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HYPERGROUPS DEFINED FROM A LINEAR SPACE

by

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Abstract. In this paper certain types of hypergroups are studied, which are defined via a vector space. We introduce the notion of the **generalized canonical polysymmetrical hypergroup** and we prove that the hypergroups considered here, are of this type. We also introduce the notions of the **broken** and **extended convexity** for subsets of a linear space. These notions are connected with the semi-sub-hypergroups of the hypergroups which are defined from the linear space. Based on the above hypergroups we define the **generalized canonical polysymmetrical hyperring.**

A

One of us, motivated by the work of Prenowitz and Jantosciak [4], [20], [21], [22], [23], [24], [25] studied the Join Space (which they have introduced and studied, but only in association to Geometries) as an abstract algebraic hypercompositional structure, i.e. an algebraic structure with at least one hypercomposition among its operations, and named it (since it is a hypergroup), **Join Hypergroup**, that is a commutative hypergroup (H, .) [10], which is characterized by the axiom

$$(x/y)\cap(z/w)\neq\varnothing \Rightarrow (xw)\cap(yz)\neq\varnothing \qquad (J)$$

where, for every x, $y\in H$, $x/y = x:y = \{t \mid x \in ty\}$ is the *quotient* of x with y (see [11], [12]). In this work he used the example which appears in Proposition 1. This example plays a determinative role in the applications that are developed there and it also connects the linear spaces directly with the hypercompositional structures.

Proposition 1. Let (V, +) be a linear space over an ordered field (F, +, .). Then V endowed with the hypercomposition:

$$x \dot{+} y = \{\kappa x + \lambda y \mid \kappa,\lambda \in F_+^*,\ \kappa + \lambda = 1\} \tag{1}$$

($F^* = F - \{0\}$, $F_+ = \{\kappa \in F \mid \kappa \geqslant 0\}$, $F_+^* = \{\kappa \in F \mid \kappa > 0\}$) becomes a join hypergroup.

This join hypergroup, which is defined via the linear space (V, +), was named, by the author of [11], [12] **attached hypergroup of V.** From now on we shall use this term for all the hypergroups that will be defined via V. In order to distinguish one from the other we shall use the number of the relation which defines the hypercomposition (or the number of the corresponding Proposition).

The above mentioned connection of the theory of linear spaces with the theory of hypercompositional structures is mainly shown by the following two Propositions, that are also contained in [11] and [12].

Proposition 2. The convex subsets of V are semi-sub-hypergroups of its attached hypergroup defined from (1) and vice versa.

Proposition 3. The linear subspaces of V are the closed sub-hypergroups of its attached hypergroup (1) and vice versa.

We remark that a sub-hypergroup h of a hypergroup (H, .) is called **closed from the right** (in H), (resp. **from the left**), if for every $x \in H - h$ holds $xh \cap h = \varnothing$ (resp. $hx \cap h = \varnothing$). h is called **closed** (in H) if it is closed from the right and from the left. Also the next Proposition is valid: **h is closed from the right, if and only if from the relation $xh \cap h \neq \varnothing$ derives that $x \in h$ (resp. from the left)** (see [6], [7], [13], [16]).

Using the above Propositions as well as some others that derive from them, we get a great number of Properties of the linear spaces. Some of these are already known, while others are new ones. All these Properties form a class of Properties of the linear spaces that derive from their connection with the hypercompositional structures.

Since the notation of the hypercomposition is additive, we denote by $x \dot{-} y$ the quotient x/y, and so

$$x \dot{-} y = \{\kappa x + \lambda y \mid \kappa,\lambda \in F,\ \lambda < 0 < \kappa,\ \kappa + \lambda = 1\} \tag{2}$$

$$y \dot{-} x = \{\kappa x + \lambda y \mid \kappa,\lambda \in F,\ \lambda < 0 < \kappa,\ \kappa + \lambda = 1\} \tag{2'}$$

Thus we have the following geometrical figure:

$x \div y$ $\qquad$ $x \dot{+} y$ $\qquad$ $y \div x$ $\qquad$ $(=\{\kappa x+\lambda y \in V\})$

x $\qquad$ y

(with $\lambda<0<\kappa$ $\qquad$ $0<\kappa,\lambda$ $\qquad$ $\kappa<0<\lambda$ $\qquad$ and $\kappa+\lambda=1$)

B

Besides the theory that is developed in [11] and [12] about the attached hypergroup (1) of V, we additionaly remark the following:

a) **The attached hypergroup (1) does not have a neutral element, unless $V=\{0\}$.**

Because if e were such an element, then, for every $x \in V$, we would have $x \in e \dot{+} x$ and thus there would exist κ, $\lambda \in F_+^*$ with $\kappa+\lambda=1$ such that $x=\kappa e+\lambda x$, and so $\kappa e=(1-\lambda)x=\kappa x$, that is $x=e$. And if $x=0$ then $e=0$ and therefore $V=\{0\}$. In the following we will consider $V \neq \{0\}$.

b) **For $0 \in V$ and for every $x \in V$ we have**

$$0 \dot{+} x=\{\lambda x \mid \lambda \in F_+^*,\ \lambda<1\} \tag{3}$$

so for $x \neq 0$ holds $x \notin 0 \dot{+} x$, while $0 \dot{+} 0=0$ (besides it is generally valid that $x \dot{+} x=x$)

c) **For every $x \in V$, there exists $x' \in V$ such that**

$$0 \in x \dot{+} x' \tag{4}$$

Indeed there are $\kappa,\lambda \in F_+^*$ with $\kappa+\lambda=1$ such that $\kappa x+\lambda x'=0$. So

$$x'=-(\kappa/\lambda)x=-\rho x, \quad \rho \in F_+^* \tag{5}$$

The elements $x'=-\rho x$, $\rho \in F_+^*$, are called **opposites** or **symmetricals** of x *(with regard to 0)* and their set, denoted by $S(x)$, that is the set

$$S(x)=\{x' \in V \mid 0 \in x+x'\} \tag{6'}$$

i.e. the set

$$S(x)=\{-\rho x \mid \rho \in F_+^*\} \tag{6}$$

is the **symmetrical** (set) of x.

For every $\rho \in F_+^*$ we have:

$$S(-\rho x)=\{-\rho_1(-\rho x)\mid \rho_1\in F_+^{*}\}$$

thus

$$S(-\rho x)=\{\rho' x\mid \rho'\in F_+^{*}\} \tag{7}$$

and so

$$S(-x)=\{\rho x\mid \rho\in F_+^{*}\} \tag{7'}$$

[where $-x$ is the opposite of x in the space $(V, +)$]
Therefore for every x', $x''\in S(x)$ derives that

$$S(x')=S(x'')=S(S(x)) \tag{8}$$

[obviously $S(S(x))=\cup_{x'\in s(x)}S(x')$] and also if $x'\in S(x)$ and x^{*}, $x^{**}\in S(x')$ we have:

$$S(x^{*})=S(x^{**})=S(S(x')) \tag{8'}$$

So for every $x\in V$ and $x^{*}\in S(S(x))$ holds:

$$S(x)=S(x^{*}) \tag{9}$$

Obviously we also have:

$$S(0)=0 \quad \text{(instead of } \{0\}) \tag{10}$$

Here let us mention that, when nothing opposes it, we make no distinction between the elements and their corresponding singletons.

d) Making the proper alterations in the definition (1) of the hypercomposition $x\dot{+}y$ in V, we can make 0 the neutral element of $(V, \dot{+})$, while $(V, \dot{+})$ *remains a hypergroup*. Indeed if we keep the definition (1) for every $x,y\neq 0$, we have only two possible cases:

$$\text{i)}\ x\dot{+}0=0\dot{+}x=x \tag{11}$$

In this case 0 is a scalar element of V with regard to the hypercomposition $x\dot{+}y$. Generally an element s of a hypergroupoid (H, .) is called **scalar** if for every $x\in H$ the products s.x, x.s are singletons [6], [7], [13], [16].

$$\text{ii)}\ x\dot{+}0=0\dot{+}x=\{\lambda x\mid \lambda\in F_+^{*},\ \lambda\leqslant 1\} \tag{12}$$

for every $x\in V$. Thus:

i) If $x\dot{+}0=0\dot{+}x=x$, for every $x\in V$, the structure $(V, \dot{+})$ becomes a hypergroup. The only cases that have to be verified are the sums $(x \dot{+} 0) \dot{+} y$, $(0 \dot{+} 0) \dot{+} x$ and $0 \dot{+} V$. This hypergroup is not a join one because if it were it should have been canonical [1], [11], [12], but it is not so because of (6), that is because of the uniqueness of the opposite element.

So we can see that the axion J is not valid in general, although it obviously holds for every $x, y, z, w \in V^*$ $(= V - \{0\})$. Thus, if for example, $y = \lambda x$, $\lambda > 0$, we have:

$0 \dot{-} x = \{-\rho' x \mid \rho' \in F_+^*\}$,

$0 \dot{-} y = \{-\rho(\lambda x) \mid \rho \in F_+^*\} = \{-\rho'' \in F_+^*\}$

So $(0 \dot{-} x) \cap (0 \dot{-} y) \neq \varnothing$, while $(0 + y) \cap (0 \dot{+} x) = \{y\} \cap \{x\} = \varnothing$, for $\lambda \neq 1$.

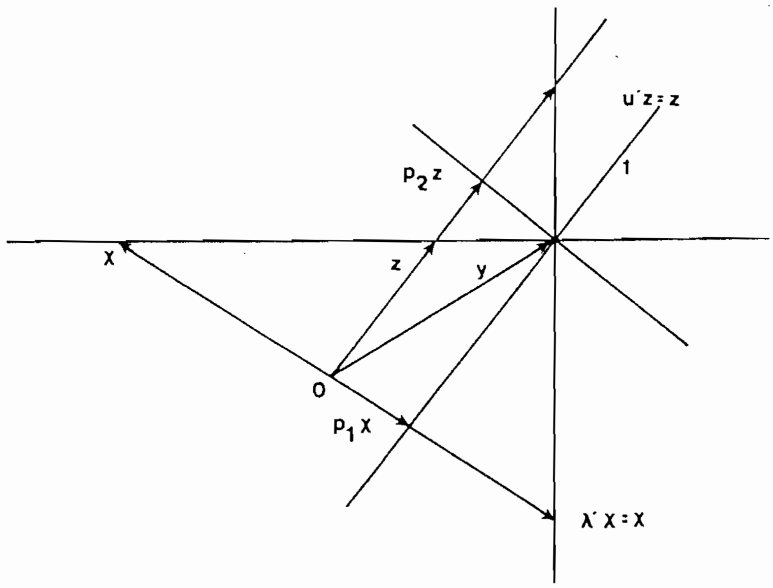
In this case the hypercompositional structure $(V, +)$ satisfies the first four axioms of the *canonical polysymmetrical hypergroups* (C.P.H.) (see Appendix). As far as the last axiom of those hypergroups concerns, it is valid only under conditions. Indeed, if $z \in x \dot{+} y$, then there exist $\kappa, \lambda \in F_+^*$, with $\kappa + \lambda = 1$, such that $z = \kappa x + \lambda y$, so $y = (1/\lambda) z - (\kappa/\lambda) x = \kappa' z + \lambda' x$ with $\kappa' = (1/\lambda) > 0$, $\lambda' = -(\kappa/\lambda) < 0$, $\kappa' + \lambda' = 1$, i.e. we have:

$$z \in x \dot{+} y \Leftrightarrow y \in z \dot{-} x \quad (= z : x)$$

On the other hand we have:

$$\kappa' z \in S(-z) = S(S(z)), \ \lambda' x \in S(x)$$

and therefore, if $\kappa' z = z^*$, $\lambda' x = x'$ we have:



$$z \in x \dot{+} y \Rightarrow (\exists z^{\star} \in S(S(z))(\exists x' \in S(x))[y \in z^{\star} \dot{+} x'] \qquad (13)$$

which can also be seen in the above figure of the corresponding to V affine space [where $\kappa' > \rho_2$ (>0) and $|\lambda'| > \rho_1$ (>0), that is not for every $\kappa', \lambda' \in F_+^{\star}$ i.e. not for every $x' \in S(x)$ and $z^{\star} \in S(S(z))$].

It is obvious that the relation (13) generalizes the corresponding relations which are mentioned as axioms in the definitions of the C.H. and the C.P.H. [13], [16], [18].

ii) If $x \dot{+} 0 = 0 \dot{+} x = \{\lambda x \mid \lambda \in F_+^{\star}, \lambda \leqslant 1\}$ for every $x \in V$, the structure $(V, \dot{+})$ is also a hypergroup, which satisfies the axion J of the join hypergroups and the relation (13).

From the above derives that we can consider the following hypercompositional algebraic structure which obviously generalizes the canonical polysymmetrical hypergroup.

Definition 1. We call **generalized canonical polysymmetrical hypergroup (G.-C.P.H.)** a set H endowed with a hypercomposition $x+y$ which satisfies the axioms:

I. $x+y=y+x$, for every $x,y \in H$

II. $(x+y)+z=x+(y+z)$, for every $x,y,z \in H$

III. $(\exists 0 \in H)$ $(\forall x \in H)$ $[x \in 0+x]$

IV. $(\forall x \in H)$ $(\exists x' \in H)$ $[0 \in x+x']$.

The element x' is called **opposite** or **summetrical** of x (with regard to 0) and their set

$$S(x) = \{x' \in H \mid 0 \in x + x'\}$$

is the **symmetrical (set)** of x (with regard to 0).

Obviously $0 \in S(0)$

V. $S(0)=0$ (It gives the uniqueness of O, which is called **zero** of H).

VI. For every $x,y,z \in H$,

$z \in x+y \Rightarrow (\exists z^{\star} \in S(S(z))(\exists x' \in S(x))[y \in z^{\star}+x']$

[obviously $S(S(z)) = \cup_{z' \in S(z)} S(z')$]

Yet from the definition derives that $x+H=H$, for every $x \in H$, i.e. the reproductivity of the hypercomposition and so the structure $(H, +)$ jastifies the name hypergroup.

Remark 1. With regard to the terminology we follow here we remark that J.

Mittas, in various papers considered some types of completely regular hypergroups (according to Fr. Marty's definition) which satisfy at least one more axion similar to the axion VI of this definition [10]. These hypergroups were named by him *polysymmetrical* ([3], [15], [18], [19]). In order to avoid confusion we must note that all these hypergroups in the commutative case satisfy the axioms of the above definition. Special types of such hypergroups are the *canonical* [**non proper** polysymmetrical, since for every element x the $S(x)$ is a singleton, denoted by $-x$, with a unique *scalar neutral* element, and in which the axiom VI becomes, as we have mentioned, $z \in x + y \Rightarrow \Rightarrow y \in z - x$], the *canonical polysymmetrical* [**proper**, when there exist elements x with $S(x)$ non singleton, having *scalar neutral* element and in which the axiom VI becomes $z \in x + y \Rightarrow (\exists x' \in S(x))[y \in z + x']$ and which are generalized by the above definition]. In the class of G.-C.P.H. belongs the polysymmetrical hypergroup [15] (proper), with non scalar neutral element 0 and in which for every $x' \in S(x)$ holds $x + x' = 0$. This hypergroup satisfies the axiom VI as follows:

$$z \in x + y \Rightarrow (\forall x' \in S(x))[y \in z + x'].$$

Analogous results are also valid for the *polysymmetrical hyperrings* [15] as well as for the *canonical polysymmetrical hyperrings* and the *canonical polysymmetrical hyperfields* [18] (in language abuse since these structures are not hyperrings and hyperfields in the sence of [5], [8], [14], [17]).

The study of G.-C.P.H. in detail is going to be the subject of a future paper of ours.

From what we have mentioned above, derive the following Propositions, for the linear space we are dealing with:

Proposition 4. V, endowed with the hypercomposition

$$x \dot{+} y = \{\kappa x + \lambda y \mid \kappa, \lambda \in F_+^{\star},\ \kappa + \lambda = 1\}$$

for every $x, y \in V^{\star}$, and

$$x \dot{+} 0 = 0 \dot{+} x = x \quad \text{for every } x \in V \tag{14}$$

is a G.-C.P.H., but not a join one.

Proposition 5. V, endowed with the hypercomposition

$$x \dot{+} y = \{\kappa x + \lambda y \mid \kappa, \lambda \in F_+^{\star},\ \kappa + \lambda = 1\} \quad \text{for every } x, y \in V^{\star}$$

and $x \dot{+} 0 = 0 \dot{+} x = \{\lambda x \mid \lambda \quad F_+^*, \ \lambda \leqslant 1\}$ for every $x \in V$ (15)

is a Join G.-C.P.H.

Remark 2. a) The above relations (6), (6′), - (9) are valid for the symmetrical elements of those Polysymmetrical hypergroups

b) For every $x, y \in V^*$ and for both cases [as it is in the join hypergroup (1) for every $x, y \in V$ (see [11], [12])] we have:

$$x \neq y \qquad x, y \notin (x \dot{+} y) \cup (x \dot{-} y) \cup (y \dot{-} x)$$

while

$$x \dot{+} x = x - x = x$$

c) We can verify corresponding Propositions to the significant for the several applications Propositions 2 and 3, for both cases of these hypergroups in the following way:

Proposition 6. i) The convex subsets of V which do not contain the element 0 are semi-sub-hypergroyps of the attached hypergroup (14) and vice versa (for its semi-sub-hypergroups that do not contain 0).

ii) As far the attached hypergroup (15) concerns, the above property (i) is also valid, without any restriction for the existence or not, of the element 0.

Proposition 7. i) The linear subspaces of V are closed sub-hypergroups of the attached hypergroup (14). The converse does not hold.

ii) In the attached hypergroup (15) not only the above is valid, but the converse holds as well.

Especially for the linear subspaces of V, considered as hypergroups of their attached hypergroups in both cases we have

a) they contain 0 (anyhow every closed sub-hypergroup of any hypergroup contains all its units [6], [7], [13], [16])

b) for every element of x it contains its symmetrical S(x) and

c) the subspace itself is a G.-C.P.H. with regard to the induced from the subspace hypercomposition.

Definition 2. Every sub-hypergroup h of a G.-C.P.H. (H, +) is called **generalized canonical polysymmetrical sub-hypergroup (G.-C.P.S.-H.)** of H if h itself is G.-C.P.H. with regard to the restriction of the hypercomposition of H in h and if it has the same zero element (i.e. $0 \in h$) and moreover if for every $x \in h$, holds $S(x) \subseteq h$.

So we have the Proposition:

Proposition 8. i) The linear subspaces of V are G.-C.P.S.-H. of its attached hypergroups (14) and (15)

ii) For the hypergroup of (15), the converse of the Proposition holds as well.

Concerning the G.-C.P.S.-H., which as we have mentioned before, are going to be the subject of a coming paper, the Proposition holds:

Proposition 9. Every G.-C.P.S.-H of a G.-C.P.H. (H, +), is a closed subhypergroup of H and vice versa.

Proof. Let $h \subseteq H$ be a G.-C.P.S.-H of H. It is obvious that if for every $x \in H$, $(x+h) \cap h \neq \varnothing$ is valid, then there will exist $y, z \in h$ such that $z \in x+y$. So applying the axiom VI of the definition 1, we will have $x \in z^{\star} + y'$ for proper $y' \in S(y)$ and $z^{\star} \in S(S(z))$. But since $y, z \in h$ and h is a G.-C.P.S.-H. of H, we have that $S(y) \subseteq h$, $S(z) \subseteq h$, and $S(S(z)) \subseteq h$. Thus $y', z^{\star} \in h$, and $z^{\star} + y' \subseteq h$, and therefore $x \in h$, and so h is closed.

Conversely now, let h be closed. Then $0 \in h$ and so, for every $x \in h$ and for every $x' \in S(x)$ we will have $(x'+h) \cap h \neq \varnothing$, thus $x' \in h$ and $S(x) \subseteq h$. Therefore h is a G.-C.P.S.-H. of H.

Remark 3. The above definition 2 is the same with the definition which is used in the case of C.P.H. One can prove the more general case: **The sets of the subhypergroups of a C.P.H. which are inversible, closed, canonical polysymmetrical and also which contain for every x its symmetrical S(x), coincide [18].**

Another hypergroup which is directly connected with the attached (1), as well as with the addition of two vectors $x, y \in V$ derives if we consider the next hypercomposition in V:

$$x \underset{\bullet}{+} y = \{\kappa x + \lambda(x+y), \kappa y + \lambda(x+y) \mid (\kappa, \lambda) \in F_+ \times F_+^{\star}, \kappa + \lambda = 1\} \qquad (16)$$

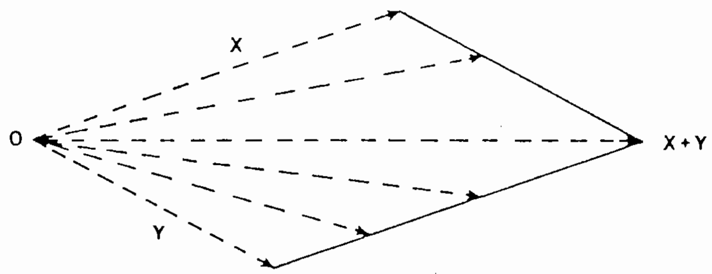
so

$$x \underset{\bullet}{+} y = \{x + \lambda y, y + \lambda x \mid \lambda \in F_+^{\star}, \lambda \leqslant 1\} \qquad (16')$$

or using the hypercomposition (1)

$$x \underset{\bullet}{+} y = [x \overset{\bullet}{+} (x+y)] \cup [y \overset{\bullet}{+} (x+y)] \cup \{x+y\} \qquad (16'')$$

In the respective affine space of V the geometric figure which presents this hypercomposition, is the set of the vectors with origin 0 that fill the parallelogram with sides x and y, and end to the opposite sides of x and y.



Obviously

$$x + y \in x \underset{\cdot}{+} y \tag{17}$$

while in general we have that

$$x \neq y \Rightarrow x, y \notin x \underset{\cdot}{+} y \tag{18}$$

Indeed $x \in x \underset{\cdot}{+} y$ if and only if either $x = x + \lambda y$ (so $\lambda y = 0$ i.e. $y = 0$, since $\lambda \neq 0$), or $x = \lambda x + y$ (so $y = (1-\lambda)x$, i.e. $y = \kappa x$ with $0 \leqslant \kappa < 1$). Thus for every $x \in V$ we have:

$$x \in 0 \underset{\cdot}{+} x \tag{19}$$

or more generally

$$y = \kappa x \quad \text{with} \quad 0 \leqslant \kappa < 1 \Rightarrow x \in x \underset{\cdot}{+} y \tag{20}$$

Also it is obvious that for every $x, y \in V$ holds:

$$x \underset{\cdot}{+} y = y \underset{\cdot}{+} x \tag{21}$$

Having concluded the above discussion we observe that the reproductivity of the hypercomposition $x \underset{\cdot}{+} y$ is almost obvious, while the associativity can be proved without any essential difficulties, by distinguishing all the different cases [for the either colinear $x, y, z \in V$ or not]. It derives that the structure $(V, \underset{\cdot}{+})$ is a commutative hypergroup with a non scalar neutral element, the 0. Apparently it is easy to show that there are symmetrical elements for every $x \in V$, which verify the above relations (6), (6′) - (13). Moreover if we denote by $x \dot{-} y$ the quotient $x : y = \{z \in V \mid x \in y \underset{\cdot}{+} z\}$, we see that

$$x \dot{-} y = \{x + \lambda(-y),\ (x-y)/\lambda \mid \lambda \in F_+^*,\ \lambda \leqslant 1\} \tag{22}$$

Also the axiom J of the join hypergroups is also valid, and so we have the Proposition:

Proposition 10. V, endowed with the hypercomposition (16), or equivalently with the (16′) or (16″), i.e. the hypercompositions of the attached hypergroup of (V, +) is a join G.-C.P.H.

Now concerning the respective Propositions of 2 and 3 of the attached hypergroup (1) - which led to the applications in the linear spaces - we firstly have the following notion, which is analogous to the notion of the convexity in V.

Definition 3. A subset V′ of a linear space V is called **brokenly convex** when for every $x, y \in V'$, holds:

i) $x + y \in V'$

and ii) $x \underset{\cdot}{+} y \subseteq V'$

which means that the segments $x\ x+y$ and $x+y\ y$, considered as sets of points, are subsets of V′.

Based on the above, we have the next Proposition, which is analogous to Proposition 1:

Proposition 11. The brokenly convex subsets of V are semi-sub-hypergroups of the attached hypergroup of (V, +) and vice versa.

On the other hand, if $V' \subseteq V$ is a linear subspace of V, then, obviously, it is a semi-sub-hypergroup of V, such that [because of (22)] $x \dot{-} y \quad V'$, for every $x, y \in V'$. Thus V′ is a closed subhypergroup of V [11], [12] and so, according to Proposition 9, a G.-C.P.S.-H. Therefore we have the Proposition:

Proposition 12. The linear subspaces of V are G.-C.P.S.-H. (and thus closed ones) of the attached hypergroup of (V, $\underset{\cdot}{+}$) and vice versa.

Two other interesting types of G.-C.P.H. derive from the hypercompositions (14) and (15). Indeed, if we change them in such a way that the sum $x + y$ will denote, geometrically, not only the set of points of the segment xy (for $x \neq y$), but the set of all the points of the line through x,y ($x \neq y$), we get the following Propositions:

Proposition 13. V, endowed with the hypercomposition:

$$x \dot{+} y = \{\kappa x + \lambda y \mid \kappa, \lambda \in F^*,\ \kappa + \lambda = 1\} \quad \text{for every } x, y \in V^*$$

and
$$x \dot{+} 0 = 0 \dot{+} x = x, \quad \text{for every } x \in V \tag{23}$$

becomes a non join G.-C.P.H.

Proposition 14. V, endowed with the hypercomposition:

$$x \dot{+} y = \{\kappa x + \lambda y \mid \kappa, \lambda \in F^*,\ \kappa + \lambda = 1\} \quad \text{for every } x, y \in V^*$$

and
$$x \dot{+} 0 = 0 \dot{+} x = \{\lambda x \mid \lambda \in F^*,\ \lambda \leqslant 1\} \quad \text{for every } x \in V \tag{24}$$

becomes a join G.-C.P.H.

The proof of the above two Propositions is fairly simple, because it derives by verifying the axioms.

From these Propositions, and in correspondence with Proposition 1 we have:

Proposition 15. V, endowed with the hypercomposition:

$$x \dot{+} y = \{\kappa x + \lambda y \mid \kappa, \lambda \in F^*,\ \kappa + \lambda = 1\} \tag{25}$$

for every $x, y \in V^*$, becomes a join G.-C.P.H.

Remark 4. a) The relations (6), (6′) - (9) also hold for the polysymmetrical hypergroups of Propositions 13 and 14.

b) What is mentioned in Remark 2b is valid also for these hypergroups as well as for the hypergroup (25). Yet for these hypergroups we have the additional relation $x \dot{+} y = x \div y = y \div x$ for every $x, y \in V$.

From the introduction of the hypercompositions (23) and (24), we are led to the considetation of the following notion of the extended convexity.

Definition 4. A subset V′ of a linear space V, is called **extendedly convex,** if $x \dot{+} y \subseteq V'$ for every $x, y \in V'$, with $x \neq y$, i.e. when the entire line through the points x and y, in the respective geometrical representation, considered as a set of points, is contained in V′.

Proposition 16. The extendedly convex subsets of V are semi-sub-hypergroups of the attached hypergroup of (23), (24) and (25) and vice versa.

Yet we have the following Proposition, that is analogous to 3, 7 and 12, which refers to the linear subspases of V.

Proposition 17. The linear subspaces of V are closed sub-hypergroups of its attached hypergroup (23), (24) and (25). More precicely, as far as the hypergroups (23) and (24) concerns, the linear subspaces are G.-C.P.S.-H. and vice versa.

Remark 5. It is obvious that combining Propositions 13, 14 and 15 with Proposition 10 we also get other attached hypergroups of V with analogous results on their semi-sub-hypergroups (i.e. Propositions analogous to 2, 6, 11 and 16). Now these semi-sub-hypergroups are the extendedly and brocenly convex subsets of V. Also there are similar Propositions to Propositions 3, 7, 12 and 17, for the linear subspaces.

C

If instead of the linear space (V, +) over an ordered field (F, +, .), we consider an associative algebra, or particularly a linear algebra (A, +, .) [2], [9] it is easy to find out that the multiplication in A is bilaterally distributive in all the types of hypergroups that we have considered above. Thus derive hypercompositional structures having the addition as hypercomposition and the multiplication as composition. These structures are correponding with the hyperrings and in the cases of the hypergroups of the Propositions 4, 5, 10, 13, 14 they can be named **generalized canonical polysymmetrical hyperrings.** Thus the notion of the canonical polysymmetrical hyperring, which is mentioned in [15], is generalized. Of course, if $A^* = A - \{0\}$ is a multiplicative group, then we will have the **general canonical polysymmetrical hyperfield**, a structure which is more general than the one which is given in [18]. Thus, according to these conclusions, using a commutative algebra, we can deduce Propositions for the generalized canonical polysymmetrical hyperrings, analogous to Propositions 4, 5, 10, 13, 14.

APPENDIX

The C.P.H. is a special case of (commutative) *polysymmetrical hypergroups* [3], [19], a notion which derives from the study of matrixes with elements from a hyperring (hypermatrixes) [5], [8], [14], [17]. Independently of this theory the definition of the canonical polysymmetrical hypergroups is the following: We call **canonical polysymmetrical hypergroup (C.P.H.)** an algabraic hypercompositional structure (H, +) which satisfies the axioms:

I. $x + y = y + x$, for every $x, y \in H$

II. $(x + y) + z = x + (y + z)$, for every $x, y, z \in H$

III. $(\exists 0 \in H)(\forall x \in H)[0 + x = x]$

(The element 0, which is obviously unique, is the zero element of H.)

IV. $(\forall x \in H)(\exists x' \in H)[0 \in x + x']$.

The element x' is called **opposite** or **symmetrical** of x and theit set $S(x) = \{x' \in H \mid 0 \in x + x'\}$ is the **symmetrical** (set) of x.

V. For every $x, y, z \in H$,
$z \in x + y \Rightarrow (\exists\, x' \in S(x))\,[y \in z + x']$

Apparently a C.P.H. differs from a canonical hypergroup (C.H.) [13], [16] in the fact that in the C.H., for every x   H, S(x) is the singleton $\{x'\}$, and so if we denote the unique opposite element x' of x with $-x$, we have for axiom V:

$$z \in x + y \;\Rightarrow\; y \in z - x$$

On the otjer hand, it easily derives from axiom III that for every $x \in H$, holds $x + H = H$. This, along with axiom II, justify the name hypergroup to the structure $(H, +)$, [3], [6], [7], [13], [16], [18], [19].

When Fr. Marty introduced the notion of the hypergroup [10], he used the quotients $\frac{x}{|y}$, $\frac{x}{y|}$, which are equal to each other, when the hypergroup is commutative:

$$\frac{x}{|y} = \frac{x}{y|} = \frac{x}{y} = \{z \in H \mid x \in z + y\}$$

Also it is always true that $\frac{x}{|y} \neq \varnothing$, $\frac{x}{y|} \neq \varnothing$ and so we have the converse of the axiom V:

$$(\forall (z, x) \in H^2)\;\; (\exists\, y \in H)\,[z \in x + y]$$

Thus the axiom V of the commutative polysymmetrical hypergroups, which is mentioned in [3], [18], [19], is not necessary to the definition of the C.P.H. (which however is a property of the canonical polysymmetrical hypergroups that derive from the hypermatrixes).

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