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HOMOMORPHIC RELATIONS ON HYPERRINGOIDS AND JOIN HYPERRINGS

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Abstract This paper is a study of the Join Hyperringoid, which is a hypercompositional structure that has appeared recently. Here appear the homomorphic relations and a special type of such relations, the congruence ones. Moreover, the homomorphisms of the join hyperringoids are being studied, along with the homomorphisms of the Fortified Join Hyperringoids.

1. INTRODUCTION

The **hyperringoid** is a hypercompositional structure that has been introduced by G. Massouros and J. Mittas for the study of the theory of Automata and Languages [4]. The hyperringoid is a triplet (Y,+, .), for which the following axioms are valid:

I. (Y,+) is a hypergroup
ii. (Y,.) is a semigroup
iii. the composition "." is bilaterally distributive with regard to the hypercomposition "+"

If the hypergroup (Y,+) is a join one, then the hyperringoid is called **join**. The join hypergroup is a commutative hypergroup in which the Prenowitz's join axiom is also valid, i.e., it holds that:

(J) $(a:b) \cap (c:d) \neq \varnothing \Rightarrow (a+d) \cap (b+c) \neq \varnothing$, for every $a,b,c,d \in H$

where $a:b = \{x \in H \mid a \in x+b\}$ is the induced from "+" hypercomposition [1].

An important join hyperringoid for the theory of Languages, is the **B-hyperringoid** [6], in which the hypercomposition is defined as follows:

$$a + b = \{a, b\}$$

We shall begin the study of the congruence relations starting with certain Propositions which hold in more general hypercompositional structures, the hypergroupoids. So let (H,+) and (H′,+) be two hypergroupoids with hyperoperations defined in the entire sets and always giving non void result, i.e., $a+b \neq \varnothing$, for every two of their elements a, b. Then:

Definition 1.1. A binary relation $R \subseteq H \times H'$ is called **homomorphic** if for every $(a_1,b_1) \in R$ and $(a_2,b_2) \in R$ holds:
$(\forall x \in a_1+a_2)(\exists y \in b_1+b_2)\ [(x,y) \in R]$ and $(\forall y' \in b_1+b_2)(\exists x' \in a_1+a_2)\ [(x',y') \in R]$ (D_1)
or equivalently for every $x \in a_1+a_2$ and for every $y \in b_1+b_2$ holds:
$[\{x\} \times (b_1+b_2)] \cap R \neq \varnothing$ and $[(a_1+a_2) \times \{y\}] \cap R \neq \varnothing$ (D_1')

From the definition it derives that the inverse binary relation R^{-1} is also a homomorphic one. Moreover, when R defines a mapping $\varphi : H \to H'$, then, if $a,b \in H$ for every $x \in a+b$, we have $\varphi(x) \in \varphi(a)+\varphi(b)$, and therefore $\varphi(a+b) \subseteq \varphi(a)+\varphi(b)$. Also, for every $y \in \varphi(a)+\varphi(b)$ there exists $x \in a+b$ such that $\varphi(x) = y$, thus $\varphi(a)+\varphi(b) \subseteq \varphi(a+b)$. Consequently the condition $\varphi(a_1+a_2) = \varphi(a_1)+\varphi(a_2)$ is being verified and so the Proposition:

Proposition 1.1. *If a homomorphic relation between two hypergroupoids defines a mapping, then it is a normal homomorphism.*

We remind that, according to the terminology which has been introduced by M. Krasner, a mapping φ from the hypergroupoid H to the power-set $\wp(H')$ of the hypergroupoid Ç′ is called **homomorphism** if $\varphi(x+y) \subseteq \varphi(x)+\varphi(y)$ for every $x,y \in H$. A homomorphism is called **strong** if the above relation holds as an equality. Moreover, if φ is a mapping from H to H′ for which $\varphi(x+y) \subseteq \varphi(x)+\varphi(y)$, then φ is a **strict homomorphism**. Lastly, if for a strict homomorphism holds $\varphi(x+y) = \varphi(x)+\varphi(y)$, then we have a **normal** or **good homomorphism.**

For the homomorphic relations and the normal homomorphisms we give the Propositions:

Proposition 1.2. *If R, S are homomorphic relations between the hypergroupoids H′, H and H, H″ respectively, then their composition SR is a homomorphic relation between H′ and H″.*

Next let R be a homomorphic relation between the hypergroupoids (H,+) and (H′,+). If $h' \subseteq H'$ is a subhypergroupoid of H′ and h the image of h′ under R^{-1}, then:

Proposition 1.3. *If h′ is stable under the hypercomposition, then h is stable as well.*
P r o o f. Let $x,y \in h$. It will be proved that $x + y \subseteq h$. Indeed, since $x,y \in h$ then there exist t_1, t_2 from h′, such that (x,t_1), (y,t_2) belong to R. But since R is a homomorphic relation, it derives that for every $w \in x + y$ holds:

$$[\{w\} \times (t_1 + t_2)] \cap R \neq \varnothing.$$

But h′ is stable with regard to the hypercomposition and therefore $t_1 + t_2$ is a subset of h′. Thus, for every w from x + y, there exists t from h′ such that (w,t) ∈ R. So w ∈ h and therefore, for every y from h, x + y is a subset of h. Consequently h is stable.

Corollary 1.1 *The inverse image of a semi-subhypergroup through a homomorphism between two hypergroups is a semi-subhypergroup.*

2. Homomorphic Relations in the Join Hypergroups

As it is known, from the general theory of the hypergroups, a subhypergroup h of a hypergroup H is closed in H if a:b ⊆ h for every a,b ∈ h [2]. Thus, in a closed subhypergroup h of a join hypergroup H, the axiom (J) is being verified in h. Moreover if a subhypergroup h of H is a join hypergroup itself, then it is called **join subhypergroup** of H. Therefore the closed subhypergroups of H are its join subhypergroups. For the following, let H be a join hypergroup, h a join subhypergroup of H and E a hypergroupoid with hyperoperation defined for every two elements of E and always giving non void result. If R is a homomorphic relation from H to E with the property: y = y′, when (x,y), (x,y′) belong to R, then:

Proposition 2.1. *The image h′ of h through R is a subhypergroup of E. Also if all the elements of h′: h′ are images through R of elements of H, then the elements of h′ satisfy the join axiom inside E, but not necessarily inside h′.*

P r o o f. Let (x,y) ∈ R with x ∈ h and y ∈ h′ and let's consider the hypersum y + t, t ∈ h′. For t ∈ h′ there exists v ∈ h such that (v,t) ∈ R. Consequently for every b ∈ y + t there exists a ∈ x + v such that (a,b) ∈ R, and therefore y + t ⊆ h′. Thus y + h′ ⊆ h′. Next let t ∈ h′. Then (v,t) ∈ R for some v ∈ h. Now, for v, there exists a ∈ h such that v ∈ x + a. Let b be an element of h′ such that (a,b) ∈ R. Then:

$$[\{v\} \times (y + b)] \cap R \neq \varnothing$$

so t ∈ y + b and therefore h′ ⊆ y + h′. Thus h′= y + h′. Also it can be proved that for every three elements a′, b′ and c′ from h′ the associativity holds and so h′ is a hypergroup. Next let's assume that all the elements of h′:h′ are images, through R, of elements of H. Suppose that for the elements a′,b′,c′,d′ of h′ holds:

$$(a':b') \cap (c':d') \neq \varnothing$$

We remark that the a′:b′ and c′:d′ are not necessarily subsets of h′. If t ∈ a′:b′ and t ∈ c′:d′, then a′ ∈ b′ + t and c′ ∈ d′ + t. Next we choose the elements v ∈ H, and b, d ∈ h in such a way that the pairs (b,b′), (d,d′) and (v,t) belong to R. Then for every x ∈ b + v and y ∈ d + v holds:

$$[\{x\} \times (b' + t)] \cap R \neq \varnothing \text{ and } [\{y\} \times (d' + t)] \cap R \neq \varnothing.$$

Therefore, for the a', c' which belong to $b' + t$ and $d' + t$ respectively, there exist a, c, such that the pairs (a,a') and (c,c') belong to R and also $a \in b + v$ and $c \in d + v$. Then $v \in (a:b) \cap (c:d)$ and thus $(a:b) \cap (c:d) \neq \varnothing$. From the last relation, and since H is a join hypergroup, it derives that $(a + d) \cap (c + b) \neq \varnothing$. Now let w be an element of the intersection $(a + d) \cap (c + b)$. Then:

$$[\{w\} \times (a' + d')] \cap R \neq \varnothing \text{ and } [\{w\} \times (c' + b')] \cap R \neq \varnothing$$

So there exists w' which belongs to the hypersums $(a'+ d')$ and $(c'+ b')$ such that $(w,w') \in R$. Thus

$$(a' + d') \cap (c' + b')] \neq \varnothing$$

Therefore it has been proved that the join axiom is being verified for the elements of h', not necessarily inside it, but inside E.

Corollary 2.1. *Let φ be a normal epimorphism from the join hypergroup H on the hypergroupoid E. Then E is a join hypergroup and the image through φ of every join subhypergroup of H is a subhypergroup of E.*

A homomorphic relation which is also an equivalence relation will be named **congruence relation.**

Proposition 2.2. *Every congruence relation R on a hypergroup H is a normal equivalence relation and therefore the set H/R is a hypergroup if we define the hypercomposition:*

$$C_x \bullet C_y = \{ C_z \,/\, z \in x + y \}$$

where C_a is the class of an arbitrary element $a \in H$.

Proposition 2.3. *If the hypergroup H is join, then H/R is also a join hypergroup.*

3. Homomorphic Relations and Homomorphisms in the Join Hyperringoids

Let Y and Y′ be two hyperringoids and let $R \subseteq Y \times Y'$ be a binary relation from Y to Y′.

Definition 3.1. R will be called **homomorphic relation**, if it satisfies the axioms of the Definition 1.1. and moreover if for every $(a_1,b_1) \in R$ and $(a_2,b_2) \in R$ holds:

$$(a_1a_2,\, b_1b_2) \in R \qquad (D_2)$$

The notion of the homomorphism, as well as the different special types of homomorphisms that exist in the hypergroups are also being defined in the hyperringoids, with the use of the additional axiom:

$$\varphi(x.y) = \varphi(x) \,.\, \varphi(y)$$

for every x, y from the domain of φ. For the following, let K and K′ be two join hyperringoids and E a hyperringoid. Then:

Proposition 3.1. *If φ is a strict homomorphism between K and K′, then the inverse image through φ, of a join subhyperringoid of K′, is a join subhyperringoid of K.*

Proposition 3.2. *If R is a homomorphic relation between K and E, then the image of every join subhyperringoid of K is a subhyperringoid of E.*

Corollary 3.1. *Let φ be a strong epimorphism from K to E, then E is a join hyperringoid and the image, through φ of every join subhyperringoid of K is a subhyperringoid of E.*

Proposition 3.3. *Every congruence relation R over E is a normal equivalence relation and therefore the set E/R is a hyperringoid with the following hypercomposition and composition:*

$$C_x \bullet C_y = \{\, C_z \in E/R \;/\; z \in x + y \,\}$$
$$C_x \,.\, C_y = C_{zy}$$

Proposition 3.4. *If E is a join hyperringoid, then E/R is also a join hyperringoid.*

Proposition 3.5. *Let A be a bilateral hyperidealoid of K. If we define in K a relation R as follows:*

$$(\kappa,\lambda) \in R \text{ if } (\kappa : \lambda) \cap A \neq \varnothing \text{ and } (\kappa : \lambda) \cap A \neq \varnothing$$

Then R is a homomorphic relation.

P r o o f. Let $(\kappa_1, \lambda_1) \in R$ and $(\kappa_2, \lambda_2) \in R$. Then from the definition of R we have: $(\kappa_1 : \lambda_1) \cap A \neq \varnothing$, $(\lambda_1 : \kappa_1) \cap A \neq \varnothing$ and $(\kappa_2 : \lambda_2) \cap A \neq \varnothing$, $(\lambda_2 : \kappa_2) \cap A \neq \varnothing$
So there exist x, x′ belonging to A and such that $x \in \kappa_1 : \lambda_1$ and $x' \in \lambda_1 : \kappa_1$. From here it derives that $\kappa_1 \in x + \lambda_1$ and $\lambda_1 \in x' + \kappa_1$, from where $\kappa_1 \in \lambda_1 + A$ (1) and $\lambda_1 \in \kappa_1 + A$ (2). Similarly, $\kappa_2 \in \lambda_2 + A$ (3) and $\lambda_2 \in \kappa_2 + A$ (4). From (1) and (3) we have $\kappa_1 + \kappa_2 \subseteq (\lambda_1 + \lambda_2) + A$ i.e. for every $a \in \kappa_1 + \kappa_2$ there exists $b \in \lambda_1 + \lambda_2$ such that $a \in b + A$, or equivalently $(a{:}b) \cap A \neq \varnothing$, from where, due to the definition of R, $(a,b) \in R$. So $[\,\{a\} \times (\lambda_1 + \lambda_2)\,] \cap R \neq \varnothing$ for every $a \in \kappa_1 + \kappa_2$. Similarly, from (2) and (4) it derives that for every $b \in \lambda_1 + \lambda_2$ there exists $a \in \kappa_1 + \kappa_2$ such that $(a,b) \in R$. Thus $[\,(\kappa_1 + \kappa_2) \times \{b\}\,] \cap R \neq \varnothing$.

Moreover, from the relations $\kappa_1 \in x + \lambda_1$ and $\kappa_2 \in y + \lambda_2$ it derives that $\kappa_1.\kappa_2 \in (x + \lambda_1).(y + \lambda_2)$ and due to the Properties III.1.1 of [5]

$$(x + \lambda_1).(y + \lambda_2) \subseteq x.y + x.\lambda_2 + \lambda_1.y + \lambda_1.\lambda_2$$

Therefore $\kappa_1.\kappa_2 \in x.y + x.\lambda_2 + \lambda_1.y + \lambda_1.\lambda_2$ But, because of the multiplicative property of A, we have $x.\lambda_2, \lambda_1.y, x.y \in A$, so $\kappa_1.\kappa_2 \in \lambda_1.\lambda_2 + A$ or $(\kappa_1.\kappa_2 : \lambda_1.\lambda_2) \cap A \neq \varnothing$ thus $(\kappa_1.\kappa_2 , \lambda_1.\lambda_2) \in R$ and so R is a homomorphic relation.

Proposition 3.6. *Let R be a congruence relation over K. Then the mapping φ from K to K/R which is defined as follows:*

$$\varphi(x) = C_x \text{ for every } x \in K$$

is a normal homomorphism from K on K/R.

Proposition 3.7. *Let φ be a normal epimorphism of K on K΄. We define in K a relation R as follows:*

$$(x,y) \in R \text{ if and only if } \varphi(x) = \varphi(y)$$

Then R is a congruence relation in K and K/R is isomorphic to K΄.

P r o o f. Obviously the relation R is an equivalence relation and with not much difficulty it can be proved that it is also homomorphic. Next let C_a be the equivalence class of R which is defined from a. If σ is the mapping from K/R to K΄ which is defined by $\sigma(C_a) = \varphi(a)$, then σ is well defined, 1-1 and it maps K/R to K΄. Also

$$\sigma(C_a.C_b) = \sigma(C_{ab}) = \varphi(ab) = \varphi(a).\varphi(b) = \sigma(C_a).\sigma(C_b)$$

and
$$\sigma(C_a + C_b) = \sigma\{C_x \mid x \in a + b\} = \{\varphi(x) \mid x \in a + b\}$$
$$= \varphi(a + b) = \varphi(a) + \varphi(b) = \sigma(C_a) + \sigma(C_b)$$

Therefore σ is indeed an isomorphism.

Corollary 3.2. *Let φ be a normal homomorphism from K to K΄. Then there exists a congruence relation R in K, a natural epimorphism $\pi : K \to K/R$ and a monomorphism $\psi : K/R \to K΄$ such that $\varphi = \psi \bullet \pi$.*

Next we observe that if an equivalence relation R in a hyperringoid E satisfies the property:

$$xRy \text{ and } w \in E \Rightarrow xwRyw \text{ and } wxRwy \qquad [D_2΄]$$

then it satisfies the axiom $[D_2]$ of the Definition 3.1. If a relation satisfies $[D_2΄]$, then it is called **compatible** to the composition. It is possible though that an equivalence relation satisfies only one of the conditions of the second part of $[D_2΄]$. We will call such a relation **right** or **left compatible** to the composition respectively.

Theorem 3.1. *Let L be a subset of E. We define in E a relation R_L as follows:*
$$x R_L y \Leftrightarrow (\forall a,b \in E)[x.a \in L \Leftrightarrow y.a \in L \;\; (i) \;\; \text{and} \;\; b.x \in L \Leftrightarrow b.y \in L \;\; (ii)]$$

Then R_L is an equivalence relation in E compatible to the composition. If R_L satisfies only (i), [symb. R_L'] or only (ii) [symb. $'R_L$] then it is right or left compatible respectively. If E is a B-hyperringoid, then R_L is a congruence relation.

P r o o f. Obviously this relation is reflexive and symmetric, and it is not very difficult to prove that it is transitive as well. Next let $x_1R_Ly_1$ and $x_2R_Ly_2$. Suppose that for some $b \in E$ holds $b(x_1x_2) \in L$ or equivalently $(bx_1)x_2 \in L$ Then, since $x_2R_Ly_2$ we have $(bx_1)y_2 \in L$ or equivalently $b(x_1 y_2) \in L$ and so $x_1x_2R_Lx_1y_2$. Similarly $x_1y_2R_Ly_1y_2$, and thus $x_1x_2R_Ly_1y_2$, that is the axiom [D_2]. Next let E be a B-hyperringoid. If $w \in x_1 + x_2$, then $w \in \{x_1, x_2\}$. Thus $[\{w\} \times (y_1 + y_2)] \cap R_L = [\{w\} \times \{y_1, y_2\}] \cap R_L$ and therefore this intersection is non void. Similarly, for $z \in y_1 + y_2$ we have $[(x_1 + x_2) \times \{z\}] \cap R_L \neq \varnothing$. Thus the axiom [$D_1$] of Definition 1.1. is being satisfied and so the Theorem.

Corollary 3.3. *If E is a B-hyperringoid, then the quotient E/R_L is a B-hyperringoid as well.*

Now, let's suppose that the subset L of E is a union of classes with regard to an equivalence relation R. Then, if R is right compatible with regard to the multiplication, from xRy, it derives that xaRya for every $a \in E$. Therefore the classes $(xa)_R$ and $(ya)_R$ are equal for every $a \in E$ and since L is a union of classes, it derives that:

$$xa \in L \Leftrightarrow ya \in L \quad \text{for every } a \in E$$

So, according to Theorem 3.1, the above relation defines an equivalence relation R_L' in E, for which $xRy \Rightarrow x R_L' y$, and consequently every class of R is contained in a class of R_L'. Therefore every class of R_L' is a union of one or more classes of R and so $rk(R_L') \leq rk(R)$. Respective results we get when R is a right compatible or a compatible relation. Thus:

Theorem 3.2. *If there exists an equivalence relation R in E compatible to the multiplication, with regard to which L is a union of classes, then $rk(R_L) \leq rk(R)$ and therefore, if $rk(R) < \infty$ then $rk(R_L) < \infty$. Respective properties hold for R_L' and $'R_L$, if R is right or left compatible with regard to the multiplication.*

A special case of join hyperringoid is the **Join Hyperring** [7], in which the additive hypergroup is a **Fortified Join Hypergroup** [7], i.e. a join hypergroup (H,+) that also satisfies the axioms:

FJ_1 There exists a unique neutral element, denoted by 0, the zero element of H, such

that for every $x \in H$ holds: $x \in x+0$ and $0+0 = 0$

and

FJ_2 For every $x \in H\setminus\{0\}$, there exists one and only one element $x' \in H\setminus\{0\}$, the opposite
or symmetrical of x, denoted by -x, such that: $0 \in x+x'$ Also -0 = 0.

In the following we will present a few Propositions which refer to the homomorphisms of the join hyperrings. If Y, Y′ are two join hyperrings and φ is a normal homomorphism from Y to Y′, then, as usual [8], we define the kernel of ö, denoted by kerφ, to be the subset $\varphi^{-1}(\varphi(0))$ of Y and we denote the homomorphic image φ(Y) of Y, with Imφ. In accordance now to what holds in the case of the normal homomorphisms of the fortified join hypergroups [3], in the join hyperrings holds:

Proposition 3.8.

i. kerφ is a subhyperringoid of Y
ii. Imφ is a subhyperringoid of Y′, which generally does not contain the element 0′ ∈ Y′, but φ(0) is neutral element in Imφ
iii. If T is a join subhyperring of Y which contains the kernel of φ, and if φ is an epimorphism, then φ(T) is a join subhyperring of Y′.

Proposition 3.9. *If Y is an integral join hyperring, then kerφ is a symmetrical hyperideal of Y.*

P r o o f. It has been proved (see Proposition 2.5 of [3]) that the set $[\ker ö] = -\varphi^{-1}(\varphi(0)) \cup \varphi^{-1}(\varphi(0))$ is a symmetrical subhypergroup. And since Y is an integral join hyperring, if φ(x) = φ(0), then for φ(-x) we have:

$$\varphi(-x) = \varphi(x)\varphi(-1) = \varphi(0)\varphi(-1) = \varphi(0)$$

Thus $\varphi(-x) \in \ker\varphi$ and therefore $[\ker\varphi] = \ker\varphi$. Now if $x \in \ker\varphi$ and w is an arbitrary element of Y, then φ(xw) = φ(0). Consequently $xw \in \ker\varphi$ and so kerφ is a symmetrical hyperideal.

The study of the homomorphisms in the case of the fortified join hypergroups [3] has shown that if φ is a normal homomorphism (and much more a homomorphism), then its kernel does not necessarily contain the opposite of every element it consists of. Thus a new type of homomorphism, the **complete homomorphism** was introduced, for which $-x \in \ker\varphi$ for every $x \in \ker\varphi$. As it has been proved in the previous Proposition, this relation holds when Y is an integral join hyperring and so:

Proposition 3.10. *Every normal homomorphism with domain an integral join hyperring, is complete.*

Also we have the Proposition:

Proposition 3.11. *If φ is a complete and normal homomorphism from Y to Y′ with the property φ(0) = 0, then:*

i. kerφ is a symmetrical hyperideal of Y
ii. Imφ is a symmetrical subhyperring of Y´
iii. if T is a symmetrical subhyperring of Y, then $\varphi(T)$ is a symmetrical subhyperring of Y´

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