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Strong identities and fortification in transposition hypergroups

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Strong identities and fortification in transposition hypergroups

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Abstract

An element e in a hypergroup H is a *strong* identity if $x \in ex = xe \subseteq x \cup e$. The elements of H separate into two classes, the set $A = \{x \in H \,|\, ex = xe = x \cup e\}$, including e, of *attractive* elements and the set $C = \{x \in H - e \,|\, ex = xe = x\}$ of *canonical* elements. If H is a transposition hypergroup then A is shown to be a closed subhypergroup of *essentially indistinguishable* elements. The structure of H is then determined, for A can be *contracted* into e leaving the "resulting" $C \cup e$, which is a polygroup under the relativized hyperoperation. Therefore, H can be reconstructed from A and $C \cup e$, as H is isomorphic to the *expansion* of the polygroup $C \cup e$ by the transposition hypergroup A through e. The study of transposition hypergroups containing a strong identity separates into the study of polygroups and the study of transposition hypergroups of all attractive elements.

A *fortified* transposition hypergroup H is defined and shown to contain a unique strong identity. Moreover, each nonidentity element is shown to have unique nonidentity left and right inverses that are identical. For H consisting of all attractive elements, the subhypergroups K that are *symmetric*, $K = K^{-1}$, are studied. The *double cosets* of K, the sets $K \setminus (x/K) = (K \setminus x)/K$ if $x \notin K$, otherwise K, partition H. The resulting quotient space $H : K$ of double cosets is proven to be a fortified transposition hypergroup in which K is the strong identity and every member is attractive.

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1. Introduction and basics

Many algebraic hypergroups are extensive, the elements being joined are members of the resulting join. Moreover, it is known, and straightforward to verify, that the extensive enlargement of any hypergroup is also a hypergroup. An identity element e of an extensive hypergroup must be a member of every join in which it is a factor, that is, $e \in ex$ and $e \in xe$ for every x. Minimal such identities, where $ex = xe = \{x, e\}$, are considered. More generally, a study of any hypergroup, but for the most part, a transposition hypergroup, which contains an element e, herein called a *strong* identity, that satisfies $x \in ex = xe \subseteq \{x, e\}$ is made.

If a transposition hypergroup with a strong identity has the property that each nonidentity element has unique nonidentity right and left inverses which are identical, it is said to be a *fortified* transposition hypergroup. When commutative, such hypergroups have application to the theory of languages and automata and have been known and studied as fortified join hypergroups (see [8], [7] and [9]). Fortified transposition hypergroups are examined up through the study of quotient structures in this presentation.

To conclude this section, the basics needed for this work are reviewed.

Hypergroups. Let $(H, \cdot)$ be an algebraic hypergroup. So $\cdot$ is a hyperoperation or join operation in H, given $a, b \in H$, the join $a \cdot b$, or just ab, is a nonempty subset of H. For subsets A and B of H, the join $AB = \bigcup\{ab \mid a \in A, b \in B\}$. Notationally, aB is used for $\{a\}B$ and Ba for $B\{a\}$. Furthermore, $(H, \cdot)$ satisfies the two axioms,

(Reproduction) $aH = Ha = H$ for all $a \in H$;
(Associativity) $a(bc) = (ab)c$ for all $a, b, c \in H$.

Extension Hyperoperations. The join operation $\cdot$ has two inverses *right extension* $/$ and *left extension* $\backslash$ defined by

$$a/b = \{x \mid a \in xb\} \quad \text{and} \quad b \backslash a = \{x \mid a \in bx\}. \tag{1}$$

It is convenient to use the relational notation $A \approx B$ (read A meets B) to assert that subsets A and B have an element in common, that is, that $A \cap B \neq \emptyset$. Then, as a singleton $\{a\}$ is identified with its member a, the notation $a \approx A$ or $A \approx a$ is used as a substitute for $a \in A$. Thus (1) becomes the easy to manipulate,

$$x \approx a/b \text{ if and only if } a \approx xb \text{ and } x \approx b \setminus a \text{ if and only if } a \approx bx\,.$$

As is done with any hyperoperation, by definition

$$A/B = \bigcup\{a/b \mid a \in A,\ b \in B\} \text{ and } B \setminus A = \bigcup\{b \setminus a \mid a \in A,\ b \in B\}.$$

Then $A \approx B/C$ if and only if $B \approx AC$, and $A \approx C \setminus B$ if and only if $B \approx CA$.

Transposition Hypergroups. The hypergroup H is a *transposition* hypergroup if it satisfies the axiom,

$$\text{(Transposition)} \quad b \setminus a \approx c/d \text{ implies } ad \approx bc \text{ for all } a, b, c, d \in H\,.$$

Clearly the transposition axiom extends to sets, that is, $B \setminus A \approx C/D$ implies $AD \approx BC$. A theory of transposition hypergroups is given in [5].

Duality. Two statements of the theory of hypergroups are *dual* statements (see [5]) if each results from the other by interchanging the order of the join operation $\cdot$, that is, interchanging any join ab with the join ba. Observe that each of the axioms, reproduction and associativity, is self-dual. The extensions $/$ and $\setminus$ have dual definitions, hence must be interchanged in a construction of a dual statement. Observe then that the transposition axiom is self-dual. Therefore, a *principle of duality* holds for the theory of hypergroups and the theory of transposition hypergroups.

Given a theorem, the dual statement, which results from the interchanging of the order of the join operation $\cdot$ (and necessarily interchanging $/$ and $\setminus$), is also a theorem.

Equivalences. For any equivalence relation θ on the hypergroup H, let a_θ denote the equivalence class of element a and $A_\theta = \bigcup\{a_\theta \mid a \in A\}$. Let $H : \theta$ denote the family of equivalence classes, that is, $H : \theta = \{a_\theta \mid a \in H\}$. The induced hyperoperation $\circ$ on $H : \theta$ is given by

$$a_\theta \circ b_\theta = \{x_\theta \mid x \in a_\theta b_\theta\}\,.$$

The equivalence θ is known as a *regular* or a *type 2* equivalence if $a_\theta b_\theta \subseteq (ab)_\theta$, and as a *congruence* or a *type 3* equivalence if $(ab)_\theta \subseteq a_\theta b_\theta$. It is well-known for θ either regular or a congruence, that $(H : \theta, \circ)$ is a hypergroup (see [3] and [1]). The hypergroup $H : \theta$ is known as the *factor* or *quotient* hypergroup of H *modulo* θ. If θ is a regular equivalence and H is a transposition hypergroup, then it is known that $H:\theta$ is a transposition hypergroup (see [5]). Each of the notions regular equivalence and congruence is a self-dual notion.

Elementary Algebra. The two results that underlie the algebra of transposition hypergroups (see [5], Propositions 1 and 2) are stated here. They will be used without citation throughout this paper.

Proposition. *In a hypergroup,*

$$(B \setminus A)/C = B \setminus (A/C);$$
$$(A/B)/C = A/CB \text{ and dually } C \setminus (B \setminus A) = BC \setminus A;$$
$$A \neq \emptyset \text{ implies } B \subseteq (A/B) \setminus A, \text{ and dually, implies } B \subseteq A/(B \setminus A).$$

Proposition. *In a transposition hypergroup,*

$$A(B/C) \subseteq AB/C \text{ and dually } (C \setminus B)A \subseteq C \setminus BA;$$
$$A/(B/C) \subseteq AC/B \text{ and dually } (C \setminus B) \setminus A \subseteq B \setminus CA.$$

2. Strong identities

The object of study in this section is defined.

Definition. An element e of a hypergroup H is a *strong* identity if

$$x \approx ex = xe \subseteq x \cup e \quad \text{for all} \quad x \in H. \tag{2}$$

The definition obviously implies that a strong identity e satisfies $ee = e$. Then e is *idempotent* and *central* in H. Note that (2) is self-dual, so that the notion of a strong identity is a self-dual notion. A hypergroup H with a strong identity e has a natural partition. Let

$$A = \{x \in H \,|\, ex = xe = x \cup e\} \text{ and } C = \{x \in H - e \,|\, ex = xe = x\}.$$

Then $H = A \cup C$ and $A \cap C = \emptyset$. Observe that each of the subsets A and C is self-dual.

Definition. A member of A is an *attractive* element and a member of C is a *canonical* element.

See [7] for the origin of the terminology. Note that e is then attractive but not canonical.

2.1 *Examples*

Some examples of a hypergroup with a strong identity are given. Each one is chosen to be a transposition hypergroup.

Example 1. Let H be a nonempty set and $\cdot$ be union (of sets), that is, $x \cdot y = x \cup y$. Then H is a commutative transposition hypergroup (join space), known as the *minimal extensive* hypergroup and also called a B(iset)-hypergroup. The verification is treated in [9]. In H, each element is a strong identity, and every element is attractive with respect to a given strong identity.

Example 2. Let H be a quasicanonical hypergroup (polygroup). Then H is a transposition hypergroup with a unique strong identity, its scalar identity. Moreover every nonidentity element of H is canonical.

Example 3. Let H be a quasicanonical hypergroup. Let $\cdot$ be the *extensive* enlargement of the hyperoperation on H, so the common enlargement of the hyperoperation on H and the set union hyperoperation of Example 1, precisely said, $x \cdot y = xy \cup x \cup y$ where xy is the hyperproduct in H. Then H under $\cdot$ is easily seen to be a hypergroup. Denoting the right and left extensions that are inverse to $\cdot$ by $\cdot/$ and $\cdot\backslash$, respectively, it is immediate that

$$x \cdot/ y = \begin{cases} x/y \cup x, & \text{if } x \neq y; \\ H, & \text{if } x = y \end{cases} \quad \text{and} \quad y \cdot\backslash x = \begin{cases} y \backslash x \cup x, & \text{if } x \neq y; \\ H, & \text{if } x = y\,. \end{cases}$$

Transposition, since it holds for a quasicanonical hypergroup, can now be verified in a straightforward case-by-case argument. Then H under $\cdot$ is a transposition hypergroup. It contains a unique strong identity, the scalar identity of the quasicanonical H, and every element of H is now attractive. The above construction appears in [7] for the commutative case.

Example 4. Let H be a dilated B-hypergroup (see [7]). That is, H is a set with a distinguished element e. Furthermore, $\cdot$ is the set union hyperoperation of Example 1 enlarged so that e is in the hypercomposition of an element with itself. Thus,

$$x \cdot y = \begin{cases} x \cup y, & \text{if } x \neq y; \\ x \cup e, & \text{if } x = y\,. \end{cases}$$

That H is a commutative hypergroup is easily verified. Then noting that

$$x/y = y \setminus x = \begin{cases} H, & \text{if } x = y; \\ x \cup y, & \text{if } x \neq y \text{ and } x = e; \\ x, & \text{if } x \neq y \text{ and } x \neq e\,, \end{cases}$$

transposition follows without difficulty. Therefore, H is a join space with the unique strong identity e, and in which, every element is attractive. It may also be observed that H is not isomorphic to a hypergroup constructed as in Example 3 if the order of H is at least three.

Example 5. Let K be a quasicanonical hypergroup with scalar identity e. Let L be a transposition hypergroup, disjoint from $K - e$, all of whose elements are attractive with respect to a strong identity also denoted by e. Let $H = (K - e) \cup L$ be the *expansion* of K by L *through* the idempotent e of K as defined in [4]. Therefore, the hyperoperation $\cdot$ on H is given by

$$x \cdot y = \begin{cases} xy, & \text{if } x, y \in L; \\ \sigma^{-1}(\sigma(x)\sigma(y)), & \text{otherwise} \end{cases}$$

where $\sigma : H \to K$ is given by

$$\sigma(x) = \begin{cases} e, & \text{if } x \in L; \\ x, & \text{otherwise.} \end{cases}$$

From [4] it follows that H under $\cdot$ is a hypergroup in which the extensions $\cdot/$ and $\cdot\setminus$ satisfy

$$x \cdot/ y = \begin{cases} x/y, & \text{if } x, y \in L\,; \\ \sigma^{-1}(\sigma(x)/\sigma(y)), & \text{otherwise} \end{cases}$$

and

$$y \cdot\setminus x = \begin{cases} y \setminus x, & \text{if } x, y \in L\,; \\ \sigma^{-1}(\sigma(y) \setminus \sigma(x)), & \text{otherwise.} \end{cases}$$

Then in a manner that modifies the proof of the transposition axiom for join spaces in [4] to accommodate noncommutative joins, it can be verified that H is a transposition hypergroup. Furthermore, e is a strong identity in H for which L is the set of attractive elements and $K-e$ is the set of canonical elements.

2.2 *Uniqueness*

Let H be a hypergroup with a strong identity e. Example 1 makes it clear that e need not be unique. Let $E = \{e \in H \,|\, e \text{ satisfies } (2)\}$ be the set of strong identities of H.

Theorem 1. *Let $u, v \in E$. Then $uv = vu = u \cup v$.*

Recall that a subset K of a hypergroup H is a *subhypergroup* if $xK = Kx = K$ for each $x \in K$. A subhypergroup K of H is *central* if $xy = yx$ for each $x \in K$ and $y \in H$.

Corollary 1. *The set E of strong identities is a central subhypergroup of H.*

Moreover, E is a B-hypergroup (Example 1) contained in H.

Now the terms *attractive* element and *canonical* element would seem to depend upon the member of E with respect to which they are defined. This is not the case – the concepts are independent of the chosen strong identity as is obvious from the next theorem.

Theorem 2. *Let $u, v \in E$. Then $ux = xu = x \cup u$ implies $vx = xv = x \cup v$.*

Proof. Suppose $ux = xu = x \cup u$. Then

$$v \approx vu \subseteq v(x \cup u) = v(xu) = (vx)u \subseteq vx \cup u\,.$$

Hence, whether $v = u$ or not, $v \approx vx = xv$, and so $vx = xv = x \cup v$. □

A hypergroup with a unique strong identity can be obtained from H by factoring E out. Let ε be the equivalence relation on H whose equivalence classes are given by

$$x_\varepsilon = \begin{cases} E, & \text{if } x \in E\,; \\ x, & \text{otherwise.} \end{cases}$$

Then ε is regular as $x_\varepsilon y_\varepsilon \subseteq (xy)_\varepsilon$ is easy to establish. Hence, the family $H : e$ of equivalence classes forms a hypergroup under the induced hyperoperation.

Theorem 3. *The quotient hypergroup $H : \varepsilon$ has the unique strong identity E and is a transposition hypergroup if H is.*

2.3 *Attractive and canonical elements*

Let H be a transposition hypergroup containing a strong identity e. The partition of H into A and C is studied.

Theorem 4. *If* $x \neq e$ *then* $x/e = e \setminus x = x$.

Proof. $y \approx x/e$ implies $x \approx ye \subseteq y \cup e$. Since $x \neq e$, clearly $y = x$. Thus, $x/e = x$ must follow. By duality, $e \setminus x = x$ holds. □

Theorem 5. $A = e/e = e \setminus e$.

Proof. $x \approx A$ is equivalent to $xe \approx e$, and so to $x \approx e/e$. Dually, $A = e \setminus e$ holds. □

Recall that a subset S of a hypergroup is *closed* if $x, y \approx S$ implies $x/y \subseteq S$ and $y \setminus x \subseteq S$.

Theorem 6. A *is closed.*

Proof. It is enough to show $A/A \subseteq A$. For then by duality, $A \setminus A \subseteq A$ holds. Thus, it follows that A is closed. By the elementary algebra of transposition hypergroups,

$$\begin{aligned} A/A &= (e/e)/(e \setminus e) = e/(e \setminus e)e \subseteq e/(e \setminus ee) = e/(e \setminus e) = e/(e/e) \\ &\subseteq ee/e = e/e = A. \end{aligned}$$

□

It is known that a closed subset of a hypergroup is a subhypergroup (see [5]).

Corollary 2. A *is a closed subhypergroup of* H.

Corollary 3. A *is a transposition hypergroup with strong identity* e *in which every element is attractive.*

Theorem 7. *If* $a, b \approx A$ *then* $a \cup b \subseteq ab$.

Proof. $a \approx ae \subseteq a(b \cup e) = a(be) = (ab)e = ab \cup e$. Thus, whether $a = e$ or not, $a \approx ab$. Duality, $b \approx ab$. □

Corollary 4. *If* $a, b \approx A$ *then* $a \approx a/b$ *and* $a \approx b/a$.

Corollary 5. *If* $a \approx A$ *then* $A = a/a = a \setminus a$.

Proof. Let $x \approx A$. Then $a \approx xa$ yields $x \approx a/a$. Thus $A \subseteq a/a$. Since A is closed, $A = a/a$. Dually, $A = a \setminus a$. □

The theorem states that, in a transposition hypergroup that contains a strong identity, the subhypergroup A of attractive elements is *extensive*, an enlargement of a B-hypergroup. Its second corollary implies that A contains no proper closed subhypergroups.

Theorem 8. *If $c \approx C$ and $a \approx A$ then $ca = ac = c$.*

Proof. Let $x \approx ca$. Then $c \approx x/a$. Since A is closed, $x \not\approx A$, so $x \approx C$. But then $a \approx c \setminus x$, and so $e \approx ae \subseteq (c \setminus x)e \subseteq c \setminus xe = c \setminus x$. Therefore, $x \approx ce = c$. Since ca is nonempty, $ca = c$ holds. Dually, $ac = c$ holds. □

The theory in [4] applies. A subhypergroup K of hypergroup is a subhypergroup of *operationally equivalent* elements if $xy = Ky$ and $yx = yK$ whenever $x \approx K$ and $y \not\approx K$.

Corollary 6. *A is a subhypergroup of operationally equivalent elements.*

Theorem 9. *Let $c \approx C$. Then $A \approx cz$ implies $A \subseteq cz$, and $A \approx zc$ implies $A \subseteq zc$.*

Proof. Suppose $A \approx cz$. Let $a \approx A \cap cz$. Then $e \approx ea \subseteq ecz = cz$. Let $x \approx A$. Then $x \approx xe \subseteq xcz = cz$, and so $A \subseteq cz$. Dually, $A \approx zc$ implies $A \subseteq zc$. □

A subhypergroup K of a hypergroup is a subhypergroup of *inseparable* element if $K \approx xy$ implies $K \subseteq xy$ whenever $x \not\approx K$ or $y \not\approx K$.

Corollary 7. *A is a subhypergroup of inseparable elements.*

A subhypergroup K of a hypergroup is a subhypergroup of *essentially indistinguishable* elements if K is a subhypergroup of operationally equivalent and inseparable elements.

Corollary 8. *A is a subhypergroup of essentially indistinguishable elements.*

2.4 *Structure results*

Let H be a transposition hypergroup containing a strong identity e. The theory in [4] for subhypergroups of essentially indistinguishable elements and the theory in [5] for transposition hypergroups is employed to deduce the structure of H.

As in [4], let α be the equivalence relation on H whose equivalence classes are A and the elements (singletons) of C. Hence,the class of x, denoted by x_α, is given by

$$x_\alpha = \begin{cases} A, & \text{if } x \in A\,; \\ x, & \text{if } x \in C\,. \end{cases}$$

Then Theorems 8 and 9 imply α satisfies $x_\alpha y_\alpha = (xy)_\alpha$, and so α is both regular and a congruence. Therefore, the family $H : \alpha$ of equivalence classes of α is a transposition hypergroup, which in [4] is called the quotient of H modulo the *contraction* of the subhypergroup A of essentially indistinguishable elements *into* the idempotent element A (of $H : \alpha$). Furthermore, $(ex)_\alpha = (xe)_\alpha = x_\alpha$, that is, e is a *scalar identity* for α as defined in [5]. Hence, Proposition 15 of [5] applies and yields the next result.

Theorem 10. *The quotient hypergroup $H : \alpha$ is a quasicanonical hypergroup (polygroup) whose scalar identity is $e_\alpha = A$.*

Recall that a subhypergroup N of a hypergroup H is *normal* if $xN = Nx$ for each $x \in H$ and *reflexive* if $x \setminus N = N/x$ for each $x \in H$. Note that, in a transposition hypergroup H with a strong identity e, the closed subhypergroup A satisfies

$$xA = Ax = \begin{cases} A, & \text{if } x \in A\,; \\ x, & \text{if } x \in C\,. \end{cases} \tag{3}$$

Thus, A is normal. Then by Proposition 9 of [5], since A is closed, A is reflexive. A notion of *equivalence modulo* a nonempty reflexive closed subhypergroup is studied in [5] for elements of a transposition hypergroup. By definition, y and z of H are *equivalent modulo A* means $yA \approx Az$. But by (3), the condition $yA \approx Az$ is equivalent to the disjunctive condition $y, z \in A$ or $y = z$, which is the defining condition of the equivalence relation α. Hence, for equivalence modulo A, the class of x, denoted by x_A and called the *coset* of A determined by x, is equal to x_α. The hypergroup, denoted by $H : A$, of cosets x_A is the hypergroup $H : \alpha$. The next theorem summarizes.

Theorem 11. *A is a reflexive closed subhypergroup of H, α is equivalence modulo A and the quasicanonical hypergroup $H : \alpha = H : A$.*

Through abuse of notation, let the canonical homomorphism of H onto $H : \alpha = H : A$ be denoted by α, that is, $\alpha(x) = x_\alpha = x_A$

for each $x \in H$. Obviously, α is the identity mapping on C and sends every member of A onto the scalar identity of $H : \alpha$. Let $\circ$ denote the hyperoperation on $H : \alpha$. Then, unless $x, y \in A$ holds,

$$\alpha^{-1}(\alpha(x) \circ \alpha(y)) = \alpha^{-1}(x_\alpha \circ y_\alpha) = (x_\alpha y_\alpha)_\alpha = xy\,.$$

Therefore, the hypergroup H can be reconstructed from the quasicanonical hypergroup $H : \alpha$ and the transposition hypergroup A of all attractive elements. The method of reconstruction is that of Example 5 with $K = H : \alpha$, $L = A$ and $\sigma = \alpha$. This is the content of Theorem 1 in [4].

Theorem 12. *The hypergroup H is the expansion of the hypergroup $H : A$ by the hypergroup A through the idempotent element $e_A = A$ of $H : A$.*

Consider the set $C \cup e$, of canonical elements with e adjoined, under the hyperoperation, denoted by $\cdot$, which is that on H relativized, that is, $x \cdot y = xy \cap (C \cup e)$ for $x, y \in C \cup e$. Consider also the canonical homomorphism α restricted to $C \cup e$. It is easy to see that α is an iso-morphism of $C \cup e$ under $\cdot$ with the quasicanonical hypergroup $H : \alpha$. The quasicanonical hypergroup $C \cup e$ is referred to as having been *contracted* from H.

Structure Theorem. *A transposition hypergroup H containing a strong identity e is isomorphic to the expansion of the quasicanonical hypergroup $C \cup e$ by the transposition hypergroup A of all attractive elements through the idempotent e.*

It is now clear that every transposition hypergroup containing a strong identity is embraced by Example 5. It is also apparent how transposition hypergroups that contain a strong identity may be studied.

Summary Remark. The study of transposition hypergroups that contain a strong identity separates into two parts, (i) the study of quasicanonical hypergroups and (ii) the study of transposition hypergroups composed of all attractive elements.

3. Fortified transposition hypergroups

Attention is directed toward the kind of transposition hypergroup defined next.

Definition. A transposition hypergroup H is *fortified* if H contains an element e which satisfies the axioms,

(a) $ee = e$;

(b) $x \approx ex = xe$ for every $x \in H$;

(c) for every $x \in H - e$ there exists a unique $y \in H - e$ such that $e \approx xy$, and furthermore, y satisfies $e \approx yx$.

By axiom (a), the element e is idempotent, and by axiom (b), is a central (two-sided) identity in H. Then, speaking with respect to identity e, axiom (c) states that each nonidentity element has a unique nonidentity right inverse which also happens to be a left inverse. But it is easy to show that a nonidentity left inverse must also be unique. For given $e \approx xy$, where $x, y \in H - e$, it follows that y is uniquely determined in $H - e$ by x, and that $e \approx yx$ holds. Hence, x is uniquely determined in $H - e$ by y. A nonidentity element has unique nonidentity right and left inverses which are identical.

Therefore, axiom (c) is self-dual. Obviously, so are (a) and (b). The principle of duality holds for a fortified transposition hypergroup (FTH).

3.1 *Identity*

Let H be a fortified transposition hypergroup with respect to identity e. For $x \in H - e$, the notation x^{-1} is used for the unique member of $H - e$ that satisfied axiom (c). Then $e \approx xx^{-1}$ and $e \approx x^{-1}x$, and clearly, $(x^{-1})^{-1} = x$. The next result is then evident.

Theorem 13. *Let $x \in H - e$. Then $e \approx xy$ or $e \approx yx$ implies $y \approx x^{-1} \cup e$.*

Next the role of e can be clarified.

Theorem 14. *e is a strong identity for H.*

Proof. It suffices to show that $ex \subseteq x \cup e$. For $x = e$ the inclusion holds. Let $x \neq e$. Suppose $y \approx ex$. Then $e \setminus y \approx x$. But $e \approx xx^{-1}$ implies $e/x^{-1} \approx x$. Thus, $e \setminus y \approx e/x^{-1}$, and transposition yields $e = ee \approx yx^{-1}$. By the previous theorem, $y \approx x \cup e$. Therefore, the theorem holds. □

It is now apparent that e is unique. For if u is an identity distinct from e, there would exist d distinct from u such that $u \approx ed$. But on the contrary, $ed \subseteq d \cup e$ holds.

Theorem 15. *A fortified transposition hypergroup is a transposition hypergroup with a unique strong identity.*

A fortified transposition hypergroup H, therefore, consists of canonical elements C and attractive elements A. Moreover, by the Structure Theorem, H is the expansion through e of the contracted hypergroup $C \cup e$ by the hypergroup A. Clearly, $C \cup e$, being quasi-canonical is a fortified transposition hypergroup. Since A is closed, A contains x^{-1} if A contains x. Hence it follows easily, in view of Corollary 3, that A is a fortified transposition hypergroup of all attractive elements. The study of such hypergroups is next.

Observe that the hypergroups given in Examples 3 and 4 are fortified transposition hypergroups of all attractive elements.

3.2 *Algebra*

Let H be a fortified transposition hypergroup with the strong identity e for which every element of H is attractive. Then, of course,

$$ea = ae = a \cup e \quad \text{for all } a \in H.$$

The algebraic results of Section 2.3 must hold. These results are summarized.

$$\begin{aligned} &a \cup b \subset ab \quad \text{for all } a, b \in H. \\ &a \subseteq a/b \text{ and } a \subseteq b \setminus a \quad \text{for all } a, b \in H. \\ &a/a = a \setminus a = H \quad \text{for all } a \in H. \\ &a/e = e \setminus a = a \quad \text{for all } a \in H - e. \end{aligned} \tag{4}$$

Moreover, every nonidentity element a of H has unique nonidentity right and left inverses which are equal and denoted by a^{-1}. For convenience, the definition $e^{-1} = e$ is made. For any subset A of H, let $A^{-1} = \{a^{-1} \mid a \in A\}$.

The next result is now clear.

Theorem 16. *If $e \neq a$ then $e/a = ea^{-1} = a^{-1} \cup e = a^{-1}e = a \setminus e$.*

The theorem can be generalized for a nonempty set.

Corollary 9. *If A is nonempty and $e \not\approx A$ then $e/A = eA^{-1} = A^{-1} \cup e = A^{-1}e = A \setminus e$.*

Now comes a result that deals with the extent to which the property known as *reversibility* holds.

Theorem 17. *If* $a \neq b$ *then* $ab^{-1} = a/b \cup b^{-1}$ *and* $b^{-1}a = b \setminus a \cup b^{-1}$.

Proof. Assume $a \neq b$. If $a = e$ then $b \neq e$, so the previous theorem applies. If $b = e$ then $a \neq e$, so the last result of (4) applies.

Assume $a \neq e$ and $b \neq e$. Then $a \neq b$ implies $e \not\approx ab^{-1}$. Thus, by the previous theorem and (4),

$$a/b \cup b^{-1} \subseteq a/(e/b^{-1}) \cup b^{-1} \subseteq ab^{-1}/e \cup b^{-1} = ab^{-1} \cup b^{-1} = ab^{-1}.$$

On the other hand,

$$ab^{-1} \subseteq a(e/b) \subseteq ae/b = (a \cup e)/b = a/b \cup e/b = a/b \cup b^{-1} \cup e.$$

Since $e \not\approx ab^{-1}$, it follows that $ab^{-1} \subseteq a/b \cup b^{-1}$. Therefore, the first equality is established. The second equality follows by duality. □

Corollary 10. *If* A *is nonempty and* $A \not\approx B$ *then*

$$AB^{-1} = A/B \cup B^{-1} \quad \textit{and} \quad B^{-1}A = B \setminus A \cup B^{-1}.$$

Since reversibility is of some interest, it may be useful to remark that the result of the theorem cannot be improved to $ab^{-1} = a/b$ for distinct nonidentity elements a and b. Example 3, beginning with a spherical join space (see [2]) of more than three elements, gives a fortified transposition hypergroup of all attractive elements in which a counterexample is effected for any such a and b provided a and b^{-1} are distinct also.

Inverse of joins and extensions are considered next.

Theorem 18. *If* $a \neq b^{-1}$ *then* $(ab)^{-1} = b^{-1}a^{-1}$.

Proof. If $a = e$ or $b = e$, the result clearly holds. Assume $a \neq e$ and $b \neq e$. Given $a \neq b^{-1}$, note that $e \not\approx ab$. By the previous results,

$$\begin{aligned}(ab)^{-1} \cup e &= e/ab = (e/b)/a = (b^{-1} \cup e)/a \\ &= b^{-1}/a \cup e/a = b^{-1}/a \cup a^{-1} \cup e = b^{-1}a^{-1} \cup e.\end{aligned}$$

Since $e \not\approx ab$, then $e \not\approx (ab)^{-1}$ and $e \not\approx b^{-1}a^{-1}$. Hence, the theorem is established. □

Corollary 11. *If* $A \not\approx B^{-1}$ *then* $(AB)^{-1} = B^{-1}A^{-1}$.

Observe that the theorem also holds when $a = b = e$. Hence, the corollary holds when $A \cap B^{-1} = e$. That $(aa^{-1})^{-1} = aa^{-1}$ may fail to hold is shown by Example 2.2 in [8].

Theorem 19. $(a/b)^{-1} \cup b = b/a \cup a^{-1}$ *and* $(b \setminus a)^{-1} \cup b = a \setminus b \cup a^{-1}$.

Proof. If $a = b$ then the result is clear by (4). If $a \neq b$ then by the previous two theorems,

$$(a/b)^{-1} \cup b = (a/b \cup b^{-1})^{-1} = (ab^{-1})^{-1} = ba^{-1} = b/a \cup a^{-1}.$$

Duality gives the rest. □

Corollary 12. *If* A *and* B *are nonempty then*

$$(A/B)^{-1} \cup B = B/A \cup A^{-1} \quad \textit{and} \quad (B \setminus A)^{-1} \cup B = A \setminus B \cup A^{-1}.$$

3.3 *Symmetric subhypergroups*

Let H be a fortified transposition hypergroup with the strong identity e for which every element of H is attractive. Since $a/a = H$ for any $a \in H$, there are no nonempty proper closed subhypergroups in H. The study of subhypergroups that are closed with respect to inverses becomes of interest.

Definition. A subset A is *symmetric* if $a \in A$ implies $a^{-1} \in A$. A subhypergroup K that is symmetric is a *symmetric* subhypergroup.

Note that A is symmetric if and only if $A^{-1} = A$. Since $A \subseteq xA$ and $A \subseteq Ax$ by (4), it follows that if A is closed under join then A is a subhypergroup. Hence, K is a symmetric subhypergroup if and only if $KK = K$ and $K^{-1} = K$. Finally, observe, that e is a symmetric subhypergroup and that $e \approx K$ for any nonempty symmetric subhypergroup K.

The notions of being symmetric and of being a symmetric subhypergroup are self-dual notions.

Three results that lead to the notion of cosets are given.

Theorem 20. *Let* K *be a symmetric subhypergroup. If* $a \not\approx K$ *then*

(a) $a/K \not\approx K$ *and* $K \setminus a \not\approx K$;

(b) $aK = a/K \cup K$ *and* $Ka = K \setminus a \cup K$.

Proof. Let $a \not\approx K$. (a) Since $a/K \approx K$ yields $a \approx KK = K$, certainly $a/K \not\approx K$ holds. (b) Corollary 10 is employed to conclude that $aK = aK^{-1} = a/K \cup K^{-1} = a/K \cup K$. The rest in each of (a) and (b) follows by duality. □

Theorem 21. *Let K be a symmetric subhypergroup. If $a \not\approx K$ then*

$$(a/K)K = aK \quad \textit{and} \quad K(K \setminus a) = Ka\,.$$

Proof. By (4) and the previous theorem, $aK \subseteq (a/K)K \subseteq aKK = aK$. Duality yields the other part. □

Theorem 22. *Let K be a symmetric subhypergroup. If $a \cup b \not\approx K$ then*

$$a/K \approx b/K \quad \textit{implies} \quad a/K = b/K,$$

and

$$K \setminus a \approx K \setminus b \quad \textit{implies} \quad K \setminus a = K \setminus b\,.$$

Proof. By the previous two theorems, $a/K \approx b/K$ implies, under the assumption $b \not\approx K$, that $a \approx (b/K)K = bK = b/K \cup K$. Then the assumption $a \not\approx K$ yields $a \approx b/K$, so that $a/K \subseteq (b/K)/K = b/KK = b/K$. By symmetry, $b/K \subseteq a/K$, hence $a/K = b/K$. Duality gives the rest. □

A useful result concerning inverses is given.

Theorem 23. *Let K be symmetric subhypergroup. If $a \not\approx K$ then*

(a) $K/a = Ka^{-1}$ *and* $a \setminus K = a^{-1}K$;

(b) $(a/K)^{-1} = K \setminus a^{-1}$ *and* $(K \setminus a)^{-1} = a^{-1}/K$.

Proof. The results are trivial if $K = \emptyset$. Suppose K is nonempty.

(a) Now $a^{-1} \approx e/a \subseteq K/a$ holds. Thus, since $a \not\approx K$ is given, Corollary 10 implies $Ka^{-1} = K/a \cup a^{-1} = K/a$. By duality, (a) is established.

(b) From $a \not\approx K$ follows $a^{-1} \not\approx K$. Hence, Corollary 12, the previous part and Theorem 20 imply $(a/K)^{-1} \cup K = K/a \cup a^{-1} = Ka^{-1} = K \setminus a^{-1} \cup K$. But $(a/K)^{-1} \not\approx K$ and $K \setminus a^{-1} \not\approx K$ hold, so surely, $(a/K)^{-1} = K \setminus a^{-1}$. The second part of (b) follows by duality, or by the first part. □

It is noted that part (b) of the theorem holds even if $a \approx K$.

3.4 *Cosets*

Now cosets may be defined for a nonempty symmetric subhypergroup in a fortified transposition hypergroup H of attractive elements.

Definition. Let $a \in H$ and let K be a nonempty symmetric subhypergroup. Then $a_{\overleftarrow{K}}$, the *left coset* of K determined by a, and dually, $a_{\overrightarrow{K}}$, the *right coset* of K determined by a, are given by

$$a_{\overleftarrow{K}} = \begin{cases} K, & \text{if } a \approx K\,; \\ a/K, & \text{if } a \not\approx K \end{cases} \qquad \text{and} \qquad a_{\overrightarrow{K}} = \begin{cases} K, & \text{if } a \approx K\,; \\ K \backslash a, & \text{if } a \not\approx K\,. \end{cases}$$

Notice that $a \in a_{\overleftarrow{K}}$, and dually, $a \in a_{\overrightarrow{K}}$. The results of the previous subsection assure that distinct left cosets, dually, right cosets, are disjoint. Hence, $H : \overleftarrow{K} = \{a_{\overleftarrow{K}} \mid a \in H\}$ and $H : \overrightarrow{K} = \{a_{\overrightarrow{K}} \mid a \in H\}$, the families of left and right cosets, respectively, are each partitions of H.

For $A \subseteq H$, let $A_{\overleftarrow{K}} = \bigcup\{a_{\overleftarrow{K}} \mid a \in A\}$ and $A_{\overrightarrow{K}} = \bigcup\{a_{\overrightarrow{K}} \mid a \in A\}$. Some results of the last subsection are now recast for cosets.

Theorem 24. *Let K be a nonempty symmetric subhypergroup. Then*

$$a_{\overleftarrow{K}}K = aK = a_{\overleftarrow{K}} \cup K \quad \text{and} \quad Ka_{\overrightarrow{K}} = Ka = a_{\overrightarrow{K}} \cup K\,.$$

Proof. If $a \approx K$ then the results are clear since every part of each equality equals K. If $a \not\approx K$ then use Theorem 21 and Theorem 20 (b). □

Corollary 13. *Let K be a nonempty symmetric subhypergroup. Then if A is nonempty,*

$$A_{\overleftarrow{K}}K = AK = A_{\overleftarrow{K}} \cup K \quad \text{and} \quad KA_{\overrightarrow{K}} = KA = A_{\overrightarrow{K}} \cup K\,.$$

The equivalence relation whose classes are the left cosets, dually, the right cosets, is neither necessarily regular nor necessarily a congruence. A variant of the congruence property holds, however.

Theorem 25. *Let K be a nonempty symmetric subhypergroup. Then*

$$(ab)_{\overleftarrow{K}} \subseteq a_{\overleftarrow{K}}b_{\overleftarrow{K}} \cup K \quad \text{and} \quad (ab)_{\overrightarrow{K}} \subseteq a_{\overrightarrow{K}}b_{\overrightarrow{K}} \cup K\,.$$

Proof. By the previous corollary and theorem, and by (4)

$$\begin{aligned} (ab)_{\overleftarrow{K}} &\subseteq abK \subseteq a_{\overleftarrow{K}}bK = a_{\overleftarrow{K}}(b_{\overleftarrow{K}} \cup K) \\ &= a_{\overleftarrow{K}}b_{\overleftarrow{K}} \cup a_{\overleftarrow{K}}K = a_{\overleftarrow{K}}b_{\overleftarrow{K}} \cup a_{\overleftarrow{K}} \cup K = a_{\overleftarrow{K}}b_{\overleftarrow{K}} \cup K\,. \end{aligned}$$

Duality gives the other part. □

Corollary 14. *Let* K *be a nonempty symmetric subhypergroup. Then*

$$(AB)_{\overleftarrow{K}} \subseteq A_{\overleftarrow{K}} B_{\overleftarrow{K}} \cup K \quad \textit{and} \quad (AB)_{\overrightarrow{K}} \subseteq A_{\overrightarrow{K}} B_{\overrightarrow{K}} \cup K .$$

The inverse of a left coset is a right coset, and conversely.

Theorem 26. *Let* K *be nonempty symmetric subhypergroup. Then*

$$(a_{\overleftarrow{K}})^{-1} = (a^{-1})_{\overrightarrow{K}} \quad \textit{and} \quad (a_{\overrightarrow{K}})^{-1} = (a^{-1})_{\overleftarrow{K}} .$$

Proof. If $a \approx K$ then the results are obvious. If $a \not\approx K$ then use Theorem 23 (b). □

Corollary 15. *Let* K *be nonempty symmetric subhypergroup. Then*

$$(A_{\overleftarrow{K}})^{-1} = (A^{-1})_{\overrightarrow{K}} \quad \textit{and} \quad (A_{\overrightarrow{K}})^{-1} = (A^{-1})_{\overleftarrow{K}} .$$

Each of the families $H : \overleftarrow{K}$ and $H : \overrightarrow{K}$ of cosets do not necessarily form a hypergroup, as associativity may fail for the induced hyperoperation. A hypergroup is formed, however, by the family of *double* cosets, which are studied next by using the results concerning cosets.

3.5 *Double cosets*

Let H be a fortified transposition hypergroup of attractive elements.

Definition. Let $a \in H$ and let K be a nonempty symmetric subhypergroup. Then a_K, the *double coset* of K determined by a, is given by

$$a_K = \begin{cases} K, & \text{if } a \approx K ; \\ K \setminus (a/K) = (K \setminus a)/K, & \text{if } a \not\approx K . \end{cases}$$

Observe that the notion of being a double coset is a self-dual notion. Notice that $a \in a_K$, that $a_{\overleftarrow{K}} \subseteq a_K$ and $a_{\overrightarrow{K}} \subseteq a_K$, and that $a_K = (a_{\overleftarrow{K}})_{\overrightarrow{K}} = (a_{\overrightarrow{K}})_{\overleftarrow{K}}$.

For $A \subseteq H$, let $A_K = \bigcup\{a_K \mid a \in A\}$. Then $A_K = (A_{\overleftarrow{K}})_{\overrightarrow{K}} = (A_{\overrightarrow{K}})_{\overleftarrow{K}}$ holds also.

Let $H : K = \{a_K \mid a \in H\}$ denote the family of double cosets. That $H : K$ is a partition of H is a consequence of the next result.

Theorem 27. *Let K be a nonempty symmetric subhypergroup. Then*

$$a_K \approx b_K \quad \textit{implies} \quad a_K = b_K\,.$$

Proof. By assumption, $(a_{\overleftarrow{K}})_{\overrightarrow{K}} \approx (b_{\overleftarrow{K}})_{\overrightarrow{K}}$. Hence $a_{\overleftarrow{K}} \approx (b_{\overleftarrow{K}})_{\overrightarrow{K}} = (b_{\overrightarrow{K}})_{\overleftarrow{K}}$. It follows that $a_{\overleftarrow{K}} \subseteq (b_{\overrightarrow{K}})_{\overleftarrow{K}} = (b_{\overleftarrow{K}})_{\overrightarrow{K}}$. Therefore, $a_K = (a_{\overleftarrow{K}})_{\overrightarrow{K}} \subseteq (b_{\overleftarrow{K}})_{\overrightarrow{K}} = b_K$. By symmetry, $b_K \subseteq a_K$, and so, equality holds. □

In view of $a_K = (a_{\overleftarrow{K}})_{\overrightarrow{K}} = (a_{\overrightarrow{K}})_{\overleftarrow{K}}$, the results of the last section are applicable for double cosets.

Theorem 28. *Let K be a nonempty symmetric subhypergroup. Then*

$$K\,a_K = K\,a_{\overleftarrow{K}} = a_K \cup K = K\,a\,K = a_{\overrightarrow{K}}K = a_K K\,.$$

Proof. By Corollary 13, it follows that

$$K\,a_K = K(a_{\overleftarrow{K}})_{\overrightarrow{K}} = K\,a_{\overleftarrow{K}} = (a_{\overleftarrow{K}})_{\overrightarrow{K}} \cup K = a_K \cup K\,.$$

Duality gives the remainder except for the equality with KaK. But

$$KaK = K(a_{\overleftarrow{K}} \cup K) = Ka_{\overleftarrow{K}} \cup K = Ka_{\overleftarrow{K}}\,,$$

so that the theorem is established. □

Corollary 16. *Let K be a nonempty symmetric subhypergroup. Then if A is nonempty,*

$$KA_K = KA_{\overleftarrow{K}} = A_K \cup K = KAK = A_{\overrightarrow{K}}K = A_K K\,.$$

As was the case with cosets, the equivalence relation with the double cosets as classes need be neither regular nor a congruence. The analogue of Theorem 25 for double cosets is valid.

Theorem 29. *Let K be a nonempty symmetric subhypergroup. Then*

$$(ab)_K \subseteq a_K b_K \cup K\,.$$

Proof. By Corollary 14, it follows that

$$\begin{aligned}(ab)_K &= ((ab)_{\overleftarrow{K}})_{\overrightarrow{K}} \subseteq (a_{\overleftarrow{K}} b_{\overleftarrow{K}} \cup K)_{\overrightarrow{K}} = (a_{\overleftarrow{K}} b_{\overleftarrow{K}})_{\overrightarrow{K}} \cup K_{\overrightarrow{K}} \\ &\subseteq (a_{\overleftarrow{K}})_{\overrightarrow{K}} (b_{\overleftarrow{K}})_{\overrightarrow{K}} \cup K \cup K = a_K b_K \cup K\,.\end{aligned}$$

□

Corollary 17. *Let K be a nonempty symmetric subhypergroup. Then*

$$(AB)_K \subseteq A_K B_K \cup K\,.$$

Corollary 18. *Let K be a nonempty symmetric subhypergroup. Then*

$$K \approx A_K B_K \quad \textit{implies} \quad (A_K B_K)_K = A_K B_K \cup K\,.$$

Corollary 19. *Let K be a nonempty symmetric subhypergroup. Then*

$$K \not\approx A_K B_K \quad \textit{implies} \quad (A_K B_K)_K = A_K B_K\,.$$

The inverse of a double coset is a double coset.

Theorem 30. *Let K be a nonempty symmetric subhypergroup. Then*

$$(a_K)^{-1} = (a^{-1})_K\,.$$

Proof. By Corollary 15, it follows that

$$(a_K)^{-1} = ((a_{\overleftarrow{K}})_{\overrightarrow{K}})^{-1} = ((a_{\overleftarrow{K}})^{-1})_{\overleftarrow{K}} = ((a^{-1})_{\overrightarrow{K}})_{\overleftarrow{K}} = (a^{-1})_K\,. \quad \square$$

Corollary 20. *Let K be a nonempty symmetric subhypergroup. Then*

$$(A_K)^{-1} = (A^{-1})_K\,.$$

3.6 *The quotient hypergroup*

Let H be a fortified transposition hypergroup of attractive elements. Here it is shown that the family $H : K$ of double cosets of a nonempty symmetric subhypergroup K forms a hypergroup, indeed, one that is also a fortified transposition hypergroup of attractive elements.

Let $\circ$ be the induced hyperoperation on $H : K$. It is easy to prove (Proposition 3.1 of [3]) that associativity holds for $\circ$ if and only if

$$((a_K b_K)_K\, c_K)_K = (a_K (b_K c_K)_K)_K\,.$$

A theorem establishes this equality.

Theorem 31. *Let K be a nonempty symmetric subhypergroup. Then*

$$((a_K b_K)_K\, c_K)_K = (a_K b_K c_K)_K = (a_K (b_K\, c_K)_K)_K\,.$$

Proof. The first equality is shown to hold.

Suppose $K \not\approx a_K b_K$. Then Corollary 19 yields $(a_K b_K)_K = a_K b_K$, and the rest is trivial.

Suppose $K \approx a_K b_K$. Then Corollary 18 yields $(a_K b_K) = a_K b_K \cup K$. Hence, Theorem 28 gives

$$\begin{aligned}(a_K b_K)_K c_K &= (a_K b_K \cup K) c_K = a_K b_K c_K \cup K c_K \\ &= a_K b_K c_K \cup c_K \cup K = a_K b_K c_K \cup K.\end{aligned}$$

Since $K \approx a_K b_K \subseteq a_K b_K c_K$, then $K \subseteq (a_K b_K c_K)_K$ holds. Thus,

$$((a_K b_K)_K\, c_K)_K = (a_K b_K c_K \cup K)_K = (a_K b_K c_K)_K \cup K = (a_K b_K c_K)_K.$$

The second equality is a consequence of duality. The theorem holds. □

Theorem 32. *Let K be a nonempty symmetric subhypergroup. Then $H : K$ is a hypergroup.*

Proof. Associativity in $H : K$ is a consequence of the previous theorem. Reproduction in $H : K$ is an easy consequence of reproduction in H. □

For a nonempty symmetric subhypergroup K, properties typical of a fortified transposition hypergroup of attractive elements hold for the hypergroup $H : K$.

Theorem 33. *In the hypergroup $H : K$,*

(a) $K \circ a_K = a_K \circ K = \{a_K, K\}$;

(b) $\{a_K, b_K\} \subseteq a_K \circ b_K$;

(c) $K \in a_K \circ (a^{-1})_K$.

Proof. Assertion (a) is a consequence of Theorem 28. Assertions (b) and (c) are obvious. □

Another property of a fortified transposition hypergroup of attractive elements is proven for the hypergroup $H : K$. Let ϕ and $\between$ be, respectively, the right and left extension hyperoperations on $H : K$.

Theorem 34. *In the hypergroup $H : K$, if $a_K \neq b_K$ then*

$$a_K \circ (b^{-1})_K = a_K \phi\, b_K \cup (b^{-1})_K \text{ and } (b^{-1})_K \circ a_K = b_K \between a_K \cup (b^{-1})_K.$$

Proof. The following statements are equivalent.

$$x_K \in a_K \phi\, b_K\,; \quad a_K \in x_K \circ b_K\,; \quad a_K \approx x_K b_K\,; \quad x_K \approx a_K / b_K.$$

Since $a_K \neq b_K$ is given, obviously, $a_K \not\approx b_K$, so Corollary 10 applies, and with Theorem 30 gives

$$a_K(b^{-1})_K = a_K(b_K)^{-1} = a_K/b_K \cup (b_K)^{-1} = a_K/b_K \cup (b^{-1})_K\,.$$

Hence, the statements given next are equivalent.

$$x_K \in a_K \circ (b^{-1})_K\,; \quad x_K \approx a_K(b^{-1})_K\,;$$
$$x_K \approx a_K/b_K \cup (b^{-1})_K\,; \quad x_K \in a_K \oslash b_K \cup (b^{-1})_K\,.$$

Therefore, the first assertion is established. The second assertion follows by duality. □

Next it is shown that the quotient hypergroup $H : K$ is a transposition hypergroup.

Theorem 35. *Transposition holds in the hypergroup* $H : K$*.*

Proof. Suppose $b_K \obslash a_K \approx c_K \oslash d_K$. That $a_K \circ d_K \approx b_K \circ c_K$ is shown.

If $a_K = b_K$ or $c_K = d_K$ then Theorem 33 (b) yields the result to be shown.

Assume $a_K \neq b_K$ and $c_K \neq d_K$. Then the previous theorem implies

$$a_K \in b_K \circ (c_K \oslash d_K) \subseteq b_K \circ c_K \circ (d^{-1})_K$$

and

$$c_K \in (b_K \obslash a_K) \circ d_K \subseteq (b^{-1})_K \circ a_K \circ d_K\,.$$

Hence, it follows that $a_K \oslash (d^{-1})_K \approx b_K \circ c_K$ and $(b^{-1})_K \obslash c_K \approx a_K \circ d_K$ hold.

If $a_K \neq (d^{-1})_K$ or if $c_K \neq (b^{-1})_K$, then the previous theorem applies again and either yields $a_K \oslash (d^{-1})_K \subseteq a_K \circ d_K$ or $(b^{-1})_K \obslash c_K \subseteq b_k \circ c_K$, and so done in either case.

Finally, assume $a_K = (d^{-1})_K$ and $c_K = (b^{-1})_K$. Then Theorem 33 (c) gives the result. □

The quotient hypergroup of a fortified transposition hypergroup of all attractive elements modulo the double cosets of a nonempty symmetric subhypergroup is also a fortified transposition hypergroup of all attractive elements.

Theorem 36. *The transposition hypergroup* $H : K$ *is a fortified transposition hypergroup with the strong identity* K *for which every member of* $H : K$ *is attractive.*

Proof. That K is a strong identity for which every member of $H : K$ is attractive follows immediately from Theorem 33 (a). Hence, axioms (a) and (b) for a fortified transposition hypergroup hold.

Consider axiom (c). Let $x_K \neq K$. It is immediate that $(x^{-1})_K \neq K$. Theorem 33 (c) gives $K \in x_K \circ (x^{-1})_K$ and $K \in (x^{-1})_K \circ x_K$. The existence assertion of axiom (c) holds. For the uniqueness assertion of axiom (c), suppose $K \in x_K \circ y_K$ where $y_K \neq K$. Then $K \approx x_K y_K$, so that $y_K \approx x_K \setminus K$. Theorem 23, Theorem 30 and Theorem 28 yield

$$y_K \approx (x_K)^{-1}K = (x^{-1})_K K = (x^{-1})_K \cup K.$$

Hence, $y_K = (x^{-1})_K$. The uniqueness assertion is established, and so is the theorem. □

In any hypergroup H, a subhypergroup K is normal if and only if $x/K = K \setminus x$ for each $x \in H$ (see [5]). Thus, for the nonempty symmetric subhypergroup K, if K is normal then it follows easily that $a_{\overleftarrow{K}} = a_{\overrightarrow{K}} = a_K$. Therefore, a corollary holds.

Corollary 21. *If* K *is normal then* $H : \overleftarrow{K} = H : \overrightarrow{K}$ *is a fortified transposition hypergroup with the strong identity* K *for which every element is attractive.*

3.7 *Quasicanonical hypergroups*

Recall that, by the Structure Theorem for transposition hypergroups with a strong identity, the study of fortified transposition hypergroups separates into that of those containing only attractive elements and that of those that are quasicanonical. The theory presented above for fortified transposition hypergroups of attractive elements simplifies a great deal for quasicanonical hypergroups. A quasicanonical hypergroup may be characterized as a transposition hypergroup with a scalar identity, an element e such that $ex = xe = x$ for each element x (see [5]).

Let H be a quasicanonical hypergroup. Since $a/b = ab^{-1}$ and, dually, $b \setminus a = b^{-1}a$ in H, if follows that a subhypergroup is symmetric if and only if it is closed. Let K be a nonempty symmetric subhypergroup of H. Then K is *invertible*, $a \approx bK$ implies $b \approx aK$ and, dually, $a \approx Kb$

implies $b \approx Ka$. Hence, left and right cosets of K take the respective forms aK and Ka, and give partitions of H. Moreover, $abK \subseteq aKbK$ clearly holds, and so the equivalence whose classes are the left cosets is a congruence (dually, for the right cosets).

The family of left cosets under the induced hyperoperation $\circ$ thus forms a hypergroup. In this quotient hypergroup, it is easy to see that right extension $/$ (but not necessarily left extension $\backslash$) reduces to a join, that is, $aK / bK = aK \circ b^{-1}K$. Therefore, transposition holds. Indeed, $bK \backslash aK \approx cK / dK$ yields $aK \approx bK \circ (cK / dK) = bK \circ cK \circ d^{-1}K$. Thus follows $aK / d^{-1}K \approx bK \circ cK$, which gives $aK \circ dK \approx bK \circ cK$. The quotient hypergroup is not necessarily quasicanonical. Although the left coset K is a scalar right identity, $aK \circ K = aK$, and a left identity, $aK \approx K \circ aK$, it need not be a left scalar. Of course, the dual results hold for the right cosets.

Double cosets have the form KaK and also partition H. The equivalence having the double cosets as classes is again a congruence, $KabK \subseteq KaKKbK$. Hence, a quotient hypergroup of double cosets results. The quotient hypergroup is easily seen to be a transposition hypergroup as $KaK / KbK = KaK \circ Kb^{-1}K$ and $KbK \backslash KaK = Kb^{-1}K \circ KaK$ both hold. Using the characterization of a quasicanonical hypergroup given above, one has the next result.

Theorem 37. *If H is a quasicanonical hypergroup and K a nonempty symmetric subhypergroup then $H : K$, the quotient hypergroup of double cosets, is a quasicanonical hypergroup with the scalar identity K, where $(KaK)^{-1} = Ka^{-1}K$.*

A corollary that appears in [6] results if K is normal.

Corollary 22. *If K is normal then the quotient hypergroup of left (right) cosets is a quasicanonical hypergroup.*

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