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HYPERGROUPS AND GEOMETRY

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Cet article présente une manière d'aborder la géométrie récemment développée et basée sur la théorie des structures hypercompositionnelles. On analyse certaines notions, comme la dépendance affine, avec les nouveaux outils fournis par cette théorie. On obtient des résultats relevant pour ce domaine et quelques théorèmes connus de géométrie qui sont des conséquences des théorèmes plus généraux valables pour les hypergroupes.

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It is very well known that there exists a close relation between Algebra and Geometry. So, as it should be expected, this relation also appeared between Geometry and the Theory of the Hypergroups, which is a branch of Algebra that has been introduced by F. Marty, in 1934 [5]. It really is of exceptional interest that the axioms of the hypergroup are related directly to certain Euclid's postulates [2].

The hypergroup is a pair (H, \cdot) where H is a non empty set and " \cdot " is a hypercomposition in H , i.e. a mapping of $H \times H$ to the set of the subsets of H satisfying the associative and reproductive axioms, that is:

- i. $(ab)c = a(bc)$, for every $a, b, c \in H$;
- ii. $aH = Ha = H$, for every $a \in H$.

It can be proved that the result of the hypercomposition is always a nonempty set [4]. Indeed, let $ab = \emptyset$. Then $H = aH$ [reproductivity] = $a(bH)$ [reproductivity] = $(ab)H$ [associativity] = $\emptyset H = \emptyset$, absurd.

The hypercomposition also defines the two induced hypercompositions [5]:

$$a : b = \{x \in H \mid a \in xb\} \text{ and } a..b = \{y \in H \mid a \in by\}$$

If the hypercomposition is commutative, then $a : b = a..b$. Also it can be proved that the reproductive axiom is equivalent to the nonemptiness of the sets $a : b$ and $a..b$, for every $a, b \in H$ [8]. Indeed, Ha is a subset of H (resp. $aH \subseteq H$). Let $x : a \neq \emptyset$ for every $x \in H$. Then there exists $y \in H$ such that $y \in x : a \Rightarrow x \in ya$. Thus $x \in Ha$ and so $H \subseteq Ha$. Conversely now. Let $aH = H$ for every $a \in H$. Then for every $b \in H$ there exists $x \in H$ such that $b \in ax$ and so $x \in b..a$. Thus $b..a \neq \emptyset$ for every $a, b \in H$. Similarly from the relation $Ha = H$ derives that $b : a \neq \emptyset$.

Now let's come to Euclid's postulates. According to the first one:

"Ἡκίσθω ἀπὸ παντός σημείου ἐὰν ᾧσιν σημεῖον εὐθείαν γραμμὴν ἀγαγεῖν"

(Let the following be postulated: to draw a straight line from any point to any point)

So if two points a, b are considered, then ab (as a segment of a straight line) represents a set of points. Thus a hypercomposition has been defined in the set of the points. Next, according to the second postulate:

"καὶ ἀεσπερασμένην εὐθείαν κατὰ τὸ συνεχές ἐσ' εὐθείας ἐμβαλεῖν"

(To produce a finite straight line continuously in a straight line)

the sets $a : b$ and $b : a$ are nonempty. Therefore, the second axiom of the hypergroup is also valid. Besides it is easy to prove that the associativity holds in the set of the points. It is only necessary to keep in mind the definition of the equal figures given by Euclid in the "Common Notions":

"τὰ πῶ αὐτῶ ἴσα καὶ ἀλλήλοισ ἴσιν ἴσα"

(Things which are equal to the same thing are also equal to one another)

So the set of the points is a hypergroup. Moreover, through similar reasoning, it can be proved that any Euclidean space of dimension n can become a hypergroup. Indeed:

PROPOSITION 1. *Let $(V, +)$ be a linear space over an ordered field $(F, +, \cdot)$. Then V , with the hypercomposition:*

$$xy = \{ux + \lambda y | u, \lambda \in F_+^*, u + \lambda = 1\}$$

becomes a hypergroup.

This hypergroup is called attached hypergroup. In fact, several hypergroups can be attached to a vector space [9], but this paper deals only with the one mentioned in this Proposition.

The notions of the semi-subhypergroup, the subhypergroup and the closed subhypergroup are fundamental for the study of Geometry with the use of the theory of the Hypergroups. A subset h of a hypergroup H is called semi-subhypergroup if $ab \subseteq h$ for every $a, b \in h$. h is called subhypergroup of H if $ah = ha = h$, for every $a \in h$. Also a subhypergroup h is a right closed subhypergroup of H if $ah \cap h = \emptyset$, for every $a \in H \setminus h$. Analogous is the definition of the left closed subhypergroup of H . It has been proved that h is right closed in H if and only if $a : b \subseteq h$ and left closed, if and only if $a..b \subseteq h$, for every $a, b \in h$ [8]. Especially, in the

case of the join hypergroups, every closed subhypergroup h of a join hypergroup H creates a partition in H . The classes of this partition are the sets $h : a, a \in H$ [7]. The notation $[A]$ (resp. $\langle A \rangle$) is used for the semi-subhypergroup (resp. the closed subhypergroup) deriving from an arbitrary nonempty subset A of H .

The first one who has introduced in Geometry the theory of the hypergroups is W. Prenowitz (see [10], [11], [12], [13]). For this purpose he has used a properly defined hypergroup, which he named join space. The axioms of the join space are the following [13]:

- i. $ab = ba$
- ii. $aa = a$
- iii. $a : a = a$
- iv. if $(a : b) \cap (c : d) \neq \emptyset$, then $ad \cap bc \neq \emptyset$

The commutative hypergroup that satisfies the last axiom (iv) has been called Join [7].

Several geometrical notions can be described with the use of the language of the hypercompositional structures. So let's start with the notion of convexity. It is known that a figure is called convex, if it contains the segment joining each pair of its points. As mentioned above, the set of the points of the plane, as well as the set of the points of any vector space V over an ordered field, with the hypercomposition defined there, becomes a hypergroup. From this point of view, i.e. with the use of the hypercomposition, a subset E of V is convex if $ab \subseteq E$, for every $a, b \in E$. But a subset E of a hypergroup that has this property is a semi-subhypergroup. Thus:

PROPOSITION 2. *The convex subsets of V are semi-subhypergroups of its attached hypergroup.*

Consequently the properties of the convex sets of a vector space are simple applications of the properties of the semi-subhypergroups, or the subhypergroups of a hypergroup, and more precisely, the attached hypergroup. So this study, except from the fact that it leads to remarkable results, it also gives the opportunity to generalize the already known theorems of the vector spaces in sets with fewer axioms than the ones of the vector spaces. Let's see an example now, starting with the relation of the affinity dependence, which is a basic notion in the vector spaces, to the attached hypergroup [8].

PROPOSITION 3. *In a vector space V over an ordered field F , the elements $a_i, i = 1, \dots, k$ are affinely dependent if and only if there exist distinct integers $s_1, \dots, s_n, t_1, \dots, t_m$ that belong to $\{1, \dots, k\}$ such that*

$$[a_{s_1}, \dots, a_{s_n}] \cap [a_{t_1}, \dots, a_{t_m}] \neq \emptyset$$

is the respective attached hypergroup.

In a simple hypergroup now, i.e. a hypergroup in which no extra axioms hold, the following theorem is valid [8]:

THEOREM 1. *Let H be a hypergroup in which every set with cardinality greater than n has two disjoint subsets A, B such that $[A] \cap [B] \neq \emptyset$. If $(y_i)_{i \in I}$ with $\text{card } I \geq n$ is a finite family of semi-subhypergroups of H , in which the intersection of every n elements is non void, then all the sets y_i have a non-void intersection.*

The combination of Proposition 3, and this Theorem gives the Corollary:

COROLLARY 1. *Helly's Theorem.*

In a finite family $(C_i)_{i \in I}$ of convex sets in R^d , with $d + 1 \leq \text{card } I$, if any $d + 1$ of the sets C_i have a nonempty intersection, then all the sets C_i have nonempty intersection.

During his study on the join spaces, Prenowitz has introduced a new axiom, which he named "Exchange Postulate":

$$\text{If } c \in \langle a, b \rangle \text{ and } c \neq a, \text{ then } \langle a, b \rangle = \langle a, c \rangle$$

So the join spaces that satisfy this axiom were named "Exchange Spaces" [13]. The above axiom enabled him to develop a theory of linear independence and dimension of type familiar to the classical geometry. A small generalization of the theory of the exchange spaces appears in [14], where Prenowitz and Jantosciak left the axiom $xx = x$ out of the axioms of the join spaces. On the other hand, a remarkable generalization of this theory has been achieved by Freni, who has developed the notions of the independence, the dimension etc. in a hypergroup H that satisfies only the axiom:

$$[x \in \langle A \cup \{y\} \rangle, x \notin \langle A \rangle] \Rightarrow [y \in \langle A \cup \{x\} \rangle], \text{ for every } x, y \in H \text{ and } A \subseteq H.$$

Freni has called these hypergroups cambiste [3]:

DEFINITION 1. *A subset B of a hypergroup H is called free or independent if either $B = \emptyset$, or $\forall x \in B, x \notin \langle B \setminus \{x\} \rangle$, otherwise it is called non free or dependent. B generates H if $\langle B \rangle = H$ and then B is a set of generators of H . If H has a finite set of generators, then H is called a finite type hypergroup. A free set of generators is a basis of H .*

Among the results reached by Freni are:

THEOREM 2. *Let B be a nonempty subset of a cambiste hypergroup H . B is a basis of H if and only if:*

(i) B is a maximal free set, and (ii) B is a minimal set of generators of H

THEOREM 3. *Every cambiste hypergroup has at least one basis.*

THEOREM 4. *All the bases of a cambiste hypergroup have the same cardinality.*

DEFINITION 2. The dimension of a cambiste hypergroup H (denoted by $\dim H$) is the cardinality of any basis of H .

A special case of a cambiste hypergroup is the homiogene one [6]:

DEFINITION 3. A hypergroup H will be called homiogene if from the relation $x \in \langle y \rangle$, where x is not the scalar neutral element of H (if there exists such element), derives that $\langle x \rangle = \langle y \rangle$.

The homiogene hypergroup is a hypergroup with many and interesting properties [6], which extend beyond the scope of this paper. We will refer though to certain theorems of a special type of homiogene hypergroup, the strongly homiogene one. Corollaries of these theorems are well known theorems of the convex sets.

DEFINITION 4. Let H be a join hypergroup which satisfies the following axioms

- i) H is a homiogene hypergroup
- ii) $[x] = \langle x \rangle$, for every $x \in H$
- iii) $\langle x, y \rangle = \langle x \rangle \cup \langle y \rangle \cup \langle x \rangle \cdot \langle y \rangle \cup \langle x \rangle : \langle y \rangle \cup \langle y \rangle : \langle x \rangle$, for every $x, y \in H$

Then H will be called strongly homiogene hypergroup.

THEOREM 5. *Let Ω be a strongly homiogene hypergroup with $\dim \Omega = n$. If $x \in x_1 \dots x_{n+1}$ then there exists a proper subset $\{x_{u_1}, \dots, x_{u_\mu}\}$ of $\{x_1, \dots, x_{n+1}\}$, such that $[x] \subseteq [x_{u_1}] \dots [x_{u_\mu}]$*

COROLLARY 2. *Let Ω be a strongly homiogene hypergroup and let E be a subset of Ω with $\dim \langle E \rangle = n$. If $x \in [E]$ then there exists a subset A of E with $\text{card } A \leq n$, such that $[x] \subseteq [A]$*

COROLLARY 3. *Caratheodory's Theorem*

Let x be an element of the convex hull of a subset M of R^d , with $\dim(\text{aff } M) = n$. Then x belongs to a convex combination of at most $n + 1$ points from M .

Also Caratheodory's Theorem appears in [1] and [15] for a kind of join Geometry.

THEOREM 6. *Let Ω be a strongly homiogene hypergroup with $\dim \Omega = n$. If $x \in x_1 \dots x_{n+1}$ and if $y \in \langle x_1, \dots, x_{n+1} \rangle$ with $y \neq x_i$, $1 \leq i \leq n + 1$, then there exists a subset $\{\omega_{u_1}, \dots, \omega_{u_\mu}\}$ of $\{y, x_1, \dots, x_{n+1}\}$ which contains at most $n - 1$ elements from x_1, \dots, x_{n+1} and such that $[x] \subseteq [\omega_{u_1}] \dots [\omega_{u_\mu}]$.*

COROLLARY 4. *Let Ω be a strongly homiogene hypergroup, let E be a subset of Ω with $\dim \langle E \rangle = n$ and let $x, y \in [E]$. Then there exists a set $A \subseteq E$ such that*

card $A \leq n - 1$ and $[x] \subseteq [A \cup \{y\}]$.

From this Theorem we conclude that some of the $\eta + 1$ elements can be chosen arbitrarily from the closed subhypergroup which they generate. Thus we have a generalization of the Theorem of Caratheodory when the hypergroup is the R^d .

THEOREM 7. *Let Ω be a strongly homiogene hypergroup with $\dim \Omega = n$ and let $E \subseteq \Omega$, $X \subseteq [E]$ and $\text{card } X \geq 2$. Then $X \subseteq [A]$ for some set $A \subseteq E$ with $\text{card } A \leq (n - 1)(\text{card } X)$.*

In the case of R^d , the above Theorem is an extension of the result of Caratheodory's Theorem, when the single element x is replaced by a finite set X , with $\text{card } X \geq 2$.

Finally we will give the description of the notion of the internal point of a convex set, with the use of the theory of the hypergroups. Let E be a semi-subhypergroup of a hypergroup H . w is called internal element of E , if for every $x \in H$ there exists $y \in H$, such that $w \in xy$ [15]. In the case of the cambiste hypergroups though there can appear another, equivalent definition, which is very useful. So if H is a cambiste hypergroup with $\dim H = n$ and E a semi-subhypergroup of H , x will be called internal element of E , if for every closed subhypergroup h of H with $\dim h = n - 1$ holds:

$$h : a \cap E \neq \emptyset \text{ and } h : b \cap E \neq \emptyset$$

where $h : a$ and $h : b$ are the two disjoint classes mod h . It is worth mentioning that this definition can be generalized to any hypergroup, provided that $h : a \cap E \neq \emptyset$ and $h : b \cap E \neq \emptyset$ is valid for every closed subhypergroup h of H , where $h : a$ and $h : b$ are two disjoint classes mod h .

In the strongly homiogene hypergroups now, the following are also valid:

PROPOSITION 4. *If k is an interior element of a semi-subhypergroup $[E]$ of a strongly canonical hypergroup Ω , with $\dim \Omega = n$, then k is interior to $[A]$, where A is a subset of E with $\text{card } A \leq n^2$.*

THEOREM 8. *Let k be an interior element of a semi-subhypergroup E of a strongly homiogene hypergroup Ω with $\dim \Omega = n$. Then k is interior element of a semi-subhypergroup of E , which is generated from at most $2n$ elements.*

COROLLARY 5. *Steinitz's Theorem*

Any point interior to the convex hull of a set E in a d -dimensional euclidian space is interior to the convex hull of some subset of E containing at most $2d$ points.

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