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TRANSPOSITION HYPERGROUPS WITH IDEMPOTENT IDENTITY

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ABSTRACT. This paper studies transposition hypergroups T with idempotent identity e. Some of their fundamental properties are presented. Hypergroups are strictly e-regular, if for each x in T the set of the left inverses is equal to the set of the right inverses. The elements of these hypergroups are separated into two classes: the set $A = \{x \in T | e \in ex = xe\}$, including e, of attractive elements and the set of non-attractive elements. A study of these elements is also conducted.

1. Introduction

In 1934 F. Marty, in order to study problems in non-commutative algebra, such as cosets determined by non-invariant subgroups, generalized the notion of the group, thus defining the *hypergroup* [25, 26, 27]. An *operation* or *composition* in a non void set H is a function from $H \times H$ to H, while a *hyperoperation* or *hypercomposition* is a function from $H \times H$ to the powerset $P(H)$ of H. An algebraic structure that satisfies the axioms

i. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for every $a, b, c \in H$ (associativity) and
ii. $a \cdot H = H \cdot a = H$ for every $a \in H$ (reproductivity).

is called *group* if "·" is a composition [42] and *hypergroup* if "·" is a hypercomposition [25]. When there is no likelihood of confusion "·" can be omitted. If A and B are subsets of H, then AB signifies the union $\bigcup_{(a,b)\in A\times B} ab$, in particular $A = \emptyset \vee B = \emptyset \Leftrightarrow AB = \emptyset$. Ab and aB have the same meaning as $A\{b\}$ and $\{a\}B$. In general, the singleton $\{a\}$ is identified with its member a. In [25] F. Marty also defined the two *induced hypercompositions* (right and left division) that follow from the hypercomposition of the hypergroup, i.e.

$$\frac{a}{|b} = \{x \in H | a \in xb\} \text{ and } \frac{a}{b|} = \{x \in H | a \in bx\}$$

It is obvious that, if the hypergroup is commutative, then the two induced hypercompositions coincide. For the sake of notational simplicity, a/b or $a : b$ is used to denote the right division (as well as the division in commutative hypergroups) and $b\backslash a$ or $a..b$ is used to denote the left division [17, 31].

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F. Marty's life was short, as he died in a military mission during World War II. [25, 26, 27] are the only works on hypergroups he left behind. However, several relevant papers by other authors began appearing shortly thereafter (e.g. Krasner [20, 21], Kuntzmann [23] etc). It is worth mentioning here that the hypergroup, which is a very general structure, was progressively enriched with additional axioms, either more or less powerful, thus creating a significant number of specific hypergroups e.g. [2, 3, 12, 17, 18, 30, 34, 36, 37, 48, 49, 54, 55, 56, 57, 68, 69]. Moreover some of these hypergroups constituted a constructive origin for the development of other new hypercompositional structures (e.g. see [22, 28, 39, 51, 53, 58]). Thus, W. Prenowitz enriched hypergroups with an axiom, in order to use them in the study of geometry [61, 62, 63, 64, 65]. More precisely, he introduced into the commutative hypergroup, the *transposition axiom*

$$a/b \cap c/d \neq \emptyset \text{ implies } ad \cap bc \neq \emptyset \text{ for all } a, b, c, d \in H$$

and named this new hypergroup *join space* [64, 65]. Prenowitz was followed by others, such as J. Jantosciak [16], D. Freni [12, 13], Ch. Massouros [29, 31, 32, 33] etc. For the sake of terminology unification, join spaces are also called *join hypergroups* and numerous authors wrote dozens of papers on the study of this structure (e.g. R. Ameri [1], J. Chvalina [9, 10], P. Corsini [4, 5, 6], I. Cristea [6, 11], Š. Hošková [10, 14], A. Iranmanesh [15], A. Kehagias [19], V. Leoreanu [5, 24], Ch. Massouros [34, 38, 67], G. Massouros [44, 45, 49, 50], I. Mittas [49], J. Nieminen [59, 60], I. Rosenberg [66], Ch. Tsitouras [67], M. M. Zahedi [1] etc. For an extensive bibliography on this issue see [7] and [8]). It has been proven that these hypergroups also comprise a useful tool in the study of languages and automata [44, 45, 47, 52]. Later on, J. Jantosciak generalized the above axiom in an arbitrary hypergroup as follows:

$$b\backslash a \cap c/d \neq \emptyset \text{ implies } ad \cap bc \neq \emptyset \text{ for all } a, b, c, d \in H.$$

He named this particular hypergroup *transposition hypergroup* [17]. Subsequently, in [40], this axiom was also introduced into H_V-groups and, therefore, *transposition H_V-groups* were defined. Clearly, if A, B, C and D are subsets of H, then $B\backslash A \cap C/D \neq \emptyset$ implies that $AD \cap BC \neq \emptyset$. In what follows, the relational notation $A \approx B$(read A meets B) is used to assert that sets A and B have a non-void intersection. The study of transposition hypergroups is not as extensive as that of join hypergroups. This paper contributes in this direction by analyzing transposition hypergroups with idempotent identity.

2. **Algebraic Calculus**

Consequences of the hypergroup's definition axioms are [29, 31, 41, 42]:

i. $ab \neq \emptyset$, for all a, b in H
ii. $a/b \neq \emptyset$ and $a\backslash b \neq \emptyset$, for all a, b in H
iii. $H = H/a = a/H$ and $H = a\backslash H = H\backslash a$, for all a in H
iv. the nonempty result of the induced hypercompositions is equivalent to the reproductive axiom.

It has been proven in [17, 31] that in any hypergroup the following properties are valid:

Proposition 2.1. *In any hypergroup*

i. $(a/b)/c = a/(cb)$ *and* $c\backslash(b\backslash a) = (bc)\backslash a$ *(mixed associativity)*
ii. $(b\backslash a)/c = b\backslash(a/c)$
iii. $b \in (a/b)\backslash a$ *and* $b \in a/(b\backslash a)$

Corollary 2.2. *In any hypergroup*H*, if* A, B, C *are non-empty subsets of* H*, then:*

i. $(A/B)/C = A/(CB)$ *and* $C\backslash(B\backslash A) = (BC)\backslash A$
ii. $(B\backslash A)/C = B\backslash(A/C)$
iii. $B \subseteq (A/B)\backslash A$ *and* $B \subseteq A/(B\backslash A)$

An element e is called *right identity*, if $x \in x \cdot e$ for all x in H. If $x \in e \cdot x$ for all x in H, then x is called *left identity*, while x is called *identity* if it is both right and left identity. If equality $e = ee$ is valid for an identity e, then e is called *idempotent identity*. A hypergroup H is called *semi-regular*, if every $x \in H$ has at least one right and one left identity.

Proposition 2.3. *If* e *is a right identity in* H*, then* $x \in x/e$ *for all* $x \in H$ *and if* e *is a left identity in* H*, then* $x \in e\backslash x$ *for all* $x \in H$*. If* e *is an identity in* H*, then* $x \in x/e$ *and* $x \in e\backslash x$ *for all* $x \in H$*.*

Corollary 2.4. *If* X *is a non-empty subset of* H*, then* $X \subseteq X/e$*, if* e *is a right identity in* H *and* $X \subseteq e\backslash X$ *if* e *is a left identity in* H*.* $X \subseteq X/e$ *and* $X \subseteq e\backslash X$ *are valid when* e *is an identity in* H

If $x = x \cdot e = e \cdot x$ for all x in H, then e is a *scalar identity*. When a scalar identity exists in H, then it is unique. An identity e is a *strong identity*, if $e \in x \cdot e = e \cdot x \subseteq \{e, x\}$ for all x in H. The strong identity need not be unique [18]. Both scalar and strong identities are idempotent identities.

Proposition 2.5. *If* e *is a strong identity in* H *and* $x \neq e$*, then* $x/e = e\backslash x = x$*.*

Proof. Let $t \in x/e$. Then $x \in te \subseteq \{t, e\}$. Since $x \neq e$, it follows that $t = x$. Thus $x/e = x$. Similarly, it can be proven that $e\backslash x = x$. □

Corollary 2.6. *If* e *is a strong identity in* H *and* X *is a non-empty subset of* H*not containing* e*, then* $X/e = e\backslash X = X$*.*

Proposition 2.7. *If* e *is a scalar identity in* H*, then* $x/e = e\backslash x = x$*.*

Corollary 2.8. *If* X *is a non-empty subset of* H *and if* e *is a scalar identity in* H*, then* $X/e = e\backslash X = X$*.*

A subset h of H is called a *subhypergroup* of H, if $xh = hx = h$ for all $x \in h$. A subhypergroup h of H is *central* if $xy = yx$ for all $x \in h$ and $y \in H$.

Proposition 2.9. *If* H *is a hypergroup with strong identities, then the set* E *of these identities is a central subhypergroup of* H*.*

An element x' is called *right e-inverse* or *right e-symmetric* of x, if a right identity $e \neq x'$ exists such that $e \in x \cdot x'$. The definition of the *left e-inverse* or *left e-symmetric* is analogous to the above, while x' is called *e-inverse* or *e-symmetric* of x, if it is both right and left inverse with regard to the same identity e. If e is an identity in a hypergroup H, then the set of left inverses of $x \in H$, with regard to e, will be denoted by $S_{el}(x)$, while $S_{er}(x)$ will denote the set of right inverses of $x \in H$ with regard to e. The intersection $S_{el}(x) \cap S_{er}(x)$ will be denoted by $S_e(x)$. A semi-regular hypergroup H is called *regular*, if it has at least one identity e and if each element has at least one right and one left e-inverse. H is called *strictly e-regular*, if for the identity e the equality $S_{el}(x) = S_{er}(x)$ is valid for all $x \in H$. In a strictly e-regular hypergroup, the inverses of x are denoted by $S_e(x)$ and when there is no likelihood of confusion, e can be omitted. H has *semistrict e-regular structure*, if $S_{el}(x) \cap S_{er}(x) \neq \emptyset$ for any $x \in H$ is true for the identity e. Obviously in commutative hypergroups only strict e-regular structures exist.

Proposition 2.10. *If e is an identity in a hypergroup H, then* $S_{el}(x) = e/x - \{e\}$ *and* $S_{er}(x) = x \backslash e - \{e\}$.

Corollary 2.11. *If* $S_{el}(x) \cap S_{er}(x) \neq \emptyset$, $x \in H$, *then* $x \backslash e \cap e/x \neq \emptyset$.

Proposition 2.12. [17] *The following are true in any transposition hypergroup:*

i. $a(b/c) \subseteq ab/c$ *and* $(c \backslash b)a \subseteq c \backslash ba$
ii. $a/(c/b) \subseteq ab/c$ *and* $(b \backslash c) \backslash a \subseteq c \backslash ba$

Proposition 2.13. *The following are true in any transposition hypergroup:*

i. $(b \backslash a)(c/d) \subseteq (b \backslash ac)/d = b \backslash (ac/d)$
ii. $(b \backslash a)/(c/d) \subseteq (b \backslash ad)/c = b \backslash (ad/c)$
iii. $(b \backslash a) \backslash (c/d) \subseteq (a \backslash bc)/d = a \backslash (bc/d)$

Proof.

(i) Let $x \in (b \backslash a)(c/d)$. There exists $y \in b \backslash a$ such that $x \in y(c/d) \subseteq (yc)/d$. Thus there exists $r \in yc$ such that $x \in r/d$ or $r \in xd$. Therefore $xd \approx yc$. Consequently $xd \approx (b \backslash a)c \subseteq b \backslash (ac)$ which implies that $x \in (b \backslash ac)/d$. Hence $(b \backslash a)(c/d) \subseteq (b \backslash ac)/d$.

(ii) Let $x \in (b \backslash a)/(c/d)$. There exists, $y \in b \backslash a$ such that $x \in y/(c/d) \subseteq (yd)/c$. Thus there exists $r \in yd$ such that $x \in r/c$ or $r \in xc$ Therefore $xc \approx yd$. Consequently $xc \approx (b \backslash a)d \subseteq b \backslash (ad)$ which implies that $x \in (b \backslash ad)/c$. Hence $(b \backslash a)/(c/d) \subseteq (b \backslash ad)/c$.

(iii) Let $x \in (b \backslash a) \backslash (c/d)$. There exists $y \in c/d$ such that $x \in (b \backslash a) \backslash y \subseteq a \backslash by$. Thus there exists $r \in by$ such that $x \in a \backslash r$ or $r \in ax$. Therefore $ax \approx by$. Consequently $ax \approx b(c/d) \subseteq (bc)/d$ which implies that $x \in a \backslash (bc/d)$. Hence $(b \backslash a) \backslash (c/d) \subseteq a \backslash (bc/d)$.

Corollary 2.14. *The following is true in any transposition hypergroup*

$$(b \backslash a)(c/d) \cup (b \backslash a)/(d/c) \cup (a \backslash b) \backslash (c/d) \subseteq (b \backslash ac)/d = b \backslash (ac/d)$$

Proposition 2.15. [29, 34] *The following are true in any join hypergroup*

(1) $a\,(b/c) \cup b\,(a/c) \cup a/\,(c/b) \cup b/\,(c/a) \subseteq ab/c$,
(2) $(a/b)\,(c/d) \cup (a/d)\,(c/b) \cup (a/b)/(d/c) \cup (a/d)/(b/c) \cup (c/d)/(b/a) \cup (c/b)/(d/a) \subseteq ac/bd$.

In [17] and then in [18] a principle of duality is established in the theory of hypergroups and in the theory of transposition hypergroups as follows:

Given a theorem, the dual statement which results from the interchanging of the order of the hypercomposition "·" (and necessarily interchanging of the left and the right division), is also a theorem.

Since we are working in transposition hypergroups, this principle is used throughout this paper. In what follows, it is assumed that the identities are bilateral and idempotent. Examples of such transposition hypergroups, some of which are connected to the theory of languages and automata, can be found in [18, 35, 36, 44, 45, 50, 52, 54].

3. Some Fundamental Properties

Proposition 3.1. *If H is a transposition hypergroup with an identity e and $z \in xy$, then:*

i. $ey \approx x'z$, *for all* $x' \in S_{el}(x)$,
ii. $xe \approx zy'$, *for all* $y' \in S_{er}(y)$.

Proof. $z \in xy$ implies that $x \in z/y$ and that $y \in x\backslash z$. Let $x' \in S_{el}(x)$ and $y' \in S_{er}(y)$. Then $e \in x'x$ and $e \in yy'$. Thus $x \in x'\backslash e$ and $y \in e\backslash y'$. Therefore $x'\backslash e \approx z/y$ and $x\backslash z \approx e/y'$. Hence, because of transposition, $ey \approx x'z$ and $xe \approx zy'$ □

Proposition 3.2. *Let H be a transposition hypergroup with a strong identity e and x, y, z elements in H, such that $x, y, z \neq e$ and $z \in xy$. Then:*

i. *if* $S_{el}(x) \cap S_{el}(z) = \emptyset$, *then* $y \in x'z$, *for all* $x' \in S_{el}(x)$,
ii. *if* $S_{er}(y) \cap S_{er}(z) = \emptyset$, *then* $x \in zy'$, *for all* $y' \in S_{er}(y)$.

Proof. According to Proposition 3.1, $z \in xy$ implies that $ey \approx x'z$ for all $x' \in S_l(x)$. Since e is strong $ey = \{e, y\}$. Hence $\{e, y\} \approx x'z$. But $S_{el}(x)$ and $S_{el}(z)$ are disjoint. Thus $e \notin x'z$, therefore $y \in x'z$. The rest follows by duality. □

Proposition 3.3. *If H is a transposition hypergroup with a scalar identity e, then, for any $x \neq e$ in H, the quotients e/x and $x\backslash e$ are singletons and equal to each other.*

Proof. Let x be in H. Because of reproduction, there exist x' and x'', such that $e \in x'x$ and $e \in xx''$. Thus $x \in x'\backslash e$ and $x \in e/x''$. Hence, $x'\backslash e \approx e/x''$. Therefore, because of transposition, $ex'' \approx x'e$ is valid. Since e is a scalar identity, the following is true: $x'' = ex''$and $x'e = x'$. Thus $x' = x''$. However, $x' \in e/x$ and $x'' \in x\backslash e$. Therefore, e/x and $x\backslash e$ are equal and since this argument applies to any $y', y'' \in H$, such that $e \in y'x$ and $e \in xy''$, it follows that e/x and $x\backslash e$ are singletons. □

Hence, if H is a transposition hypergroup with a scalar identity e, then the inverse of x is equal to $e/x = x\backslash e$ for all $x \neq e$ in H.

Corollary 3.4. [17] *A hypergroup is a quasicanonical hypergroup (or polygroup) [30] if and only if it is a transposition hypergroup with a scalar identity.*

Corollary 3.5. [29] *A hypergroup is a canonical hypergroup [54] if and only if it is a join hypergroup with a scalar identity.*

Propositions 3.4 and 3.5 are consequences of Proposition 2.6:

Proposition 3.6. *If a hypergroup H has semistrict e-regular structure, then $x\backslash e \approx e/x$ for all $x \in H$*

Corollary 3.7. *If H is a transposition hypergroup with semistrict e-regular structure, then $ex \approx xe$ for all $x \in H$.*

Proposition 3.8. *A hypergroup is strictly e-regular if and only if $x\backslash e = e/x$*

Proposition 3.9. *If H is a strictly e-regular hypergroup, then $ex = xe$ for all $x \in H$.*

Proof. Suppose that for some $x \in H$ there exists $s \in H$ such that $s \in ex$ and $s \notin xe$. Since $s \in ex$ it follows that $x \in e\backslash s$. Since $s \notin xe$ it derives that $x \notin s/e$. Thus $s\backslash e \neq e/s$, which is absurd per Proposition 3.5 □

4. The Attractive Elements

Let e be an identity element in a hypergroup H and x an element in H. Then, x will be called *right e-attractive*, if $e \in ex$, while it will be called *left e-attractive* if $e \in xe$. If x is both left and right e-attractive, then it will be called *e-attractive*. When there is no likelihood of confusion, then e can be omitted. When the identity is strong, then $ex = xe = \{e, x\}$ is valid, if x is attractive; if x is non-attractive, then $ex = xe = x$ is valid. In the case of strong identity, non-attractive elements are called canonical. See [50] for the origin of the terminology.

Proposition 4.1. *In a hypergroup H, $e\backslash e$ is the set of right e-attractive elements of H and e/e is the set of left e-attractive elements of H*

Proof. Suppose that x is a right attractive element in H. Then $e \in ex$. Thus $x \in e\backslash e$. Also if $x \in e\backslash e$, then $e \in ex$. Hence $e\backslash e$ consists of the right attractive elements of H. The rest follows per duality. □

Proposition 4.2. *If x is not a right (resp. left) e-attractive element in a hypergroup with idempotent identity e, then ex consists of non-right (resp. left) e-attractive elements.*

Proof. Suppose that $y \in ex$ and assume that y is attractive. Then $ey \subseteq e\,(ex) = (ee)\,x = ex$. Since $e \in ey$, it follows that $e \in ex$, which is absurd. □

Proposition 4.3. *If x is a right (resp. left) e-attractive element in a transposition hypergroup with idempotent identity e, then all the elements of xe are right (resp. left) e-attractive.*

Proof. Let $y \in xe$, then $x \in y/e$. Also, per Proposition 4.1, $x \in e\backslash e$ is valid. Thus $e\backslash e \approx y/e$, which implies that $ee \approx ey$. Therefore, $e \in ey$. □

Proposition 4.4. *If x is a right (resp. left) attractive element in a transposition hypergroup with idempotent identity e, then its right (resp. left) inverses are also right (resp. left) attractive elements.*

Proof. Since $e \in ex$, it follows that $x \in e\backslash e$. Moreover, if x' is a right inverse of x, then $e \in xx'$. Therefore, $x \in e/x'$. Consequently, $e\backslash e \approx e/x'$. Transposition gives $ee \approx ex'$ and since e is idempotent, $e \in ex'$. Thus, x' is right attractive. □

Corollary 4.5. *If x is not a right (resp. left) attractive element in a transposition hypergroup with idempotent identity e, then its right (resp. left) inverses are also not right (resp. left) attractive elements.*

Proposition 4.6. *In a hypergroup H for any $x \neq e$ it is true that: $e/x \subseteq \{e\} \cup S_{el}(x)$ and $x\backslash e \subseteq \{e\} \cup S_{er}(x)$*

Proof. $y \in e/x$, if and only if $e \in yx$. This means that either $y \in S_{el}(x)$ or $y = e$, if x is right attractive. Hence, $e/x \subseteq \{e\} \cup S_{el}(x)$. The rest follows per duality. □

The following is a straightforward consequence of Proposition 4.5:

Proposition 4.7. *Let H be a strictly e-regular hypergroup, where e is a strong identity. Then:*

i. $x\backslash e = eS(x) = \{e\} \cup S(x) = S(x)e = e/x$ *for any attractive element* $x \neq e$
ii. $x\backslash e = e/x = S(x)$ *for any non attractive element* x

Corollary 4.8. *. If X is a non-empty subset of H and if $e \notin X$ and if X contains an attractive element, then $X\backslash e = eS(X) = \{e\} \cup S(X) = S(X)e = e/X$, while $S(X) = X\backslash e = e/X$, if X consists of non–attractive elements.*

Proposition 4.9. *If x is not a right (resp. left) e-attractive element in a hypergroup H with strong identity e, then $xS_{er}(x)$ (resp. $S_{el}(x)x$) contains all the right (resp. left) attractive elements.*

Proof. Let z be an arbitrary right attractive element. Then, from reproductivity $xH = Hx = H$, it follows that there exists $y \in H$, such that $z \in xy$. Next the following is valid: $z \in xy \Rightarrow ez \subseteq e(xy) = (ex)y = xy$. Since $e \in ez$, it follows that $e \in xy$. Hence, $y \in S_{er}(x)$. □

The following is a direct consequence of Proposition 3.6:

Proposition 4.10. *If a hypergroup has strict e-regular structure, then right and left attractive elements coincide*

In what follows T will denote a strictly e-regular transposition hypergroup where e is an idempotent identity. In T let A denote the set of attractive elements and C the set of non-attractive ones. Then $H = A \cup C$ and $A \cap C = \emptyset$.

Proposition 4.11. *If x and y are an attractive and non–attractive element respectively in T, then xy and yx consist of non–attractive elements.*

Proof. According to Proposition 4.2, ye consists of non-attractive elements. Next, because of Corollary 4.1, the inverses of x are all attractive elements, thus no inverse of x belongs to ye. Also $e \notin ye$. This implies that e cannot be in the result of hypercomposition $x\,(ye)$. Since $x\,(ye) = (xy)\,e$, we have that $e \notin (xy)\,e$. Hence, no attractive elements exist in set xy. □

Corollary 4.12. *If x, y are attractive elements in T, then $x/y \subseteq A$ and $y\backslash x \subseteq A$.*

Proposition 4.13. *If x is a non-attractive element in T, then $A \subseteq xC \cap Cx$.*

Proof. Per the reproductive axiom, if y is an attractive element, then there exist elements $z, w \in T$, such that $y \in xz$ and $y \in wx$. Elements z and w cannot be attractive. If they were, sets xz and wx would contain only non-attractive elements, per Proposition 4.9. Thus, they could not contain y, which is an attractive element. Thus z and w are non-attractive elements, QED. □

Corollary 4.14. *Set C of non-attractive elements of $T$is not stable under the hypercomposition.*

Proposition 4.15. *The result of the hypercomposition of two attractive elements in T contains only attractive elements.*

Proof. Let x, y be two attractive elements and suppose that z is a non-attractive element which belongs to xy. Then, $z \in xy$ implies that $x \in z/y$. Moreover, if x' is an inverse of x, then $x \in x'\backslash e$. Thus, $z/y \approx x'\backslash e$ which, because of transposition, implies that $ey \approx x'z(1)$. However, x is an attractive element, and, according to Proposition 4.4, x' is also attractive; therefore, $x'z$ does not contain the identity and it consists only of non-attractive elements (Prop. 4.9). Moreover, since y is an attractive element, ey consists only of attractive elements per Proposition 4.3. This leads to intersection $(ey) \cap (x'z)$ being an empty set, which contradicts (1). □

From Proposition 4.11 above and Proposition 4.9 it follows that:

Corollary 4.16. *If either x or y are non-attractive elements, then $x/y \subseteq C$ and $y\backslash x \subseteq C$.*

Proposition 4.17. *If the identity of T is strong, then:*

i. *the result of the hypercomposition of two attractive elements contains these two elements (see also [18, 45, 49]),*
ii. *the result of the hypercomposition of an attractive element with a canonical element is the canonical element (see also [45, 49]).*

Proof. (i) Let x, y be two attractive elements. Then, $xe = \{x, e\}$ and $ye = \{y, e\}$. Thus, $x \in xe \subseteq x\,\{y, e\} = x\,(ye) = (xy)\,e = xy \cup \{e\}$. Hence, $x \in xy$, whether x equals e or not. Duality yields $y \in xy$.

(ii) Let x be an attractive element and y a canonical element. Then, $e \in y\backslash y$ and $e \in e/x$. Therefore, the implications $y\backslash y \approx e/x \Rightarrow yx \approx ye \Rightarrow y \in yx$ are valid.

y is the only canonical element in yx. Indeed, suppose that $y' \in C$ and $y' \in yx$. Then, $x \in y\backslash y'$. Moreover, $x \in e/e$. Therefore, $y\backslash y' \approx e/e \Rightarrow ye \approx ey' \Rightarrow y = y'$. Next, it will be proven that none of the attractive elements belong to yx. Indeed, suppose that $x' \in A$ and $x' \in yx$. Then, $ex' \subseteq e\,(yx) = (ey)\,x = yx$. Thus, $\{e, x'\} \subseteq yx$. Therefore $e \in yx$. Hence, $y \in S\,(x)$, which, because of Proposition 4.4, is absurd. □

Corollary 4.18. *If the identity of T is strong, then:*

i. $x \in x/y$ *and* $x \in y\backslash x$, *for all* $x, y \in A$,
ii. $A = x/x = x\backslash x$, *for all* $x \in A$.

Proposition 4.19. *If the identity of T is strong and*

i. *x, y are two attractive elements in T, such that $e \notin xS\,(y)$, then $xS\,(y) = x/y \cup S\,(y)$ and $S\,(y)\,x = y\backslash x \cup S\,(y)$,*
ii. *x, y are two elements in T and any of these is non-attractive, then $xS\,(y) = x/y$ and $S\,(y)\,x = y\backslash x$.*

Proof.
Since $e \in yS\,(y)$, it follows that $y \in e/S\,(y)$. Therefore, $x/y \subseteq x/\,(e/S\,(y))$.

(i) If x, y are two attractive elements in T, using Proposition 2.7, Corollary 2.3, Proposition 4.4 and Proposition 4.12 sequentially, we get:

$$x/y \cup S\,(y) \subseteq x/\,(e/S\,(y)) \cup S\,(y) \subseteq xS\,(y)\,/e \cup S\,(y) = xS\,(y) \cup S\,(y) = xS\,(y)\,.$$

On the other hand, according to Proposition 4.6, $e/y = \{e\} \cup S\,(y)$. Hence, using Proposition 2.7, we have: $xS\,(y) \subseteq x\,(e/y) \subseteq xe/y = \{x, e\}\,/y = x/y \cup e/y = x/y \cup \{e\} \cup S\,(y)$. Since e is not in $xS\,(y)$, it follows that $xS\,(y) \subseteq x/y \cup S\,(y)$. Thus, $xS\,(y) = x/y \cup S\,(y)$. Dually, $S\,(y)\,x = y\backslash x \cup S\,(y)\,.$

(ii) Suppose that x is non-attractive. Then, using Proposition 2.7 and Corollary 2.3, we get: $x/y \subseteq x/\,(e/S\,(y)) \subseteq xS\,(y)\,/e = xS\,(y)$. Also, $xS\,(y) \subseteq x\,(e/y) \subseteq xe/y = x/y$. Hence, $xS\,(y) = x/y$. Next, suppose that y is non-attractive. According to Proposition 4.7, all the attractive elements belong to $S\,(y)\,y$ and, since $x/y = \{z \in T : \ x \in zy\}$, it follows that $S\,(y) \subseteq x/y$. Also, because of Proposition 4.6, $S\,(y)$ equals e/y. Using Proposition 2.7, we get:

$$xS\,(y) = x\,(e/y) \subseteq xe/y \subseteq \{x, e\}\,/y = x/y \cup e/y = x/y \cup S\,(y) = x/y$$

Moreover, as was proven above, inclusion $x/y \subseteq xS\,(y)$ is valid. Therefore, $xS\,(y) = x/y$. Dually, $S\,(y)\,x = y\backslash x$, QED.

Corollary 4.20. *If the identity of T is strong and:*

(i) *X, Y are non-empty subsets of $A \subseteq T$ and $e \notin XS\,(Y)$, then $XS\,(Y) = X/Y \cup S\,(Y)$ and $S\,(Y)\,X = Y\backslash X \cup S\,(Y)$,*
(ii) *if X or Y are non-empty subsets of $C \subseteq T$, then $XS\,(Y) = X/Y$ and $S\,(Y)\,X = Y\backslash X$.*

When identity is strong, properties of attractive elements, are developed in [18, 44, 48, 49].

Proposition 4.21. *Let x, y be two attractive elements in T, such that $e \notin xy$. Then, $eS(xy) = eS(y)S(x)$.*

Proof. Since x, y are attractive elements, xy consists of attractive elements per Proposition 4.11. Therefore, Corollary 4.2 yields $e/xy = S(xy) \cup \{e\} = eS(xy)$. Also, using mixed associativity, we get that $e/xy = (e/y)/x$. Thus, we have:

$$S(xy) \cup \{e\} = e/xy = (e/y)/x = [S(y) \cup \{e\}]/x = S(y)/x \cup e/x = S(y)/x \cup S(x) \cup \{e\}.$$

Next, per Proposition 4.13, $S(y)/x \cup S(x) = S(y)S(x)$. Hence, $S(y)/x \cup S(x) \cup \{e\} = S(y)S(x) \cup \{e\}$ is valid. Therefore, $S(xy) \cup \{e\} = S(y)S(x) \cup \{e\}$, which, per Corollary 4.2, yields $eS(xy) = eS(y)S(x)$. □

Corollary 4.22. *If $e \notin xy$ and $e \notin S(y)S(x)$, then $S(xy) = S(y)S(x)$.*

Proof. Since $e \notin xy$ implies that $e \notin S(xy)$ and since $e \notin S(y)S(x)$, from equality $eS(xy) = eS(y)S(x)$ it follows that $S(xy) = S(y)S(x)$. □

Corollary 4.23. *If X, Y are non empty subsets of $A \subseteq T$, then:*

i. $eS(XY) = eS(Y)S(X)$, *if* $e \notin XY$,
ii. $S(XY) = S(Y)S(X)$, *if* $e \notin XY$ *and* $e \notin S(Y)S(X)$.

Remark 4.24. As was proven in [49], when $y = x^{-1}$ equality $(xy)^{-1} = y^{-1}x^{-1}$ may not be valid in the case of fortified join hypergroups. Since fortified join hypergroups belong to the class of transposition hypergroups with idempotent identity, it follows that $S(xy) = S(y)S(x)$ may not be valid, if $e \in xy$. Also, from examples 2.5 and 2.6 in [35], it becomes evident that this equality fails to hold when $e \in S(y)S(x)$.

Naturally, next issues on the transposition hypergroups with idempotent identity that should be dealt with, are the study of their subhypergroups, the study of the cosets modulo them and the study of their homomorphisms. Afterwards the topic that can be focus on is the existence of isomorphism theorems not only analogous to the ones on the quasicanonical hypergroups (or polygroups) [30], and their generalizations in transposition hypergroups with quotients modulo closed subhypergroups [17] but also modulo symmetric subhypergroups when only attractive elements exist.

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