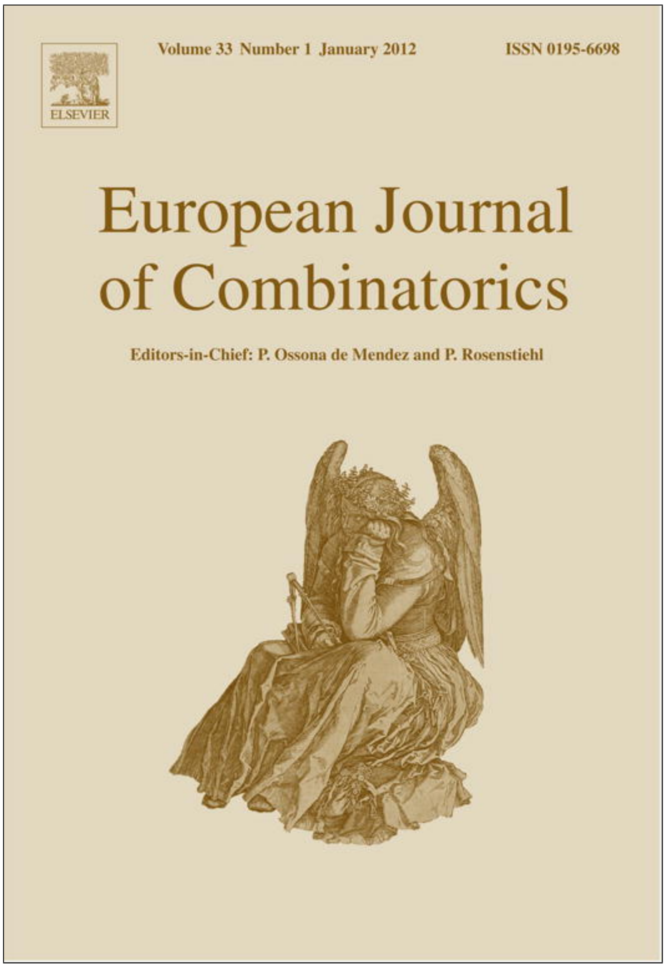
Volume 33 Number 1 January 2012
ISSN 0195-6698
ELSEVIER
European Journal of Combinatorics
Editors-in-Chief: P. Ossona de Mendez and P. Rosenstiehl

European Journal of Combinatorics 33 (2012) 1777–1786

Contents lists available at SciVerse ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Enumeration of Rosenberg-type hypercompositional structures defined by binary relations

Ch. Tsitouras, Ch.G. Massouros

TEI of Chalkis, Department of Applied Sciences, GR34400, Psahna, Greece

ARTICLE INFO

Article history:
Received 5 January 2011
Received in revised form
12 January 2012
Accepted 28 February 2012

ABSTRACT

Every binary relation ρ on a set H, $(card(H) > 1)$ can define a hypercomposition and thus endow H with a hypercompositional structure. In this paper, binary relations are represented by Boolean matrices. With their help, the hypercompositional structures (hypergroupoids, hypergroups, join hypergroups) that emerge with the use of the Rosenberg's hyperoperation are characterized, constructed and enumerated using symbolic manipulation packages. Moreover, the hyperoperation given by $x \circ x = \{z \in H |\ (z, x) \in \rho\}$ and $x \circ y = x \circ x \cup y \circ y$ is introduced and connected to Rosenberg's hyperoperation, which assigns to every (x, y) the set of all z such that either $(x, z) \in \rho$ or $(y, z) \in \rho$.

1. Introduction

A hypercomposition in a non-empty set H is a function from $H \times H$ to the powerset $P(H)$ of H. This notion was introduced in mathematics, alongside the notion of the hypergroup, by Marty in 1934, during the 8th Congress of the Scandinavian Mathematicians held in Stockholm [11].

The axioms that endow the pair $(H, \cdot)$ with the hypergroup structure, where H is a non-empty set and "$\cdot$" is a hypercomposition on H, are:

(i) $a(bc) = (ab)c$ for all $a, b, c \in H$ (associativity);
(ii) $aH = Ha = H$ for all $a \in H$ (reproductivity).

If only (i) is valid, then $(H, \cdot)$ is called a semihypergroup, while it is called a quasihypergroup if only (ii) holds. In a hypergroup, the result of the hypercomposition is always a non-empty set. Indeed,

E-mail addresses: tsitoura@teihal.gr (Ch. Tsitouras), masouros@teihal.gr (Ch.G. Massouros).
URLs: http://users.ntua.gr/tsitoura/ (Ch. Tsitouras), http://www.teihal.gr/gen/profesors/massouros/index.htm (Ch.G. Massouros).

0195-6698/$ – see front matter © 2012 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ejc.2012.03.032

suppose that for two elements a, b in H it holds that $ab = \emptyset$. Then $H = aH = a(bH) = (ab)H = \emptyset H = \emptyset$, which is absurd. Thus if $(H, \cdot)$ is non-associative and reproductive, then the empty set can be among the results of the hypercomposition. $(H, \cdot)$ is called a hypergroupoid if $xy \neq \emptyset$ for all x, y in H; otherwise it is called a partial hypergroupoid.

Furthermore Marty [11] defined the two *induced hypercompositions* (the left and the right division) that result from the hypercomposition, i.e.

$$a/b = \{x \in H | a \in xb\} \quad \text{and} \quad b \setminus a = \{y \in H | a \in by\}.$$

When "$\cdot$" is commutative, $a/b = b \setminus a$ is valid. In a hypergroup, a/b and $a \setminus b$ are non-empty for all a, b in H and this is equivalent to the reproductive axiom [14]. A *transposition hypergroup* [10] is a hypergroup $(H, \cdot)$ that satisfies a postulated property of transposition, i.e. $(b \setminus a) \cap (c/d) \neq \emptyset \Rightarrow (ad) \cap (bc) \neq \emptyset$. A *join space* or *join hypergroup* is a commutative transposition hypergroup. It is worth mentioning here that the hypergroup, which is a very general structure, has been progressively enriched with additional axioms, either more or less powerful, thus creating a significant number of specific hypergroups such as the above mentioned transposition and join ones, their fortifications [10,17,19], the canonical and quasicanonical ones [13], etc., with many widespread applications; see e.g. [12,15,16].

Several papers have been written on the construction of hypergroups, since hypergroups are much more varied than groups. For example, for each prime number p, there is only one group, up to isomorphism, with cardinality p, while there are a very large number of non-isomorphic hypergroups. Specifically, there are 3999 non-isomorphic hypergroups with three elements [23].

Among others, Rosenberg [22], Corsini [1,2], Corsini and Leoreanu [3], Cristea [4], Cristea and Stefanescu [6,7], and De Salvo and Lo Faro [8,9] studied hypercompositional structures defined in terms of binary relations. Corsini constructed in [1] partial hypergroupoids by introducing into a non-empty set H the hypercomposition

$$x \cdot y = \{z \in H | (x, z) \in \rho \text{ and } (z, y) \in \rho\},$$

where ρ is a binary relation on H. Obviously, such a partial hypergroupoid is a hypergroupoid if for each pair of elements x, y in H there exists z in H such that $(x, z) \in \rho$ and $(z, y) \in \rho$. In [20] it is proven that this hypercomposition generates only one semihypergroup and only one quasihypergroup which coincide with the total hypergroup. Also in [20], it is computed that this hypercomposition generates 2, 17, 304 and 20 660 non-isomorphic hypergroupoids of order 2, 3, 4, 5 respectively.

De Salvo and Lo Faro introduced in a non-empty set H the hypercomposition

$$x \cdot y = \{z \in H | (x, z) \in \rho \text{ or } (z, y) \in \rho\},$$

where ρ is a binary relation on H. They characterized in [8] the relations ρ, which give quasihypergroups, semihypergroups and hypergroups.

On the other hand, Rosenberg introduced in a non-empty set H the hypercomposition

$$x \bullet x = \{z \in H | (x, z) \in \rho\} \quad \text{and} \quad x \bullet y = x \bullet x \cup y \bullet y,$$

where ρ is a binary relation on H, and studied the structure that emerged [22]. This structure is further studied in the present paper and was also studied in [3] and [5].

This paper deals with the Rosenberg-type hypercompositional structures, the properties of their generative binary relations and their representations using Boolean matrices. Some conclusions from [22] are restated with the use of Boolean matrices, in order to develop Mathematica programs, which enumerate the hypergroupoids, the hypergroups and the join hypergroups with two, three, four and five elements. During the preparation of this paper, the authors became familiar with [5], where an extensive program written in C# computes the Rosenberg hypergroups with two, three and four elements. As regards hypergroups with two, three and four elements (which both papers enumerate), the results are the same, even though they are obtained (in software and size) through completely different computational methods.

2. The Rosenberg-type hypercompositional structures

Let H be a non-empty set and ρ a binary relation on H. As usual,

$$\rho^2 := \{(x, y) \mid (x, z), (z, y) \in \rho, \text{ for some } z \in H\}.$$

An element x of H is called an outer element of ρ if $(z, x) \notin \rho^2$ for some $z \in H$; otherwise x is called an inner element. The domain of ρ is the set

$$D = \{x \in H | (x, z) \in \rho, \text{ for some } z \in H\}$$

and the range of ρ is the set

$$R = \{x \in H | (z, x) \in \rho, \text{ for some } z \in H\}.$$

Rosenberg introduced in H the hypercomposition

$$x \bullet x = \{z \in H | (x, z) \in \rho\} \quad \text{and} \quad x \bullet y = x \bullet x \cup y \bullet y$$

and he observed that $H_\rho = (H, \bullet)$ is a hypergroupoid if and only if H is the domain of ρ and that H_ρ is a quasihypergroup if and only if H is the domain and the range of ρ. He also proved:

Proposition 1 ([22]). *H_ρ is a semihypergroup if and only if:*

(i) *H is the domain of ρ;*
(ii) $\rho \subseteq \rho^2$;
(iii) $(a, x) \in \rho^2 \Rightarrow (a, x) \in \rho$ *whenever x is an outer element of ρ.*

From the last two elements of this proposition it follows that, whenever x is an outer element of ρ for some $a \in H_\rho$, then (a, y) is in ρ if and only if (a, y) is in ρ^2. Thus, one can easily observe that Proposition 1 is equivalent to:

Proposition 2. *H_ρ is a semihypergroup if and only if:*

(i) *H is the domain of ρ;*
(ii) $(y, x) \in \rho^2 \Leftrightarrow (y, x) \in \rho$ *for all $y \in H$ whenever x is an outer element of ρ.*

Thus:

Proposition 3. *H_ρ is a hypergroup if and only if:*

(i) *H is the domain and the range of ρ;*
(ii) $(y, x) \in \rho^2 \Leftrightarrow (y, x) \in \rho$ *for all $y \in H$ whenever x is an outer element of ρ.*

On the other hand, the binary relation ρ can define in H another hypercomposition, which is the following:

$$x \circ x = \{z \in H | (z, x) \in \rho\} \quad \text{and} \quad x \circ y = x \circ x \cup y \circ y.$$

Proposition 4. *If ρ is symmetric, then the hypercompositional structures $(H, \bullet)$ and $(H, \circ)$ coincide.*

One can easily observe that $(H, \circ)$ is a hypergroupoid if and only if H is the range of ρ. For $(a, b) \in \rho$, a is called a predecessor of b and b a successor of a [22]. Following Rosenberg's terminology, an element x will be called a predecessor outer element of ρ if $(x, z) \notin \rho^2$ for some $z \in H$.

The following two propositions are proven in a similar way to Propositions 1 and 2.

Proposition 5. *$(H, \circ)$ is a semihypergroup if and only if:*

(i) *H is the range of ρ;*
(ii) $(x, y) \in \rho^2 \Leftrightarrow (x, y) \in \rho$ *for all $y \in H$ whenever x is a predecessor outer element of ρ.*

Proposition 6. *$(H, \circ)$ is a hypergroup if and only if:*

(i) *H is the domain and the range of ρ;*
(ii) $(x, y) \in \rho^2 \Leftrightarrow (x, y) \in \rho$ *for all $y \in H$ whenever x is a predecessor outer element of ρ.*

From the definitions of the two above hypercompositions it follows that the hypercompositional structures constructed through them are always commutative. Since "$\bullet$" is commutative, the two induced hypercompositions "/" and "\" coincide. The same holds true for the hypercomposition "$\circ$".

Proposition 7. *If $H_\rho = (H, \bullet)$ is a hypergroup, then*

$$x/y = \begin{cases} H, & \text{if } (y,x) \in \rho \\ x \circ x, & \text{if } (y,x) \notin \rho \end{cases}$$

for all x, y in H.

Proof.

$$\begin{aligned} x/y &= \{v \in H | x \in v \bullet y\} = \{v \in H | x \in v \bullet v \cup y \bullet y\} \\ &= \{v \in H | (v,x) \in \rho \text{ or } (y,x) \in \rho\} \end{aligned}$$

which is equal to H if $(y,x) \in \rho$ or equal to $x \circ x$ if $(y,x) \notin \rho$. □

Corollary 1. *If ρ is reflexive, then $x/x = H$, for each $x \in H$.*

The following is a direct result of Proposition 7:

Proposition 8. *Let x, y, z, w be in H. If $x/y \cap w/z \neq \emptyset$, then there are three possible cases:*

(i) *$x/y \cap w/z = H$ when $(y,x) \in \rho$ and $(z,w) \in \rho$;*
(ii) *$x/y \cap w/z = x \circ x$ when $(y,x) \notin \rho$ and $(z,w) \in \rho$ or*
$x/y \cap w/z = w \circ w$ when $(y,x) \in \rho$ and $(z,w) \notin \rho$;
(iii) *$x/y \cap w/z = x \circ x \cap w \circ w$ when $(y,x) \notin \rho$ and $(z,w) \notin \rho$.*

Lemma 1. *If ρ is reflexive, then the transposition axiom is fulfilled in* (i) *and* (ii) *of Proposition 8.*

Proof. (i) Consider the intersection $x \bullet z \cap w \bullet y$. We have $x \bullet z \cap w \bullet y = (x \bullet x \cup z \bullet z) \cap (w \bullet w \cup y \bullet y)$. Since $(y,x) \in \rho$, it follows that $x \in y \bullet y$. Also $(x,x) \in \rho$, because ρ is reflexive. Thus $x \in x \bullet x$. Consequently $x \bullet z \cap w \bullet y \neq \emptyset$. The proof of (ii) is similar. □

Lemma 2. *If $\rho^2 = \rho$, then the transposition axiom is fulfilled in cases* (i) *and* (ii) *of Proposition 8.*

Proof. (i) Consider the intersection $x \bullet z \cap w \bullet y$, which is equal to $(x \bullet x \cup z \bullet z) \cap (w \bullet w \cup y \bullet y)$. Suppose that $(z,w) \in \rho$. Since H is the domain and the range of ρ, there exists $t \in H$ such that $(w,t) \in \rho$. Thus, $t \in w \bullet w$. Next, $(z,t) \in \rho^2$, because $(z,w) \in \rho$ and $(w,t) \in \rho$. But $\rho^2 = \rho$; hence $(z,t) \in \rho$ and therefore $t \in z \bullet z$. Consequently, $t \in x \bullet z \cap w \bullet y$, so the intersection is non-void. The proof of (ii) is similar. □

Corollary 2. *If ρ is transitive, then the transposition axiom is fulfilled in cases* (i) *and* (ii) *of Proposition 8.*

Proof. If ρ is transitive, then $\rho^2 \subseteq \rho$. Since H_ρ is a hypergroup, it holds that $\rho \subseteq \rho^2$. Thus, $\rho^2 = \rho$. □

Proposition 9. *If ρ is compatible (i.e. reflexive and symmetric), then the transposition axiom is valid in H_ρ.*

Proof. Since ρ is reflexive, according to Lemma 1 the transposition axiom is valid in cases (i) and (ii) of Proposition 8. Now, for case (iii), suppose that $x/y \cap w/z \neq \emptyset$. Since $x/y \cap w/z = x \circ x \cap w \circ w$, it follows that the intersection $x \circ x \cap w \circ w$ is non-empty. But $x \circ x \cap w \circ w = x \bullet x \cap w \bullet w$, because ρ is symmetric. Thus, $x \bullet x \cap w \bullet w \neq \emptyset$. Next, the inclusion $x \bullet x \cap w \bullet w \subseteq x \bullet z \cap w \bullet y$ holds true. Hence, $x \bullet z \cap w \bullet y \neq \emptyset$ and so the transposition axiom is valid. □

Also from the above lemmas, it follows that we have:

Proposition 10. *If ρ is reflexive or transitive and the implication*

$$x \circ x \cap w \circ w \neq \emptyset \Rightarrow x \bullet x \cap w \bullet w \neq \emptyset$$

holds true for all x, w in H, then the transposition axiom is valid in H_ρ.

The implication $x \circ x \cap w \circ w \neq \emptyset \Rightarrow x \bullet x \cap w \bullet w \neq \emptyset$ means that a pair of elements with a common predecessor have a common successor.

Proposition 11. *If* $(y, x) \in \rho$ *and* $x \bullet x$ *contains an outer element, then*

$$x/y \cap w/z \neq \emptyset \Rightarrow x \bullet z \cap w \bullet y \neq \emptyset.$$

Proof. Let $(y, x) \in \rho$ and let t be an outer element in $x \bullet x$. Then, $(x, t) \in \rho$. Therefore, $(y, t) \in \rho^2$. But t is an outer element, so $(y, t) \in \rho$. Thus, $t \in y \bullet y$. □

The two hypercompositions "•" and "∘" can be viewed as in graphs. A directed graph consists of a finite set V whose members are called vertices and a subset A of $V \times V$ whose members are called arcs. Thus, A is a binary relation in V and so, through A, the two hypercompositions "•" and ∘ can be defined. Then, $x \bullet x$ consists of all vertices z for which an arrow exists pointing from x to z, while $x \circ x$ consists of all vertices z for which an arrow exists pointing from z to x (see also [18]).

3. Boolean matrices and finite hypergroupoids

The Boolean domain $B = \{0, 1\}$ becomes a semiring under the addition

$$0 + 1 = 1 + 0 = 1 + 1 = 1, \qquad 0 + 0 = 0$$

and the multiplication

$$0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0, \qquad 1 \cdot 1 = 1.$$

This semiring is called a *binary Boolean semiring*. A *Boolean matrix* is a matrix with entries from the binary Boolean semiring. Every binary relation ρ in a finite set H with $cardH = n \neq 0$ can be represented by a Boolean matrix M_ρ and conversely every $n \times n$ square Boolean matrix defines on H a binary relation. Indeed, let H be the set $\{a_1, \ldots, a_n\}$. Then, an $n \times n$ Boolean matrix is constructed as follows: the element (i, j) of the matrix is 1 if $(a_i, a_j) \in \rho$ and it is 0 if $(a_i, a_j) \notin \rho$ and vice versa. Hence, in every set with n elements, 2^{n^2} partial hypergroupoids can be defined. The element a_k of H is an outer element of ρ if the kth column of M_{ρ^2} has a zero entry. If all the entries of the kth column are 1, then a_k is an inner element of ρ. Moreover, $M_{\rho^2} = (M_\rho)^2$. A square Boolean matrix is called *total* if all its entries are equal to 1. A Boolean matrix is called *good* if its square is the total matrix [1], i.e. the good matrices are the square roots of the total matrix [21]. A *basic* Boolean matrix is a good matrix which is converted to one that is not good through the replacement of any unit entry with 0 [21]. It is proven that all the good matrices are generated from the basic ones [21]. An $n \times n$ Boolean matrix which has all the entries of its ith row and its ith column equal to 1, $i = 1, \ldots, n$, is called the minimum basic matrix [21].

Let H_ρ denote the above mentioned partial hypergroupoid, which is defined by a binary relation ρ through the hypercomposition "• ". Then, the propositions of the previous paragraph can be restated using Boolean matrices. Thus:

Theorem 1. H_ρ *is a hypergroupoid if and only if* M_ρ *has no zero rows.*

Theorem 2. H_ρ *is a quasihypergroup if and only if* M_ρ *has no zero rows and no zero columns.*

From Proposition 2 it follows that:

Theorem 3. H_ρ *is a semihypergroup if and only if:*

(i) M_ρ *consists only of non-zero rows;*
(ii) *if a column of the matrix* M_{ρ^2} *has a zero entry, then it coincides with the same column of* M_ρ*.*

Also from Proposition 3, it follows that:

Theorem 4. H_ρ *is a hypergroup if and only if:*

(i) M_ρ *consists only of non-zero rows and non-zero columns;*
(ii) *whenever a column of the matrix* M_{ρ^2} *has a zero entry, it coincides with the same column of* M_ρ*.*

Since the square roots of the total Boolean matrices consist only of non-zero rows and non-zero columns [21], it follows that:

Theorem 5. *The square roots of the total Boolean matrices produce Rosenberg hypergroups.*

Moreover, from Proposition 10 it follows that:

Theorem 6. *A hypergroup H_ρ is a join one if:*

(i) *all the elements on the main diagonal of M_ρ are equal to 1 or $M_\rho = M_{\rho^2}$;*
(ii) *the entrywise product of two rows a_{i*} and a_{j*} of M_ρ contains a non-zero entry whenever the entrywise product of the corresponding columns a_{*i} and a_{*j} contains a non-zero entry.*

More generally, if $(a_{i*})\left(a_{j*}\right)$ is the entrywise product of the two rows a_{i*} and a_{j*}, then, from Proposition 8, it follows that:

Theorem 7. *A hypergroup H_ρ is a join one if and only if*

(i) *whenever an entry (j, i) is 1 , then the row vector $(a_{i*} + a_{l*})\left(a_{j*} + a_{k*}\right)$ is not the zero one, for all the row vectors a_{l*}, a_{k*} of M_ρ;*
(ii) *the entrywise product of two rows a_{i*} and a_{j*} of M_ρ contains a non-zero entry whenever the entrywise product of the corresponding columns a_{*i} and a_{*j} contains a non-zero entry.*

Corollary 3. *The Rosenberg hypergroup which is produced from the minimum basic matrix is a join one.*

Relevant propositions hold true for the hypercompositional structures which are defined by a binary relation ρ through the hypercomposition "$\circ$ "; e.g., from Proposition 6 it follows that:

Theorem 8. *$(H, \circ)$ is a hypergroup if and only if:*

(i) *M_ρ consists only of non-zero rows and non-zero columns;*
(ii) *whenever a row of the matrix M_{ρ^2} has a zero entry, it coincides with the same row of M_ρ.*

Thus, a principle of duality holds between the two hypercompositions "$\bullet$" and "$\circ$":

Given a theorem, the dual statement resulting from the interchanging of one hypercomposition with the other is also a theorem.

Hence:

Theorem 9. *The hypergroup $(H, \circ)$ is a join one, if and only if:*

(i) *whenever an entry (i, j) is 1 , then the column vector $(a_{*i} + a_{*l})(a_{*j} + a_{*k})$ is not the zero one, for all column vectors a_{*l}, a_{*k} of M_ρ;*
(ii) *the entrywise product of two columns a_{*i} and a_{*j} of M_ρ contains a non-zero entry whenever the entrywise product of the corresponding rows a_{i*} and a_{j*} contains a non-zero entry.*

Next, we come to the question of whether two hypergroupoids generated by binary relations are isomorphic or not. The answer has been given in [20] by the following proposition and theorem:

Proposition 12. *If, in the Boolean matrix M_ρ, the ith and jth rows are interchanged and, at the same time, the corresponding ith and jth columns are interchanged as well, then the resulting new matrix, like the initial one, produces isomorphic hypergroupoids.*

Theorem 10. *If the Boolean matrix M_σ results from M_ρ on interchanging rows and the corresponding columns, then the hypergroupoids H_σ and H_ρ are isomorphic.*

4. Mathematica packages

The Mathematica [24] packages follow below.

4.1. Counting all hypergroups

The function `Good[di]` returns the Boolean matrices that form a hypergroupoid:

```
Good[di_] :=
Module[{c, i1, z},
        c = Tuples[Tuples[{0, 1}, di], di];
        z = Table[Min[Flatten[
                  - c[[i1]] + Sign[c[[i1]].c[[i1]]]]]*2^(di*di)
                  + Length[Position[c[[i1]], Table[0, {i2, 1, di}]]]],
            {i1, 1, 2^(di*di)}];
        Return[c[[Flatten[Position[z, 0]]]]]
];
```

For example, the eight Boolean matrices of second order that give hypergroupoids are the following:

```
In[1]:=Good[2]
Out[1]:={{{0, 1}, {0, 1}}, {{0, 1}, {1, 1}},
         {{1, 0}, {0, 1}}, {{1, 0}, {1, 0}},
         {{1, 0}, {1, 1}}, {{1, 1}, {0, 1}},
         {{1, 1}, {1, 0}}, {{1, 1}, {1, 1}}}
```

The results of the enumeration of hypergroupoids of order 2, 3, 4, 5 are as follows:

```
In[2]:= Length[Good[2]]
Out[2]= 8
In[3]:= Length[Good[3]]
Out[3]= 236
In[4]:= Length[Good[4]]
Out[4]= 28023
In[5]:= Length[Good[5]]
Out[5]= 13419636
```

The code that follows constructs a hypergroupoid from a Boolean matrix:

```
HyperGroupoid[a_List, order_] :=
   Table[Table[Complement[
               Union[Sign[a[[i1]] + a[[i2]]]*Table[j3,
                                        {j3, 1, order}]
                     ], {0}],
               {i2, 1, order}],
          {i1, 1, order}];
```

Example. The 99th Boolean matrix of third order that results in a hypergroupoid is the following:

```
In[6]:=Good[3][[99]] // MatrixForm
```

$$\mathtt{Out[6]} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

The hypergroupoid resulting from the above matrix is the following:

```
In[7]:=HyperGroupoid[Good[3][[99]], 3] // MatrixForm
```

$$\mathtt{Out[7]} = \begin{pmatrix} \{1\} & \{1, 2\} & \{1, 3\} \\ \{1, 2\} & \{1, 2\} & \{1, 2, 3\} \\ \{1, 3\} & \{1, 2, 3\} & \{1, 3\} \end{pmatrix}.$$

The function `GoodH[di]` returns the Boolean matrices that form a hypergroup:

```
GoodH[di_] :=
Module[{c, i1, z, h2, outer, indexes},
       c = Tuples[Tuples[{0, 1}, di], di];
       z = Table[ Min[Flatten[-c[[i1]]
           + Sign[c[[i1]].c[[i1]]]]]*2^(di*di)
           + Length[Position[c[[i1]], Table[0, {i2, 1, di}]]]
           + Length[Position[Transpose[c[[i1]]],
                    Table[0, {i2, 1, di}]]]], {i1, 1, 2^(di*di)}];
       h2 = c[[Flatten[Position[z, 0]]]];
       outer = Table[Complement[
               Sign[di - Total[Sign[h2[[j1]].h2[[j1]]]]]*
               Table[j3, {j3, 1, di}], {0}], {j1, 1, Length[h2]}];
       indexes = Complement[Range[1, Length[h2]],
                            Flatten[Position[
                              Table[Max[
                                Sign[h2[[j1]].h2[[j1]]][[All,outer[[j1]]]]
                              - h2[[j1]][[All, outer[[j1]]]]],
                               {j1, 1, Length[h2]}], 1]]
                            ];
Return[h2[[indexes]]]
];
```

For example we get the six Boolean matrices of second order that form a hypergroup:

```
In[8]:=GoodH[2]
Out[8]:={{{0, 1}, {1, 1}}, {{1, 0}, {0, 1}}, {{1, 0}, {1, 1}},
         {{1, 1}, {0, 1}}, {{1, 1}, {1, 0}}, {{1, 1}, {1, 1}}}
```

Enumeration of hypergroups:

```
In[9]:= Length[GoodH[2]]
Out[9]= 6
In[10]:= Length[GoodH[3]]
Out[10]= 149
In[11]:= Length[GoodH[4]]
Out[11]= 9729
In[12]:= Length[GoodH[5]]
Out[12]= 2921442
```

These are the only hypergroups resulting from the hypercompositions which are defined from binary relations.

With a small modification of the above codes we found the join hypergroups of orders 2, 3, 4, and 5 to be 5, 106, 6979 and 2 122 681 respectively.

4.2. Counting non-isomorphic hypergroups

The packages that enumerate the non-isomorphic classes follow below. `IsomorphTest1` returns all isomorphisms of a matrix.

```
IsomorphTest1[a_List] :=
Module[{p, a1},
        p = Permutations[Range[1, Length[a]]];
        Return[Table[a1 = a;
                     a1 = ReplaceAll[a1, a1[[All, Table[j2,
                                     {j2, 1, Length[a1]}]]] ->
                                         a1[[All, p[[j1]]]]];
                          ReplaceAll[a1, a1[[Table[j2,
                                     {j2, 1, Length[a]}]]] ->
                                         a1[[p[[j1]]]]],
                     {j1, 1, Length[p]}]
              ]]
```

Let us examine the six permutations of the matrix

$$M_\rho = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

which are defined by corresponding binary relations producing isomorphic hypergroupoids:

```
In[13]:= IsomorphTest1[{{1, 0, 1}, {1, 1, 0}, {0, 1, 1}}]
Out[13]:= {{{1,0,1}, {1,1,0}, {0,1,1}}, {{1,1,0}, {0,1,1}, {1,0,1}},
           {{1,1,0}, {0,1,1}, {1,0,1}}, {{1,0,1}, {1,1,0}, {0,1,1}},
           {{1,0,1}, {1,1,0}, {0,1,1}}, {{1,1,0}, {0,1,1}, {1,0,1}}}
```

We now count the isomorphic classes of the hypergroupoids:

```
Cardin[d_] :=
Module[{h2, cardinalities, len, temp1, temp},
  h2 = Good[d];
  cardinalities = Table[0, {j1, 1, Factorial[d]}];
  While[Length[h2] > 0,
        temp = Union[IsomorphTest1[h2[[1]]]];
        len = Length[Union[temp]];
        cardinalities[[len]] = cardinalities[[len]] + 1;
        h2 = Complement[h2, temp]
       ];
Return[cardinalities]]
```

We then get the following:

```
In[14]:= Cardin[2]
Out[14]:= {2, 3}
In[15]:= Total[%]
Out[15]:= 5
In[16]:= Cardin[3]
Out[16]:= {3, 1, 13, 0, 0, 32}
In[17]:= Total[%]
Out[17]:= 49
In[18]:= Cardin[4]
Out[18]:= {3, 0, 2, 17, 0, 15, 0, 8, 0, 0, 0, 238,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1039}
In[19]:= Total[%]
Out[19]:= 1322
In[20]:= Cardin[5]
Out[20]= ...
In[21]:= Total[%]
Out[21]= 117534
```

By substituting line `h2=Good[di]` in the above function `Cardin[]` with line `h2=GoodH[di]`, we get the isomorphic classes of the hypergroups:

```
In[22]:= Cardin[2]
Out[22]= {2, 2}
In[23]:= Total[%]
Out[23]= 4
In[24]:= Cardin[3]
Out[24]= {3, 1, 10, 0, 0, 19}
In[25]:= Total[%]
Out[25]= 33
In[26]:= Cardin[4]
Out[26]= {3, 0, 2, 11, 0, 12, 0, 5, 0, 0, 0, 139,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 329}
In[27]:= Total[%]
Out[27]= 501
In[28]:= Cardin[5]
Out[28]= ...
In[29]:= Total[%]
Out[29]= 26409
```

Table 1
Cumulative results.

Order →	2	3	4	5
Boolean matrices (BM)	16	512	65 536	33 554 432
BM forming hypergroupoids	8	236	28 023	13 419 636
Non-isomorphic BM forming hypergroupoids	5	49	1 322	117 534
BM forming hypergroups	6	149	9 729	2 921 442
Non-isomorphic BM forming hypergroups	4	33	501	26 409
BM forming join hypergroups	5	106	6 979	2 122 681

5. Conclusion

It is proven herein that there exist numerous Rosenberg-type hypercompositional structures, the number of which is calculated with the use of Mathematica packages that are constructed for this purpose. The results of these calculations are given in the cumulative Table 1 for orders 2, 3, 4 and 5. Because of the principle of duality enunciated above, the same number of hypercompositional structures exist for the dual hypercomposition.

References

[1] P. Corsini, Binary relations and hypergroupoids, Ital. J. Pure Appl. Math. 7 (2000) 11–18.
[2] P. Corsini, On the hypergroups associated with a binary relation, Multi.-Val. Logic 5 (2000) 407–419.
[3] P. Corsini, V. Leoreanu, Hypergroups and binary relations, Algebra Universalis 43 (2000) 321–330.
[4] I. Cristea, Several aspects on the hypergroups associated with n-ary relations, An. Stiint. Univ. "Ovidius" Constanta Ser. Mat. 17 (3) (2009) 99–110.
[5] I. Cristea, M. Jafarpour, S.S. Mousavi, A. Soleymani, Enumeration of Rosenberg hypergroups, Comput. Math. Appl. 32 (2011) 72–81.
[6] I. Cristea, M. Stefanescu, Binary relations and reduced hypergroups, Discrete Math. 308 (2008) 3537–3544.
[7] I. Cristea, M. Stefanescu, C. Angheluta, About the fundamental relations defined on the hypergroupoids associated with binary relations, Electron. J. Combin. 32 (2011) 72–81.
[8] M. De Salvo, G. Lo Faro, Hypergroups and binary relations, Multi.-Val. Logic 8 (2002) 645–657.
[9] M. De Salvo, G. Lo Faro, A new class of hypergroupoids associated to binary relations, J. Mult.-Valued Logic Soft Comput. 9 (2003) 361–375.
[10] J. Jantosciak, Ch.G. Massouros, Strong identities and fortification in transposition hypergroups, J. Discrete Math. Sci. Cryptogr. 6 (2003) 169–193.
[11] F. Marty, Sur une gènèralisation de la notion de groupe, Huitième Congres des Mathématiciens Scad, Stockholm, 1934, pp. 45–59.
[12] Ch.G. Massouros, Hypergroups and convexity, Riv. Mat. Pura ed Appl. 4 (1989) 7–26.
[13] Ch.G. Massouros, Quasicanonical hypergroups, in: Proc. of the 4th Internat. Cong. on Algebraic Hyperstructures and Applications, World Scientific, 1990, pp. 129–136.
[14] Ch.G. Massouros, On the semi-subhypergroups of a hypergroup, Int. J. Math. Math. Sci. 14 (1991) 293–304.
[15] G.G. Massouros, Automata-languages and hypercompositional structures, Doctoral Thesis, NTUA 1993.
[16] G.G. Massouros, Hypercompositional structures in the theory of the languages and automata, An. Ştiinţ. Univ. Al. I. Cuza Iaşi Tom. III, Informat. (1994) 65–73.
[17] G.G. Massouros, The subhypergroups of the fortified join hypergroup, Ital. J. Pure Appl. Math. 2 (1997) 51–63.
[18] Ch.G. Massouros, G.G. Massouros, Hypergroups associated with graphs and automata, AIP Conf. Proc. 1168 (2009) 380–383.
[19] G.G. Massouros, Ch.G. Massouros, J.D. Mittas, Fortified join hypergroups, Ann. Math. Blaise Pascal 3 (1996) 155–169.
[20] Ch.G. Massouros, Ch. Tsitouras, Enumeration of hypercompositional structures defined by binary relations, Ital. J. Pure Appl. Math. 28 (2011) 43–54.
[21] Ch.G. Massouros, Ch. Tsitouras, S. Voliotis, Square roots of total Boolean matrices—enumeration issues, in: 16th International Conference on Systems, Signals and Image Processing, IWSSIP 2009, pp. 1–4.
[22] I.G. Rosenberg, Hypergroups and join spaces determined by relations, Ital. J. Pure Appl. Math. 4 (1998) 93–101.
[23] Ch. Tsitouras, Ch.G. Massouros, On enumerating hypergroups of order 3, Comput. Math. Appl. 59 (2010) 519–523.
[24] Wolfram Research, Inc., Mathematica, Version 7.0, Champaign, IL, 2008.