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Free and Cyclic Hypermodules (*).

CH. G. MASSOUROS

Summary. – *This paper is a study of the theory of hypermodules. At first an example of a hypermodule is given which leads us to some interesting applications. After that the free and the cyclic hypermodules are introduced and examined.*

1. – Introduction.

A non void set H endowed with a hypercomposition $+$ is called canonical hypergroup if the following axioms are satisfied:

i) $(x + y) + z = x + (y + z)$;

ii) $x + y = y + x$;

iii) there is an element $0 \in H$ such that for every $x \in H$ there is one and only one $x' \in H$ such as $0 \in (x + x') \cap (x' + x)$. We shall denote x' by $-x$ and we shall call it opposite of x; moreover, instead of $x + (-y)$ we shall write $x - y$;

iv) $z \in x + y$ implies $x \in z - y$.

REMARK. – It can be easily proved that 0 is unique and that $x + 0 = 0 + x = x$ for every $x \in H$. (For a study of canonical hypergroups see [6], [14].)

A non void set P endowed with a hypercomposition « $+$ » and with an internal composition « $\cdot$ » is called hyperring if:

i) $(P, +)$ is a canonical hypergroup;

ii) $(P, \cdot)$ is a multiplicative semigroup having 0 as a bilaterally absorbing element;

iii) $z(x + y) = zx + zy$, $(x + y)z = xz + yz$ (distributivity).

If $P \setminus \{0\}$ is a multiplicative group then $(P, +, \cdot)$ is a hyperfield. (The properties which derive from the structure of these mathematical objects are discussed in [7], [8], [15] while examples are given in [2], [3], [4], [5], [9], [11].)

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A commutative hyperring, in which is valid: $ab = 0 \Rightarrow a = 0$ or $b = 0$ is called an integral hyperdomain.

A subset J of a hyperring P is called a hyperideal of P if J is a subhypergroup of P and whenever $x \in P$, $k \in J$ then $kx \in J$ and $xk \in J$. A hyperideal of an integral hyperdomain P is called principal hyperideal of P, if it is generated by a single element, while P is called principal hyperideal domain, if it is an integral hyperdomain and every hyperideal of P is principal.

The notions of the hyperfield and the hyperring were introduced by M. KRASNER (see [1]). Also M. KRASNER introduced in [2] a class of hyperrings and hyperfields, the quotient hyperrings and hyperfields. Their construction is as follows: Let $(P, +, \cdot)$ be a ring and G a normal subgroup of P's multiplicative semigroup (i.e. $xG = Gx$ for every $x \in P$). Then the set $\bar{P} = P/G$ of the classes modulo G becomes a hyperring if we define the sum and the product of $\bar{P}$'s any two elements as follows:

$$xG + yG = \{(xp + yq)G\colon\ p,\ q \in G\}$$

and

$$xG\, yG = xyG\ .$$

Note that if $(P, +, \cdot)$ is a field, then $(\bar{P}, +, \cdot)$ is a hyperfield. The existence of non quotient hyperrings and hyperfields plays a very important and quite determinative role in the independence, selfsufficiency and further development of the theory of hyperrings and hyperfields. The existence of such hyperrings and hyperfields was proved in papers [3], [4], [11]. Moreover the construction methods used there endowed this theory with new interesting classes of hyperrings and hyperfields except the quotient one.

A left hypermodule over a unitary hyperring P is a canonical hypergroup M together with an external composition, $(a, m) \to am$, from $P \times M$ to M satisfying the conditions:

i) $a(m + n) = am + an$;

ii) $(a + b)m = am + bm$;

iii) $(ab)m = a(bm)$;

iv) $1m = m$ and $0m = 0$;

for all $a, b \in P$ and all $m, n \in M$. (In [10] one can find another aspect of axiom (ii). See also [15].)

There is a similar definition of a right P-hypermodule in which the elements of P are written on the right. A right and a left P-hypermodule will be called P-hypermodule from now on. Sometimes it is important to have both possibilities together i.e. a P-hypermodule. Nevertheless for the sake of simplisity we shall continue considering P-hypermodules.

We shall go on now with an example of a hypermodule which leads to some interesting applications. Let M be a P-module, where P is a unitary ring, and let G be a subgroup of the multiplicative semigroup of P, which satisfies the condition $aG\,bG = abG$, for every $a, b \in P$. Note that this condition is equivalent to the normality of G only when $P\setminus\{0\}$ is a group, which appears only in the case of division rings (see [3]). Now we introduce in M an equivalence relation $\sim$ defined as follows:

$$x \sim y \Leftrightarrow x = ty\,, \quad t \in G\,.$$

Let $\overline{M}$ be the set of the equivalence classes of M modulo $\sim$. We endow $\overline{M}$ with a hypercomposition $\dot{+}$ defined as follows:

$$\overline{x} \dot{+} \overline{y} = \{\overline{w} \in \overline{M} : \overline{w} \subseteq \overline{x} + \overline{y}\}$$

i.e. $\overline{x} \dot{+} \overline{y}$ consists of all the classes $\overline{w} \in \overline{M}$ which are contained in the setwise sum of $\overline{x}, \overline{y}$. Then $(\overline{M}, \dot{+})$ becomes a canonical hypergroup. Let now $\overline{P}$ be the quotient hyperring of P by G. Next let us consider an external composition from $\overline{P} \times \overline{M}$ to $\overline{M}$ defined as follows: $\overline{a}\,\overline{x} = \overline{ax}$ for every $\overline{a} \in \overline{P}, \overline{x} \in \overline{M}$. This composition satisfies the axioms of the hypermodule and so $\overline{M}$ becomes a $\overline{P}$-hypermodule.

We shall show next, that this hypermodule is strongly related with the analytic projective geometries as well as with the Euclidean spherical geometries.

Let M be a module over a division ring D. Since $D^* = D\setminus\{0\}$ is a multiplicative group, by using it we construct the hypermodule $\overline{M}$. Note that the zero element of $\overline{M}$ is the zero 0 of M, while the opposite of $\overline{a} \in \overline{M}$ is $\overline{a}$ itself: $\overline{a} \dot{+} \overline{a} = \{\overline{a}, 0\}$. We introduce now in M a new hypercomposition $\dotplus$ defined as follows: $\overline{x} \dotplus \overline{y} = (\overline{x} \dot{+} \overline{y}) \cup \cup \{\overline{x}, \overline{y}\}$ whenever $x \neq y$. This hypercomposition preserves the axioms of the canonical hypergroup, provided that $\overline{a} \dotplus \overline{a}$ will be defined to be the whole hypermodule $\overline{M}$ (for the proof see [5]). Now if we define the elements of $\overline{M}\setminus\{0\}$ as points and the result of the hypercomposition $\overline{x} \dotplus \overline{y}$ of any two points $\overline{x}, \overline{y}$, with $\overline{x} \neq \overline{y}$, as lines, then we form an analytic projective geometry. Furthermore all analytic projective geometries can derive using this method (see also [12]).

Next suppose that V is a vector space over an ordered field F. Let F^+ be the positive cone of F. Since F^+ is a multiplicative subgroup of F, we can construct by this the vector hyperspace $\overline{V}$ over the hyperfield $\overline{F} = \{F^+, 0, F^-\}$. Observe that this $\overline{V}$ creates exactly what is called by PRENOWITZ and JANTOSCIAK « ray join space » (see [12], [13]). Consider now a hypersphere S of V centered at 0. The map $\overline{x} \to x$ of $\overline{V}$ onto $S \cup \{0\}$ is one to one and $\overline{a} \dot{+} \overline{b}, \overline{a} \neq \overline{b}$ is mapped to the minor arc $\widehat{ab}$ of a great circle with end points a, b. We remark that $\overline{a} \dot{+} (-\overline{a}) = \{\overline{a}, -\overline{a}, 0\}$ while the two end points a, b do not belong to $\widehat{ab}$ since $\overline{a}, \overline{b} \notin \overline{a} \dot{+} \overline{b}$. But if we consider the hypercomposition $\dotplus$, then $\overline{a}, \overline{b} \in \overline{a} \dotplus \overline{b}$, the arc $\widehat{ab}$ is closed and $\overline{a} \dotplus (-\overline{a}) = \overline{V}$ i.e. two « opposite » points « generate » the whole hypersphere S, which seems more realistic since through these two points infinitely many great circles pass, which contain all the points of S. Thus every Euclidean spherical geometry can be considered as a quotient hypermodule.

We shall end this part by giving the definitions of the homomorphisms, which have been introduced by M. KRASNER.

Let M, M' be two P-hypermodules and $\mathcal{P}(M')$ the power set of M'. Then a function $\varphi\colon M \to \mathcal{P}(M')$ is called a homomorphism if:

i) $\varphi(kx) = k\varphi(x)$ for every $k \in P$, $x \in M$ and

ii) $\varphi(x + y) \subseteq \varphi(x) + \varphi(y)$ for every $x, y \in M$;

$\varphi\colon M \to \mathcal{P}(M')$ is called strong homomorphism if instead of (ii) we have:

ii)′ $\varphi(x + y) = \varphi(x) + \varphi(y)$.

A function $\varphi\colon M \to M'$ is called strict homomorphism if the above axioms (i) and (ii) are valid. $\varphi\colon M \to M'$ is called normal if (i) and (ii)′ are valid.

2. – Free hypermodules.

Let M be a P-hypermodule. Then

DEFINITION. – A linear combination of a family $(x_i\colon i \in I)$ of elements of M is a sum of the form $\sum a_i x_i$, where $a_i \in P$ and the set $\{a_i\colon a_i \neq 0\}$ is finite. $(x_i\colon i \in I)$ is linear dependent if there exists a linear combination $\sum a_i x_i$ containing 0, without being all the a_i equal to 0. Otherwise $(x_i\colon i \in I)$ is called linear independent.

DEFINITION. – A subset X of M generates M if every element of M belongs to a linear combination of elements from X.

DEFINITION. – Let N be a P-hypermodule and let X be a subset of N. Then X generates N freely if

i) X generates N and

ii) for every function ψ of X into a P-hypermodule M there exists a homomorphism $\varphi\colon N \to \mathcal{P}(M)$ such that $\varphi(x) = \{\psi(x)\}$ for every $x \in X$.

We shall call a hypermodule N free if there exists a subset X of N which generates N freely. Any set which freely generates N is called a free basis of N.

Let us construct now a free hypermodule over a hyperring P. For this purpose we consider a non void set Ω and the set P^{Ω} of the functions with domain Ω and range P. Next we choose from P^{Ω} all these functions which vanish almost everywhere. Denote this set by $E(\Omega)$. Then $E(\Omega)$ becomes a canonical hypergroup if we introduce in it a hypercomposition $+$ defined as follows:

$$f+g=\{h\in E(\Omega)\colon (\forall x\in\Omega),\ h(x)\in f(x)+g(x)\}\,.$$

We define now a map

$$*\colon \mathcal{P}(P)\times E(\Omega)\to\mathcal{P}(E(\Omega))$$

as follows:

$$*\,(A,f)=\Big\{g\in E(\Omega)\colon (\forall x\in\Omega),\ g(x)\in\bigcup_{a\in A}\{af(x)\}\Big\}\,.$$

Next with the help of this map we introduce an external composition from $P\times E(\Omega)$ to $E(\Omega)$, $(a,f)\to af$, where af is the only element of the set $*\,(\{a\},f)$. So if $A\subseteq P$ we define $Af=*\,(A,f)$.

THEOREM 1. – $E(\Omega)$ endowed with the hypercomposition « $+$ » and with the external composition «·», becomes a P-hypermodule.

PROOF. – The neutral element of the hypercomposition « $+$ » is the zero function, while the opposite of every function $f\in E(\Omega)$ is obviously $-f$, which also belongs to $E(\Omega)$. The verification of the commutativity and assosiativity presents no difficulties. Suppose now that $h\in f+g$. Then $h(x)\in f(x)+g(x)$ for every $x\in\Omega$, thus $f(x)\in h(x)-g(x)$ for every $x\in\Omega$, so $f\in h-g$. Next for every $a,b\in P$ and $f,g\in E(\Omega)$ we have:

$$a(f+g)=a\{h\in E(\Omega)\colon (\forall x\in\Omega),\ h(x)\in f(x)+g(x)\}=$$
$$=\{ah\in E(\Omega)\colon (\forall x\in\Omega),\ h(x)\in f(x)+g(x)\}=af+ag$$

also

$$(a+b)f=*\,\big((a+b),f\big)=\{g\in E(\Omega)\colon (\forall x\in\Omega),\ g(x)\in\bigcup_{c\in a+b}\{cf(x)\}\}=$$
$$=\{g\in E(\Omega)\colon (\forall x\in\Omega),\ g(x)\in(a+b)f(x)\}=$$
$$=\{g\in E(\Omega)\colon (\forall x\in\Omega),\ g(x)\in af(x)+bf(x)\}=af+bf\,.$$

We shall end the proof here since the verification of the other axioms presents no difficulties.

It is clear now, after the proof of Theorem 1, why we have been led to that « strange » definition of the external composition. This derived from the necessity of the axiom (ii) to be verified as it is established in the present article. The classical definition of the external composition does not verify this axiom (see [10]).

THEOREM 2. – There exists a subset B of $E(\Omega)$ linear independent, which generates $E(\Omega)$ and such as $\operatorname{card} B = \operatorname{card} \Omega$.

PROOF. – For every $a \in \Omega$ we define the function

$$f_a(x) = \begin{cases} 1\,, & \text{if } x = a \\ 0\,, & \text{if } x \neq a\,. \end{cases}$$

The set B of the functions f_a generates $E(\Omega)$. Also B is linear independent since if $0 \in \sum_{j=1}^{k} r_j f_{a_j}$, then for every i $(i = 1, 2, \ldots, k)$ we have $0 \in \sum_{j=1}^{k} r_j f_{a_j}\,(a_i) = \{r_i\}$ and so $r_i = 0$. Obviously $\operatorname{card} B = \operatorname{card} \Omega$.

DEFINITION. – An element x of a P-hypermodule M is called torsion free if and only if $rx = 0$ implies $r = 0$. M is called tortion free if all its elements are tortion free.

PROPOSITION 3. – Let N be a P-hypermodule and let $B = \{n_1, \ldots, n_k\}$ be a finite subset of N. Then the following are equivalent:

i) B is a free basis of N;

ii) B is linear independent and generates N;

iii) for every $n \in N$ there are uniquely defined elements $s_1, \ldots, s_k \in P$ such as $n \in \sum_{i=1}^{k} s_i n_i$;

iv) every n_i is torsion free and

$$N = Pn_1 \oplus \ldots \oplus Pn_k$$

PROOF. – (i) $\Rightarrow$ (ii). Consider the P-hypermodule $E(B)$. Then, because of the definition, for the function

$$\psi\colon B \to E(B)\,, \quad \psi(n_i) = f_{n_i} \quad (i = 1, \ldots, k)$$

there exists a homomorphism

$$\varphi\colon N \to \mathcal{P}(E(B))\,, \quad \varphi(n_i) = \{f_{n_i}\} \quad (i = 1, \ldots, k)\,.$$

Now if $0 \in \sum_{i=1}^{k} r_i n_i$, then $0 = \varphi(0) \in \varphi\Big(\sum_{i=1}^{k} r_i n_i\Big) \subseteq \sum_{i=1}^{k} r_i \varphi(n_i) = \sum_{i=1}^{k} r_i f_{n_i}$ and since the f_{n_i} $(i = 1, \dots, k)$ are linear independent $r_1 = r_2 = \dots = r_k = 0$.

(ii) $\Rightarrow$ (iii). Suppose that $n \in \Big(\sum_{i=1}^{k} s_i n_i\Big) \cap \Big(\sum_{i=1}^{k} s_i' n_i\Big)$. Then $0 \in n - n \subseteq \sum_{i=1}^{k} (s_i - s_i') n_i$ or $0 \in \sum_{i=1}^{k} c_i n_i$ for some $c_i \in s_i - s_i'$, $(i = 1, \dots, k)$ from where we get $c_1 = c_2 = \dots = c_k = 0$. Thus $0 \in s_i - s_i'$, $(i = 1, \dots, k)$ and therefore $s_i = s_i'$ $(i = 1, \dots, k)$.

(iii) $\Rightarrow$ (iv). If $kn_i = 0$ then $k = 0$, since every element belongs to a unique linear combination. It is obvious that $N = \sum_{i=1}^{k} Pn_i$: Suppose that $x \in Pn_i \cap \sum_{i \neq j} Pn_j$. Then $x = s_i n_i$ and $x \in \sum_{i \neq j} s_j n_j$. Thus $0 \in x - x \subseteq s_i n_i + \dots + (-s_i n_i) + \dots + s_k n_k$, so $s_i = 0$ $(i = 1, \dots, k)$ and therefore the conclusion.

(iv) $\Rightarrow$ (iii). If $x \in \Big(\sum_{i=1}^{k} s_i n_i\Big) \cap \Big(\sum_{i=1}^{k} s_i' n_i\Big)$ then $s_i n_i = s_i' n_i$ (see [15]) from where $0 \in (s_i - s_i') n_i$ $(i = 1, \dots, k)$ and therefore $s_i = s_i'$ $(i = 1, \dots, k)$ since the n_i are torsion free.

The proof of the statement (iii) $\Rightarrow$ (ii) is simple so we proceed to the proof that:

(ii) $\Rightarrow$ (i). For every $x \in N$ there are $s_i \in P$ such as $x \in s_1 n_1 + \dots + s_k n_k$. The s_i $(i = 1, \dots, k)$ are called coordinates of x. Observe that the coordinates of x are functions of x with values in P. Let them be denoted by $f_1, \dots, f_k$. Then we have:

$$f_i(n_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

and $f_i(x + y) \subseteq f_i(x) + f_i(y)$, $f_i(kx) = kf_i(x)$, $x, y \in N$, $k \in P$. If M is a P-hypermodule and ψ a function from B to M we define a homomorphism φ from N to $\mathcal{T}(M)$ as follows:

$\varphi(x) = f_1(x)\psi(n_1) + \dots + f_k(x)\psi(n_k)$ for every $x \in N$. φ satisfies the condition $\varphi(n_i) = \{\psi(n_i)\}$ and it is indeed a homomorphism since:

$$\begin{aligned}\varphi(x + y) &= \bigcup_{w \in x+y} \varphi(w) = \bigcup_{w \in x+y} \big(f_1(w)\psi(n_1) + \dots + f_k(w)\psi(n_k)\big) \subseteq \bigcup_{w \in x+y} \{f_1(w)\}\psi(n_1) + \\ &+ \dots + \bigcup_{w \in x+y} \{f_k(w)\}\psi(n_k) = f_1(x + y)\psi(n_1) + \dots + f_k(x + y)\psi(n_k) \subseteq f_1(x)\psi(n_1) + \\ &+ \dots + f_k(x)\psi(n_k) + f_1(y)\psi(n_1) + \dots + f_k(y)\psi(n_k) = \varphi(x) + \varphi(y)\end{aligned}$$

and

$$\varphi(kx) = k\varphi(x)\,.$$

Now let φ' be another homomorphism from N to $\mathcal{P}(M)$ such as $\varphi'(n_i) = \psi(n_i)$ $(i = 1, \ldots, k)$. Then we have for every $x \in N$:

$$x \in f_1(x)n_1 + \ldots + f_k(x)n_k \Rightarrow \varphi'(x) \subseteq \varphi'(f_1(x)n_1 + \ldots + f_k(x)n_k) \Rightarrow$$
$$\Rightarrow \varphi'(x) \subseteq f_1(x)\psi(n_1) + \ldots + f_k(x)\psi(n_k) \Rightarrow \varphi'(x) \subseteq \varphi(x) .$$

Therefore φ is the maximum homomorphism having the property which the definition of the free hypermodule demands.

PROPOSITION 4. – Let φ be the maximum homomorphism from a free hypermodule N to a hypermodule M. Then $\varphi(N)$ is a sub-hypermodule of M.

PROOF. – Let $\{n_1, \ldots, n_k\}$ be a free basis of N and x, y two elements of $\varphi(N)$. Then there exist $a, b \in N$ such as $x \in \varphi(a)$, $y \in \varphi(b)$. Thus

$$x + y \subseteq \varphi(a) + \varphi(b) = (f_1(a) + f_1(b))\varphi(n_1) + \ldots + (f_k(a) + f_k(b))\varphi(n_k) =$$
$$= \bigcup_{f_1(c) \in f_1(a) + f_1(b)} f_1(c)\varphi(n_1) + \ldots + \bigcup_{f_k(c) \in f_k(a) + f_k(b)} f_k(c)\varphi(n_k) =$$
$$= \Big\{\varphi(c) \colon c \in \sum_{f_i(c) \in f_i(a) + f_i(b)} f_i(c) n_i\Big\} \subseteq \varphi(N) .$$

Also, if $x \in \varphi(N)$, then $-x \in \varphi(N)$ and $kx \in \varphi(N)$ for every $k \in P$. Furthermore $0 \in \varphi(N)$ and the proposition is proved.

PROPOSITION 5. – Let M be a P-hypermodule, N a finitely generated free P-hypermodule and $f \colon M \to N$ a normal epimorphism. Then M has a subhypermodule F isomorphic to N such as:

$$M = F \oplus \ker f .$$

PROOF. – Let $B = \{e_1, \ldots, e_k\}$ be a basis of N. Since f is an epimorphism there are $n_i \in M$ such as $f(n_i) = e_i$, $(i = 1, \ldots, k)$. Consider a function $\psi \colon B \to M$ with $\psi(e_i) = n_i$. Since N is free, there is a maximum homomorphism $\varphi \colon N \to \mathcal{P}(M)$ such as $\varphi(e_i) = \{\psi(e_i)\} = \{n_i\}$, $(i = 1, \ldots, k)$. Then $f(\varphi(e_i)) = f(n_i) = e_i$. So if $x \in x_1 e_1 + \ldots + x_k e_k$ we have $f(\varphi(x)) = f(x_1 n_1 + \ldots + x_k n_k) = x_1 e_1 + \ldots + x_k e_k \ni x$. Thus $x \in f(\varphi(x))$ for every $x \in N$. Consider now $\varphi(N)$. As it is proved in proposition 4, this is a subhypermodule of M, which we shall denote by F. Then we have $f(F) = f(\varphi(N)) \supseteq N$, but $f(F) \subseteq N$ thus $f(F) = N$. If $n \in M$ there is an element $n' \in F$ such as $f(n) = f(n')$, from where we have $0 \in f(n) - f(n')$, or $0 \in f(n - n')$, or $(n - n') \cap \ker f \neq \emptyset$, or $n \in F + \ker f$. Thus $M = \ker f + F$. Now let $m \in \ker f \cap F$. Since $m \in F$ there is an element $x \in N$ such as $m \in \varphi(x)$ and since $m \in \ker f$, $f(m) = 0$. Let $x \in x_1 e_1 + \ldots + x_k e_k$, then $0 = f(m) \in f(\varphi(x)) = f(x_1 n_1 + \ldots + x_k n_k) = x_1 e_1 + \ldots + x_k e_k$, which means that $x_1 = x_2 = \ldots = x_k$. So $x = 0$, also $m = 0$ and therefore $\ker f \cap F = \{0\}$.

THEOREM 6. – Let M be a free hypermodule over a commutative unitary hyperring P and let $X = \{x_i\colon i \in I\}$, $Y = \{y_j\colon j \in J\}$ be two bases of it. Then $\operatorname{card} I = \operatorname{card} J$.

PROOF. – With a hyperideal A of P we shall consider the sub-hypermodule $\langle AM \rangle$ generated by all hyperideal multiplies of M and the corresponding quotient P-hypermodule $M/\langle AM \rangle$ of M by $\langle AM \rangle$. Next we observe that $M/\langle AM \rangle$ is not only a P-hypermodule but also a R/A-hypermodule if the external composition is defined as follows: $(t + A)(x + \langle AM \rangle) = tx + \langle AM \rangle$. We verify that this composition is well defined. Let $t + A = s + A$. Then $(t - s) \cap A \neq \emptyset$, thus $(t - s)x \cap \langle AM \rangle \neq \emptyset$, or $tx + \langle AM \rangle = sx + \langle AM \rangle$, or $(t + A)(x + \langle AM \rangle) = (s + A)(x + \langle AM \rangle)$. The following stage in this proof is to show that $(x_i + \langle AM \rangle\colon i \in I)$ is a free basis for the R/A-hypermodule $M/\langle AM \rangle$. Let $\sum\limits_{i \in I} (t_i + A)(x_i + \langle AM \rangle)$ be a linear combination of the family containing the zero $\langle AM \rangle$. Then $\langle AM \rangle \in \bigcup\limits_{r \in \sum_{i \in I} t_i x_i} r + \langle AM \rangle$ from where we get $\left(\sum\limits_{i \in I} t_i x_i\right) \cap \langle AM \rangle \neq \emptyset$. So there exist $s_i \in A$ such that $\left(\sum\limits_{i \in I} t_i x_i\right) \cap \left(\sum\limits_{i \in I} s_i x_i\right) \neq \emptyset$ or $0 \in \sum\limits_{i \in I} (t_i - s_i) x_i$. But since the x_i are linearly independent we get $0 \in t_i - s_i$, or $t_i = s_i$, for all $i \in I$. But $s_i \in A$, so $t_i \in A$ and therefore $t_i + A = A$ for all $i \in I$. Thus the family is linear independent. The proof that the family generates the hypermodule presents no difficulty. So we continue by choosing a maximal hyperideal of P. Then (see [15]) P/A is a hyperfield and thus the P/A-hupermodule is a vector hyperspace having $(x_i + \langle AM \rangle\colon i \in I)$ as a basis. So also is the family $(y_j + \langle AM \rangle\colon j \in J)$. But two bases of a vector hyperspace have the same cardinality (see [15]) and thus $\operatorname{card} I = \operatorname{card} J$.

LEMMA 7. – Let $x \in a_1 x_1 + \ldots + a_k x_k$ and $y \in b_1 x_1 + \ldots + b_k x_k$. Then for every $c \in a_i + b_i$ $(i = 1, \ldots, k)$ we have:

$$(x + y) \cap [(a_1 + b_1)x_1 + \ldots + c x_i + \ldots + (a_k + b_k)x_k] \neq \emptyset .$$

PROOF. – Since $x \in \sum\limits_{j=1}^{k} a_j x_j$ and $y \in \sum\limits_{j=1}^{k} b_j x_j$, there are $w \in \sum\limits_{j \neq i} a_j x_j$ and $w' \in \sum\limits_{j \neq i} b_j x_j$ such as $x \in w + a_i x_i$ and $y \in w' + b_i x_i$. From these relations we get: $a_i x_i \in x - w$ and $b_i x_i \in y - w'$, or $(a_i + b_i)x_i \subseteq (x + y) - (w + w')$. Thus for every $c \in a_i + b_i$ there are $z \in w + w'$ and $v \in x + y$ such as $c x_i \in v - z$ or $v \in c x_i + z$. Thus $(x + y) \cap [(w + w') + c x_i] \neq \emptyset$ or $(x + y) \cap [(a_1 + b_1)x_1 + \ldots + c x_i + \ldots + (a_k + b_k)x_k] \neq \emptyset$.

THEOREM 8. – Let M be a finitely generated free hypermodule over a principal hyperideal domain P and let N be a subhypermodule of M. Then N is also free and $\dim N \leqslant \dim M$.

PROOF. – Let $B = \{x_i\colon i = 1, \ldots, k\}$ be a basis of M and let $N_j = N \cap \langle x_1, \ldots, x_j \rangle$, $j = 0, 1, \ldots, k$. $N_0 = \{0\}$ and $N_k = N$. $N_1 = N \cap \langle x_1 \rangle$ is a subhypermodule of $\langle x_1 \rangle$.

The set $A = \{a : a \in P \text{ and } ax_1 \in N_1\}$ is a hyperideal of P and let a_1 be its generator. Thus $N_1 = \langle a_1 x_1 \rangle$ and therefore N_1 is $\{0\}$ or it has a basis with only one element. Suppose now that N_t has a free basis and that $\dim N_t \leqslant t$. We consider the set $A = \{a : a \in P$ and there exists $x \in N, b_1, b_2, \ldots, b_t \in P$ such that $x \in b_1 x_1 + \ldots + b_t x_t + ax_{t+1}\}$ A is a hyperideal, because if $a, a' \in A$, then there are $x, x' \in N$ such as $x \in b_1 x_1 + \ldots + b_t x_t + ax_{t+1}$ and $x' \in b_1' x_1 + \ldots + b_t' x_t + a' x_{t+1}$, from where we get $x + x' \subseteq (b_1 + b_1')x_1 + \ldots + (b_t + b_t')x_t + (a + a')x_{t+1}$. But according to the above lemma for every $c \in a + a'$, we have $(x + x') \cap [(b_1 + b_1')x_1 + \ldots + (b_t + b_t')x_t + cx_{t+1}] \neq \emptyset$. Thus $a + a' \subseteq A$. The remaining conditions can be easily prooved, so A is indeed a hyperideal. Let a_{t+1} be the generator of A. If $a_{t+1} = 0$ then $N_{t+1} = N_t$ and induction is complete. If $a_{t+1} \neq 0$, consider $y \in N_{t+1}$ such as $y \in a_1 x_1 + \ldots + a_t x_t + a_{t+1} x_{t+1}$. For every $x \in N_{t+1}$ there is $c \in P$ such as $(x - cy) \cap N_t \neq \emptyset$. Thus $N = N_t + \langle y \rangle$. But $N_t \cap \langle y \rangle = \{0\}$, so $N = N_t \oplus \langle y \rangle$ and the induction is complete.

THEOREM 9. – Every finitely generated tortion-free P-hypermodule is free with a finite basis.

PROOF. – Let M be a finitely generated tortion-free P-hypermodule and let $C = \{n_1, \ldots, n_s\}$ be a set which generates M. Consider the maximum subset $B = \{n_1, \ldots, n_k\}$ of linear independent elements of C and let N be the subhypermodule of M generated by B. Then N, because of proposition 3, is free. Now according to our hypothesis every set of the form $\{n_1, \ldots, n_k, n_i\}$, $k < i \leqslant s$ is linear dependent. Thus in the relation $0 \in r_1 n_1 + \ldots + r_k n_k + t_i n_i$ not all the coefficients are zero, and more precisely the $t_i \neq 0$, since $n_1, \ldots, n_k$ are linearly independent. Now $t_i n_i \in -(r_1 n_1 + \ldots + r_k n_k)$ so $t_i n_i \in N$ for every $i > k$. Let $t = t_{k+1}, \ldots, t_s$, then $tn_i \in N$ for every $i \leqslant s$, so $tn \in N$, for every $n \in M$ and therefore the map $f : n \to tn$ is a normal homomorphism from M to a subhypermodule of N. But since M is tortion-free the kernel of f is the $\{0\}$, thus M is isomorphic to a subhypermodule T of N. Now because of theorem 8, T is free and therefore M is also free.

3. – Cyclic hypermodules.

In this chapter we shall use notions like: a devides b, unit element, prime element, relatively prime elements, associates elements, greatest common divisor e.t.c. All these notions are connected only with the multiplicative structure of the hyperring which is the same as that of the ring. Thus their definitions remain unchanged for the case of the hyperrings and there is no need of repeating them.

DEFINITION. – If a P-hypermodule M is generated by a single element, then M is called cyclic hypermodule.

In the following R will signify a principal hyperideal domain.

Let M be an R-hypermodule. Then, if $\langle x \rangle$ is the cyclic subhypermodule of M which is generated by $x \in M$, we have $\langle x \rangle \cong R/\langle a \rangle$, where $\langle a \rangle$ is the hyperideal which vanishes x. Indeed, let $f: R \to \langle x\rangle$ be a function with $f(t) = tx$. f is a normal epimorphism between the two R-hypermodules R and $\langle x\rangle$. But $\ker f = \{t: t \in R$ and $tx = 0\}$, so $\langle a\rangle = \ker f$, thus, because of proposition 5, $R/\ker f \cong \langle x\rangle$ or $R/\langle a\rangle \cong \langle x\rangle$.

PROPOSITION 10. – Let $a, b \in R$ be relatively prime and let $c = ab$. Then $R/\langle c\rangle \cong P_1 \oplus P_2$, where $P_1 \cong R/\langle a\rangle$ and $P_2 \cong R/\langle b\rangle$.

PROOF. – $R/\langle c\rangle$ is a cyclic R-hypermodule whose generator is $1 + \langle c\rangle$. Since a, b are relatively prime there are $x, y \in R$ such as $1 \in xa + yb$ (see appendix). So if $d \in R$ then $d + \langle c\rangle \in d(xa + yb) + \langle c\rangle \subseteq dx(a + \langle c\rangle) + dy(b + \langle c\rangle) \subseteq \langle a + \langle c\rangle\rangle + \langle b + \langle c\rangle\rangle$ where $\langle a + \langle c\rangle\rangle$, $\langle b + \langle c\rangle\rangle$ are subhypermodules of $R/\langle c\rangle$ which are generated from $a + \langle c\rangle$, $b + \langle c\rangle$ respectively. Now the set $\langle a + \langle c\rangle\rangle \cap \langle b + \langle c\rangle\rangle$ has both a and b annihilators, so if $p + \langle c\rangle \in \langle a + \langle c\rangle\rangle \cap \langle b + \langle c\rangle\rangle$, then $ap \in \langle c\rangle$ and $bp \in \langle c\rangle$. Thus for every $k, s \in R$ we have $(ka + sb)p = kap + sbp \subseteq \langle c\rangle$. Since a, b are relatively prime there are $x, y \in R$ such as $1 \in xa + yb$, so 1 vanishes $\langle a + \langle c\rangle\rangle \cap \langle b + \langle c\rangle\rangle$. But 1 vanishes only $\langle c\rangle$ thus $\langle a + \langle c\rangle\rangle \cap \langle b + \langle c\rangle\rangle = \langle c\rangle$; and so $R/\langle c\rangle = \langle a + \langle c\rangle\rangle \oplus \langle b + \langle c\rangle\rangle$. But $\langle b\rangle$ vanishes $a + \langle c\rangle$ and $\langle a\rangle$ vanishes $b + \langle c\rangle$ so $\langle a + \langle c\rangle\rangle \cong R/\langle b\rangle$ and $\langle b + \langle c\rangle\rangle \cong R/\langle a\rangle$.

PROPOSITION 11. – Let p be a prime in R and M an R-hypermodule such as $M = Rx_1 \oplus \ldots \oplus Rx_k$, where the Rx_i are cyclic hypermodules of order p^{a_i} with $a_1 \leqslant \ldots \leqslant a_k$. If $m \in M$ and c is an integer with $0 \leqslant c \leqslant a_1$ such as $p^{a_1 - c}m = 0$ then $m = p^c n$ for some $n \in M$.

PROOF. – M belongs to a sum of the form $\sum_{i=1}^{k} t_i x_i$, $t_i \in R$ so $0 = p^{a_1 - c} m \in \sum_{i=1}^{k} p^{a_1 - c} t_i x_i$ from where $p^{a_1 - c} t_i x_i = 0$ for every $i = 1, \ldots, k$. Thus $p^{a_i} | p^{a_1 - c} t_i$, so $p^{a_1} | p^{a_1 - c} t_i$ for every $i = 1, \ldots, k$, from where we get $p^c | t_i$. If $t_i = p^c s_i$ then $m \in p^c \sum_{i=1}^{k} s_i x_i$ and therefore $m = p^c n$ with $n \in \sum_{i=1}^{k} s_i x \subseteq M$.

DEFINITION. – An R-hypermodule M will be called p-tortion, where p is a prime, if $p^a M = \{0\}$ for some integer a.

PROPOSITION 12. – Let M be a p-tortion R-hypermodule generated by elements $m_1, \ldots, m_k$, where m_i has order p^{a_i}, $a_1 \leqslant \ldots \leqslant a_k$. Then there exist $n_1, \ldots, n_k$, where n_i has order p^{b_i} with $b_i = a_i$ and $M = Rn_1 + \ldots + Rn_s$.

PROOF. – We shall prove this proposition by induction on $\sum_{i=1}^{k} a_i$. Let $\sum_{i=1}^{k} a_i > 0$, $a_1 > 0$ and $k > 1$. We denote by $\overline{M}$ the $\sum_{i=2}^{k} Rm_i$. Then $M = Rm_1 + \overline{M}$ and because of the induction $\overline{M} = Rn_2 \oplus \ldots \oplus Rn_k$ where the order of n_i is p^{b_i} with $b_i \leqslant a_i$. If $b_i < a_i$ for some a_i the proposition is valid. If $b_i = a_i$, $i = 2, \ldots, k$ we consider the hyperideal I which annihilates $m_1 + \overline{M} \in M/\overline{M}$. Then $I = tR$ with $t | p^{a_1}$ from where we have that the order of $m_1 + \bar{M}$ is p^c with $0 \leqslant c \leqslant a_1$, so $p^c m_1 = m^* \in M$ from where we have $p^{a_1 - c} m^* = 0$ or $p^{b_2 - c} m^* = 0$ and because of proposition 11 $m^* = p^c \bar{m}$, $\bar{m} \in \bar{M}$. Thus there is $n_1 \in m_1 - \bar{m}$ such that $p^c n_1 = 0$ from where we get $m_1 \in n_1 + \bar{m}$ and therefore $M = Rn_1 + \bar{M}$. Now if $yn_1 \in Rn_1 \cap \bar{M}$, $y \in R$ then $ym_1 \in yn_1 + y\bar{m} \subseteq \bar{M}$ so $p^c | y$, thus $yn_1 = 0$ and so $M = Rn_1 \oplus \bar{M}$.

PROPOSITION 13. – Let M be a non trivial R-hypermodule and let $dM = 0$, where $d \in R$ while it is neither zero, nor belongs to the group of units of R. Let $d = up_1^{a_1} \dots p_k^{a_k}$ where u is a unit and p_i is relatively prime. Then M can be written as a direct sum $M = M_1 \oplus \dots \oplus M_k$, where $p_i^{a_i} M_i = \{0\}$.

PROOF. – Let us denote the $d/p_i^{a_i} = u \prod_{j \neq i} p_j^{a_j}$ by d_i. Then, if M can be written as the above proposition claims we shall prove that $d_i M = M_i$. Obviously $d_i M \subseteq \subseteq d_i M_1 + \dots + d_i M_k = d_i M_i \subseteq M_i$. Also since $(d_i, p^{a_i}) = 1$ there are $r, s \in R$ such as $rd_i + sp_i^{a_i} \ni 1$. Thus if $m \in M_i$ we have: $m \in (rd_i + sp_i^{a_i})m = rd_i m + sp_i^{a_i} m = d_i(rm) \in \in d_i M$. So $M_i \subseteq d_i M \subseteq d_i M_i$. But $d_i M_i \subseteq M_i$ so $M_i = d_i M$. We shall prove now that M has a decomposition as the proposition mentions. For this purpose we define $M_i = d_i M$. Obviously $p_i^{a_i} M_i = \{0\}$. Now since $d_1, \dots, d_k$ are relatively prime there are $r_1, \dots, r_k \in R$ such as $\sum r_i d_i \ni 1$. Thus if $x \in M$, then $x = 1x \in \sum d_i(r_i x) \subseteq \sum d_i M = = \sum M_i$. Now let $y \in \left(\sum_{i \neq j} M_j\right) \cap M_i$. Then $d_i y = 0$ and $p_i^{a_i} y = 0$. If we chose proper r, s we have $rd_i + sp_i^{a_i} \ni 1$ thus $y \in (rd_i + sp_i^{a_i})y = rd_i y + sp_i^{a_i} y = 0$, therefore $y = 0$ and the proposition is proved.

PROPOSITION 14. – Let $M = M_1 \oplus \dots \oplus M_k$ where M_i are cyclic tortion hypermodules of order r_i and $(r_i, r_j) = 1$, if $i \neq j$. Then M is cyclic of order $r_1 \cdot \ldots \cdot r_k$.

PROOF. – Let $M_i = Rm_i$ and $m \in m_1 + \dots + m_k$. Also let d be the $r_1 \cdot \ldots \cdot r_k$. Now if for some $s \in R$ we have $sm = 0$ then $sm_i = 0$ for every i. Thus $d | s$ and the order of m is d. Next let us denote d/r_i by d_i. Then we have $d_i m = d_i m_i$ and since $(r_i, d_i) = 1$ there are $s, t \in R$ such as $tr_i + sd_i \ni 1$. Thus $m_i \in (tr_i + sd_i) m_i = = tr_i m_i + sd_i m_i = sd_i m_i$, so $m_i = sd_i m_i \in R(d_i m_i) = R(d_i m) \subseteq Rm$. Therefore Rm contains every m_i and so $Rm = M$.

THEOREM 15. – Let M be a finitely generated hypermodule over a principal hyperideal domain R. Then M can be expressed as an (internal) direct sum $M_1 \oplus \dots \oplus M_s$, where the M_i are non trivial cyclic subhypermodules of M of order d_i and $d_i | d_{i+1}$ $(i = 1, \dots, s-1)$.

PROOF. – Let T be tortion subhypermodule of M. Then M/T is a tortion free hypermodule and, because of theorem 9, it is also free. Thus, according to proposition 5, $M = T \oplus F$, where F is a free subhypermodule of M finitely generated. Then $T = M/F$ and so T is finitely generated. Let $T = (x_1, \dots, x_t)$. The x_i are tortion elements thus there are $r_i \in R$, $r_i \neq 0$ $(i = 1, \dots, t)$ such as $r_i x_i = 0$. Denote by r the product $r_1 \cdot \ldots \cdot r_t$. Then $r \neq 0$ and since every $w \in T$ belongs to a linear combination $\sum_{i=1}^{t} q_i x_i$, $q_i \in R$ we get $rw \in r\left(\sum_{i=1}^{t} q_i x_i\right) = \sum_{i=1}^{t} q_i r x_i = 0$ for every $w \in T$. Thus $rT = 0$. If r is a unit element, then $T = \{0\}$. Otherwise $r = up_1^{a_1} \cdot \ldots \cdot p_k^{a_k}$. But in this case, because of proposition 13, we have $T = T_1 \oplus T_2 \oplus \ldots \oplus T_k$, where T_i are finitely generated p_i-tortion hypermodules. Thus because of proposition 13:

$$\begin{array}{l} T_1 = T_{11} + \ldots + T_{1n} \\ T_2 = T_{21} + \ldots + T_{2n} \\ \vdots \quad\;\; \vdots \qquad\quad\;\; \vdots \\ T_k = T_{k1} + \ldots + T_{kn} \end{array}$$

where the T_{ij} are cyclic hypermodules of order $p_i^{a_{ij}}$ and $a_{i1} \leqslant a_{i2} \leqslant \ldots \leqslant a_{in}$. Now let $M_j = T_{ij} \oplus \ldots \oplus T_{kj}$. Then the M_j are cyclic hypermodules of order $d_j = p_1^{a_{1j}} p_2^{a_{2j}} \ldots p_k^{a_{kj}}$. Thus $d_1 | d_2 | \ldots | d_n$. Also $T = M_1 \oplus M_2 \oplus \ldots \oplus M_n$. Next the free subhypermodule F is (according to proposition 3) the direct sum of non trivial tortion free cyclic hypermodules, hence $F = M_{n+1} \oplus \ldots \oplus M_s$ and therefore:

$$M = M_1 \oplus \ldots \oplus M_n \oplus M_{n+1} \oplus \ldots \oplus M_s .$$

4. – Appendix.

In this appendix we shall show that in a principal hyperideal domain R every two elements a, b different from 0 have a greatest common divisor, which belongs to a linear combination of the form

$$ka + sb \qquad \text{with } k, s \in R .$$

Indeed, let $C(a, b)$ be the set $\bigcup_{x,y \in R} (xa + yb)$. For every $p, q \in C(a, b)$, there are $x_1, x_2, y_1, y_2 \in R$ such as $p \in x_1 a + y_1 b$, $q \in x_2 a + y_2 b$ from where we have:

$$p + q \subseteq (x_1 a + y_1 b) + (x_2 a + y_2 b) = (x_1 + x_2) a + (y_1 + y_2) b =$$
$$= \bigcup_{x \in x_1 + x_2} \{xa\} + \bigcup_{y \in y_1 + y_2} \{yb\} .$$

So $p + q \subseteq C(a, b)$ for every $r \in R$. Also $0 \in C(a, b)$ and for every $x \in C(a, b)$ we have $-x \in C(a, b)$ and $rx \in C(a, b)$ for every $r \in R$. Thus $C(a, b)$ is a hyperideal, and since R is a principal hyperideal domain, there is $d \in C(a, b)$ such as $\langle d \rangle = C(a, b)$. Yet d belongs to $ka + sb$ for some $k, s \in R$. a, b both belong to $C(a, b)$ because

$a = 1a + 0b$, $b = 0a + 1b$. As members of the hyperideal $\langle d \rangle$, both a, b are multiples of d. Now if c is a common divisor of a, b, then $a = a'c$ and $b = b'c$ for some $a', b' \in R$. Thus $d \in ka + sb = ka'c + sb'c = (ka' + sb')c$ which means that there exists $t \in ka' + sb'$ such as $d = tc$. Therefore d is a multiple of c.

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