

INTERNATIONAL JOURNAL OF MATHEMATICS AND MATHEMATICAL SCIENCES

VOLUME 8

NUMBER 4

DECEMBER 1985

CONTENTS

RESEARCH PAPERS

Convex Curves of Bounded Type, A. W. GOODMAN	625
About the Defects of Curves Holomorphic in the Half Plane, MOQBUL HOSSAIN	635
On Convergence of (μ, ν) -Sequences of Unisolvent Rational Approximants to Meromorphic Functions in C^n , C. H. LUTTERODT	641
Integral Operators of Certain Univalent Functions, O. P. AHUJA	653
On Fixed Points of Set-Valued Directional Contractions, SEHIE PARK	663
Diagonalization of a Self-Adjoint Operator Acting on a Hilbert Module, PARFENY P. SAWOROTNOW	669
On Certain Inequalities for Some Regular Functions in $ z < 1$, MILUTIN OBRADOVIC	677
Random Walk Over a Hypersphere, J. M. C. JOSHI	683
Tauberian Conditions for Conull Spaces, J. CONNOR and A. K. SNYDER	689
A Variant of a Fixed Point Theorem of Browder-Fan and Reich, V. M. SEHGAL	693
On Coefficients of Circuit Polynomials and Characteristic Polynomials, E. J. FARRELL	697
A Representation of Jacobi Functions, E. Y. DEEBA and E. L. KOH	707
On Two New Characterizations of Stieltjes Transforms for Distributions, SUNIL KUMAR SINHA	719
Methods of Constructing Hyperfields, CH. G. MASSOUROS	725
On Pairwise S-Closed Bitopological Spaces, M. N. MUKHERJEE	729
On Orderability of Topological Groups, G. RANGAN	747
Diophantine Equations and Identities, MALVINA BAICA	755
On a Subclass of Bazilevič Functions, D. K. THOMAS	779
Growth Results for a Subclass of Bazilevič Functions, RABIHA MD. EL-ASHWAH and D. K. THOMAS	785
On a Function Related to Ramanujan's Tau Function, JOHN A. EWELL	795
On k -Triad Sequences, HANSRAJ GUPTA and K. SINGH	799
A Note on Convergence of Weighted Sums of Random Variables, XIANG CHEN WANG and M. BHASKARA RAO	805
RESEARCH NOTES	
The Spaces O_M and O_C are Ultrabornological, A New Proof, JAN KUCERA	813
A Note on the Problem of Finding a $(3,9)$ -Cage, P. K. WONG	817
On Mennicke Groups of Deficiency Zero 1, MUHAMMAD A. ALBAR	821
INDEX	825
REFEREES AND ASSISTANT EDITORS	827
INDEX FOR VOLUME 8, 1985	829

METHODS OF CONSTRUCTING HYPERFIELDS

CH. G. MASSOUIROS

54 Klious St. Cholargos
Ahtens, Greece
TK 155-61

(Received February 1, 1983 and in revised form July 21, 1983)

ABSTRACT. In this paper we introduce a class of hyperfields which contains non quotient hyperfields. Thus we give a negative answer to the question of whether every hyperfield is isomorphic to a quotient $\frac{K}{G}$ of a field K by some subgroup G of its multiplicative group.

KEY WORDS AND PHRASES. *Hypercomposition, hyperfield, quotient hyperfield, hyperring.*
1980 AMS SUBJECT CLASSIFICATION CODE. 12K99.

1. INTRODUCTION

A hyperfield is a triplet $(H, +, \cdot)$ where H is a non-empty set, $+$ is a hypercomposition (i.e. a mapping whose domain is $H \times H$ and whose range is the power set of H), and \cdot an internal composition of H (i.e. for every $x, y \in H$, $x \cdot y \in H$). These two operations satisfy the following axioms:

I. Multiplicative axiom

H is an almost-group with regard to the multiplication. An almost-group is a semigroup S , which is the union $GU\{0\}$, where G is a group and 0 a bilaterally absorbing element of S . By 1 we shall denote its neutral element and by 0 its absorbing element.

II. Additive axioms

- (i) $x + y = y + x$
- (ii) $(x + y) + z = x + (y + z)$
- (iii) For every $x \in H$, there exists one and only one $x' \in H$ such that $0 \in x + x'$. x' is written $-x$ and called the opposite of x ; moreover we shall write $x - y$ instead of $x + (-y)$.
- (iv) $z \in x + y$ implies that $y \in z - x$

III. Distributive axiom

$$z \cdot (x + y) = z \cdot x + z \cdot y, \quad (x + y) \cdot z = x \cdot z + y \cdot z$$

We remark that:

- (a) By usual conventions if " \cdot " is an internal composition of a set E and X, Y are

subsets of E , $X \cdot Y$ is the set of all $x \cdot y$ such that $(x,y) \in X \times Y$. If " \cdot " is a hypercomposition in E then $X \cdot Y$ signifies the union $\bigcup_{(x,y) \in X \times Y} x \cdot y$. In both cases $x \cdot Y$ and $X \cdot y$ will have the same meaning as $\{x\} \cdot Y$ and $X \cdot \{y\}$ respectively.

(b) By virtue of axioms II.(iii) and II.(iv) one has

$$x + 0 = x \text{ for each } x \in H$$

Indeed; $0 \in x - x$ so $x \in x + 0$. Let's suppose that some y different than x belongs to $x + 0$. Then $0 \in x - y$ which is absurd because of II.(iii). So $x + 0 = x$.

Also, because $+$ is a hypercomposition, the distributive axiom gives:

$(a + b)(c + d) \subseteq ac + ad + bc + bd$. (properties of hyperfields can be found in [3], [5])

(c) If we replace the multiplicative axiom I by

I'. H is a multiplicative semigroup having a bilaterally absorbing element 0 . we obtain a more general structure $(H, +, \cdot)$ which is called hyperring. (see [4], [6])

The concept of the hyperfield has been introduced by M. Krasner in his study [1].

The hyperfield that occurs there was obtained by considering a quotient of the type

$\frac{K}{\pi_\rho}$ where K is a valued field and π_ρ an equivalence relation defined from the

field's valuation. This hyperfield was called the residual of $K(\text{mod } \pi_\rho)$.

2. THE QUOTIENT HYPERFIELD.

Let F be a field and G a subgroup of F 's multiplicative group. Then the multiplicative classes modulo G in F form a partition of F . We shall denote by $\overline{F} = \frac{F}{G}$ the set of the classes of this partition. A multiplication can be introduced in \overline{F} by defining the product $\overline{x} \cdot \overline{y}$ of two classes $\overline{x}, \overline{y}$ to be the class which results from their setwise product. It can also be proved (see [2]) that if we add $\overline{x}, \overline{y}$ as subsets of F , the set that results is a union of classes modulo G . This enables us to define the sum $\overline{x} + \overline{y}$ of $\overline{x}, \overline{y}$ to be the set of all classes $\overline{z} \in \overline{F}$ which are contained in the setwise sum of $\overline{x}, \overline{y}$. \overline{F} endowed with these two operations becomes a hyperfield ([2]) called quotient hyperfield of F by G .

If F is a ring and G a normal subgroup of F^* ($:= F \setminus \{0\}$) (that is a subset of R , such that (G, \cdot) is a group, and $xG = Gx$ for all x in R) then $\overline{F} = \frac{F}{G}$ will be a hyperring called the quotient hyperring of F by G .

The question that Krasner sets ([2]) is whether there are other hyperfields besides the quotient hyperfields. Indeed it is quite important to find out how rich the class of all hyperfields is, for if there were only quotient hyperfields then a good part of hyperfield theory could have been deduced directly from corresponding results in field theory.

At first one can notice that the residual hyperfields are always quotient hyperfields. Next we shall prove that every subhyperfield of a quotient hyperring is a quotient of a field by a multiplicative group. Indeed let $\frac{R}{G}$ be a hyperring and $\frac{F}{G}$ a subhyperfield of $\frac{R}{G}$. Then

LEMMA 1. If uG is the unit of $\frac{F}{G}$, then uG is a multiplicative group.

PROOF. Let $x = u\gamma$ be an element of uG . Since $uG = Gu$ we have:
 $xuG = (u\gamma)uG = u(u\gamma')G = u^2G \subseteq uG \cdot uG = uG$. So the class xuG is a subset of uG ,
 thus $xuG = uG$. In the same way we can show that $uGx = uG$ and therefore uG is
 indeed a group.

Now if we denote by $[[FG]]$ the subring of R which is generated by the set
 FG , we observe that $[[FG]] = uG [[FG]] = [[FG]] uG$.

LEMMA 2. The unit of uG is also the unit of $[[FG]]$

PROOF. Let e be the unit of uG . If $s \in [[FG]]$ then there exist $s' \in$
 $[[FG]]$ and $\gamma \in G$ such that $s = s'(u\gamma)$. Thus we have: $se = (s'(u\gamma))e = s'((u\gamma)e) =$
 $= s'(u\gamma) = s$. Similarly $es = s$ and so e is the unit of $[[FG]]$.

PROPOSITION 1. Let R be a ring, G a normal multiplicative subgroup of R^*
 and $\frac{R}{G}$ the quotient hyperring of R by G . Then every subhyperfield of $\frac{R}{G}$ is the
 quotient of a subfield of R by some subgroup of this subfield's multiplicative group.

PROOF. Let $\frac{F}{G}$ be a subhyperfield of $\frac{R}{G}$ and let uG be its unit. We observe
 that $\frac{F}{G} = \frac{[[FG]]}{uG}$. Now since $\frac{[[FG]]}{uG}$ is a hyperfield, for every $t \in [[FG]]$ there
 exists an element $t' \in [[FG]]$ such that $tG \cdot t'G = uG$, and therefore there exist
 $\gamma, \gamma' \in G$ such that $(t\gamma)(t'\gamma') = e$, where e is both the unit of G and the unit
 of $[[FG]]$. Thus $\gamma t' \gamma'$ is the inverse element of t and therefore $[[FG]]$ is a
 field.

3. NON QUOTIENT HYPERFIELDS.

We shall construct a class of hyperfields which contains hyperfields that are not
 isomorphic to quotient ones. For this purpose we consider a commutative multiplica-
 tive almost-group (H, \cdot) in which we introduce a hypercomposition $+$ defined as
 follows:

$$\begin{aligned} x + y &= \{x, y\} && \text{if } x, y \neq 0 \text{ and } x \neq y \\ x + x &= H \setminus \{x\} && \text{if } x \neq 0 \end{aligned}$$

and $x + 0 = 0 + x = x$ for every $x \in H$

Then:

PROPOSITION 2. The triplet $(H, +, \cdot)$ is a hyperfield.

PROOF. At first one can notice that the opposite of x is x itself since
 $0 \in x + x$. Next it is obvious that the hypercomposition is commutative and that the
 associativity is valid when one of x, y, z is equal to 0. Now if $x, y, z \neq 0$ and
 distinct, then we have:

$$x + (y + z) = x + \{y, z\} = (x + y)U(x + z) = \{x, y\}U\{x, z\} = \{x, y, z\}$$

A similar calculation shows $(x + y) + z = \{x, y, z\}$. Finally if two of x, y, z are
 equal to each other; for example $x = y$, we have:

$$\begin{aligned} x + (y + z) &= x + (x + z) = x + \{x, z\} = (x + x)U(x + z) = [H \setminus \{x\}] U\{x, z\} = H \\ \text{and} \quad (x + y) + z &= (x + x) + z = [H \setminus \{x\}] + z \supseteq (z + z)U(w + z) = \\ &= [H \setminus \{z\}] U\{w, z\} = H, \text{ with } w \neq z \end{aligned}$$

Now as far as the axiom II.iv. is concerned we have:

(i) if $y = x$ then $x + y = x + x = H \setminus \{x\}$. This means that for every $z \in H \setminus \{x\}$
 we must have $x \in z + x$ which is in fact true.

(ii) If $y \neq x$, 0 then $x + y = \{x, y\}$, thus $z = x$ or $z = y$. If $z = x$ then $x \in z + y = \{z, y\}$ and if $z = y$ then $x \in z + y = H \setminus \{y\}$. So this axiom is valid as well.

Finally it is straight forward to verify the distributive axiom.

THEOREM. The class of the hyperfields constructed as above contains elements that do not belong to the class of quotient hyperfields.

PROOF. We choose an almost group (H, \cdot) with $\text{card } H > 3$ and such that $x^2 = 1$ for every $x \in H \setminus \{0\}$. Next we consider the hyperfield $(H, +, \cdot)$ which is constructed according to the above method and we suppose that this hyperfield is isomorphic to a quotient hyperfield $(\frac{F}{G}, \dagger, \cdot)$. Then for $\frac{F}{G}$ the following must be valid:

(i) $xG \cdot xG = G$ or $x^2G = G$ for every $x \in F$

(ii) $G = -G$

(iii) $G \dagger G = (\frac{F}{G}) \setminus G$ from which $G + G = F \setminus G$

Because of (i) G must contain all the squares of F . But if F is not of characteristic 2 then every element of F can be written as a difference of two squares:

$$x = \left(\frac{x+1}{2}\right)^2 - \left(\frac{x-1}{2}\right)^2 = \left(\frac{x+1}{2}\right)^2 + (-1)\left(\frac{x-1}{2}\right)^2$$

So we have $G - G = G + G = F$. This contradicts (iii).

Now if the characteristic of F is 2 then the sum of two squares is always a square; thus $G \subseteq G + G$, which contradicts again (iii). So H cannot be isomorphic to any quotient hyperfield.

REMARKS. (a) This theorem gives something more, that is, H cannot be isomorphic even to a subhyperfield of a quotient hyperring.

(b) Two other classes of hyperfields completely different can be found in [7]. The problem of those hyperfields' isomorphism to quotient hyperfields is also discussed there but no final answer is given.

REFERENCES

1. KRASNER, M. Approximation des corps values complete de caracteristique $p \neq 0$ par ceux de caracteristique 0. Colloque d'Algebre Superieure (Bruxelles, Decembre 1956), CBRM, Bruxelles 1957.
2. KRASNER, M. A class of hyperrings and hyperfields. Internat. J. Math. & Math. Sci. **6** (1983) 307-312.
3. MITTAS, J. : Sur les hyperanneaux et les hypercorps Mathematica Balkanica, t. 3. (1973) 368-382.
4. MITTAS, J. Hyperanneaux et certains de leur proprietes C. R. Acad. Sc. Paris, Serie A t. **269** (1969) 623-626.
5. STRATIGOPOULOS, D. Hyperanneaux, hypercorps, hypermodules, hyperspaces vectoriels et leurs proprietes elementaires, C. R. Acad. Sc. Paris, Serie A, t. **269** (1969) 489-492.
6. STRATIGOPOULOS, D. Hyperanneaux artiniens, centralisateur d'un hypermodule et theoreme de densite, C. R. Acad. Sc. Paris, Serie A, t. **269** (1969) 889-891.
7. STRATIGOPOULOS, D. and MASSOUROS, Ch. G. : On a class of fields Mathematica Balkanica, t. **12**.