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ON THE HYPERCOMPOSITIONAL STRUCTURES

CHRISTOS G. MASSOUROS

54, Klious st., 155 62 Cholargos, Athens, GREECE.

ABSTRACT.

This paper presents the several types of hyperrings and hyperfields along with the generating causes that gave the whole spectrum of these structures. Starting from Marty's hypergroup we go on to Krasner's hyperfield, to Mittas' superring, to Corsini's feeble hyperring, and we end up to the fortified join hyperringoids. We also give a hitherto open question in the theory of fields that derived from the theory of the hyperfields and we point out the structures that have not been studied thoroughly yet.

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The **hypercompositional structures** are algebraic structures equipped with **multivalued** compositions. So the result of the composition of two elements is not one element only, but it is a subset of the set on which this multivalued composition has been defined. Such compositions are named hyperoperations, or hypercompositions. The theory of the hypercompositional structures has been introduced in algebra, through the notion of the hypergroup, by F. Marty in 1934 [9].

The **hypergroup** is a nonempty set H with a hypercomposition "." that satisfies the axioms:

i. $(ab)c = a(bc)$ for every $a,b,c \in H$ (associative axiom)

ii. $aH = Ha = H$ for every $a \in H$ (reproductive axiom)

Several problems in the non commutative algebra led in a natural way to the introduction of these axioms. Let's mention for instance the cosets in groups that derive from non invariant subgroups (e.g.[27]). The hypergroup, being a very general and abstract structure, can accept the introduction of other, more or less strict axioms, leading thus to a number of special hypergroups. Thus a lot of branches of this hypercompositional structure appeared, each one of which having its own special difficulties (see [18]).

The first one who has introduced a hypercompositional structure, part of which is a particular hypergroup, is M. Krasner, who, in 1956, with his paper [7], has defined the hyperfield. Krasner's hyperfield was born as a result of a clearly mathematical necessity. More precisely, in [7], M. Krasner has used the hyperfield as the proper algebraic tool, in order to define a certain approximation of a complete valued field by sequences of such fields.

A **hyperfield** is a triplet (H,+,.), where H is a nonempty set, "+" a hypercomposition and "." a composition, for which the following axioms are valid:

I. Multiplicative axiom.

(H,.) is an almost-group with regard to the multiplication (i.e. the union of a multiplicative group H^* with a bilaterally absorbing element, denoted by 0).

II. Additive axioms.

1. $x+y = y+x$ (commutativity)
2. $(x+y)+z = x+(y+z)$ (associativity)
3. for every $x \in H$, there exists one and only one $x' \in H$ such that $0 \in x+x'$. x' is called the opposite of x and it is denoted by $-x$.
4. $z \in x+y$ implies $y \in z-x$.

III. Distributive axiom.

$z(x+y) = zx + zy, \quad (x+y)z = xz + yz$

A structure which is related to the hyperfield in the same way as the ring is related to the field, has been named **hyperring** by M. Krasner [8]. The additive part of this structure has been named **canonical hypergroup** by J. Mittas, who has also studied it in great depth and has presented a great number of interesting results (e.g. see [20]).

The certain type of hyperfield that firstly appeared in Algebra, was the residual hyperfield. Naturally thus there arised the question:

Are there other hyperfields than the residual ones?

M Krasner has generalized this question, with constructions of more general hyperfields, the quotient hyperfiels, the class of which contains the residual ones. Furthermore he has constructed the quotient hyperrings [8]. Thus the question finally became:

Are there hyperrings and hyperfields other than the quotient ones?

The answer to this question was very significant, not only because it would give a measure of how rich is the class of these algebraic structures, but mostly because it would prove the independence and selfsufficiency of the whole theory of the hyperfields and the hyperrings from the corresponding theory of the fields and the rings. Indeed the quotient hyperrings derive from the rings, in the following way:

Let F be a ring and G an invariant subgroup of the multiplicative semigroup of the ring. Then the quotient set F/G becomes a hypergroup if we define:

$$xG + yG = \{(xp+yq)G \mid p,q \in G\} \quad \text{and} \quad xGyG = xyG$$

Now, if F is a field, then F/G is a hyperfield. Therefore, if the only hyperrings and the only hyperfields were the quotient ones, then many results of this theory could have derived directly from corresponding results of the theory of the rings and the fields.

Answers to this question have been given as follows:

In the beginning J. Mittas has proved for a hypergroup that in order to be valuated, it must be superiorly canonical i.e. it must satisfy the axioms:

S_1. if for $x,y \in H$ holds $x \in x + y$, then $x + y = x$

S_2. for every $x,y,z,w \in H$, such that $(x+y) \cap (z+w) \neq \emptyset$

either $x+y \subseteq z+w$, or $z+w \subseteq x+y$ is valid

S_3. for every $x,y,z,w \in H$ such that $0 \notin x+y$ and $z,w \in x+y$ holds $z-z = w-w$

S_4. if $x \in z-z$ and $y \notin z-z$, then $x-x \subseteq y-y$

Afterwards, with the use of non valuated canonical hypergroups he has constructed hyperfields which are not residual hyperfields of valuated fields by a multiplicative equivalence. Let's see for instance the following hyperfield, constructed by J. Mittas.

THEOREM 1. *Let $(H, .)$ be a totally ordered multiplicative almost-group with 0 being its minimum element. If in H the hypercomposition:*

$$x+y = \max\{x, y\}, \quad \text{if } x \neq y \quad \text{and} \quad x+x = \{z \in H \mid z \leq x\} = [0, x]$$

is introduced, then $(H,+)$ is a canonical but not superiorly canonical hypergroup, and $(H,+,.)$ is a hyperfield non isomorphic to a residual hyperfield of valuated field with a multiplicative equivalence.

Next, Ch. Massouros in [10] and after that, more generally in [11], and [14] has proved that there exist hyperfields that do not belong to the class of the quotient hyperfields. The relevant Theorems are:

THEOREM 2. *Let Θ be a non unitary multiplicative group and let $(H^*,.)$ be its direct product with the multiplicative group $\{-1, 1\}$. Consider the almost-group $(H,.) = (H^* \cup \{0\},.)$. A hypercomposition can be introduced in* *it defined as follows:*

for every $(x,i) \in H$

$$(x,i) + (x,i) = H \setminus \{(x,i),\ (x,-i),\ 0\}$$

$$(x,i) + (x,-i) = H \setminus \{(x,i),\ (x,-i)\}$$

$$(x,i) + 0 = 0 + (x,i) = (x,i), \quad 0 + 0 = 0$$

and for every $(y,j) \neq (x,i),\ (x,-i),\ 0$

$$(x,i) + (y,j) = \{(x,i),\ (y,j),\ (x,-i),\ (y,-j)\}$$

Then the triplet $H(\theta) = (H,+,.)$ *is a hyperfield, and if* θ *is a periodic group, then* $H(\theta)$ *is not isomorphic to a quotient hyperfield.*

THEOREM 3. *Let* θ *be a multiplicative group with card* $G > 2$. *Consider the almost-group* $(H,.) = (\theta \cup \{0\},.)$. *If the hypercomposition:*

$$x+0 = 0+x = x \qquad \text{for every } x \in H$$

$$x+x = H \setminus \{x\} \qquad \text{for every } x \in \theta \;.$$

$$x + y = \{x,\ y\} \qquad \text{for every } x,y \in \theta, \text{ with } x \neq y$$

is introduced in H, *then* $H(\theta) = (H,+,.)$ *becomes a hyperfield, and if* θ *is a periodic group, then* $H(\theta)$ *does not belong to the class of quotient hyperfields.*

Having proved the existence of non quotient hyperfields, and using such, the existence of non quotient hyperrings can also be proved. Relatively we have the Theorem [11]:

THEOREM 4. *The direct sum* S *of the hyperrings* S_i, $i \in I$, *is not isomorphic to a subhyperring of a quotient hyperring if at least one of the* S_i, *is a non quotient hyperfield.*

Moreover A. Nakassis, in his paper [26] proves the existence of non quotient hyperrings and non quotient hyperfields with the use of a counting lemma which associates the cardinality of the classes defined by the multiplicative subgroup G with the cardinality of the hypersums xG+yG.

LEMMA 1. *Let R be a ring and P an equivalence relation that induces a hyperring structure in R/P with P(0) = {0}. Assume that a hyperring H is embeddable in R/P and assume that there are two elements x and y in H such that for every $z \in x+y$, $(z-z) \cap (y-y) = \{0\}$. If y' is the image of y in R/P, then the cardinality of y' (considered as a subset of R) can not exceed the cardinality of x+y.*

The construction of the non quotient hyperfield introduced by Nakassis is the following:

THEOREM 5. *Let (T,.) be an almost-group with card T > 4. Next, in T a hypercomposition "+" is being introduced:*

$x+0 = 0+x = x$ *for every* $x \in T$

$x+x = \{x,0\}$ *for every* $x \in T^*$

$x+y = y+x = T \setminus \{0,x,y\}$ *for every* $x, y \in T^*$, *with* $x \neq y$

Then (T,+,.) is a hyperfield and if the cardinality of the group T^* is chosen properly, then it does not belong to the class of the quotient hyperfields.

With a similar construction to the above one, he managed to show that there exist hyperrings which are not embeddable into partition hyperrings.

In Nakassis' hyperfield the hypersum of two elements does not contain the participating elements. There exist though hyperfields in which the hypersum of two elements always contains those elements, as in the Example:

EXAMPLE. Assume that in the almost-group (H,.) constructed in Theorem 2, we also introduce the hypercomposition:

$(x,i)+(y,j) = \{(x,i), (y,j)\}$ if $(x,i) \neq (y,j)$, $i,j \in \{-1, 1\}$

$(x,1)+(x,-1) = H$

$(x,i)+0 = 0+(x,i) = (x,i)$

Then (H,+,.) is a hyperfield.

It has been proved that in a hyperfield H the two participating elements are contained in their hypersum if and only if the difference x-x equals to H for every $x \neq 0$ [13]. Such hyperfields which have the property to be generated by the difference of every non zero element from itself, are characterized as **monogene** hyperfields. In [13] has been proved that there exist monogene hyperfields that are isomporphic to quotient ones. The question though of whether there exist monogene hyperfields that are not quotient hyperfields is still open.

The study of this problem has led to another problem in the theory of the fields [13]. Indeed, lets assume that a monogene hyperfield is isomorphic to a quotient hyperfield K/G. Then in K/G the equality $xG-xG = K/G$ holds for every $xG \neq 0$. Thus $G-G = K/G$, and so for K the equality $G-G = K$ must be valid. Therefore the question arises:

Which fields can be written as a difference of a subgroup of their multiplicative group from itself, and which are these subgroups?

This problem has been answered for the case of certain finite fields in [14].

In 1973, J. Mittas generalized the notion of the hyperring, with the notion of the **superring** that he has introduced and studied in [21]. In this new structure the additive part remains a canonical hypergroup, but the multiplication is a hypercomposition and so the multiplicative part of this structure is a semi-hypergroup, instead of semigroup. Moreover he has proved that all the superfields are hyperfields. Analogous structure is the **D-hyperring** (H,+,.), introduced by M. DeSalvo [4] in 1985, where the hypergroup is not a canonical one, as in the superring, but it is just an abelian hypergroup. Its multiplicative part is still a semi-hypergroup and there also appears the new axiom:

$H\omega_H \cup \omega_H H \subseteq \omega_H$, where ω_H denotes the heart of the abelian hypergroup (H,+).

Furthermore, another generalization of the hyperring has been introduced in 1975 by P. Corsini in his paper [1], where he defines the **feeble hyperrings**. A **feeble hyperring** ⟨A,+,.⟩ is a regular commutative hypergroup (A,+) equipped with a hyperoperation "." that, for every $(a,b,c) \in A^3$, satisfies the axioms:

i. $a.(b+c) \subseteq a.b+a.c+\omega_A$
ii. $(b+c).a \subseteq b.a+c.a+\omega_A$
iii. $(a.b).c \subseteq a.(b.c)+\omega_A$
iv. $\forall (a,b) \in A^2 \ \exists \ (a.b)' \in P^*(A) : a.b+(a.b)' \subseteq \omega_A$
   where ω_A is the heart of A.

More extensive references on this structure can be found in [2].

In 1982 R. Rota introduced the notion of the **multiplicative hyperring** [29], in which, unlike Krasner's hyperring, the multiplication is a hyperoperation and the addition is a (single-valued) operation. More precisely the triplet (P,+,.) is called **multiplicative hyperring** if:

i. (P,+) is an abelian group
ii. (P,.) is a semi-hypergroup
iii. $x(y+z) \subset xy+xz$, $(y+z)x \subset yx+zx$ $\forall x,y,z \in P$
iv. $x(-y) = (-x)y = -(xy)$ $\forall x,y \in P$

This hypercompositional structure has been studied in depth by Rota [29], [30] and Procesi [28].

In 1991 G. G. Massouros and J. Mittas, motivated from the theory of Languages and Automata, introduced the notions of the **hyperringoid**, of the **join hyperringoid** and of the **fortified join hyperringoid** or **join hyperring** [15]. Indeed the set of the words A^* over an alphabet A, as it is known, is equipped with a composition, the concatenation of the words. With regard to this composition and with the consideration of the empty word λ, A^* is a monoid, having λ as neutral element. In order to describe the language accepted by a deterministic automaton we use the regular expressions. The definition of the regular expressions over the alphabet A presumes the introduction of the bisets {x,y} from A^*. This suggests the definition of a hypercomposition in A^*:

$$x+y = \{x,y\}$$

A^*, equipped with this hypercomposition becomes a join hypergroup, that is a commutative hypergroup satisfying the axiom: $a:b \cap c:d \neq \emptyset \Longrightarrow ad \cap cb \neq \emptyset$, for every $a,b,c,d \in H$. Furthermore it has been proved that the concatenation is billaterally distributive with regard to this hypercomposition. Thus, in a natural way, there derived the multiplicative-hyperadditive structure, which was named hyperringoid.

A **hyperringoid** is a non void set Y with an operation "." and a hyperoperation "+" that satisfy the axioms:

i. (Y,+) is a hypergroup

ii. (Y,.) is a semigroup

iii. the operation is billaterally distributive to the hyperoperation.

If (Y,.) is a join hypergroup, then the hyperringoid is called **join hyperringoid**. The special join hyperringoid which derives from the theory of languages has been named **B-hyperringoid**.

The necessity of the consideration of the "null word" in the Theory of Languages, has led to the introduction of a non scalar neutral element in the join hypergroup with regard to which every element has a unique opposite. Thus the **fortified join hypergroup** was defined [16]. The null word is billaterally absorbing with regard to the concatenation. The hyperringoid whose hypercompositional part is a fortified join hypergroup, was named **fortified join hyperringoid**, or **join hyperring**. The fortified join hyperringoid which derives from all the words, the null word included, was named **B-hyperringoid.** These structures have been studied by G. G. Massouros and the results of his work have been applied into the Theory of Languages and Automata [16].

Apart from the above, there have also been introduced other types of hyperrings and hyperfields, such as the **polysymmetrical hyperrings** and **hyperfields** [19] as well as the **hypernear-rings** [3], [5] (in the sence of Krasner). The additive part of the first two is a polysymmetrical hypergroup while it is a quasi-canonical hypergroup, i.e. a non commutative canonical hypergroup in the third one. The multiplicative part of all of them is a semi-group. These structures have not been studied thoroughly yet

Other structures that have been developed on the basis of the ones mentioned up to now, are the **hypermodules** [12] and the **vector hyperspaces** [22], the **feeble hypermodules** [1], [31], the **hypermoduloids**, the **supermoduloids** [17] and the **hyperagebra of Boole** [23]. Moreover the theory of the **hyperlattices** [6] and the **superlattices** [24] has been developed. Lastly J. Mittas has introduced the **hypercompositional analysis** [25].

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