H$ with regard to e. The intersection $S_{el}(x) \cap S_{er}(x)$ will be denoted by $S_e(x)$.

Proposition 22. *If e is an identity in a left almost-hypergroup H, then*

$$S_{er}(x) = (e/x) \cdot \cdot \{e\} \text{ and } S_{er}(x) = (x\backslash e) \cdot \cdot \{e\}.$$

Proof. $y \in e/x$, if and only if $e \in yx$. This means that either $y \in S_{el}(x)$ or $y = e$, if x is right attractive. Hence, $e/x \subseteq \{e\} \cup S_{el}(x)$. The rest follows in a similar way. $\quad\square$

Corollary 12. *If $S_{el}(x) \cap S_{er}(x) \neq \varnothing$, $x \in H$, then $x\backslash e \cap e/x \neq \varnothing$.*

Proposition 23. *If H is a transposition left almost-hypergroup with an identity e and $z \in xy$, then:*
i. $ey \cap x'z \neq \varnothing$, *for all* $x' \in S_{el}(x)$,
ii. $xe \cap zy' \neq \varnothing$, *for all* $y' \in S_{er}(y)$.

Proof. $z \in xy$ implies that $x \in z/y$ and that $y \in x\backslash z$. Let $x' \in S_{el}(x)$ and $y' \in S_{er}(y)$. Then $e \in x'x$ and $e \in yy'$. Thus $x \in x'\backslash e$ and $y \in e\backslash y'$. Therefore $x'\backslash e \cap z/y \neq \varnothing$ and $x\backslash z \cap e/y' \neq \varnothing$. Hence, because of the transposition, $ey \cap x'z \neq \varnothing$ and $xe \cap zy' \neq \varnothing$. $\quad\square$

Proposition 24. *Let H be a transposition left almost-hypergroup with a strong identity e and let x, y, z be elements in H such that $x, y, z \neq e$ and $z \in xy$. Then:*
i.   *if* $S_{el}(x) \cap S_{el}(z) = \varnothing$, *then* $y \in x'z$, *for all* $x' \in S_{el}(x)$,
ii.   *if* $S_{er}(y) \cap S_{er}(z) = \varnothing$, *then* $x \in zy'$, *for all* $y' \in S_{er}(y)$.

Proof. (i) According to Proposition 23, $z \in xy$ implies that $ey \cap x'z \neq \varnothing$ for all $x' \in S_l(x)$. Since e is strong $ey = \{e, y\}$. Hence $\{e, y\} \cap x'z \neq \varnothing$. But $S_{el}(x)$ and $S_{el}(z)$ are disjoint. Thus $e \notin x'z$, therefore $y \in x'z$. Analogous is the proof of (ii). $\quad\square$

6. Substructures of the Left/Right Almost-Hypergroups

There is a big variety of substructures in the hypergroups, which is much bigger than the one in the groups. Analogous is the variety of the substructures which are revealed here in the case of the left/right almost-hypergroups. For the consistency of the terminology [4,13,33,55,91–99], the terms semisub-left/right almost-hypergroup, sub-

left/right almost-hypergroup, etc. will be used in exactly the same way as the prefixes sub- and semisub- are used in the cases of the groups and the hypergroups, e.g., the terms subgroup, subhypergroup are used instead of hypersubgroup, etc. The following research is inspired by the methods and techniques used in [93–97].

Let H be a left almost-hypergroup. Then,

Definition 12. *A non-empty subset K of H is called semisub-left-almost-hypergroup (semisub-LA-hypergroup) when it is stable under the hypercomposition, i.e., when it has the property $xy \subseteq K$ for all $x, y \in K$.*

Definition 13. *A semisub-LA-hypergroup K of H is a sub-left-almost-hypergroup (sub-LA-hypergroup) of H if it satisfies the reproductivity, i.e., if the equality $xK = Kx = K$ is valid for all $x \in K$.*

Example 9. *In the left almost-hypergroup, which is described in the Cayley Table 19, $\{5, 6, 7\}$ is a semisub-LA-hypergroup while $\{1, 2\}$, $\{3, 4\}$, $\{1, 2, 3, 4\}$, $\{1, 2, 5, 6, 7\}$ and $\{3, 4, 5, 6, 7\}$ are sub-LA-hypergroup.*

Table 19. Left almost-hypergroup.

	1	2	3	4	5	6	7
1	{1,2}	{1,2}	{1,2,3,4}	{1,2,3,4}	{1,2,5,6,7}	{1,2,5,6,7}	{1,2,5,6,7}
2	{1}	{1,2}	{1,2,3,4}	{1,2,3,4}	{1,2,5,6,7}	{1,2,5,6,7}	{1,2,5,6,7}
3	{1,2,3,4}	{1,2,3,4}	{3,4}	{4}	{3,4,5,6,7}	{3,4,5,6,7}	{3,4,5,6,7}
4	{1,2,3,4}	{1,2,3,4}	{3}	{3,4}	{3,4,5,6,7}	{3,4,5,6,7}	{3,4,5,6,7}
5	{1,2,5,6,7}	{1,2,5,6,7}	{3,4,5,6,7}	{3,4,5,6,7}	{5,7}	{7}	{6,7}
6	{1,2,5,6,7}	{1,2,5,6,7}	{3,4,5,6,7}	{3,4,5,6,7}	{6,7}	{7}	{7}
7	{1,2,5,6,7}	{1,2,5,6,7}	{3,4,5,6,7}	{3,4,5,6,7}	{6,7}	{6,7}	{6,7}

Proposition 25. *If a semisub-LA-hypergroup K of H is stable under the induced hypercompositions, then K is a sub-LA-hypergroup of H.*

Proof. We have to prove the reproductivity. Let $x \in K$. Obviously $xK \subseteq K$ and $Kx \subseteq K$. Next let y be an element of K. Then $x \backslash y = \{z \in H \mid y \in xz\}$ is a subset of H. Therefore, there exists an element $z \in K$ such that $y \in xz \subseteq xK$. Thus $K \subseteq xK$. Similarly, $y/x \subseteq K$ yields $K \subseteq Kx$. □

Proposition 26. *If K is a sub-LA-hypergroup of H, then $H \cdot \cdot K \subseteq (H \cdot \cdot K)s$ and $H \cdot \cdot K \subseteq s(H \cdot \cdot K)$, for all $s \in K$.*

Proof. Let r be an element in $H \cdot \cdot K$ which does not belong to $(H \cdot \cdot K)s$. Because of the reproductive axiom, $r \in Hs$ and since $r \notin (H \cdot \cdot K)s$, r must be a member of Ks. Thus, $r \in Ks \subseteq KK = K$. This contradicts the assumption and so $H \cdot \cdot K \subseteq (H \cdot \cdot K)s$. The second inclusion follows similarly. □

Proposition 27. *If K is a sub-LA-hypergroup of H, $A \subseteq K$ and $B \subseteq H$, then*

i.   *$A(B \cap K) \subseteq AB \cap K$ and*
ii.   *$(B \cap K)A \subseteq BA \cap K$.*

Proof. Let $t \in A(B \cap K)$. Then $t \in ax$, with $a \in A$ and $x \in B \cap K$. Since x lies in $B \cap K$, it derives that $x \in B$ and $x \in K$. Hence $ax \subseteq aB$ and $ax \subseteq aK = K$. Thus $ax \subseteq AB \cap K$ and therefore $t \in AB \cap K$. The second inclusion follows similarly. □

The next definition introduces the notion of the closed sub-LA/RA-hypergroups. The significance of the closed subhypergroups which is mentioned in [4,13,33,55,91–96,99] remains exactly the same in the case of the sub-LA/RA-hypergroups.

Definition 14. *A sub-LA-hypergroup K of H is called right closed in H if it is stable under the right induced hypercomposition, that is if $a/b \subseteq K$ for any two elements a and b in K. Similarly, K is called left closed if $b \backslash a \subseteq K$, for all $a, b \in K$. K is closed when it is both right and left closed.*

Proposition 28.

i. *K is a right closed sub-LA-hypergroup of H, if and only if $xK \cap K = \varnothing$, for every $x \in H \cdot \cdot K$*
ii. *K is a left closed sub-LA-hypergroup of H, if and only if $Kx \cap K = \varnothing$, for every $x \in H \cdot \cdot K$*
iii. *K is a closed sub-LA-hypergroup of H, if and only if $xK \cap K = \varnothing$ and $Kx \cap K = \varnothing$, for every $x \in H \cdot \cdot K$*

Proof. (i) Suppose that $xK \cap K \neq \varnothing$ for some $x \in H$. So there exist $y, z \in K$ such that $y \in xz$ or equivalently $x \in y/z$. Since K is right closed $y/z \subseteq K$, therefore $x \in K$. Conversely now. Let $y, z \in K$. Then for every $x \in y/z$ we have that $y \in xz$ or equivalently $xK \cap K \neq \varnothing$. Consequently $x \in K$, thus $y/z \subseteq K$. The proof of (ii) is similar, while (iii) derives directly from (i) and (ii). $\square$

Proposition 29.

i. *A sub-LA-hypergroup K of H is right closed in H, if and only if*

$$(H \cdot \cdot K)s = H \cdot \cdot K, \text{ for all } s \in K.$$

ii. *A sub-LA-hypergroup K of H is left closed in H, if and only if*

$$s(H \cdot \cdot K) = H \cdot \cdot K, \text{ for all } s \in K.$$

iii. *A sub-LA-hypergroup K of H is closed in H, if and only if*

$$s(H \cdot \cdot K) = (H \cdot \cdot K)s = H \cdot \cdot K, \text{ for all } s \in K.$$

Proof. (i) Let K be right closed in H. Suppose that z lies in $H \cdot \cdot K$ and assume that $zs \cap K \neq \varnothing$. Then, there exists an element y in K such that $y \in zs$, or equivalently, $z \in y/s$. Therefore $z \in K$, which is absurd. Hence $(H \cdot \cdot K)s \subseteq H \cdot \cdot K$. Next, because of Proposition 26, $H \cdot \cdot K \subseteq (H \cdot \cdot K)s$ and therefore $H \cdot \cdot K = (H \cdot \cdot K)s$. Conversely now. Suppose that $(H \cdot \cdot K)s = H \cdot \cdot K$ for all $s \in K$. Then $(H \cdot \cdot K)s \cap K = \varnothing$ for all $s \in K$. Hence $x \notin rs$ and so $r \notin x/s$ for all $x, s \in K$ and $r \in H \cdot \cdot K$. Therefore $x/s \cap (H \cdot \cdot K) = \varnothing$ which implies that $x/s \subseteq K$. Thus K is right closed in H. (ii) follows in a similar way and (iii) is an obvious consequence of (i) and (ii). $\square$

Proposition 30.

i. *If K is a sub-LA-hypergroup of H and*

$$Kx \cap (H \cdot \cdot K)x = \varnothing$$

for all $x \in H$, then K is right closed in H.
ii. *If K is a sub-LA-hypergroup of H and*

$$xK \cap x(H \cdot \cdot K) = \varnothing$$

for all $x \in H$, then K is left closed in H.

Proof. (i) From the reproductive axiom we have:

$$K \cup (H \cdot \cdot K) = H = Hx = Kx \cup (H \cdot \cdot K)x$$

According to the hypothesis $Kx \cap (H \cdot \cdot K)x = \varnothing$, which implies that $K \cap (H \cdot \cdot K)x = \varnothing$ when $x \in K$. Therefore, $H = K \cup (H \cdot \cdot K)x$ is a union of disjoint sets. Thus $(H \cdot \cdot K)x = H \cdot \cdot K$. So, per Proposition 29, K is right closed in H. Similar is the proof of (ii). $\square$

Proposition 31.

i. If K is a right closed sub-LA-hypergroup in H, $A \subseteq K$ and $B \subseteq H$, then

$$(B \cap K)A = BA \cap K.$$

ii. If K is a left closed sub-LA-hypergroup in H, $A \subseteq K$ and $B \subseteq H$, then

$$A(B \cap K) = AB \cap K.$$

Proof. (i) Let $t \in BA \cap K$. Since K is right closed, for any element y in $B \cdot \cdot K$, it is valid that $yA \cap K \subseteq yK \cap K = \varnothing$. Hence $t \in (B \cap K)A \cap K$. But $(B \cap K)A \subseteq KK = K$. Thus $t \in (B \cap K)A$. Therefore $BA \cap K \subseteq (B \cap K)A$. Next the inclusion becomes equality because of Proposition 27. (ii) derives in a similar way. $\square$

Proposition 32.

i. If K is a right closed sub-LA-hypergroup in H, $A \subseteq K$ and $B \subseteq H$, then

$$(B \cap K)/A = (B/A) \cap K.$$

ii. If K is a left closed sub-LA-hypergroup in H, $A \subseteq K$ and $B \subseteq H$, then

$$(B \cap K)\backslash A = B\backslash A \cap K.$$

Proof. (i) Since $B \cap K \subseteq B$, it derives that $(B \cap K)/A \subseteq B/A$. Moreover $A \subseteq K$ and $B \cap K \subseteq K$, thus $(B \cap K)/A \subseteq K$. Hence $(B \cap K)/A \subseteq (B/A) \cap K$. For the reverse inclusion now suppose that $x \in (B/A) \cap K$. Then, there exist $a \in A$ and $b \in B$ such that $x \in b/a$ or equivalently $b \in ax$. Since $ax \subseteq K$ it derives that $b \in K$ and so $b \in B \cap K$. Therefore $b/a \subseteq (B \cap K)/A$. Thus $x \in (B \cap K)/A$. Hence $(B/A) \cap K \subseteq (B \cap K)/A$. (ii) derives in a similar way. $\square$

Although the non-void intersection of two sub-LA-hypergroups is stable under the hypercomposition, it is usually not a sub-LA-hypergroup since the reproductive axiom is not always valid in it.

Proposition 33. *The non-void intersection of any two closed sub-LA-hypergroups of H is a closed sub-LA-hypergroup of H.*

Proof. Let K, M be two closed LA-subhypergroups of H and suppose that x, y are two elements in $K \cap M$. Then $xy \subseteq K$ and $xy \subseteq M$. Therefore $xy \subseteq K \cap M$. Next, since K, M are closed LA-subhypergroups of H, x/y and $y\backslash x$ are subsets of $K \cap M$. Thus, because of Proposition 25, $K \cap M$ is a closed LA-subhypergroup of H. $\square$

Corollary 13. *The set of the closed sub-LA-hypergroups of H which are containing a non-void subset of H, is a complete lattice.*

Proposition 34. *If K is a closed sub-LA-hypergroups of H and $x \in K$, then:*

$$x/K = K/x = K = K\backslash x = x\backslash K$$

Proof. Since K is closed $x/K \subseteq K$ and $K/x \subseteq K$. Let $y \in K$. Because of the reproductivity $x \in yK$ or equivalently $y \in x/K$. Therefore $x/K = K$. Moreover since K is a LA-subhypergroups of H, $yx \subseteq K$. Thus $y \in K/x$. So $K/x = K$. The equalities $K = K\backslash x = x\backslash K$ follow in a similar way. $\square$

Corollary 14. *In any left almost-hypergroup, K is a closed sub-LA-hypergroup if and only if:*

$$K/K = K\backslash K = K$$

Definition 15. *A sub-LA-hypergroup M of H is called right invertible if $x/y \cap M \neq \varnothing$ implies $y/x \cap M \neq \varnothing$, while it is called left invertible if $y\backslash x \cap M \neq \varnothing$ implies $x\backslash y \cap M \neq \varnothing$. If M is right and left invertible, then it is called invertible.*

Direct consequence of the above definition is the following proposition:

Proposition 35.

i.   M is a right invertible sub-LA-hypergroup of H, if and only if:

$$x \in My \Rightarrow y \in Mx, \ x, y \in H$$

ii.   M is a left invertible sub-LA-hypergroup of H, if and only if:

$$x \in yM \Rightarrow y \in xM, \ x, y \in H$$

Proposition 36. *If K is an invertible sub-LA-hypergroup of H, then K is closed.*

Proof. Let $x \in K/K$. Then $K \cap xK \neq \varnothing$ and $x\backslash K \cap K \neq \varnothing$. Since K is invertible $K\backslash x \cap K \neq \varnothing$. Thus $x \in KK$. But K is a LA-subhypergroup of H, so $KK = K$. Therefore $x \in K$. Hence $K/K \subseteq K$. Similarly, $K\backslash K \subseteq K$ and so the Proposition. $\square$

Definition 16. *If H has an identity e, then a sub-LA-hypergroup K of H is called symmetric if $x \in K$ implies $S_{el}(x) \cup S_{er}(x) \subseteq K$.*

Proposition 37. *To any pair of symmetric sub-LA-hypergroups K and M of a LA-hypergroup H there exists a least symmetric sub-LA-hypergroup $K \vee M$ containing them both.*

Proof. Let U be the set of all symmetric subhypergroups R of H which contain both K and M. The intersection of all these symmetric subhypergroups R of H is a symmetric subhypergroup with the desired property. $\square$

7. Fortification in Transposition Left Almost-Hypergroups

The transposition left almost-hypergroups can be fortified through the introduction of neutral elements. Next, we will present two such hypercompositional structures.

Definition 17. *A transposition LA-hypergroup H, is left fortified if it contains an element e which satisfies the axioms,*

i.   e is a left identity and $ee = e$

ii.   for every $x \in H \cdot \cdot \{e\}$ there exists a unique $y \in H \cdot \cdot \{e\}$ such that $e \in xy$ and $e \in yx$

For $x \in H \cdot \cdot \{e\}$ the notation x^{-1} is used for the unique element of $H \cdot \cdot \{e\}$ that satisfies axiom (ii). Clearly $\left(x^{-1}\right)^{-1} = x$. The next results are obvious.

Proposition 38.
i　　If $x \neq e$, then $e \backslash x = x$.
ii　　$e \backslash e = H$.

Proposition 39. *Let $x \in H \cdot \cdot \{e\}$. Then $e \in xy$ or $e \in yx$ implies $y \in \left\{x^{-1}, e\right\}$.*

Now the role of the identity in the transposition left almost-hypergroups can be clarified.

Proposition 40. *The identity e of the transposition left almost-hypergroup is left strong.*

Proof. It suffices to prove that $ex \subseteq \{e, x\}$. For $x = e$ the inclusion is valid. Let $x \neq e$. Suppose that $y \in ex$. Then $x \in e \backslash y$. But $e \in xx^{-1}$, hence $x \in e/x^{-1}$. Thus, $(e \backslash y) \cap \left(e/x^{-1}\right) \neq \varnothing$. The transposition axiom gives $e = ee = yx^{-1}$. By the previous proposition, $y \in \{e, x\}$. Therefore, the proposition holds.　□

Proposition 41. *The identity e of the transposition left almost-hypergroup is unique.*

Proof. Suppose that u is an identity distinct from e. Then, there would exist the inverse of e, i.e., an element v distinct from u such that $u \in ev$, which is absurd because $ev \subseteq \{e, v\}$. □

Proposition 42. *If H consists of attractive elements only and $e \neq x$ then*

$$e/x = ex^{-1} = \left\{e, x^{-1}\right\} = x \backslash e.$$

Proof. Since e is a left strong identity, $ex^{-1} = \left\{e, x^{-1}\right\}$ is valid. Moreover

$$e/x = \{z \in H \mid e \in zx\} = \left\{e, x^{-1}\right\} \text{ and } x \backslash e = \{z \in H \mid e \in xz\} = \left\{e, x^{-1}\right\}. □$$

Corollary 15. *If A is a non-empty subset of H and $e \notin A$ then:*

$$e/A = eA^{-1} = A^{-1} \cup \{e\} = A \backslash e$$

Definition 18. *A transposition polysymmetrical left almost-hypergroup is a transposition left almost-hypergroup P that contains a left idempotent identity e which satisfies the axiom:*

for every $x \in P \cdot \cdot \{e\}$ there exists at least one element $x' \in P \cdot \cdot \{e\}$, a symmetric of x , such that $e \in x \, x'$ which furthermore satisfies $e \in x'x$.

The set of the symmetric elements of x is denoted by $S(x)$ and it is called the symmetric set of x.

Example 10. *Cayley Table 20 describes a transposition polysymmetrical left almost-hypergroup in which, the element 1 is a left idempotent identity.*

Table 20. Transposition polysymmetrical left almost-hypergroup.

○	1	2	3
1	{1}	{1,2}	{1,3}
2	{1,3}	{1,2,3}	{1,3}
3	{1,2}	{1,2}	{1,2,3}

Proposition 43. *$e\backslash x$ always contains the element x.*

Corollary 16. *If X is non-empty, then $X \subseteq e\backslash X$.*

Proposition 44. *Let $x \neq e$. Then*

i.   *$S(x) \cup \{e\} = e/x$ and $S(x) = x\backslash e$, if x is attractive*
ii.   *$eS(x) = e/x$, if e is left strong identity and x is attractive*
iii. *$S(x) = x\backslash e = e/x$, if x is non attractive.*

Corollary 17. *Let X be a non-empty set and $e \notin X$. Then*

i.   *$S(X) \cup \{e\} = e/X$ and $S(X) = X\backslash e$, if X contains an attractive element*
ii.   *$eS(X) = e/X$, if e is left strong identity and X contains an attractive element*
iii. *$S(X) = X\backslash e = e/X$, if X consists of non-attractive elements.*

Proposition 45. *For every element x of a transposition polysymmetrical left almost-hypergroup it holds $ex \subseteq \{e\} \cup S(S(x))$, while for every $x' \in S(x)$ it holds: $(ex) \cap S(x') \neq \varnothing$.*

Proof. Let $y \neq e$ and $y \in ex$, then $x \in e\backslash y$. Moreover, for every $x' \in S(x)$ it holds $x \in e/x'$. Consequently $e/x' \cap e\backslash y \neq \varnothing$, so, per transposition axiom, $ee \cap yx' \neq \varnothing$, that is $e \in yx'$ and thus $y \in S(x') \subseteq S(S(x))$.   □

Proposition 46. *If $x \neq e$, is a right attractive element of a transposition polysymmetrical left almost-hypergroup, then $S(x)$ consists of left attractive elements.*

Proof. Let $e \in ex$. Then $x \in e\backslash e$. Moreover, if x' is an arbitrary element from $S(x)$, then $e \in xx'$. Therefore $x \in e/x'$. Consequently $e\backslash e \cap e/x' \neq \varnothing$. Per transposition axiom $ee \cap x'e \neq \varnothing$. So $e \in x'e$. Thus x' is an attractive element.   □

Corollary 18. *If x is a non-attractive element, then $S(x)$ consists of non-attractive elements only.*

Corollary 19. *If $x \neq e$, is an attractive element of a transposition polysymmetrical left almost-hypergroup, then $S(x)$ consists of attractive elements only.*

Proposition 47.

i.   *If x is a right attractive element of a transposition polysymmetrical left almost-hypergroup, then all the elements of xe are right attractive.*
ii.   *If x is a left attractive element of a transposition polysymmetrical left almost-hypergroup, then all the elements of ex are left attractive.*

Proof. Assuming that x is a right attractive element we have that $e \in ex$, which implies that $x \in e\backslash e$. Additionally, if z is an element in xe, then $x \in z/e$. Thus $(z/e) \cap (e\backslash e) \neq \varnothing$. Per

transposition axiom, $(ee) \cap (ez) \neq \varnothing$ and therefore $e \in ez$, i.e., z is right attractive. Similar is the proof of (ii). $\square$

8. Conclusions and Open Problems

In [70,93] and later in [4], with more details, it was proved that the group can be defined with the use of two axioms only: the associativity and the reproductivity. Likewise, the left/right almost-group is defined here and their existence is proved via examples. The study of this structure reveals a very interesting research area in abstract algebra.

This paper focuses on the study of the more general structure, i.e., of the left/right almost-hypergroup. The enumeration of these structures showed that they appear more frequently than the hypergroups. Indeed, in the case of the hypergroupoids with three elements, there exists one hypergroup in every 1740 hypergroupoids, while there is one non-trivial purely left almost-hypergroup in every 612 hypergroupoids. The same holds for the non-trivial purely right almost-hypergroups, as it is proved in this paper that the cardinal number of the left almost-hypergroups is equal to the cardinal number of the right almost-hypergroups, over a set E. Moreover, there is one non-trivial left and right almost-hypergroup in every 5735 hypergroupoids. Considering the trivial cases as well, i.e., left and right almost-hypergroups which are also non-commutative hypergroups, there exists one left almost-hypergroup in every 453 hypergroupoids. This frequency, which is nearly 4 times higher than that of the hypergroups, justifies a more thorough study of these structures.

Subsequently, these structures were equipped with more axioms, the first one of which is the transposition axiom:

$$b\backslash a \cap c/d \neq \varnothing \text{ implies } ad \cap bc \neq \varnothing, \text{ for all } a,b,c,d \in H$$

The transposition left/right almost-hypergroup is studied here.

In [4] though, the reverse transposition axiom was introduced:

$$ad \cap bc \neq \varnothing \text{ implies } b\backslash a \cap c/d \neq \varnothing, \text{ for all } a,b,c,d \in H$$

The study of the reverse transposition left/right almost-hypergroup is an open problem.

Additionally, open problems for algebraic research are the studies of the properties of all the structures which are introduced in this paper (weak left/right almost-hypergroup, left/right almost commutative hypergroup, join left/right almost hypergroup, reverse join left/right almost hypergroup, weak left/right almost commutative hypergroup) as well as their enumerations. Especially, for the enumeration problems, it is worth mentioning those, which are associated with the rigid hypercompositional structures, that is hypercompositional structures whose automorphism group is of order 1. The conjecture is that there exists only one rigid left almost-hypergroup (and one rigid right almost-hypergroup), while it is known that there exist six such hypergroups, five of which are transposition hypergroups [81].

Appendix A

The following is the Mathematica [100] package that implements the axioms of left inverted associativity, right inverted associativity, associativity and reproductivity for testing whether a magma is a LA-hypergroup, a RA-hypergroup, a LRA-hypergroup or a non-commutative hypergroup.

```
BeginPackage["LRHtest`"];
Clear["LRHtest`*"];


LRHtest::usage = "LRHtest[groupoid] returns
                    1: for Left Asso
                    2: for Right Asso
                    3: for Left+Right Asso
                    4: for Hypergroup
                    5: for Hypergroup + Left Asso
                    6: for Hypergroup + Right Asso
                    7: for Hypergroup + Left+Right Asso"


Begin["`Private`"];
Clear["LRHtest`Private`*"];

LRHtest[groupoid_List] := Module[{r}, r = 0;
   If[groupoid != Transpose[groupoid] && ReproductivityTest[groupoid],
     If[LeftAs[groupoid], r = 1];
     If[RightAs[groupoid], r = r + 2];
     If[Asso[groupoid], r = r + 4]]; Return[r]];


LeftAs[groupoid_List] :=
  Not[MemberQ[
    Flatten[Table[
      Union[Flatten[
        Union[Extract[groupoid,
          Distribute[{groupoid[[i, j]], {k}}, List]]]]] ==
        Union[Flatten[
          Union[Extract[groupoid,
            Distribute[{groupoid[[k, j]], {i}}, List]]]]], {i, 1,
        Length[groupoid]}, {j, 1, Length[groupoid]}, {k, 1,
        Length[groupoid]}], 2], False]];

RightAs[groupoid_List] :=
  Not[MemberQ[
    Flatten[Table[
      Union[Flatten[
        Union[Extract[groupoid,
          Distribute[{{i}, groupoid[[j, k]]}, List]]]]] ==
        Union[Flatten[
          Union[Extract[groupoid,
            Distribute[{{k}, groupoid[[j, i]]}, List]]]]], {i, 1,
        Length[groupoid]}, {j, 1, Length[groupoid]}, {k, 1,
        Length[groupoid]}], 2], False]];

Asso[groupoid1_List] :=
```

```
   Not[MemberQ[
     Flatten[Table[
       Union[Flatten[
         Union[Extract[groupoid1,
           Distribute[{groupoid1[[i, j]], {k}}, List]]]]] ==
       Union[Flatten[
         Union[Extract[groupoid1,
           Distribute[{{i}, groupoid1[[j, k]]}, List]]]]], {i, 1,
       Length[groupoid1]}, {j, 1, Length[groupoid1]}, {k, 1,
       Length[groupoid1]}], 2], False]];

ReproductivityTest[groupoid_List] :=
  Min[Table[
      Length[Union[Flatten[Transpose[groupoid][[j]]]]], {j, 1,
       Length[groupoid]}]] == Length[groupoid] &&
   Min[Table[
      Length[Union[Flatten[groupoid[[j]]]]], {j, 1,
       Length[groupoid]}]] == Length[groupoid];

End[];
EndPackage[];

Use of the package:
for checking a magma, for instance the following one:

{1}      {1}      {2,3}
{1}      {1}      {2,3}
{1,2,3}  {2,3}    {1,3}

write in Mathematica:

In[1]:=LRHtest[{{{1}, {1}, {2, 3}}, {{1}, {1}, {2, 3}}, {{1, 2, 3}, {2,3}, {1, 3}}}]

And the output is:

Out[1]=2

where number 2 corresponds to «RA-hypergroup»
```

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