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ON THE THEORY OF HYPERRINGS AND HYPERFIELDS

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§ 1. Introduction

A hyperfield is an algebraic structure $(H,+,\cdot)$, where H is a non empry set, $\cdot$ is an internal composition of H (that is a mapping of $H\times H$ into H) and $+$ is a hypercomposition of H (that is a mapping with domain $H\times H$ and range $P(H)$, i.e. the set of all subsets of H).

This structure satisfies the axioms:

I. Multiplicative axiom:

H is an almost-group*) with regard to the multiplication. By 1 we shall denote its neutral element and by 0 its absorbing element

II. Additive axioms:

i) $x+y=y+x$ (commutativity);

ii) $x+(y+z)=(x+y)+z$ (associativity);

iii) for every $x\in H$ there exists one and only one $x'\in H$ such that $0\in x+x'$ (x' is written $-x$ and called the opposite of x ; moreover, instead of $x+(-y)$ we shall write $x-y$) ;

*) An almost-group is a semigroup S , which is the union $G\cup\{0\}$, where G is a group and 0 a bilaterally absorbing element of S .

iv) $z \in x+y$ implies $y \in z-x$.

III. Distributive axiom:

$$z(x+y)=zx+zy\,; \quad (x+y)z=xz+yz.$$

We note that:

a) If $\cdot$ is an internal composition of a set E and X, Y are subsets of E, as usual $X.Y$ signifies the set $\{x.y \mid (x,y)\in X\times Y\}$. Now if $\cdot$ is a hypercomposition in E, $X.Y$ will be the union $\bigcup\limits_{(x,y)\in X\times Y} x\cdot y$. In both cases $X.y$ and $x.Y$ will have the same meaning as $X.\{y\}$ and $\{x\}.Y$ respectively.

b) By virtue of axioms II. (iii) and II. (iv) it holds that $x+0=x$ for each $x\in H$.

c) If we replace the multiplicative axiom I by

I'. H is a multiplicative semigroup having a bilaterally absorbing element, 0,

we obtain a more general structure $(H,+,\cdot)$ which is called hyperring (for example, see [5, 9]).

A non-empty subset S of a hyperring R closed under the addition and the multiplication of R and which is itself a hyperring under these operations is called a subhyperring of R.

d) H being endowed with a hypercomposition satisfying the additive axioms (II) forms a canonical hypergroup (for a study of canonical hypergroups see [7, 8]).

The notion of the hyperfield was introduced by M.Krasner in his study [1]. The author there uses the hyperfield as the proper algebraic tool in order to define a certain approximation of a complete valued field, by sequences of such fields. The construction of this hyperfield is done as follows:

Let K be a valued field and let $|\cdot\cdot|$ be its valuation. Let ρ be a real or a semi-real number [2] of species $-$, such as $0\leq\rho<1$ and let π_ρ be the equivalence relation in K, that is defined as fol-

lows
$$a \equiv 0 \Leftrightarrow 0 \equiv a \Leftrightarrow a = 0,$$

if
$$a \neq 0, \quad b \equiv a \Leftrightarrow \left|\frac{b}{a} - 1\right| \leq \rho \Leftrightarrow |b-a| \leq \rho|a|.$$

The classes mod π_ρ are circles $C_\xi = C(\xi, \rho|\xi|)$ of center $\xi \in K$ and radius $\rho|\xi|$. The set K/π_ρ of these classes becomes a hyperfield if we define the product of K/π_ρ 's two elements to be their setwise product and their sum to be the set of classes which are contained in their setwise sum. Krasner named this hyperfield residual hyperfield of K (mod π_ρ).

Further development of the theory of hyperfields and hyperrings made it imperative to determine how rich the class of hyperfields and hyperrings is. Thus the question was raised: can all hyperfields be isomorphic to the residual ones? (for example see [12 , p.3]). This question became more general after the construction of the quotient hyperfields by M.Krasner (see [3]), which contain the class of residual hyperfields. So M.Krasner in [3] sets the problem whether all hyperfields are isomorphic to quotient ones. Also, he raises the analogous question for the hyperrings. The answer to these questions becomes even more important for the following additional reason: If quotient hyperfields and hyperrings were the only existing ones, then several conclusions of their theory (for example see [6, 10, 11, 12]) could be arrived at, much more directly, using the theory of classical algebraic structures (i.e. fields, rings, modules, e.t.c.).

In the following parts of this article we shall give the negative answer to the above questions presenting constructions which can produce non quotient hyperfields and hyperrings. Before proceeding though, let us refer to the way of constructing the quotient hyperrings and hyperfields:

Consider a ring R and a normal subgroup W of R's multiplicative semigroup (i.e. $xW = Wx$ for every $x \in R$). It can be proved that the multiplicative classes $\bar{x} = xW$ form a partition of R. Let R/W be the set of these classes. Then R/W becomes a hyperring if we define the product of R/W's two elements to be their setwise product and their sum to be the set of the classes which are contained in their setwise sum. If R is a field then R/W is a hyperfield.

§ 2. Non quotient hyperfields

Now we shall present a class of hyperfields and we shall prove that it contains elements that do not belong to the class of quotient hyperfields.

PROPOSITION 1. Let G be a non unitary multiplicative group and let $(H^*, .)$ be its direct product with the multiplicative group $\{-1, 1\}$. Consider the almost group $(H, .) = (H^* \cup \{0\}, .)$. A hypercomposition $\dot{+}$ can be introduced in it defined as follows: for every $(x, i) \in H$

$$(x,i) \dot{+} (x,i) = H \setminus \{(x,i), (x,-i), 0\},$$
$$(x,i) \dot{+} (x,-i) = H \setminus \{(x,i), (x,-i)\},$$
$$(x,i) \dot{+} 0 = 0 \dot{+} (x,i) = (x,i),$$

and $(x,i) \dot{+} (w,j) = \{(x,i), (w,j), (x,-i), (w,-j)\}$ for every $(w,j) \neq (x,i), (x,-i), 0$. Then the triplet $H(G) = (H, \dot{+}, .)$ is a hyperfield.

PROOF. Obviously the hypercomposition $\dot{+}$ is commutative with neutral element O. Now for every $x = (a,i), a \in G, i = \pm 1$, we denote by $\bar{x}$ the $(a,-i)$. Then $\bar{x}$ is the opposite of x, because $0 \in x \dot{+} \bar{x}$. Next as far as the proof of the associativity is concerned we have:

i) if any two of x, y, z are different between them, none of them is the opposite of the others and $x, y, z \neq 0$, then

$$x\dotplus(y\dotplus z)=x\dotplus\{y,z,\bar{y},\bar{z}\}=(x\dotplus y)\cup(x\dotplus z)\cup(x\dotplus\bar{y})\cup(x\dotplus\bar{z})=$$
$$=\{x,y,\bar{x},\bar{y}\}\cup\{x,z,\bar{x},\bar{z}\}\cup\{x,y,\bar{x},\bar{y}\}\cup\{x,z,\bar{x},\bar{z}\}=$$
$$=\{x,y,z,\bar{x},\bar{y},\bar{z}\}.$$

Similarly $(x\dotplus y)\dotplus z=\{x,y,z,\bar{x},\bar{y},\bar{z}\}$.

ii) Let us suppose that two of x,y,z are equal to each other; for example $x=y$, then

$$x\dotplus(x\dotplus z)=x\dotplus\{x,z,\bar{z},\bar{x}\}=(x\dotplus x)\cup(x\dotplus z)\cup(x\dotplus\bar{x})\cup(x\dotplus\bar{z})=$$
$$=(H\setminus\{x,\bar{x},0\})\cup\{x,z,\bar{x},\bar{z}\}\cup(H\setminus\{x,\bar{x}\})\cup\{x,z,\bar{x},\bar{z}\}=H.$$

On the other hand

$$(x\dotplus x)\dotplus z=(H\setminus\{x,\bar{x},0\})\dotplus z\subseteq(\bar{z}\dotplus z)\cup(w\dotplus z)=H$$

(w is supposed to be different than $z,\bar{z}$).
In a similar way we can prove the associativity when two of x,y,z are opposite.

Now for the axiom II.iv. we have:

i) If $y=x$ then $x\dotplus y=H\setminus\{x,\bar{x},0\}$. This means that for every z in $H\setminus\{x,\bar{x},0\}$, z must belong to the set $z\dotplus\bar{x}$, which is in fact true. Also this axiom is true when $y=\bar{x}$.

ii) If $y\neq x,\bar{x},0$ then $x\dotplus y=\{x,y,\bar{x},\bar{y}\}$. So, for example, if $z\in x\dotplus y$ and $z=x$ then $y\in z\dotplus x=x\dotplus\bar{x}$. Thus this axiom is also valid.

Next as far as the proof of the distributivity is concerned we have: if $x\neq 0$ and $z=\bar{y}$ then

$$x(y\dotplus\bar{y})=x(H\setminus\{y,\bar{y}\})=xH\setminus\{xy,x\bar{y}\}=H\setminus\{xy,\overline{xy}\},$$
$$xy\dotplus x\bar{y}=xy\dotplus\overline{xy}=H\setminus\{xy,\overline{xy}\}.$$

Here we can remark the following. In order to prove this axiom the equality $xH=Hx=H$ must be valid for every $x\in H^*$. This is true only when H is an almost-group. So it is not possible to get hyperrings that are not hyperfields using this construction.

The cases that have not been examined in the course of this proof don't present any substantial difficulties.

Suppose now that $H(G)$ is isomorphic to a quotient hyperfield $(F/W, \dot{+}, \cdot)$ *). Since in $H(G)$ $0 \notin 1 \dot{+} 1$ we conclude that char F can not be 2. Next let us denote by $\hat{W}$ the union $-W \dot{\cup} W$. It is clear that $\hat{W}$ is a multiplicative group. Then for 2 we have: $2 = 1 + 1 \in \hat{W} + \hat{W}$ so $2 \notin \hat{W}$. Thus $2\hat{W}$ and $\hat{W}$ are two disjoined classes and therefore $0 \notin 2\hat{W} + \hat{W}$. So $3 = 2 + 1$ can not be 0. This means that char F can not be 3 also. Now as far as 3 is concerned it belongs in the union $2\hat{W} \dot{\cup} \hat{W}$.

LEMMA 1. If $3 \in \hat{W}$, then char $F = 0$ and the multiplicative group of F/W is not periodic. **)

PROOF. Suppose that $3 \in \hat{W}$. Then $4 = 3 + 1 \in \hat{W} + \hat{W}$ so $4 \notin \hat{W}$. Also $4 = 2 + 2 \in 2\hat{W} + 2\hat{W}$ so $4 \notin 2\hat{W}$. This means that $4\hat{W}$ is a new class different than $\hat{W}$ and $2\hat{W}$. We shall show now that all the odd numbers, when F is of characteristic 0, and the odd numbers that are less than the characteristic, if the characteristic p is finite, belong in $\hat{W}$. Indeed, suppose that the odd numbers of the set $I = \{1, 2, \dots, n\}$ are elements of $\hat{W}$ (for $n = 3$ this is true). Let m be the next odd after n ($m \leq p - 1$ when F is of finite characteristic). Then $m = 2 + (m - 2)$ which means that $m \in 2\hat{W} + \hat{W} = 2\hat{W} \dot{\cup} \hat{W}$. Also $m = 4 + (m - 4)$ from which $m \in 4\hat{W} + \hat{W} = 4\hat{W} \dot{\cup} \hat{W}$. Thus $m \in \hat{W}$.

Now if the characteristic p is finite we have $p - 2 \in \hat{W}$, $2 \in 2\hat{W}$ and so $p = (p - 2) + 2$ belongs in $2\hat{W} + \hat{W}$. But $0 \notin 2\hat{W} \dot{\cup} \hat{W}$. So F can not be of finite characteristic, when $3 \in \hat{W}$. Thus we shall continue with the hypothesis that char $F = 0$.

Suppose that every two elements of the set $A = \{2^k \hat{W} \mid 0 \leq k < n\}$ are

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*) F and W are not supposed to be commutative but the prime field P of F is always commutative.

**) A group G is called periodic if for every $x \in G$ there exists an integer $n \geq 1$ such that $x^n = 1$.

different to each other (for $n=2$ this is valid). Then $2^{n+1}\hat{W} \notin A$. Indeed, the $2^{n+1}-1$ is an odd number and so it belongs to $\hat{W}$. Thus $2^{n+1}=(2^{n+1}-1)+1$ belongs to $\hat{W}+\hat{W}$ and therefore $2^{n+1} \notin \hat{W}$, so $2^{n+1}\hat{W} \neq \hat{W}$. Also if $2^{n+1}\hat{W}$ was equal to $2^{r}\hat{W}$, $1 \leq r \leq n$, then $2^{r}\hat{W}=2^{r}\hat{W}\cdot 2^{n+1-r}\hat{W}$, from where we get $2^{n+1-r}\hat{W}=\hat{W}$, which is absurd. So the element $2W$ of F/W is of infinite order, and so $(F/W)^{*}$ is not periodic.

Next let us suppose that $3 \in 2\hat{W}$. Then

LEMMA 2. All the numbers not divisible by 5 when char $F=0$ and all the numbers not divisible by 5 that are less than the characteristic p, if p is finite and different than 5, belongs to $\hat{W}$ or to $2\hat{W}$.

PROOF. When $3 \in 2\hat{W}$ we have for 4: $4=3+1 \in 2\hat{W}+\hat{W}=2\hat{W} \cup \hat{W}$. Also $4=2+2 \in 2\hat{W}+2\hat{W}$ so $4 \notin 2\hat{W}$, thus $4 \in \hat{W}$. Now for 5 we have : $5=3+2 \in 2\hat{W}+2\hat{W}$ so $5 \notin 2\hat{W}$. Also $5=4+1 \in \hat{W}+\hat{W}$ so $5 \notin \hat{W}$, and since $5 \neq 0$, $5\hat{W}$ is a new class.

Next we observe that the field F cannot be of characteristic 7 or 11. Indeed, $6=2.3 \in 2\hat{W}\cdot 2\hat{W}=4\hat{W}=\hat{W}$ and so 12 should belong in $2\hat{W}$. But $12 \equiv 5 \pmod 7$ and $12 \equiv 1 \pmod{11}$, so $12 \in 5\hat{W}$ in the first case and $12 \in \hat{W}$ in the second one. Thus we have produce a contradiction. Therefore char $F > 11$.

Now we shall show that this lemma is true for $n \leq 11$. Indeed, for 7 we have : $7=5+2 \in 5\hat{W}+2\hat{W}$ and $7=4+3 \in \hat{W}+2\hat{W}$ so $7 \in 2\hat{W}$. Further, $8=2.4 \in 2\hat{W}\cdot\hat{W}=2W$, $9=3^2=(2\hat{W})^2=\hat{W}$. But $10=2.5$ does not belong to $2\hat{W}$ or to $5\hat{W}$. Also $10=9+1 \in \hat{W}+\hat{W}$ so $10 \notin \hat{W}$. Similarly $11 \in \hat{W}$. So this proposition is true for $n \leq 11$. Suppose that it is also true for $n \leq k$. Let m be the next after k not multiple of 5 (in the case of finite characteristic, we must have $m < p$); then

$$m=5+(m-5) \text{ so } m \in 5\hat{W}+\hat{W} \text{ or } m \in 5\hat{W}+2\hat{W},$$

$$m=10+(m-10) \text{ so } m \in 10\hat{W}+\hat{W} \text{ or } m \in 10\hat{W}+2\hat{W}.$$

Thus m must belong either to $\hat{W}$ or to $2\hat{W}$, and so the lemma.

LEMMA 3. If char $F=0$ then $5W$ is of infinite order.

PROOF. Suppose that the elements of $I=\{\hat{W}, 5\hat{W},\dots,5^n\hat{W}\}$ are different from each other (this is true for $n=1$), then $5^{n+1}\hat{W}$ does not belong in I. Indeed, let $5^{n+1}\hat{W}=5^k\hat{W}$, $k\neq 0$. Then $5^{n-k+1}\hat{W}=$ $=\hat{W}$ which is absurd because of induction. Now we shall show that $5^{n+1}\hat{W}\neq\hat{W}$. For $5^{n+1}-1$ we have

$$5^{n+1}-1=(5-1)(5^n+5^{n-1}+\dots+5+1).$$

But $a=5^n+5^{n-1}+\dots+5+1$ is not a multiple of 5 so $a\in\hat{W}$ or $a\in 2\hat{W}$. Also $a=(5^n+5^{n-1}+\dots+5)+1\in 5^t\hat{W}+\hat{W}$, $1\leq t\leq n$ and thus $5^{n+1}-1\in\hat{W}$. Now we have $5^{n+1}=(5^{n+1}-1)+1$, so $5^{n+1}\notin\hat{W}$.

COROLLARY 1. If $H(G)$ is isomorphic to F/W and F has characteristic 0, then G can not be periodic.

LEMMA 4. If F is of finite characteristic p and if F_p is the field of p elements, then every $x\notin F_p\hat{W}$ is of infinite order [*].

PROOF. We choose an element x not belonging in $F_p\hat{W}$ and such as $x+1\in\hat{W}$. Then for $x-1$ we have: $x-1\in x\hat{W}+\hat{W}=x\hat{W}\dot{\cup}\hat{W}$ and $x-1=$ $=(x+1)-2\in\hat{W}+2\hat{W}=\hat{W}\dot{\cup}2\hat{W}$ so $x-1$ belongs also in $\hat{W}$. Now $x^2-1=$ $=(x-1)(x+1)\in\hat{W}$, thus $x^2=(x^2-1)+1\in\hat{W}+\hat{W}$, so $x^2\notin\hat{W}$. Also $x^2\notin x\hat{W}$.

Next using induction we shall prove that the sums $x^n+x^{n-1}+\dots$ $\dots+x^2+x+1$ belong in $\hat{W}$. This is true for $n=2$. Indeed $x^2+x+1=$ $=x(x+1)+1\in x\hat{W}+\hat{W}=x\hat{W}\dot{\cup}\hat{W}$ and

$$x^2+x+1=x^2+(x+1)\in x^2\hat{W}+\hat{W}=x^2\hat{W}\dot{\cup}\hat{W} \qquad \text{so } x^2+x+1\in\hat{W}.$$

Suppose that this is also true for $n\leq k$. Then for $n=k+1$ we have

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*) The ring $F_p[x]$ and the field $F_p(x)$ are commutative.

$$x^{k+1}+x^k+\ldots+x^2+x+1=x(x^k+x^{k-1}+\ldots+x+1)+1\in x\hat{W}+\hat{W}=x\hat{W}\dot{\cup}\hat{W}$$

and

$$x^{k+1}+x^k+\ldots+x^2+x+1=x^2(x^{k-1}+x^{k-2}+\ldots+1)+(x+1)\in x^2\hat{W}+\hat{W}=x^2\hat{W}\dot{\cup}\hat{W}$$

thus $x^{k+1}+x^k+\ldots+x+1\in\hat{W}$.

Now using again induction we shall prove that the order of x is not finite. Indeed suppose that the elements of the set $I=\{x^k|0\leq k\leq n\}$ are different between them (that is true for $n=2$). Then $x^{n+1}-1=$ $=(x-1)(x^n+x^{n-1}+\ldots+x+1)$. But $x-1\in\hat{W}$ and $x^n+x^{n-1}+\ldots+x+1\in\hat{W}$ so $x^{n+1}-1\in$ $\in\hat{W}$. Thus for x^{n+1} we have $x^{n+1}=(x^{n+1}-1)+1\in\hat{W}+\hat{W}$ so $x^{n+1}\notin\hat{W}$. This means that $x^{n+1}\neq 1$; also $x^{n+1}\neq x^k$, $1\leq k\leq n$ and so the lemma is proved.

LEMMA 5. The only possible finite characteristic of F is 5.

PROOF. Suppose that p, $p\neq 0,5$, is the characteristic of F. Then k and $p-k$ belongs in the same class modulo $\hat{W}$ because $k+(p-k)=$ $=p$. Thus 5 and $p-5$ belong in $5\hat{W}$. But $p-5$ is not a multiple of 5, so $p-5$ according to Lemma 2 belongs in $\hat{W}$ or in $2\hat{W}$. We have just produce a contradiction and so the lemma.

COROLLARY 2. If char $F=5$ and G is periodic, then H^* has order 4 and G is of order 2.

PROOF. Indeed, otherwise there would be an $x\notin F_6\hat{W}$ which according to the Lemma 4 would have infinite order.

THEOREM 1. Let G be a multiplicative non unitary group, in which the order of any element x, $x\neq 1$, is 2. Then the hyperfield $H(G)$ is not isomorphic to any quotient hyperfield.

PROOF. We shall prove it using reductio ad absurdum. Suppose that the quotient hyperfield $(F/W,\dot{+},\cdot)$ is isomorphic to $H(G)$. Then for

F/W the following must valid:

i) $x^2 W = W$ for every $x \in F$,

ii) $W \dot{-} W = F/W \setminus \{-W, W\}$ from which $W - W = F \setminus (-W \dot{\cup} W)$.

Because of (i) W must contain all the squares of F. But F, as we have already mention, can not be of characteristic 2, so every element of F can be written as a difference of two squares:

$$x = \left(\frac{x+1}{2}\right)^2 - \left(\frac{x-1}{2}\right)^2.$$

Thus $W - W = F$. This last contradicts with (ii) and so the theorem.

COROLLARY 3. If $H(G)$ is isomorphic to F/W and F has characteristic 5, then G can not be periodic.

The proof derives directly from the combination of Corollary 2 with Theorem 1.

We are now in a position to prove the following

THEOREM 2. The hyperfield $H(G)$ can never be isomorphic to any quotient hyperfield when G is a periodic group.

PROOF. From what we have discussed up to now we observe that if $H(G)$ is isomorphic to a quotient hyperfield F/W, then the only possible characteristics for F are 0 or 5. But when F is of characteristic 0 then the multiplicative group of F/W contains elements of infinite order (Lemma 1 and 3). Also if the characteristic is 5, then, according to Corollary 3, G can not be periodic. So when G is a periodic group, $H(G)$ can never be isomorphic to any quotient hyperfield.

REMARK. If we endow the almost group $(H, \cdot)$ of Proposition 1 with a hypercomposition $\hat{+}$ defined as follows: for every $(x, i), (y, j) \in H$

$$(x, i) \hat{+} (x, -i) = H,$$

$$(x, i) \hat{+} 0 = 0 \hat{+} (x, i) = (x, i),$$

and $(x, i) \hat{+} (y, j) = \{(x, i), (y, j)\}$ when $(y, j) \neq (x, -i)$, then $(H, \hat{+}, \cdot)$ becomes a hyperfield again, but the problem of the isomorphism of this hyperfield to the quotient hyperfields is still open. A study of the problems which turn up when the isomorphism of this type of hyperfields to the quotient hyperfields is examined, can be found in [13].

§ 3. Non quotient hyperrings

Proving that R/W is a hyperring (see [3]) we find out that the definition of the addition in R/W and the demonstration that it satisfies the additive axioms do not require the normality of W. On the other hand the definition of the multiplication and the proofs of the multiplicative and distributive axioms require only that $\bar{x}\bar{y}=\overline{xy}$ when $\bar{x}, \bar{y}$ are multiplied as subsets of R (i.e. for every $x, y \in R$, $xWyW=xyW$). This is equivalent to the normality of W when R is a multiplicative almost-group, which appears only when R is a field. In what follows we shall define a ring R and one of its multiplicative subgroups, G satisfying the above condition, but without being normal and we shall show that R/G is a hyperring which can not be isomorphic to a quotient hyperring of a ring by any of its normal multiplicative subgroups.

Let R_0 be an arbitrary unitary ring such that $2 \neq 0$ and let us consider its cartesian square $R=R_0 \times R_0$. We define the addition on R as that of vectors:

$$(a,b)+(a',b')=(a+a', b+b').$$

Clearly R is an abelian group with respect to this addition. The multiplication will be defined by the formula:

$$(a,b)(a',b')=(a(a'+b'), b(a'+b')).$$

This multiplication is assosiative and distributive with respect to the addition. So R endowed with this addition and multiplication becomes a ring.

An element $(a,b)\neq 0\ (=:(0,0))$ is idempotent if $a+b=1$. True,

$$(a,b)^2=(a(a+b), b(a+b))=(a.1, b.1)=(a,b).$$

In particular, $e\ (=:(1,0))$ itself has such a behavior. Moreover $-e=$ $=(-1,0)\neq e$ because $2e=(2,0)\neq(0,0)=0$. Since $(-e)^2=e^2=e$, $-e$ itself generates the multiplicative group G which is equal to $\{e,-e\}$. If $x\in R$ then we have $xG=\{xe,x(-e)\}=\{xe,-xe\}$ and if $x=(a,b)$ we get: $xe=(a,b)(1,0)=(a(1+0),b(1+0))=(a,b)=x$. This means that $xG=\{x,-x\}$. But then

$$\bar{x}\bar{y}=(xG)(yG)=\{x,-x\}\{y,-y\}=\{xy,\ x(-y),\ (-x)y,(-x)(-y)\}=$$
$$=\{xy,-xy\}=xyG=\overline{xy}.$$

So G satisfies the required condition. But G is not normal. Indeed if $x=(a,b)$ we have $xG=\{x,-x\}$ and $Gx=\{ex,-ex\}$. But

$$ex=(1,0)(a,b)=(1(a+b),\ 0(a+b))=(a+b,0)$$

and so we get $Gx\neq xG$ whenever $b=-a\neq 0.$

Now, $\bar{e}=G$ is obviously a right neutral element for the multiplication of $\bar{R}$ (just note that $\bar{x}\bar{e}=\overline{xe}=\bar{x}$), but it is not a left one; because $\bar{e}\bar{x}=\overline{ex}$ and if $x=(a,b)$ is such that $b=-a\neq 0$ one sees that $ex=(a+b,0)=(0,0)$ which is not equal to neither x nor $-x$. This last reasoning shows exactly that $\bar{R}=R/G$ can not be isomorphic to a quotient hyperring P/Q , of a ring P by a normal subgroup Q of P , because in P/Q, Q is a bilaterally neutral element.

Now we shall continue by proving that there are not only hyperrings which are not isomorphic to quotient hyperrings but there also exist hyperrings not isomorphic to any subhyperrings of a quotient hyperring. The essential role in proving this assertion will play first the existance of the non quotient hyperfields, which we have just proved, and second the property of the quotient hyperrings according to which every subhyperfield of a quotient hyperring R/W is the quotient of a subfield F of R by some subgroup of F's multiplicative group (see [4] or Appendix).

THEOREM 3. *The direct sum S of the hyperrings S_i, $i\in I$, is not isomorphic to a sub-*

hyperring of a quotient hyperring if at least one of the S_i is a non quotient hyperfield.

PROOF. Let S_k, $k \in I$, be a non quotient hyperfield. Then the subset K of S which consists of all $\{a_i\}_{i\in I}$ where $a_i = 1$ for $i \neq k$ and $a_k \in S_k$ is a non quotient subhyperfield of S. Now, if S is isomorphic to a quotient hyperring R/G, then there must be a subhyperfield of R/G isomorphic to K. But according to the above mentioned property, every subhyperfield of a quotient hyperring is a quotient hyperfield. Thus K should be a quotient hyperfield. But this is absurd. So S does not belong to the class of quotient hyperrings.

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Appendix

A question that naturally comes up is: do the subhyperfields of the quotient hyperrings belong to the class of quotient hyperfields or not? The answer to this question will be given in the proposition that follows.

Let R/W be a hyperring and F/W a subhyperfield of R/W. Now if we suppose that uW is the unit of F/W then we can see that uW is a group. Indeed, let $x = uw$ be an element of uW. Then

$$xuW = (uw)uW = u(uw')W = u^2W = uWuW = uW.$$

So the class xuW is a subset of uW, thus $xuW = uW$. In the same way we can show that $uWx = uW$ and therefore uW is indeed a group.

LEMMA. The unit of uW is also the unit of F.

PROOF. Let e be the unit of uW. If $s \in F$ then there exist $s' \in F$ and $w \in W$ such that $s = s'(uw)$. Thus we have:

$$se = (s'(uw))e = s'((uw)e) = s'(uw) = s.$$

Similarly $es=s$ and so e is the unit of F.

PROPOSITION. Let R be a ring, W a normal multiplicative subgroup of R^* and R/W the quotient hyperring of R by W. Then every subhyperfield of R/W is the quotient of a subfield F of R by some subgroup of F^*.

PROOF. Let F/W be a subhyperfield of R/W and let uW be its unit. Now since F/W is a hyperfield, for every $t\in F^*$ there exists an element $t'\in F^*$ such that $tW.t'W=uW$, and therefore there exist $w,w'\in G$ such that $(tw)(t'w')=e$, where e is both the unit of uW and the unit of F. Thus $wt'w'$ is the inverse element of t and therefore F is a field. So the subhyperfield F/W of R/W is the quotient of the field F by the multiplicative subgroup uW of F.

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