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HYPERGROUPS AND CONVEXITY

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ABSTRACT. In this paper we study the hypergroups in general, and also a specific category of hypergroups the join hypergroups. Yet we study their applications in the theory of linear spaces and we get, as simple corollaries, several theorems of these spaces.

0. Introduction

In 1934 Fr. Marty, generalizing the notion of the *group,* defined the *hypergroup* [11] which is based on the notion of the *hypercomposition* (see also [9], [12], [18], [21]). Still, through the definition that he introduced, (which, after all, gives only the assosiativity and the regenerativity of the hypercomposition) he defined the *left* ($\frac{a}{|b}$) and the *right* ($\frac{a}{b|}$) *division* of two elements a,b of a hypergroup (H,.), as induced hypercompositions. That is, for every a, b $\in$ H, the following is valid:

$$a.b \subseteq H\,, \frac{a}{|b} \subseteq H \qquad \frac{a}{b|} \subseteq H$$

where

$$x \in \frac{a}{|b} <==> a \in x.b \ , \ y \in \frac{a}{b|} <==> a \in b.y$$

so

$$\frac{a}{|b} = \{ x \in H \ / \ a \in x.b \} \ , \ \frac{a}{b|} = \{ y \in H \ / \ a \in b.y \}$$

Next, for the division, we will use the following symbolism, instead of the "fractional" one

$$a{:}b \ (= \frac{a}{|b}) \ , \ a..b \ (= \frac{a}{b|})$$

As we know (see [18]) for every a,b $\in$ H the a.b is non void. It is also proved easily that

$$a{:}b \neq \emptyset \ , \ a..b \neq \emptyset$$

Also, almost directly, derives that *the regenerativity of the hypercomposition in the definition of the hypergroup* (i.e. the property a.H = H.a = H, for every a $\in$ H) *is equivalent to the non void of the above "quotients"* (indeed, a.H $\subseteq$ H (resp. H.a $\subseteq$ H) and for every x $\in$ H we have x:a $\neq \emptyset$, so:

$$(\exists \ y \in H) \ [x \in a.y], \text{ therefore } x \in a.H ==> H \subseteq a.H)$$

Obviously if the hypergroup is commutative, then

$$a{:}b = a..b$$

and so, this quotien will be signified by a:b. On the other hand Marty proved that *if one of the two divisions is a composition (i.e. it is defined in a unique way), then the other one is also a composition and the hypergroup becomes a group.* Moreover, he considered hypergroups that have a bilaterally unit element (at least one), which he named *regular hypergroups* and regular ones (H,.) such that for every a $\in$ H there exists (at least one) a′ $\in$ H with the property e $\in$ a.a′ $\cap$ a′.a, where e is a unit of H (a′ is called inverse of a in regard to e). He called them *completely regular hypergroups* [11]. A special form of a completely regular hypergroup having a unit element, which is scalar[1] (and so unique) and a unique inverse for everyone of its elements, is the additive part of the *hyperfield*[2]. This hypergroup has been studied by J. Mittas (see [16], [17], [18], [19]) who named it *canonical*[3] (and it will be refered to as C-hypergroup in the following). The axioms that this hypergroup (H,.) satisfies (with the multiplicative signification of the hypercomposition) are:

For every x,y,w $\in$ H

CH_1 $x \cdot y = y.x$

CH_2 $x \cdot (y \cdot w) = (x \cdot y) \cdot w$

CH_3 *there exists an element 1 $\in$ H such that*[4] *$1 \cdot x = x$ (the unit of H).*

CH_4 *for every element x $\in$* H there is one and only one x′ *such as $1 \in x \cdot x'$. x′ is*

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[1] An element s of a hypermonoid (H,.) is called *scalar*, if for every x $\in$ H the products s.x, x.s are singleton (see [9], [16], [18]).

written x^{-1} *and called the inverse of* x.
CH_5 $w \in x \cdot y \Longrightarrow x \in w \cdot y^{-1}$

From the above definition derives directly the property

$CH_{4'}$ $x.H = H$ *for every* $x \in H$

(which along with CH_2, justifies the characterization of the structure (H,.) as hypergroup). In the case of the canonical hypergroups, as it is known, the proposition is valid (see [16], [18]):

For every $x,y,z,t \in H$, *from* $x.y \cap z.t \neq \emptyset$ *derives that* $x.z^{-1} \cap t.y^{-1} \neq \emptyset$.

And since one can easily observe that in such a hypergroup is valid $a:b = a.b^{-1}$, the above property can be written:

$$x.y \cap z.t \neq \emptyset \Longleftrightarrow x:z \cap t:y \neq \emptyset \quad \text{for every } x,y,z,t \in H$$

Also it can be prooved that *the property* $CH_{4'}$ *and the*

$$CH_{5'}\ x:z \cap t:y \neq \emptyset \Longrightarrow x.y \cap z.t \neq \emptyset$$

are equivalent with the axioms CH_4 *and* CH_5. Indeed, because of $CH_{4'}$, derives that for every x there is a $x' \in H$ such as $e \in x.x'$, that is x' is the inverse of x. On the other hand if x'' is another inverse of x, then $e \in x.x' \cap x.x''$, so $x \in e:x' \cap e:x''$, thus $e:x' \cap e:x'' \neq \emptyset$ and therefore, because of $CH_{5'}$, $e.x'' \cap e.x' \neq \emptyset$ from where $x' = x''$. So $x^{-1} = e:x$ and $(x^{-1})^{-1} = x$. Now for the axiom CH_5 we have: $w \in x.y \Longrightarrow w \in x.(y^{-1})^{-1} \Longrightarrow (y^{-1})^{-1} \in w:x \Longrightarrow e:y^{-1} \cap w:x \neq \emptyset \Longrightarrow$ $\Longrightarrow x.e \cap w.y^{-1} \neq \emptyset \Longrightarrow x \in w.y^{-1}$.
Another, hypercompositional structure, which is more general than the canonical hypergroup is the *join space* (H,.) (see [6], [27], [28]) which is defined by the axioms:
JS_1 $ab \neq \emptyset$

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[2] The *hyperfield* is a hypercompositional structure, analogous to the field, which was introduced in 1956, by M. Krasner with the addition being hypercomposition and with the multiplication, composition (see [8]). Connected to the hyperfield are some other hypercompositional structures, such as the hyperring, the hypermodule etc. (e.g. see [10], [13], [14], [15], [17], [19], [20], [22]) which were also introduced by M. Krasner.

[3] On the other hand, another type of hypergroup, which differs from the canonical as to the point that it is not commutative (so the axiom CH_5 obviously becomes $w \in x.\Psi \Longrightarrow x \in w.\Psi^{-1}$ and $\Psi \in x^{-1}w$) was introduced by P. Bonansinga - P. Corsini [1] who named it quasicanonical. Also S. Commer studied this hypergroup (under the name *polygroup*) having as an incentive the theory of binary relations and graphs and he studied it only in conection with these. S. Ioulidis independently studied the polygroup as part of the theory of the hypergroups and in connection with the canonical hypergroup (see [5]).

[4] When nothing opposes it, we make no distinction between the elements and their corresponding singletons. (Thus, for the result of 2.x we write x, instead of {x} (see [9], [16], [18]).

JS_2 $ab = ba$
JS_3 $(ab)c = a(bc)$
JS_4 $(a:b) \cap (c:d) \neq \emptyset \implies ad \cap bc \neq \emptyset$
JS_5 $a:b \neq \emptyset$

for every $a,b,c,d \in H$

The last axiom, according to the above, is equivalent to
$JS_{5'}$ $aH = H$ *for every* $a \in H$
while JS_1, as we have mentioned above, derives from the JS_3 and $JS_{5'}$. Comparing these axioms with the ones of the canonical hypergroup we observe that:

$$JS_2 \equiv CH_2,\ JS_3 \equiv CH_3,\ JS_4 \equiv CH_{5'},\ JS_5 \equiv CH_6$$

and therefore *every canonical hypergroup is a join space* and moreover *every join space is a commutative hypergroup, of a certain form which is qualified with the axiom:*

$$JH \equiv JS_4$$

The join space was used by W. Prenowitz and J. Jantosciak for the foundation and study of geometries (see [6], [23], [24], [25], [26], [27], [28]) and was regarded only in connection with them (i.e. endowing it with more axioms). In a part of this article, this structure is studied independently from Geometries, and from another point of view, as it is considered to be an abstract algebraic hypercompositional structure named by the distinct name *Join hypergroup* (from now on it will be written J-hypergroup in this text). Also applications of J-hypergroups. As well as of hypergroups in general, in the theory of linear spaces are studied, and from the relevant theory which is developed there derive, as simple corollaries, remarkable theorems, and even generalized ones, of these spaces and espesially in regard to their convex subsets (e.g. the theorems of Helly, Kakutani, Stone (see [2], [7], [28])).

1. Semi-Sub-Hypergroups and closed Sub-Hypergroups of a Hypergroup

In the beginning let us consider a linear space V over an ordered field F. As it is known [2], [7], a non void subset A of V is called c o n v e x, if for every $a,b \in$ $\in A$, with $a \neq b$, the segment which is defined from a and b belongs to A. Obviously this segment (open) is the set of points $\{\lambda a + \mu b,\ \lambda, \mu > 0,\ \lambda + \mu = 1\}$. So if a hypercomposition a.b is defined in V as follows:

$$a.b = \{\lambda a + \mu b \ /\ \lambda, \mu \in F,\ \lambda, \mu > 0,\ \lambda + \mu = 1\} \qquad \text{if } a \neq b$$

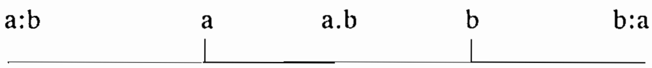
and
$a.a = a$
then the set V becomes a commutative hypemonoid (V,.), for the convex sets A of which holds:

$$a, b \in A \implies a.b \subseteq A$$

As far as the hypercomposition a.b concerns, it can be proved directly, and with no essential difficulty that:

Proposition 1.1 *The hypermonoid (V,.) is a commutative hypergroup.*

This hypergroup (V,.) is called *attached hypergroup* of the linear space V. For the induced hypercomposition of this hypergroup, holds:
$a{:}a = a$, for every $a \in V$.
and for every $a, b \in V$ with $a \neq b$
$a{:}b = \{ka + \lambda b \ / \ k,\lambda \in F,\ k>0,\ \lambda<0,\ k + \lambda = 1\}$
$b{:}a = \{ka + \lambda b \ / \ k,\lambda \in F,\ k<0,\ \lambda>0,\ k + \lambda = 1\}$
and using a geometric image [6]:



Also it is easy to verify that the axiom JH of the J-hypergroups is valid.

Proposition 1.2 *The attached hypergroup of V is a J-hypergroup, which also satisfies the property:*

$$a{:}a = a \text{ for every } a \in V.$$

and so it is a Join Geometry (see [28]).

In the case of an arbitrary hypergroup (H,.) the subsets of H which have the form of the convex subsets of V, that is the non void subset E of H with the property:

$$a.b \subseteq E \text{ for every } a, b \in E$$

(i.e. the closed under the hypercomposition subsets of H) are obviously the semi-sub-hypergroups of H. (moreover, if $a.E = E.a = E$ is also valid for every $a \in E$, then they are sub-hypergroups of H (see [9], [16], [18])).

Proposition 1.3 *The convex sets of V are semi-sub-hypergroups of its attached hypergroup.*
(This remark has been made by P. Corsini in [4])

Consequence I: It derives that *the properties of the convex subsets of a linear space are simple applications of the properties of the semi-sub-hypergroups of a hypergroup, and more precisely of the attached hypergroup* (see [2], [7], [28]).
Now, concerning the semi-sub-hypergroups of (H,.), we have:

Proposition 1.4 *If E is a semi-sub-hypergroup of a hypergroup (H,.) closed under the hypercomposition ":", "..", then E is a sub-hypergroup of H.*

Proof. We must prove the regenerartive axiom. So let $x \in E$. Obviously $x.E \subseteq E$ and $E.x \subseteq E$ hold. Next let r be an element of E. Then $r..x = \{ s \in H$ where $r \in$

x.s] is a subset of E. Therefore there is s such as r ∈ x.s ⊆ x.E. Thus E ⊆ x.E. Similarly, since r:x ⊆ E we have E ⊆ E.x, and so the Proposition.
Let H be an arbitrary hypergroup and h a sub-hypergroup of H, closed under the left division, i.e. a:b ⊆ h, for every a.b ∈ h (*divisionaly closed from the let*). Also let for some x ∈ H, xh ∩ h ≠ ∅. So there are a,b ∈ h such that x.b ∩ [a] ≠ ∅ that is a ∈ x.b, thus x ∈ a:b and since a:b ⊆ h we have x ∈ h. Analogous relations hold in the case that h is closed under the right division. Therefore we have the proposition:

Proposition 1.5 *Every sub-hypergroup h of a hypergroup H, divisionally closed from the left (resp. from the right) is a closed sub-hypergroup of H from the right*[1] *(resp. from the left).*
But the inverse holds as well, that is:

Proposition 1.6 *Every right closed (resp. left closed) sub-hypergroup h of H is divisionally closed from the left (resp. from the right).*

Proof. Let a, b ∈ h. Then for every x ∈ a:b we have a ∈ xb, so h ∩ xh ≠ ∅. And since h is right closed, x∈ h. Thus a:b ⊆ h.

Corollary 1.1 *A sub-hypergroup of a hypergroup is closed if and only if it is divisionally closed both from the left and the right.*

Corollary 1.2 *In a commutative hypergroup every sub-hypergroup is divisionally closed if and only if it is a closed sub-hypergroup.*
It derives that the linear subspace which is defined from the points a, b of the linear space V, considered as a subset of its attached hypergroup (i.e. the set h = [a,b] ∪ ∪ a.b ∪ a:b ∪ b:a) is a closed sub-hypergroup of it. Therefore we easily come to the conclusion:

Proposition 1.7 *Every linear subspace of a linear space, considered as a subset of its attached hypergroup is a closed sub-hypergroup of it.*

Consequence II. It derives that *the properties of the linear subspaces of a linear space are simple applications of the properties of the closed sub-hypergroups of a hypergroup, and more precisely of the attached hypergroup.*

Next we present some useful for the following theory examples:

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[1] A sub-hypergroup h of a hypergroup H is called *closed from the right* (in H), (resp. *from the left*), if for every a ∈ H - h holds ah ∩ h = ∅ (resp. ha ∩ h = ∅). h is called *closed* if it is closed from the right and from the left. Also holds the proposition: *h is closed from the right if and only if from the relation ah ∩ h ≠ ∅ derives that a ∈ h (resp. from the left).* (see [9], [16], [18]).

Examples

1. In a non void set H we define a hypercomposition "." in the following way:

$$a.b = \{a,b\} \quad \text{for every } a,b \in H \text{ with } a \neq b$$
$$a.a = H \quad \text{for all } a \in H$$

Then H endowed with the above hypercomposition becomes a commutative hypergoup, and as far as the induced hypercomposition ":" conserns, is valid

$$a:b = \{x \in H \;/\; a \in x.b\} = \{a,b\}$$
$$a:a = \{x \in H \;/\; a \in x.a\} = H$$

One can easily verify that this is a J-hypergroup which has not no trivial semi-sub-hypergroups.

2. Let us consider the union $Q = \cup_{i \in I} A_i$, where the $(A_i,.)$ are hypergroups. In Q we introduce a hypercomposition "∘" defined as follows:

$$a \circ b = a.b \text{ if } a,b \text{ belong to the same hypergroup } A_i$$

and $a \circ b = A_i \cup A_j$ if $a \in A_i$, $b \in A_j$ and $i \neq j$

Then one can prove that $(Q,\circ)$ is a hypergroup. Now if the hypergroups A_i are J-hypergroups then Q is a J-hypergroup.

3. In a group G we introduce a hypercomposition "." defined as follows

$$a.b = \{a,b,ab\} \text{ for every } a,b \in G$$

Then (G,.) becomes a hypergroup. In this hypergroup, for the hypercompositions ":", ".." we have

$$a:b = \{x \in G \;/\; a \in x.b\} = \{a,ab^{-1}\}$$
$$a..b = \{x \in G \;/\; a \in b.x\} = \{a,b^{-1}a\}$$

When G is commutative, then the hypergroup which derives is commutative as well. Also it can be proved that (G,.) is a J-hypergroup and that every sub-group (resp. semi-sub group) of G is a closed sub-hypergroup (resp. semi-sub-hypergroup) of the produced hypergroup. So derives a great number of commutative multiplicative (resp. additive) hypergroups of numbers with corresponding closed sub-hypergroups, the "minimum" of which is $E = \{-1, 1\}$ (proper) (non proper is the $E = \{1\}$, resp. $E = \{0\}$).

Let (H,.) be a hypergroup. As far as the indused hypercomposition concerns, we have the following simple but useful properties:

Proposition 1.8 *For every $a,b,c \in H$*, we have:

$$\text{i) } (a:b):c = a:(c.b)$$
$$\text{and ii) } (a..b)..c = a..(b.c)$$

Proof. i) Let $x \in (a:b):c$. Then there exists $y \in a:b$ such as $x \in y:c$ or $y \in x.c$. So $a:b \cap x.c \neq \emptyset$. Thus there is $w \in x.c \cap a:b$, so $a \in w.b$ from where we get that $a \in (x.c).b$ or $a \in x.(c.b)$. But this means that there is $z \in c.b$ such as $a \in x.z$ or $x \in a:z$, and therefore x ϵ a:(c.b). Conversely now. Let x ϵ a:(c.b). Then there exists y ϵ c.b such as x ϵ a:y or a ϵ x.y. So a ϵ x.(c.b) or a ϵ (x.c).b. But this means that there is w ϵ x.c such as a ϵ w.b or w ϵ a:b. Thus x.c $\cap$ a:b $\neq \varnothing$. Let z ϵ x.c $\cap$ a:b. Then x ϵ z:c and therefore x ϵ (a:b):c. The proof of the other equality is analogous to this one.

Corollary 1.3 *For the arbitrary subsets A,B,C of H, is valid:*

i) (A:B):C = A:(C.B)
and ii) (A..B)..C = A..(B.C)

Proposition 1.9 *For every a, b ϵ H, we have:*

i) b ϵ a:(a..b)
and ii) b ϵ a..(a:b)

Proof. i) We have a:b = a:b. Thus a ϵ (a:b).b. This means that there is x ϵa:b such as a ϵ x.b or equivalently b ϵ a..x. So b ϵ a..(a:b). Similarly one can prove (ii).

Proposition 1.10 *For every a ϵ H we have:*

i) H = H:a = a:H
ii) H = H..a = a..H

Proof. i) Obviously H:a $\subset$ H. Now let x ϵ H. Then a.x $\subseteq$ H from where x ϵ H:a. Therefore H $\subseteq$ H:a and so H = H:a. Similarly can be proved that H = a:H. Analogous is the proof of (ii).
Now, for the semi-sub-hypergroups and the closed hypergroups we have:

Proposition 1.11 *Let A,B be two semi-sub-hypergroups (resp. closed sub-hypergroups) of H. If A $\cap$ B is non void, then it is a semi-sub-hypergroup (resp. closed sub-hypergroup) of H.*

Proof. Let x,y ϵ A $\cap$ B. Then x.y is a subset of both A and B. Thus x.y $\subseteq$ A $\cap$ $\cap$ B and so A $\cap$ B is a semi-sub-hypergroup of H. Now if A,B are closed sub-hypergroups of H then x:y as well as x..y are also subsets of A $\cap$ B, so according to proposition 1.4 A $\cap$ B is a closed sub-hypergroup of H.

Corollary 1.4 *The set of the semi-sub-hypergroups (resp. the set of the closed sub-hypergroups) which are containing a non void subset E of H, is a complete lattice.*

From the above Corollary derives that in a given non void subset E of an arbitrary hypergroup, corresponds, by the mapping of closure, the minimum under inclusion semi-sub-hypergroup [E], which contains E. [E] is the *generated by E semi-sub-hypergroup* of H, which contains it. In the same way <E> is the *generated by E* *closed sub-hypergroup* which contains E. Moreover, as a matter of simplicity, if $E = \{a_1, a_2, ..., a_n\}$ we shall write:

$$[a_1, a_2, ..., a_n] \text{ and } <a_1, a_2, ..., a_n>$$

instead of

$$[\{a_1, a_2, ..., a_n\}] \text{ and } <\{a_1, a_2, ..., a_n\}>$$

Remark 1.1
For the attached J-hypergroup of V and for every a, b ϵ V with $a \neq b$ we have:

$$c \in <a,b> \implies c \in a.b \text{ or } b \in c.a \text{ or } a \in b.c$$

Proposition 1.12 *The Join Geometry which is defined by the attached hypergroup (see proposition 1.2) of a linear space over an ordered field, is descriptive (see [6]).*
Now refering to the monogene semi-sub-hypergroups (i.e. semi-sub-hypergroups which are generated from singletons) we have the propositions:

Proposition 1.13 *Let H be a hypergroup and a ϵ H. Then*

$$[a] = a^1 \cup a^2 \cup ... \cup a^k \cup ...$$

where $a^1 = \{a\}$, $a^2 = a.a$ *and* $a^i = a.a^{i-1}$

Proof. Since [a] is a semi-sub-hypergroup which is generated by a, we have a ϵ [a]. But from the last relation we get $a.a = a^2 \subseteq [a]$, as well as $a^{i-1}.a = a^i \subseteq [a]$ for every i ϵ N. Therefore

$$[a] \equiv a^1 \cup a^2 \cup ... \cup a^k \cup ... \quad (1)$$

Here we remark the following: If for some term of the above relation is valid $a^v \subseteq a^1 \cup a^2 \cup ... \cup a^{v-1}$ then $a^{v+p} \subseteq a^1 \cup a^2 \cup ... \cup a^{v-1}$ for every p ϵ N. (the proof is in [29] and [30], p. 54).
We proceed with the proof by naming R the right member of relation (1). We shall show that R is a semi-sub-hypergroup of H. Indeed, let x,y ϵ R. Then there are n,m ϵ ϵ N such that $x \in a^n$ and $y \in a^m$. So we have $x.y \subseteq a^n.a^m = a^{n+m} \subseteq R$. Therefore R is the least semi-sub-hypergroup of H wich contains a and which is generated by it. Thus R = [a]

Remars 1.2

a. If the term a^0 appears during some algebraic operations on a relation, it will signify that the element a does not participate any longer to this relation.

b. If X, Ψ are two semi-sub-hypergroups of a commutative hypergroup H, then X.Ψ is semi-sub-hypegroup of H. Indeed, let a,b ϵ X.Ψ. Then:

$$a.b \subseteq (X.\Psi).(X.\Psi) = (X.X).(\Psi.\Psi) = X.\Psi$$

So if $[a_1]$, $[a_2]$, ... , $[a_k]$ are monogene semi-sub-hypergroups of a hypergroup H then the set

$$[a_1]. \dots . [a_k]$$

is a semi-sub-hypergroup of H.

Theorem 1.1 *Let us consider the subset E of the commutative hypergroup H, with $E = \{a_1, a_2, \dots, a_n\}$. Then*

$$[E] = ([a_1] \cup \dots \cup [a_n]) \cup$$
$$\cup([a_1].[a_2] \cup \dots \cup [a_{n-1}][a_n]) \cup$$
$$\cup \dots \cup$$
$$\cup([a_1]. \dots .[a_n])$$

Proof. Let A be the second part of the above equality. Since [E] is the sub-hypergroup which is generated from $a_1, \dots, a_n$ we have $a_1, \dots, a_n \in [E]$ and one can easily see that [E] must contain all the sets of the form $[a_i]. \dots .[a_j]$ with $1 \leq i \leq$ $\leq j \leq n$. Thus $A \subseteq [E]$. Next, we shall prove that A is a semi-sub-hypergroup of H. Indeed let $x,y \in R$. Then

$$x \in [a_{k1}]. \dots . [a_{k_v}], \text{ with } 1 \leq k_1 < \dots < k_v \leq n$$
$$\text{and } y \in [a_{s1}]. \dots .[a_{sm}], \text{ with } 1 \leq s_1 < \dots < s_m \leq n$$

from where

$$x.y \subseteq ([a_{k1}]. \dots .[a_{k_v}]).([a_{s1}]. \dots .[a_{sm}])$$

and if we reset the indexes

$$x.y \subseteq [a_{t1}]. \dots .[a_{tp}], \text{ with } 1 \leq t_1 < \dots < t_p \leq n$$

from where is obvious that $x.y \subseteq A$. Thus A is a semi-sub-hypergroup of H, and moreover the least one that is generated by $\{a_1, \dots, a_n\}$. Therefore $[E] \subseteq A$ and so the Theorem.

Proposition 1.14 *In every commutative hypergroup H the set $\Gamma_{i=1}^{n}$ $[a_i] = [a_1]$. $[a_2]. \dots .[a_n]$ is a semi-sub-hypergroup of H that absorbs every element of*

$$[a_1, a_2, \dots, a_n]$$

Proof. In the beginning we observe that the set $\Gamma_{i=1}^{n}$ $[\alpha_i]$ $[a_i]$ absorbs all the monogene hypergroups $[a_i]$, $1 \leq \iota \leq n$. Indeed for every $\iota \in \{1,2,\dots,n\}$ we have:

$([a_1].[a_2]. \dots .[a_n]).[a_i] = [a_1].[a_2]. \dots .[a_n]$

From this relation derives that $\Gamma_{i=1}^{n}[a_i]$ absorbs every set of the form $[a_{i1}]. \dots .[a_{ik}]$ with $1 \leq \iota_1 < \dots < \iota_k \leq n$. Next we have

$$\Gamma_{i=1}^{n}[a_i] \,.\, [a_1,a_2,\dots,a_n] = ([a_1].[a_2]. \dots [a_n]).([a_1] \cup \dots \cup [a_n] \cup$$
$$\cup [a_{i1}].[a_{i2}]. \dots [a_{ik}] \cup \dots \cup [a_1].[a_1]. \dots .[a_n] =$$
$$= [a_1].[a_2]. \dots .[a_n] \cup \dots \cup [a_1].[a_2]. \dots .[a_n] = \Gamma_{i=1}^{n} [a_i]$$

Let us consider again the linear space V and let us take its attached hypergroup (V,.). Let us suppose that for the elements $a_1, \dots, a_n$ of (V,.) there are distinct integers $p_1, \dots, p_v, k_1, \dots, k_m$ such as

$$[a_{p1}, \dots, a_{pv}] \cap [a_{k1}, \dots, a_{km}] \neq \emptyset$$

Then because of the Theorem 1.1 the sub-hypergroups of the form $[a_{p1}, \dots, a_{pv}]$ can be represented from the combinations of the monogene sub-hypergroups $[a_p]$. So some intersections of these combinations are non void. Also since we are working in the attached hypergroup of a linear space $[a_i] = a_i$. Thus, according to the above syllogisms, there are

$$\{i_1, \dots, i_k\} \subseteq \{p_1, \dots, P_v\}$$
$$\text{and } \{j_1, \dots, j_t\} \subseteq \{k_1, \dots, k_m\}$$

such as

$$a_{i1} \cdot \dots \cdot a_{ik} \cap a_{j1} \cdot \dots \cdot a_{jt} \neq \emptyset$$

Now let x be an element which belongs to this last intersection. Then $x = r_1 a_{i1} + \dots + r_k a_{ik}$ with $r_1 + \dots + r_k = 1$ and $x = s_1 a_{j1} + \dots + s_t a_{jt}$ with $s_1 + \dots + s_t = 1$. Thus we have

$$s_1 a_{j1} + \dots + s_t a_{jt} + (-r_1) a_{i1} + \dots + (-r_k) a_{ik} = 0$$

with $s_1 + \dots + s_t + (-r_1) + \dots + (-r_k) = 0$ but without all the coefficients s,r equal to 0. Thus the elements $a_1, \dots, a_n$ are affinely dependent[1]. Next suppose that the elements $a_1, \dots, a_n$ of (V,.) are attinely dependent. Then

$$[a_{p1}, \dots, a_{pv}] \cap [a_{k1}, \dots, a_{km}] = \emptyset$$

where $p_1, \dots, p_v, k_1, \dots, k_m$ are distinct integers.

Proposition 1.15 *In a linear space V over an ordered filed F, the elements a_i, $i = 1, \dots, n$ are affinely dependent if and only if there are distinct integers $s_1, \dots, s_n, t_1, \dots, t_m$ that belong to $\{1, 2, \dots, \nu\}$ such that*

$$[a_{s1}, \dots, a_{sn}] \cap [a_{t1}, \dots, a_{tm}] \neq \emptyset$$

in the respective attached hypergroup.

Theorem 1.2 *Let H be a hypergroup in which every set with cardinality greater than n has two disjoint subsets A,B such as $[A] \cap [B] \neq \emptyset$. If $(Y_i)_{i \in I}$, with card $I \geq n$ is a finite family of semi-sub-hypergroups of H, in which the intersection of every n elements is non void, then all the sets Y_i have a non void intersection.*

Proof. The Theorem is trivial when card I = n. So let card I > n. We shall show first that the intersection of every n+1 of the Y_i, $i \in I$, is non void. We shall prove it for the Y_i, $1 \leq i \leq n+1$. Let $x_i \in \bigcap_{j \neq i} Y_j$, $1 \leq i \leq n+1$. Then $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1} \in Y_i$, so $[x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}] \subseteq Y_i$. The set $\{x_i, 1 \leq i \leq n+1\}$ because of our hypothesis has, two disjoint subsets A,B such that $[A] \cap [B] \neq \emptyset$. The elements of this intersection, belong to all the semi-sub-hypergroups $[x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}]$, $1 \leq i \leq n+1$. Thus these elements belong to the intersection of all these semi-sub-

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[1] An *affine combination* of points $x_1, \dots, x_n$ from a linear space W over an ordered field F is linear combination $\lambda_1 x_1 + \dots + \lambda_n x_n$, where $\lambda_1 + \dots + \lambda_n = 1$. An n-family $(x_1, \dots, x_n)$ of points from W is said to be *affinely independent* if a linear combination $\lambda_1 x_1 + \dots + \lambda_n x_n$, with $\lambda_1 + \dots + \lambda_n = 0$ can only have the value 0 when $\lambda_1 = \dots = 0$. An n-family which is not affinely independent is said to be *affinely dependent*, (see[2]).

hypergroup and therefore $\bigcap_{1\le i\le n+1}[Y_i] \neq \varnothing$. Next, suppose that the intersection of every (card I) - 1 semi-sub-hypergroups is non void. Let $x_j \in \bigcap_{j\neq i} Y_j$, $i \in I$. Then $X_i = \{x_j, j \in I-\{i\}\} \subseteq Y_i$ and therefore $[X_i] \subseteq Y_i$. Now according to our suppositions the set $X = \{x_i, i \in I\}$ has two disjoint subsets A,B such that $[A] \cap [B] \neq \varnothing$. But every semi-sub-hypergroup which is generated by a subset of X with card X = (card I) - 1 contains one of the two semi-sub-hypergroups [A], [B]. So the elements of the intersection of these two semi-sub-hypergroups [A], [B], belong to all the semi-sub-hypergroups $[X_i]$, $1\le i\le n+1$. Thus these elements belong to the intersection of all the semi-sub-hypergroups $[X_i]$, $i \in I$ and therefore $\bigcap_{i\in I}[X_i] \neq \varnothing$. But $\bigcap_{i\in I}[X_i] \subseteq \bigcap_{i\in I} Y_i$. Consequently the semi-sub-hypergroups Y_i, $i \in I$ have non void intersection.

Now from the combination of Theorem 1.2 and Proposition 1.15 derives the Corollary, which is the *Helly's Theorem* (see [2], [7], [28]):

Corollary 1.5 *let us consider a finite family* $(C_i)_{i\in I}$ *of convex sets in* R^d, *with* $d+1 \le \le$ *card I. Then, if any* $d+1$ *of the sets* C_i *have a non empty intersection, all the sets* C_i *have non empty intersection.*

2. Join Hypergroups

In the beginning as far as the induced hypercomposition concerns we have the following simple but useful properties:

Proposition 2.1 *In every J-hypergroup, the following are valid:*

i) $(a{:}d).b \cup (b{:}d).a \cup a{:}(d{:}b) \cup b{:}(d{:}a) \subseteq (a.b){:}d$

ii) $(a{:}b).(d{:}q) \cup (a{:}q).(d{:}b) \cup (a{:}b){:}(q{:}d) \cup (a{:}q){:}(b{:}d) \cup (d{:}q){:}(b{:}a) \cup (d{:}b){:}(q{:}a) \subseteq$
$\subseteq (a.d){:}(b.q)$

Proof. i) Let $x \in (a{:}d).b$. Then there is $y \in a{:}d$ such as $x \in y.b$ or $y \in x{:}b$. Thus $x{:}b \cap \cap a{:}d \neq \varnothing$ (1). Similarly we have that $x \in (b{:}d).a$ gives $x{:}a \cap b{:}d \neq \varnothing$ (2). Next if $x \in a{:}(d{:}b)$ there is $y \in d{:}b$ such as $x \in a{:}y$ from where we get $a \in x.y$ or $y \in a{:}x$. So $a{:}x \cap d{:}b \neq \varnothing$ (3). Similarly from $x \in b{:}(d{:}a)$ we have that $b{:}x \cap d{:}a \neq \varnothing$ (4). Now each one of the (1), (2), (3), (4) gives that $x.d \cap a.b \neq \varnothing$ and so there is a $w \in a.b$ such that $w \in x.d$ from which we get $x \in w{:}d$ and therefore in every case we have that $x \in (a.b){:}d$. Thus (i) has been proved.

ii) Let $x \in (a{:}b).(d{:}q)$. So there exists $y \in a{:}b$ such as $x \in y.(d{:}q) \subseteq (y.d){:}q$. Thus there is $r \in y.d$ such that $x \in r{:}q$ or $r \in x.q$. Therefore $x.q \cap y.d \neq \varnothing$. Concequently $x.q \cap \cap (a{:}b).d \neq \varnothing$ or $x.q \cap ((a.d){:}b) \neq \varnothing$. So there are $s \in x.q$, $p \in a.d$ such that $s \in p{:}b$ or $p \in s.b$. From the last relations we get $a.d \cap s.b \neq \varnothing$ or $a.d \cap x.q.b \neq \neq \varnothing$. Therefore we can choose $t \in a.d$ and $w \in q.b$ such as $t \in x.w$ or $x \in t{:}w$ from where $x \in (a.d){:}(q.b)$. Similarly $(a{:}q).(d{:}b) \subseteq (a.d){:}(b.q)$. Now let $x \in (a{:}b){:}(q{:}d)$.

Then there is $y \in a:b$ such that $x \in y:(q:d)$ from where we have: $x \in (y.d):q$, $x.q \cap y.d \neq \emptyset$, $x.q \cap (a:b).d \neq \emptyset$, $x.q \cap ((a.d):b) \neq \emptyset$. Thus there is $r \in x.q$ and $s \in a.d$ such as $r \in s:b$ or $s \in b.r$. Therefore $q.x.b \cap a.d \neq \emptyset$ and so there exists $t \in q.b$ and $w \in a.d$ such that $w \in x.t$ or $x \in w:t$ from where we have $x \in (a.d):(q.b)$. Working in a similar way we succeed in prooving (ii).
Next, using the above Proposition, one can show the following Corollary.

Corollary 2.1 *If A,B,C,D are subsets of a J-hypergroup H, then the following relations hold.*

$$i)\ (A:D).B \cup (B:D).A \cup A:(D:B) \cup B:(D:A) \subseteq (A.B):D$$

$$(ii)\ (A:B).(C:D) \cup (A:D).(C:B) \cup (A:B):(D:C) \cup (A:D):(B:C) \cup (C:D):(B:A) \cup$$
$$\cup (C:B):(D:A) \subseteq (A.C):(B.D)$$

Let H be a J-hypergroup and let h be a closed sub-hypergroup of H. Then the sets $h:x$, $x \in H$ define a partition in H. Indeed let $h:x \cap h:y \neq \emptyset$. Then $h.x \cap h.y \neq$ $\neq \emptyset$, from where we have that $y \in (h.x):h$. So $h:y \subseteq h:((h.x):h) \subseteq (h.h):(h.x) =$ $h:(h.x) = (h:h):x = h:x$ (appling successively coroll. 2.1, 1.3 and prop. 1.10). Thus $h:x \subseteq h:y$. Similarly $h:x \subseteq h:y$ and so $h:x = h:y$. So we have the proposition:

Proposition 2.2 *Let h be a closed sub-hypergroup of a J-hypergroup (H,.). Then for every x,y ∈ H we have:*

$$h:x \cap h:y \neq \emptyset \implies h:x = h:y$$

and the sub-hypergroup h defines a partition in H.

For the equivalence relation which derives from the above partition and which is denoted by mod h, or simply (h) we have:

$$x \equiv y\ (h) \iff h:x = h:y$$

Remarks 2.1

a. For every $a \in H$ there is $b \in H$ such as $a \in h:b$. Also obviously holds: $a \in h:b$ $\iff b \in h:a$.

b. Generally is vallid $a \notin h:a$.

Moreover we have the proposition:

Proposition 2.3 *The equivalence relation (mod h) is a normal one (see [18]).*

Proof. Indeed for the classes h:a of H (mod h) is valid:

$$(h:a).(h:b) \subseteq (h.h):(a.b) = h:(a.b)$$

Also if $a \equiv a'$ (h), $b \equiv b'$ (h) we have
$h:a = h:a' \implies (h:a):b = (h:a'):b \implies h:(a.b) = h:(a'.b)$ (the last equality derives because of prop. 1.8). Moreover $h:b = h:b' \implies h:(b.a') = h:(b'.a')$ and

so $h{:}(b.a) = h{:}(b'.a')$. Thus $(h{:}a).(h{:}b) \subseteq h{:}(a.b) = h{:}(a'.b')$. So $(h{:}a).(h{:}b) \subseteq$ $\subseteq \bigcup_{d \in a.b} h{:}d = \bigcup_{d' \in a'.b'} h{:}d'$.

Next we endow the quotient set H/h with a hypercomposition defined as follows: $(h{:}a)*(h{:}b) = \{h{:}u;\ u \in a.b\} = (h{:}(a.b))/h$ [1].

This hypercomposition is commutative and associative. Yet, one can easily verify that h is the neutral element in H/h.

Also we have:

$(h{:}a)*H/h = (h{:}a)*\{h{:}u;\ u \in H\} = \bigcup_{u \in H} (h{:}(a.u))/h = \{h{:}v;\ v \in a.u,\ u \in H\} =$
$= \{h{:}v;\ v \in a.H = H\} = H/h$.

So this hypercomposition is regenerative. Now for the induced from "*" hypercomposition, which will be denoted by "|", holds:

$$(h{:}a) \mid (h{:}b) = \{h{:}u;\ (h{:}a) \in (h{:}u)*(h{:}b)\} = \{h{:}u;\ a \in u.b\} = \{h{:}u;\ u \in a{:}b\}$$

Thus if we suppose that $(h{:}a) \mid (h{:}b) \cap (h{:}d) \mid (h{:}q) \neq \emptyset$, then $a{:}b \cap d{:}q \neq \emptyset$ and therefore $a.q \cap d.b \neq \emptyset$. So $(h{:}a)*(h{:}q) \cap (h{:}d)*(h{:}b) \neq \emptyset$. Consequently the axioms CH_1, CH_2, CH_3, $CH_{4'}$, $CH_{5'}$ of the canonical hypergroup are valid and so we have the proposition:

Proposition 2.4 *If h is a closed sub-hypergroup of a J-hypergroup H, then the set H/h of the classes mod h endowed with the hypercomposition "*" becomes a C-hypergroup.*

Remarks 2.2

a. The inverse of an element h:a is the class h:(h:a). We shall show first, that the set $h{:}(h{:}a) = \bigcup_{t \in h{:}a} h{:}t$ is one class. Let $t_1, t_2 \in h{:}a$, then $t_1.a \cap h \neq \emptyset$ and $t_2.a \cap$ $\cap\, h \neq \emptyset$. So $a \in h{:}t_1$ and $a \in h{:}t_2$. Thus $h{:}t_1 \cap h{:}t_2 \neq \emptyset$. and therefore $h{:}t_1 =$ $= h{:}t_2$. Next, because of Proposition 1.9, $a \in h{:}(h{:}a)$. So if $x \in h{:}a$, then $a.x \cap h \neq$ $\neq \emptyset$, thus $h \in (h{:}a)*(h{:}a))$.

b. If H has a scalar unit element e, then $\langle e \rangle = \{e\}$ is a closed sub-hypergroup of H and so the quotient $H/\{e\} \approx H$ is a C-hypergroup. Thus we have the proposition:

Proposition 2.5 *A J-hypergroup is a C-hypergroup if and only if it has a scalar unit.* (This Proposition was originally proved by P. Corsini in [4]). So a J-hypergroup differs from a C-hypergroup, only to the point that it does not have a scalar unit. Now, it is known that an inversible[2] sub-hypergroup of an arbitrary hypergroup is closed. In the case of a J-hypergroup H, for an inversible sub-hypergroup h of it, we have the proposition:

Proposition 2.6 *The set of the classes H/h forms a C-hypergroup.*

Proof. It is known that an inversible sub-hypergroup h defines a partition in H, the

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[1] If A is a union of classes (mod h) then A/h signifies the set of these classes.

classes of which are the a.h, a ϵ H. Obviously (a.h).(b.h) = $\bigcup_{d \epsilon a.b}$ (d.h) and since the equivalence relation which is defined by this partition is normal (see [18]), we may endow the quotient set H/h with the hypercomposition:

$$(a.h)*(b.h) = \{u.h;\ u \in a.b\} = ((a.b).h)/h$$

This hypercomposition is commutative and associative. Yet, one can easily verify that h is its neutral element. Also we have:

$$(a.h)*H/h = (a.h)*\{u.h;\ u \in H\} = \bigcup_{u \in H} ((a.u).h)/h = \{v.h;\ v \in a.u,\ u \in H\} =$$
$$= \{v.h;\ v \in a.H = H\} = H/h$$

So this hypercomposition is regenerative. Now for the induced from "*" hypercomposition, which will be denoted by "|", holds:

(a.h) | (b.h) = {u.h; (a.h) ϵ (u.h)*(b.h) = {u.h; aϵu.b} = {u.h; u ϵ a:b}

Thus if we suppose that (a.h) | (b.h) ∩ (d.h) | (q.h) ≠ ∅, then a:b ∩ d:q ≠ ∅ and therefore a.q ∩ d.b ≠ ∅. So (a.h)*(q.h) ∩ (d.h)*(b.h) ≠ ∅. Consequently the axioms CH_1, CH_2, CH_3, CH_4, CH_5 of the canonical hypergroup are valid and so the proposition.

So if a J-hypergroup H has an inversible sub-hypergroup h, then h defines two partitions, one as inversible and one as closed, which coincide.

Proposition 2.7 *Let A,B be two closed sub-hypergroups of a J-hypergroup. If A ∩ B ≠ ∅, then A:B = B:A.*

Proof. Let x ϵ A ∩ B. From Proposition 1.10 we have A = x:A and B = x:B. So using Corollary 2.1 we have:

$$(A:B) = (x:A):(x:B) \subseteq (x.B):(x.A) \subseteq (B:A)$$

Proposition 2.8 *Let A, B be closed sub-hypergroups of a J-hypergroup H, and A ∩ B ≠ ∅. Then A:B is a closed sub-hypergroup of H.*

Proof. From Proposition 2.7 we have A:B = B:A. So

$$(A:B):(A:B) \subseteq (A:B):(B:A) \subseteq$$
$$\subseteq (A.A):(B.B) \subseteq$$
$$\subseteq A:B$$

Moreover (A:B).(A:B) ⊆ (A.A):(B.B) ⊆ A:B

Consequently A:B is closed under the hypercompositions ".", ":". So from proposition 1.4 derives that the regenerative axiom holds as well and therefore A:B is a closed sub-hypergroup.

Proposition 2.9 *Let E be a subset of a J-hypergroup (H,.). Then E is a closed sub-hypergroup if:*

[2] A sub-hypergroup h of a hypergroup H is called *inversible from the right* (in H), (resp. *from the left*) if for every a,a' ϵ H with ah ≠ a'h is valid ah ∩ a'h = ∅ (resp. ha ≠ ha' impies that ha ∩ ha' = ∅). h is called *inversible* if it is inversible from the right and from the left (see [9], [16], [18]).

i) E is closed under the hypercomposition ":"
and ii) x.x ⊆ E for every x ∈ E

Proof. We shall show that E is closed under the hypercomposition ".". Indeed, let a,b ∈ E. Then, because of 1.9 we have a ∈ b:(b:a). So a.b ⊆ b.(b:(b:a)) ⊆ (b.b):(b:a) ⊆ ⊆ E, and therefore the Proposition derives from 1.4.

Proposition 2.10 *If A, B are semi-sub-hypergroups of a J-hypergroup, then E = = A:B is also a semi-sub-hypergroup, while it is a closed sub-hypergroup if A = B.*

Proof. A first we shall prove that E is closed under the hypercomposition ".". Indeed, let x,y ∈ E. Then there are a,d ∈ A and b,q ∈ B such that x ∈ a:b and y ∈ d:q and so we have:

$$x.y \subseteq (a:b).(d:q) \subseteq (a.d):(b.q) \subseteq A:B = E$$

Thus E is a semi-sub-hypergroup. Now if A = B, we have

$$x:y \subseteq (a:b):(d:q) \subseteq (a.q):(b.d) \subseteq A:A = E$$

So E is divisionally closed and therefore because of corollary 1.2 it is a closed sub-hypergroup of H.
So because of the above proposition and because of the remark 1.2,b if $[a_1], \ldots, [a_k]$ are monogene sub-hypergroups, of a J-hypergroup, then the sets of the form

$$([a_{\iota_1}]. \ldots .[a_{\iota_v}]) : ([a_{j_1}]. \ldots .[a_{j_\mu}])$$

with $1 \leq \iota_1 < \ldots < \iota_v \leq k$, $1 \leq j_1 < \ldots < j_\mu \leq k$
are semi-sub-hypergroups of H.

Proposition 2.11 *Let A be subset of a C-hypergroup H. Then [A]:[A] is a closed sub-hypergroup of H, which contains A.*

Proof. Since [A] is a semi-sub-hypergroup of H, then according to proposition 2.10 [A]:[A] is a closed sub-hypergroup of H. Now for every a ∈ A we have a.[A] = [A]. So there are x,y ∈ [A] such as y ∈ a.x or a ∈ y:x from where a ∈ [A]:[A] for every a ∈ A.

Theorem 2.1 *Let H be a J-hypergroup and let $\{a_1,\ldots,a_v\} \subseteq H$. Then the closed sub-hypergroup of H, which is generated by $a_1,\ldots,a_v$ is the:*

$$<a_1,\ldots,a_v> = ([a_1]. \ldots .[a_v]):([a_1]. \ldots .[a_v])$$

Proof. We have already mentioned that the set $[a_1]. \ldots .[a_v]$ is a semi-sub-hypergroup of H while from Proposition 2.10 we know that $([a_1]. \ldots .[a_v]):([a_1]. \ldots .[a_v])$ is a closed sub-hypergroup of H. Now, in order to prove that this is generated by $\{a_1,..,a_v\}$ we just have to prove that:
$a_i \in ([a_1]. \ldots .[a_v]):([a_1]. \ldots .[a_v])$, $1 \leq \iota \leq v$. Indeed we have $[a_1]. \ldots .[a_v] = [a_1]. \ldots .[a_v]$ or equivalently, because of Proposition 1.13 we have:

$$a_i.[a_1]. \ldots .[a_v] \subseteq [a_1]. \ldots .[a_v], \ 1 \leq \iota \leq v$$

So there are $x,y \in [a_1]. \dots .[a_v]$ such that $y \in a_i.x$ or $a_i \in y:x$ from where for every $\iota \in \{1,\dots,v\}$ we have:

$$a_v \in ([a_1]. \dots .[a_v]):([a_1]. \dots .[a_v])$$

Proposition 2.12 *Let A,B be two disjoint semi-sub-hypergroups of a J-hypergroup $(H,.)$ and let x be an element such that $x.x = x$, which does not belong to the union $A \cup B$. Then $[A \cup \{x\}] \cup B = \emptyset$ or $[B \cup \{x\}] \cup A = \emptyset$.*

Proof. Let $[A \cup \{x\}] \cap B \neq \emptyset$ and $[B \cup \{x\} \cap A \neq \emptyset$. Then there is $a \in A$ such as $x.a \cap B \neq \emptyset$ from where derives that $x \in B{:}a$. In a similar way there is $b \in B$ such that $x \in A{:}b$. Thus $B{:}a \cap A{:}b \neq \emptyset$ from where $B.b \cap A.a \neq \emptyset$ and since $B.b \subseteq B$, $A.a \subseteq A$ we have that $A \cap B \neq \emptyset$.

Corollary 2.2 *If A, B are disjoint convex sets in a linear space V, and x is a point not in their union, then either the convex envelope of $A \cup \{x\}$ and B are disjoint, or else the convex envelope of $B \cup \{x\}$ and A are disjoint (*Kakutani's Lemma *(see [7])).*

Proposition 2.13 *In a J-hypergroup $(H,.)$ in which holds $x.x = x$ for every $x \in H$ let A, B be two disjoint semi-sub-hypergroups. Then there are disjoint semi-sub-hypergroups X, Ψ such that $A \subseteq X$, $B \subseteq \Psi$ and $H = X \cup \Psi$.*

Proof. Let us suppose that X, Ψ are the maximum semi-sub-hypergroups with the property $A \subseteq X$, $B \subseteq \Psi$ and $X \cap \Psi = \emptyset$. If we suppose that $X \cup \Psi \neq H$ holds, then there is $w \notin X \cup \Psi$ and according to Proposition 2.12 we have $[X \cup \{w\}] \cap \Psi = \emptyset$ or $[\Psi \cup \{w\}] \cap X = \emptyset$ which contradicts the assumtion that X, Ψ are the maximum semi-sub-hypergroups endowed with the required property. Thus $X \cup \Psi = H$.

Corollary 2.3 *If A, B are disjoint convex sets in a linear space V, there are disjoint convex sets X and Ψ such that $A \subseteq X$, $B \subseteq \Psi$ and $V = X \cup \Psi$ (Stone's Theorem (see [7])).*

Definition 2.1 We shall call the elements $a_1,a_2,\dots,a_v$ of a hypergroup H *dependent*, if:

$$[a_\iota] \cap <a_1,\dots,a_{\iota-1},a_{\iota+1},\dots,a_\iota> \neq \emptyset$$

for some ι with $1 \leq \iota \leq v$. In the contrary case, the a_i, with $\iota = 1,2 \dots,v$ will be named *independent*.

Proposition 2.14 *If in a hypergroup H the elements $a_1,..a_v$ are indipendent, then holds $[a_k] \cap [a_p] = \emptyset$ for every $k,p \in \{1,\dots,v\}$ with $k \neq p$.*

Proof. Let us suppose that we have the contrary relation $[a_k] \cap [a_p] \neq \emptyset$. Then $[a_p] \cap <a_1,...,a_{p-1},a_{p+1},...,a_v> \neq \emptyset$ which contradicts the assumption.

Proposition 2.15 *Suppose that in a hypergroup H the elements $a_1,...,a_v$ are independent. Then every subset of $\{a_1,...,a_v\}$ consists of independent elements.*

Proof. Let us assume that $v > 1$, since the Proposition obviously holds for $v=1$. Now any subset of $\{a_1,...,a_v\}$ can be derived from $\{a_1,a_2,...,a_v\}$ by deleting the proper elements every time. So it is obvious that it is enough to delete an element from $\{a_1,...,a_v\}$ and then prove the Proposition. So with no loss of the generality let us prove that the elements of $\{a_2,...,a_v\}$ are independent. The proof will be through reductio at absordum. Thus let $a_2,...,a_v$ be dependent. Then for some $\iota \in \{2,...,v\}$ we will have:

$$[a_\iota] \cap <a_2,...,a_{\iota-1},a_{\iota+1},...,a_v> \neq \emptyset$$

and so $[a_\iota] \cap <a_1,a_2,...,a_{\iota-1},a_{\iota+1},...,a_v> \neq \emptyset$

But this contradicts the supposition, according to which $a_1,...,a_v$ are independent, and so the Proposition.

Theorem 2.2 *In a J-hypergroup, the elements $a_1,...,a_n$ are dependent, if and only if there exist two disjoint subsets $\{k_1,...,k_v\}$ and $\{\lambda_1,...,\lambda_\mu\}$ of $\{1,...,n\}$ such that:*

$$([a_{k_1}]. \dots .[a_{k_v}]) \cap ([a_{\lambda_1}]. \dots .[a_{\lambda_\mu}]) \neq \emptyset$$

Proof. Let us suppose that the following elements $a_1,...,a_n$ are dependent. Then for some a_ι, $1 \leq \iota \leq n$ will hold:

$$[a_\iota] \cap <a_1,...,a_{\iota-1},a_{\iota+1},...,a_n> \neq \emptyset \quad (1)$$

But, because of Theorem 2.1 we have:

$$<a_1,...,a_{\iota-1},a_{\iota+1},...,a_n> =$$
$$= ([a_1]. \dots .[a_{\iota-1}].[a_{\iota+1}]. \dots .[a_n]) :$$
$$: ([a_1]. \dots .[a_{\iota-1}].[a_{\iota+1}]. \dots .[a_n])$$

So (i) becomes:

$$[a_\iota] \cap (([a_1]. \dots .[a_{\iota-1}].[a_{\iota+1}]. \dots [a_n]) : ([a_1]. \dots .[a_{\iota-1}].[a_{\iota+1}]. \dots .[a_n])) \neq \emptyset$$

from where we have:

$$([a_1]. \dots .[a_n]) \cap$$
$$\cap ([a_1]. \dots .[a_{\iota-1}].[a_{\iota+1}]. \dots .[a_n]) \neq \emptyset$$

Conversely now. Let that for $a_{k_1},...,a_{k_v}$ and $a_{\lambda_1},...,a_{\lambda_\mu}$ holds

$$[a_{k_1}]. \dots .[a_{k_v}] \cap [a_{\lambda_1}]. \dots .[a_{\lambda_\mu}] \neq \emptyset \quad (2)$$

and suppose that for some a_{k_p} is valid:

$$a_{k_p} \notin \{a_{\lambda_1},...,a_{\lambda_\mu}\}$$
$$\text{and } a_{k_p} \in \{a_{k_1},...,a_{k_v}\}$$

Now if $v = 1$, then (2) becomes:

$$[a_{k_p}] \cap [a_{\lambda_1}]. \dots .[a_{\lambda_\mu}] \neq \emptyset$$

But for $a_{\lambda_1},...,a_{\lambda_\mu}$ we have:

$$[a_{\lambda 1}]. \dots .[a_{\lambda\mu}] \subseteq <a_1,\dots,a_{kp-1},a_{kp+1},\dots,a_n>$$

and therefore:

$$[a_{kp}] \cap <a_1,\dots,a_{kp-1},a_{kp+1},\dots,a_n> \neq \emptyset$$

from where derives the Theorem.

Moreover, if $v > 2$, then (2) gives

$$[a_{kp}] \cap (([a_{\lambda 1}]. \dots .[a_{\lambda\mu}]): \\ : ([a_{k1}]. \dots .[a_{kp-1}].[a_{kp+1}]. \dots .[a_{kv}]) \neq \emptyset$$

and since

$$([a_{\lambda 1}]. \dots .[a_{\lambda\mu}]) : \\ : ([a_{k1}]. \dots .[a_{kp-1}].[a_{kp+1}]. \dots .[a_{kv}]) \subseteq \\ \subseteq <a_1,\dots,a_{kp-1},a_{kp+1},\dots,a_n>$$

we have:

$$[a_{kp}] \cap <a_1,\dots,a_{kp-1},a_{kp+1},\dots,a_n> \neq \emptyset$$

from where the Theorem derives.

Corollary 2.4 *In a J-hypergroup H the elements $a_1,\dots,a_n$ are independent if and only if for every two disjoint subsets $\{k_1,\dots,k_v\}$ and $\{j_1,\dots,j_m\}$ of $\{1,\dots,n\}$ holds:*

$$[a_{k1}]. \dots .[a_{kv}] \cap [a_{j1}]. \dots .[a_{jm}] = \emptyset$$

Remark 2.3

Let H be a J-hypergroup and let us suppose that for the elements $a_1,\dots,a_n$ of H there exists distinct intexes $s_1, \dots ,s_p,t_1, \dots ,t_q \in \{1, \dots ,n\}$ such that:

$$[a_{s1}, \dots ,a_{sp}] \cap [a_{t1}, \dots ,a_{tq}] \neq \emptyset$$

Then, because of the Theorem 1.1, derives that there are $k_1,\dots,k_v \in \{s_1,\dots,s_p\}$ and $j_1,\dots,j_m \in \{t_1,\dots,t_q\}$ such as

$$[a_{k1}]. \dots .[a_{kv}] \cap [a_{j1}]. \dots .[a_{jm}] \neq \emptyset$$

But because of the Theorem 2.2, from this relation we have that the elements a_1, ..., a_n are dependent.

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