



## On enumeration of hypergroups of order 3

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### ABSTRACT

In this paper we present a symbolic manipulation package that enumerates the hypergroups of order 3. It separates them into isomorphic classes and calculates their cardinality.

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### 1. Introduction

1934 was the year that Frederic Marty defined the hypergroup [1] and so the time that the theory of hypercompositional structures was born. Over the years hypercompositional structures have been used in algebra, geometry, convexity, automata theory and even in some applied sciences. To make this paper self-contained, some definitions are recalled. Thus a *partial hypergroupoid* is the pair  $(H, \cdot)$  where  $H$  is a nonempty set and “ $\cdot$ ” is a *hypercomposition* in  $H$ , i.e. a function from  $H \times H$  to the powerset  $P(H)$  of  $H$ . If the map is from  $H \times H$  to the family of the nonempty subsets of  $H$ , then  $(H, \cdot)$  is called a hypergroupoid. The axioms which endow  $(H, \cdot)$  with the *hypergroup* structure are:

- i.  $a(bc) = (ab)c$  for every  $a, b, c \in H$  (associativity);
- ii.  $aH = Ha = H$  for every  $a \in H$  (reproductivity).

In a hypergroup, the result of the hypercomposition is always a nonempty set. Indeed, let  $ab = \emptyset$ ; then  $H = aH = a(bH) = (ab)H = \emptyset = \emptyset$  which is absurd (see Mittas [2]). If only (i) is valid then  $(H, \cdot)$  is called a semihypergroup, while it is called a quasi-hypergroup if only (ii) holds.

Extend the hypercomposition “ $\cdot$ ” from  $H$  to  $P(H)$ , by setting for all  $A, B \in P(H)$ :

$$A \cdot B = \bigcup_{(a,b) \in A \times B} a \cdot b$$

where  $A \cdot b$  and  $a \cdot B$  will have the same meaning as  $A \cdot \{b\}$  and  $\{a\} \cdot B$  respectively. Also, when nothing opposes it, there is no distinction made between the elements and their corresponding singletons.

Hypergroups are much more flexible and varied than groups. For example if  $H$  is of prime cardinality  $p$ , there are a large number of non-isomorphic hypergroups on  $H$ , while, up to isomorphism, there is only one group  $Z_p$ . This becomes clear in this paper, which enumerates the hypergroups of cardinality 3.

### 2. The method

A hypergroupoid is a set  $H \neq \emptyset$  with a hypercomposition “ $\cdot$ ” which is not necessarily associative or reproductivity. As regards the notification of the elements of the hypergroupoids of order 3, it can be assumed that they share the set  $H = \{1, 2, 3\}$ .

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The hypercompositions in  $H$  are defined through the following table:

$$\begin{array}{c|ccc}
 \cdot & 1 & 2 & 3 \\
 1 & a_{11} & a_{12} & a_{13} \\
 2 & a_{21} & a_{22} & a_{23} \\
 3 & a_{31} & a_{32} & a_{33}
 \end{array} \tag{1}$$

where  $a_{ij} \subseteq H, i, j = 1, 2, 3$ . The elements  $a_{ij}$  are chosen among the seven-element set

$$\Lambda = P(H) \setminus \emptyset = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

If  $\mathcal{H}_3$  denotes the set of all hypergroupoids of third order, then for its cardinality there holds  $|\mathcal{H}_3| = 7^9 = 40\,353\,607$ . Migliorato [3] found, by computer, the total number  $N_3 = 23\,192$  of hypergroups of order 3 while Nordo [4] computed using a program written in PASCAL the number  $S_3 = 3999$  of non-isomorphic hypergroups of the same order.

The Mathematica package given in the Appendix is based on two functions, namely `ReproductivityTest[ ]` and `AssociativityTest[ ]`, which check out the corresponding properties. Their argument is a set of hypergroupoids in a list and their output is a True/False table. This package checks whether hypergroupoids of any order form a hypergroup.

The reproductivity of the hypercompositions defined in (1) can be checked, through the verification of validity of the equivalent (to this axiom) equalities:

$$\bigcup_{j=1}^3 a_{ij} = H, \quad \text{for } i = 1, 2, 3 \quad \text{and} \quad \bigcup_{i=1}^3 a_{ij} = H, \quad \text{for } j = 1, 2, 3. \tag{2}$$

The cases that successfully pass this first test (i.e. the reproductivity validity test) are going through the associativity validity test, which is checking all the 27 possible triples  $a(bc) = (ab)c$ .

### 2.1. Classes of isomorphism

A hypergroupoid of order 3 is isomorphic with another five hypergroupoids. This derives from interchanges among the elements of the set  $H$ . More precisely

- (i) keep 1                    interchange 2, 3
- (ii) keep 3                    interchange 1, 2
- (iii) change 1 by 2    change 2 by 3    change 3 by 1
- (iv) change 1 by 3    change 2 by 1    change 3 by 2
- (v) keep 2                    interchange 1, 3

So for the above matrix of hypercomposition (1) there are derived the following five isomorphic hypercompositions:

$$\begin{array}{c|ccc} \text{(i)} & 1 & 2 & 3 \\ \hline 1 & \tilde{a}_{11} & \tilde{a}_{13} & \tilde{a}_{12} \\ 2 & \tilde{a}_{31} & \tilde{a}_{33} & \tilde{a}_{32} \\ 3 & \tilde{a}_{21} & \tilde{a}_{23} & \tilde{a}_{22} \end{array} \quad \begin{array}{c|ccc} \text{(ii)} & 1 & 2 & 3 \\ \hline & \tilde{a}_{22} & \tilde{a}_{21} & \tilde{a}_{23} \\ & \tilde{a}_{12} & \tilde{a}_{11} & \tilde{a}_{13} \\ & \tilde{a}_{32} & \tilde{a}_{31} & \tilde{a}_{33} \end{array} \quad \begin{array}{c|ccc} \text{(iii)} & 1 & 2 & 3 \\ \hline & \tilde{a}_{33} & \tilde{a}_{31} & \tilde{a}_{32} \\ & \tilde{a}_{13} & \tilde{a}_{11} & \tilde{a}_{12} \\ & \tilde{a}_{23} & \tilde{a}_{21} & \tilde{a}_{22} \end{array} \quad \begin{array}{c|ccc} \text{(iv)} & 1 & 2 & 3 \\ \hline & \tilde{a}_{22} & \tilde{a}_{23} & \tilde{a}_{21} \\ & \tilde{a}_{32} & \tilde{a}_{33} & \tilde{a}_{31} \\ & \tilde{a}_{12} & \tilde{a}_{13} & \tilde{a}_{11} \end{array} \quad \begin{array}{c|ccc} \text{(v)} & 1 & 2 & 3 \\ \hline & \tilde{a}_{33} & \tilde{a}_{32} & \tilde{a}_{31} \\ & \tilde{a}_{23} & \tilde{a}_{22} & \tilde{a}_{21} \\ & \tilde{a}_{13} & \tilde{a}_{12} & \tilde{a}_{11} \end{array}$$

where  $\tilde{a}_{ij}$  are the subsets that derive from the transposition of the corresponding  $a_{ij}$ 's of the original matrix and the proper replacement of their elements. Analogously the same holds for higher orders and there exist  $n!$  isomorphisms for order  $n$ .

### 3. Examples and results

Let's assume that it must be verified whether or not the following two hypergroupoids are hypergroups:

$$\begin{array}{c|ccc}
 \cdot & 1 & 2 & 3 \\
 \hline
 1 & \{1\} & \{2\} & \{1, 2, 3\} \\
 2 & \{2\} & \{1, 2, 3\} & \{1, 2, 3\} \\
 3 & \{1, 2, 3\} & \{1, 2, 3\} & \{1, 3\} \\
 \hline
 \cdot & 1 & 2 & 3 \\
 \hline
 1 & \{1, 3\} & \{3\} & \{2\} \\
 2 & \{2\} & \{1, 2, 3\} & \{2\} \\
 3 & \{2\} & \{1\} & \{1, 2, 3\}
 \end{array}$$

and we write

```
In[1] :=<<HyperGroupTest.m
In[2] :=h1={{1}, {2}, {1, 2, 3}}, {{2}, {1, 2, 3}, {1, 2, 3}},
        {{1, 2, 3}, {1, 2, 3}, {1, 3}}};
        h2={{1, 3}, {3}, {2}}, {{2}, {1, 2, 3}, {2}}, {{2}, {1}, {1, 2, 3}}};
In[3] :=HyperGroupTest[{h1, h2}]
Out[3]= {True, False}
```

From the last line (Out[3]) it derives that only the first hypercomposition defines a hypergroup in  $H$ . It is obvious though, that the second hypercomposition satisfies the reproductivity axiom and so the corresponding hypergroupoid is a quasi-hypergroup. We can verify this by using the `ReproductivityTest` function of the package in the [Appendix](#).

```
In[4] :=ReproductivityTest[h2]
Out[4]= True
```

The package in the [Appendix](#) can handle hypergroupoids of arbitrary order. For  $H = \{1, 2, 3, 4\}$  we check the following hypercomposition:

$\cdot$	1	2	3	4
1	{1, 3, 4}	{1, 3, 4}	{2, 3, 4}	{1, 2, 3, 4}
2	{1, 2, 4}	{1, 2, 3, 4}	{1, 2, 4}	{1, 3, 4}
3	{1, 2, 4}	{1, 2, 3}	{3, 4}	{1, 2, 4}
4	{2, 3, 4}	{2, 3, 4}	{1, 2, 3, 4}	{1, 2, 3}

and we get

```
In[4] :=h3={{1, 3, 4}, {1, 3, 4}, {2, 3, 4}, {1, 2, 3, 4}},
        {{1, 2, 4}, {1, 2, 3, 4}, {1, 2, 4}, {1, 3, 4}},
        {{1, 2, 4}, {1, 2, 3}, {3, 4}, {1, 2, 4}},
        {{2, 3, 4}, {2, 3, 4}, {1, 2, 3, 4}, {1, 2, 3}}};
In[5] :=HyperGroupTest[{h3}]
Out[5]= {True}
```

In order to evaluate  $N_3$  all the 40-million hypergroupoids must be checked. First  $\Lambda$  is formed and then all the  $7^3 = 343$  triads (in variable `a3`) and  $7^6 = 343^2$  hexads (in variable `a6`) with elements from this set. Their combinations in one and two rows respectively form hypercomposition matrices of the form (1). Thus the memory requirements of a  $7^9$  length list containing the description of hypercompositions can be overcome. This can be done by writing the lines below where variable `HyperGroups3` collects the hypergroups that we find.

```
In[6] :=lambda = Drop[Subsets[{1, 2, 3}], 1];
In[7] :=a6=Tuples[lambda, 6];
In[8] :=a3 = Tuples[lambda, 3];
In[9] :=HyperGroups3 = {};
In[10] :=Do[
    temp={Partition[Join[a3[[j1]], a6[[j2]]], 3]};
    If[HyperGroupTest[temp][[1]], HyperGroups3 = Join[HyperGroups3, temp]]
    , {j1, 1, 343}, {j2, 1, {343}^2}];
In[11] :=Print[Length[HyperGroups3]]
Out[11]=23192
```

Interchanging in the above lines the function `HyperGroupTest` with the function `ReproductivityTest` found in the [Appendix](#) we counted 10323979 quasi-hypergroups of order 3. For order 2 we counted 35 quasi-hypergroups. Notice that for order 2 there are 14 hypergroups in 8 isomorphic classes.

A function that gives the  $|H|!$  hypercompositions which form isomorphic hypergroupoids is given by counting the observations in Section 2.1 for every order.

```
IsomorphTest[a_List] :=
Module[{p, a1, len},
    len = Length[a]; p = Permutations[Range[1, len]];
    Return[Table[a1 = a;
        a1 = ReplaceAll[a1, a1[[All, Table[j2, {j2, 1, len}]]] ->
            a1[[All, p[[j1]]]]];
        a1 = ReplaceAll[a1, a1[[Table[j2, {j2, 1, len}]]] ->
            a1[[p[[j1]]]]];
        a1 = ReplaceAll[a1, Flatten[Table[{p[[j1, j2]] -> j2}, {j2, 1, len}]]];
        a1 = Table[Table[a1[[k1, k2]] = Sort[a1[[k1, k2]]], {k2, 1, len}],
```

```

        {k1, 1, len}],
    {j1, 1, len!}]]
]

```

In order to count the number of the different non-isomorphic classes of hypergroups of order 3, a six-digit array, called *cardinalities*, is used by the program. Each time the routine encounters a non-isomorphic class, it drops it from *HyperGroups3*.

```

In[12]:=cardinalities = {0, 0, 0, 0, 0, 0};
In[13]:=While[Length[HyperGroups3] > 0,
    temp = Union[IsomorphTest[HyperGroups3[[1]]]];
    len = Length[Union[temp]];
    cardinalities[[len]] = cardinalities[[len]] + 1;
    HyperGroups3 = Complement[HyperGroups3, temp]
];
In[14]:=Total[cardinalities];
Out[14]=3999
In[15]:=Print[cardinalities];
Out[15]={6, 10, 244, 0, 0, 3739}

```

So we found that  $S_3 = 3999$ , and it is confirmed by the cardinalities of the isomorphic classes that  $6 \cdot 1 + 10 \cdot 2 + 244 \cdot 3 + 3739 \cdot 6 = N_3$ .

#### 4. Conclusion

Generally speaking, very few things are known about the construction of finite hypergroups. For example it is known that if  $(H, \cdot)$  is a group or a hypergroup, then the  $(H, \diamond)$  with  $a \diamond b = a \cdot b \cup \{a, b\}$  is a hypergroup [5]. Thus using Cayley's theorem a family of finite hypergroups can be constructed, based on finite groups. From the above analysis it is derived that there are  $7^9 = 40\,353\,607$  hypergroupoids of order 3, 23 192 of these are hypergroups. The group of order 3 is among them, as well as the corresponding hypergroup constructed as above. The set of 23 192 hypergroups is partitioned into 3999 equivalence classes. 3739 of the above classes consist of 6 members, 244 consist of 3 members, 10 have 2 members and the last 6 are one-member classes. The total hypergroup, that is the hypergroup in which the result of the hypercomposition consists always of all the elements of the hypergroup, is in the set  $C_1$  of the six, one-member, classes. In the same set there belongs the B-hypergroup, i.e. the hypergroup in which the result of the hypercomposition consists only of the two elements which participate in the hypercomposition [6]. Relating to the B-hypergroup are two other non-isomorphic hypergroups in which the hypercomposition is defined as follows (see [5]):

$$ab = \begin{cases} \{a, b\} & \text{if } a \neq b \\ H & \text{if } a = b \end{cases} \quad \text{and} \quad ab = \begin{cases} \{a, b\} & \text{if } a \neq b \\ H \setminus \{a\} & \text{if } a = b. \end{cases}$$

These hypergroups, when they have order 3, belong also to  $C_1$ . Generally the following holds:

**Proposition:** *Let  $H$  be an arbitrary set with more than two elements. Then the hypercompositions*

$$ab = \begin{cases} H & \text{if } a \neq b \\ a & \text{if } a = b \end{cases}, \quad ab = \begin{cases} \{a, b\} & \text{if } a \neq b \\ H \setminus \{a\} & \text{if } a = b \end{cases}, \quad ab = \begin{cases} H & \text{if } a \neq b \\ H \setminus \{a\} & \text{if } a = b \end{cases} \quad \text{and} \quad ab = \begin{cases} \{a, b\} & \text{if } a \neq b \\ H & \text{if } a = b \end{cases}$$

define in  $H$  four non-isomorphic hypergroups.

It is worth mentioning that the number of the classes of hypergroups that can be constructed by the known propositions and theorems is very small compared to the existing 3999 classes of hypergroups with three elements. Also one can note that the ratio of hypergroups to hypergroupoids is exceptionally small since we meet a hypergroup in every 1740 hypergroupoids.

#### Appendix

The Mathematica package that implements the two basic properties (associativity and reproductivity) for testing whether a hypergroupoid is indeed a hypergroup follows. It can be retrieved from the first author web pages.

```
BeginPackage["HyperGroupTest"]; Clear["HyperGroupTest'*"];

```

```
HyperGroupTest::usage = "HyperGroupTest[LookUpTable] tests if
hypergroupoid operation given LookUpTable forms a HyperGroup"

```

```
Begin["Private"]; Clear["HyperGroupTest'Private'*"];

```

```

HyperGroupTest[LookupTable_List] :=
Table[If[ReproductivityTest[LookupTable[[j]]],
If[AssociativityTest[LookupTable[[j]]], True,
False], {j,1,Length[LookupTable]}];

AssociativityTest[LookupTable1_List] :=
Module[{i,j,k,len,test},
i = 1; j = 1; k = 1; test = True; len = Length[LookupTable1];
While[test && i<=len,
test = Union[Flatten[Union[Extract[LookupTable1,
Distribute[{{LookupTable1[[i, j]], {k}}, List]]]]] ==
Union[Flatten[Union[Extract[LookupTable1,
Distribute[{{i}, LookupTable1[[j, k]], List]]]]];
k = k + 1; If[k > len,
k = 1; j = j + 1;
If[j > len, i = i + 1; j = 1];
];
];
Return[test]
];

ReproductivityTest[LookupTable1_List] :=
Union[Apply[Union, LookupTable1, 1]] == {Range[1, Length[LookupTable1]]} &&
Union[Apply[Union, Transpose[LookupTable1], 1]] == {Range[1, Length[LookupTable1]]};

End[];
EndPackage[];

```

In the package above the function `AssociativityTest[ ]` is implemented by using `While`. In the most of the tested hypercompositions the property of associativity failed after the first two or three checks. Consequently it was not necessary to go through all 27 cases for hypergroups of order 3. In contrast the function `ReproductivityTest[ ]` tested all rows and columns simultaneously according to property (2), since this does not increase computational time.

The program above can be used in order to construct hypergroups of any order. A random hypergroup of sixth order can be derived writing

```

In[16] := Do[ temp =
RandomChoice[
Complement[Subsets[{1, 2, 3, 4, 5, 6}, {3, 6}], {{}}, {6, 6}];
If[HyperGroupTest[{{temp}][[1]], Print[temp]], {j1, 1, 3000000}

```

And we get the following hypergroup:

	1	2	3	4	5	6
1	{2, 3, 4}	{1, 2, 3, 4, 5, 6}	{2, 3, 4, 5, 6}	{1, 3, 4, 6}	{1, 4, 5, 6}	{1, 2, 3, 4, 6}
2	{2, 3, 5, 6}	{1, 3, 4, 5}	{1, 3, 4}	{1, 2, 3, 5, 6}	{1, 2, 5}	{1, 3, 4, 6}
3	{1, 3, 4, 6}	{2, 3, 5, 6}	{2, 3, 5, 6}	{1, 2, 3, 4}	{1, 2, 3, 4, 5, 6}	{1, 3, 5, 6}
4	{1, 3, 4, 5, 6}	{1, 2, 3, 4, 5, 6}	{1, 2, 3}	{2, 3, 4, 5}	{1, 2, 3, 4}	{1, 2, 3, 4, 5, 6}
5	{1, 2, 5, 6}	{1, 3, 4, 6}	{2, 3, 4, 5}	{1, 2, 3, 4, 5, 6}	{2, 3, 4, 5, 6}	{1, 2, 3, 5}
6	{1, 2, 4, 5, 6}	{2, 3, 4}	{1, 2, 3, 4, 5, 6}	{1, 2, 4}	{1, 2, 4, 5, 6}	{2, 3, 4, 5}

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