Can Special Relativity Describe all the Types of Gravity of General Relativity? The Case Study of Spherical Symmetry.

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Abstract. This paper shows that contrarily to the common perception, the gravity of General Relativity (GR) can produce the gravity of Special Relativity (SR) and vice-versa. This is done via the time dilation that originates from the metric of GR in order to obtain the corresponding SR Lagrangian. The inverse procedure is also achievable and it is presented here as well. Thus, the Newtonian Gravitational Potential according to SR leads to the corresponding non-Riemannian metric of GR. In fact, the SR gravity can be extended to any kind of GR spacetime metric (including the non-Riemannian spacetimes with Finsler geometry) rather than the simple description of Einstein field equations of Riemannian GR. The Case Study of gravity with Spherical Symmetry is analytically presented. This is applied to repulsive / black holes according to Schwarzschild metric and Teleparallel gravity and also to wormholes. Finally, we prove that there exist gravitational fields where the particles have superluminal speeds according to GR and SR.

INTRODUCTION

It is considered that Special Relativity (SR) cannot explain the gravitational phenomena and only General Relativity (GR) can do this (by using curved spacetime) [1] (pp. 90, 111, 116) [2] (pp. 34, 109) [3] (p. 249). This paper proves that there exist Gravitational Scalar Generalized Potentials (GSGPs), according to SR [4] and Newtonian Physics (NPs) [5], which can produce exactly the results of GR. The way to obtain these GSGPs is the usage of the GR time dilation in the SR Lagrangian. We analytically present the case of fields with *spherical symmetry* like these of *wormholes* according to the *massless scalar field* of Einstein Field Equations (EFEs) [6] and *repeal / black holes* according to the *Schwarzschild metric* [7] and *Teleparallel gravity* [8]. Finally, we show the reverse procedure: the way to obtain GR metrics from the SR GSGPs. Thus, the usage of *Newtonian Gravitational Potential* according to SR leads to the corresponding GR time dilation, which is based on GR with *non-Riemannian metric* [4] (p. 572).

THE RELATION BETWEEN GR METRICS AND SR GRAVITATIONAL SCALAR GENERALIZED POTENTIALS

The SR has the geometry of *Minkowski spacetime* with constant *Lorentz metric*. So, the gravity can be studied as a field, which comes from the SR GSGP (V_{SR}) in the rest frame Oxyz. From this potential

$$V_{\text{SR}}=V_{\text{SR}\left(x,\dot{x},y,\dot{y},z,\dot{z}\right)}=V_{\text{SR}\left(r,\dot{r},\theta,\dot{\vartheta},\phi,\dot{\phi}\right)}\quad;\quad \dot{x}=\frac{\mathrm{d}x}{\mathrm{d}t}\quad;\quad \ldots\quad;\quad \dot{\phi}=\frac{\mathrm{d}\phi}{\mathrm{d}t},\tag{1}$$

(which is completely free of conditions), we obtain the *field strength* and the *relativistic force*

$$\vec{g} = -\nabla V_{\mathrm{SR}} \quad ; \quad \vec{F} = m\vec{g} \,, \tag{2}$$

respectively [3] (p. 342), where *m* is the inertial mass. Moreover, the corresponding four-force is [2] (pp. 329, 342):

$$f^{\,i} = \gamma_{(\vec{\upsilon}_P)} \begin{bmatrix} \frac{1}{\mathrm{c}} \vec{F} \cdot \vec{\upsilon}_P \\ \vec{F} \end{bmatrix}, \tag{3}$$

where $\vec{\upsilon}_P$ is the velocity of the test particle *P*. The four-force in any other frame (O´*x*´*y*´*z*´) is obtained via the *Lorentz transformation* (SR relativization).

Furthermore, the case of SR GSGP yields the *SR gravitational Lagrangian* and *SR Lagrangian of a free particle*

$$L_{\mathrm{SR}} = -\frac{1}{\gamma_{(\beta_P)}} m\mathrm{c}^2 - mV_{\mathrm{SR}} \quad ; \quad L_{\mathrm{SR}} = -\frac{1}{\gamma_{(\beta_P)}} m\mathrm{c}^2 \,, \tag{4}$$

correspondingly, where $\gamma_{(\beta_P)}$ is the *Lorentz γ-factor* of the test particle *P* [3] (p. 351). From equation (4i) we obtain the *SR 1^st integral of motion*, e.g. the *total SR-energy*

$$E^* = \sum_{\mu=1}^{n=3} \left(\frac{\partial L_{\mathrm{SR}}}{\partial \dot{x}^{\mu}} \right) \dot{x}^{\mu} - L_{\mathrm{SR}} \,. \tag{5}$$

The function of SR GSGP can be obtained via the *Equivalence Principle of GR*: "accelerated motions caused by the gravitational field only (free falls) take place along geodesics of the metric which corresponds to the particular gravitational field" [3] (p. 248). The above implies that the curved spacetime of GR demands *no-force*. Thus, we substitute the *Lorentz γ-factor* in the *free particle SR Lagrangian* (4) with the *GR time dilation*

$$\dot{t}_{\mathrm{GR}} = \frac{\mathrm{d}t}{\mathrm{d}\tau_{\mathrm{GR}}} \,. \tag{6}$$

This gives the formula of the *SR gravitational Lagrangian*

$$L_{\mathrm{SR}} = -\frac{1}{\dot{t}_{\mathrm{GR}}} m\mathrm{c}^2 \,, \tag{7}$$

which combined with the initial formula (4) implies the corresponding GSGP

$$V_{\mathrm{SR}(r,\dot{r},\dot{\phi})} = \mathrm{c}^2 \left(\frac{1}{\dot{t}_{\mathrm{GR}}} - \frac{1}{\gamma_{(\beta_P)}} \right) \quad ; \quad \dot{t}_{\mathrm{GR}} = \frac{1}{\frac{V_{\mathrm{SR}}}{\mathrm{c}^2} + \frac{1}{\gamma_{(\beta_P)}}} \,. \tag{8}$$

This simple procedure can be applied to *any kind of GR metric* and we obtain the SR GSGP that corresponds to a specific metric. We then solve any problem, by using only the *SR 1^st integral of motion* (5) and the Equations of motion (*Euler-Lagrange equations*)

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L_{\mathrm{SR}}}{\partial \dot{x}^{\mu}} \right) - \frac{\partial L_{\mathrm{SR}}}{\partial x^{\mu}} = 0 \quad ; \quad \mu = 1, 2, 3. \tag{9}$$

Fortunately, the results are exactly the same as GR. Besides, formula (8ii) is used in the reverse procedure. It is also noted that the extracted GSGP (8i) can be modified [5] (pp. 22, 31) in order to be more flexible for the solution of problems according to SR and NPs [4] (p. 574).

APPLICATIONS ON FIELDS WITH SPHERICAL SYMMETRY

Below, we examine the case of *fields with Spherical Symmetry*. More specifically, we find the SR GSGP of *wormholes* according to the *massless scalar field* and Einstein Field Equations (EFEs) [6] and *repulsive / black holes* according to the *Schwarzschild metric* [7] and *Teleparallel gravity* [8].

Wormholes with Spherical Symmetry which originate from the Massless Scalar Field and Einstein Field Equations

The *wormholes with spherical symmetry* according to the *massless scalar field* and EFEs, have the GR metric [6] (p. 4):

$$
\begin{aligned}
dS^2 = -c^2 d\tau_{GR}^{\ 2} = &-\left(\frac{r+c_1+\sqrt{c_1^{\ 2}-c_2}}{r+c_1-\sqrt{c_1^{\ 2}-c_2}}\right)^{-\frac{c_3}{\sqrt{c_1^{\ 2}-c_2}}} c^2 dt^2 + \left(\frac{r+c_1+\sqrt{c_1^{\ 2}-c_2}}{r+c_1-\sqrt{c_1^{\ 2}-c_2}}\right)^{\frac{c_3}{\sqrt{c_1^{\ 2}-c_2}}} dr^2 + \\
&+\left(r^2+2c_1 r+c_2\right)\left(\frac{r+c_1+\sqrt{c_1^{\ 2}-c_2}}{r+c_1-\sqrt{c_1^{\ 2}-c_2}}\right)^{\frac{c_3}{\sqrt{c_1^{\ 2}-c_2}}}\left(d\theta^2+\sin^2\theta\, d\phi^2\right)
\end{aligned}
\quad , \qquad (10)
$$

where c_1, c_2 and c_3 are arbitrary constants of integration with dimensions in length, square of length and length, respectively. From the above we obtain the corresponding *GR time dilation*

$$
\dot{t}_{GR} = \left[\begin{array}{l}
\left(\frac{r+c_1+\sqrt{c_1^{\ 2}-c_2}}{r+c_1-\sqrt{c_1^{\ 2}-c_2}}\right)^{-\frac{c_3}{\sqrt{c_1^{\ 2}-c_2}}} - \left(\frac{r+c_1+\sqrt{c_1^{\ 2}-c_2}}{r+c_1-\sqrt{c_1^{\ 2}-c_2}}\right)^{\frac{c_3}{\sqrt{c_1^{\ 2}-c_2}}} \frac{\dot{r}^2}{c^2} - \\
-\frac{r^2+2c_1 r+c_2}{c^2}\left(\frac{r+c_1+\sqrt{c_1^{\ 2}-c_2}}{r+c_1-\sqrt{c_1^{\ 2}-c_2}}\right)^{\frac{c_3}{\sqrt{c_1^{\ 2}-c_2}}}\left(\dot{\theta}^2+\sin^2\theta\ \dot{\phi}^2\right)
\end{array}\right]^{-\frac{1}{2}} . \qquad (11)
$$

The next step is the substitution of (11) in formula (7), which gives the corresponding *SR gravitational Lagrangian*

$$
L_{SR} = -mc^2\left[\begin{array}{l}
\left(\frac{r+c_1+\sqrt{c_1^{\ 2}-c_2}}{r+c_1-\sqrt{c_1^{\ 2}-c_2}}\right)^{-\frac{c_3}{\sqrt{c_1^{\ 2}-c_2}}} - \left(\frac{r+c_1+\sqrt{c_1^{\ 2}-c_2}}{r+c_1-\sqrt{c_1^{\ 2}-c_2}}\right)^{\frac{c_3}{\sqrt{c_1^{\ 2}-c_2}}} \frac{\dot{r}^2}{c^2} - \\
-\frac{r^2+2c_1 r+c_2}{c^2}\left(\frac{r+c_1+\sqrt{c_1^{\ 2}-c_2}}{r+c_1-\sqrt{c_1^{\ 2}-c_2}}\right)^{\frac{c_3}{\sqrt{c_1^{\ 2}-c_2}}}\left(\dot{\theta}^2+\sin^2\theta\ \dot{\phi}^2\right)
\end{array}\right]^{\frac{1}{2}} \qquad (12)
$$

from which we obtain the *equations of motion* in this spacetime, using (9). Finally, the substitution of (11) in formula (8i) gives the GSGP that corresponds to *wormholes with spherical symmetry* according to the *massless scalar field* and EFEs:

$$
V_{SR} = c^2\left(\left[\begin{array}{l}
\left(\frac{r+c_1+\sqrt{c_1^{\ 2}-c_2}}{r+c_1-\sqrt{c_1^{\ 2}-c_2}}\right)^{-\frac{c_3}{\sqrt{c_1^{\ 2}-c_2}}} - \left(\frac{r+c_1+\sqrt{c_1^{\ 2}-c_2}}{r+c_1-\sqrt{c_1^{\ 2}-c_2}}\right)^{\frac{c_3}{\sqrt{c_1^{\ 2}-c_2}}} \frac{\dot{r}^2}{c^2} - \\
-\frac{r^2+2c_1 r+c_2}{c^2}\left(\frac{r+c_1+\sqrt{c_1^{\ 2}-c_2}}{r+c_1-\sqrt{c_1^{\ 2}-c_2}}\right)^{\frac{c_3}{\sqrt{c_1^{\ 2}-c_2}}}\left(\dot{\theta}^2+\sin^2\theta\ \dot{\phi}^2\right)
\end{array}\right]^{\frac{1}{2}} - \frac{1}{\gamma_{(\beta_P)}}\right) . \qquad (13)
$$

Janis-Newman-Winicour Wormhole (equivalent to Wyman Wormhole)

The following values of constants [6] (p. 7):

$$
c_1 = -\frac{r_S}{2\gamma} = -\frac{GM}{\gamma c^2} \quad ; \quad c_2 = 0 \quad ; \quad c_3 = \frac{r_S}{2} = \frac{GM}{c^2} \qquad (14)
$$

(where γ is arbitrary constant,

$$
r_S = \frac{2GM}{c^2} \qquad (15)
$$

is the *Schwarzschild radius*, M is the inertial mass of the wormhole and G is the gravitational constant) imply the *Janis-Newman-Winicour wormhole* (which is equivalent to the *Wyman wormhole* [9]). Indeed, we have

$$\mathrm{d}S^2 = -\mathrm{c}^2\mathrm{d}\tau_{\mathrm{GR}}{}^2 = -\left(1-\frac{r_\mathrm{S}}{\gamma r}\right)^{\gamma}\mathrm{c}^2\mathrm{d}t^2 + \frac{1}{\left(1-\frac{r_\mathrm{S}}{\gamma r}\right)^{\gamma}}\mathrm{d}r^2 + r^2\left(1-\frac{r_\mathrm{S}}{\gamma r}\right)^{1-\gamma}\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\right); \tag{16}$$

$$\dot{t}_{\mathrm{GR}} = \left[\left(1-\frac{r_\mathrm{S}}{\gamma r}\right)^{\gamma} - \frac{1}{\left(1-\frac{r_\mathrm{S}}{\gamma r}\right)^{\gamma}}\frac{\dot{r}^2}{\mathrm{c}^2} - \frac{r^2}{\mathrm{c}^2}\left(1-\frac{r_\mathrm{S}}{\gamma r}\right)^{1-\gamma}\left(\dot{\theta}^2+\sin^2\theta\,\dot{\phi}^2\right)\right]^{-\frac{1}{2}}; \tag{17}$$

$$L_{\mathrm{SR}} = -m\mathrm{c}^2\left[\left(1-\frac{r_\mathrm{S}}{\gamma r}\right)^{\gamma} - \frac{1}{\left(1-\frac{r_\mathrm{S}}{\gamma r}\right)^{\gamma}}\frac{\dot{r}^2}{\mathrm{c}^2} - \frac{r^2}{\mathrm{c}^2}\left(1-\frac{r_\mathrm{S}}{\gamma r}\right)^{1-\gamma}\left(\dot{\theta}^2+\sin^2\theta\,\dot{\phi}^2\right)\right]^{\frac{1}{2}}; \tag{18}$$

$$V_{\mathrm{SR}} = \mathrm{c}^2\left(\left[\left(1-\frac{r_\mathrm{S}}{\gamma r}\right)^{\gamma} - \frac{1}{\left(1-\frac{r_\mathrm{S}}{\gamma r}\right)^{\gamma}}\frac{\dot{r}^2}{\mathrm{c}^2} - \frac{r^2}{\mathrm{c}^2}\left(1-\frac{r_\mathrm{S}}{\gamma r}\right)^{1-\gamma}\left(\dot{\theta}^2+\sin^2\theta\,\dot{\phi}^2\right)\right]^{\frac{1}{2}} - \frac{1}{\gamma_{(\beta_P)}}\right). \tag{19}$$

Schwarzschild Black Hole

The substitution

$$\gamma = 1 \tag{20}$$

to the *Janis-Newman-Winicour wormhole* (or equivalently,

$$c_1 = -\frac{r_\mathrm{S}}{2} = -\frac{\mathrm{G}M}{\mathrm{c}^2} \quad ; \quad c_2 = 0 \quad ; \quad c_3 = \frac{r_\mathrm{S}}{2} = \frac{\mathrm{G}M}{\mathrm{c}^2} \tag{21}$$

to the general *wormhole with spherical symmetry*), implies the *Schwarzschild black hole* [2] (p. 229), [7]. Thus, we have

$$\mathrm{d}S^2 = -\mathrm{c}^2\mathrm{d}\tau_{\mathrm{GR}}{}^2 = -\left(1-\frac{r_\mathrm{S}}{r}\right)\mathrm{c}^2\mathrm{d}t^2 + \frac{1}{1-\frac{r_\mathrm{S}}{r}}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\right); \tag{22}$$

$$\dot{t}_{\mathrm{GR}} = \left[1-\frac{r_\mathrm{S}}{r} - \frac{1}{1-\frac{r_\mathrm{S}}{r}}\frac{\dot{r}^2}{\mathrm{c}^2} - \frac{r^2}{\mathrm{c}^2}\left(\dot{\theta}^2+\sin^2\theta\,\dot{\phi}^2\right)\right]^{-\frac{1}{2}}; \tag{23}$$

$$L_{\mathrm{SR}} = -m\mathrm{c}^2\left[1-\frac{r_\mathrm{S}}{r} - \frac{1}{1-\frac{r_\mathrm{S}}{r}}\frac{\dot{r}^2}{\mathrm{c}^2} - \frac{r^2}{\mathrm{c}^2}\left(\dot{\theta}^2+\sin^2\theta\,\dot{\phi}^2\right)\right]^{\frac{1}{2}}; \tag{24}$$

$$V_{\mathrm{SR}} = \mathrm{c}^2\left(\left[1-\frac{r_{\mathrm{S}}}{r}-\frac{1}{1-\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^2}{\mathrm{c}^2}-\frac{r^2}{\mathrm{c}^2}\left(\dot{\theta}^2+\sin^2\theta\,\dot{\phi}^2\right)\right]^{\frac{1}{2}}-\frac{1}{\gamma_{(\beta_P)}}\right). \tag{25}$$

The above imply exactly the results of GR [4], even though the *SR Lagrangian* (23) is different than the corresponding *GR Lagrangian* [2] (p. 238), [4] (p. 542). For instance the *radial velocity of a particle* is [4] (p. 543):

$$\dot{r}^2 = \mathrm{c}^2\left(1-\frac{r_{\mathrm{S}}}{r}\right)^2\left[1-\frac{m^2\,\mathrm{c}^4}{E_{\mathrm{GR}}^2}\left(1-\frac{r_{\mathrm{S}}}{r}\right)\left(1+\frac{1}{\mathrm{c}^2}\frac{h_{\mathrm{GR}}^2}{r^2}\right)\right], \tag{26}$$

where the *integrals of motion* are the *GR total energy* [4] (p. 544):

$$E_{\mathrm{GR}} = \frac{1-\frac{r_{\mathrm{S}}}{r}}{\sqrt{1-\frac{r_{\mathrm{S}}}{r}-\frac{1}{1-\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^2}{\mathrm{c}^2}-\frac{r^2\dot{\phi}^2}{\mathrm{c}^2}}}\,m\mathrm{c}^2 \geq 0 \tag{27}$$

and the *GR angular momentum per mass unit* [4] (p. 546):

$$h_{\mathrm{GR}} = r^2\dot{\phi}\left[1-\frac{r_{\mathrm{S}}}{r}-\frac{1}{1-\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^2}{\mathrm{c}^2}-\frac{r^2\dot{\phi}^2}{\mathrm{c}^2}\right]^{-\frac{1}{2}}. \tag{28}$$

The differentiation of (26) gives

$$\ddot{r} = \frac{\mathrm{c}^2}{2}\left(1-\frac{r_{\mathrm{S}}}{r}\right)\left(\frac{2r_{\mathrm{S}}}{r^2}\left[1-\frac{m^2\,\mathrm{c}^4}{E_{\mathrm{GR}}^2}\left(1-\frac{r_{\mathrm{S}}}{r}\right)\left(1+\frac{1}{\mathrm{c}^2}\frac{h_{\mathrm{GR}}^2}{r^2}\right)\right]-\frac{m^2\,\mathrm{c}^4}{E_{\mathrm{GR}}^2}\left(1-\frac{r_{\mathrm{S}}}{r}\right)\left[\frac{r_{\mathrm{S}}}{r^2}\left(1+\frac{h_{\mathrm{GR}}^2}{\mathrm{c}^2 r^2}\right)-\frac{2h_{\mathrm{GR}}^2}{\mathrm{c}^2 r^3}\left(1-\frac{r_{\mathrm{S}}}{r}\right)\right]\right). \tag{29}$$

Thus, a *momentarily unmoved particle* ($\dot{r}=0\,;\dot{\phi}=0$) has

$$E_{\mathrm{GR}} = \sqrt{1-\frac{r_{\mathrm{S}}}{r}}\,m\mathrm{c}^2 \geq 0 \quad ; \quad h_{\mathrm{GR}} = 0\,. \tag{30}$$

So, (29) gives the *initial acceleration of a momentarily unmoved particle* [4] (p. 563):

$$2\ddot{r}+\left(1-\frac{r_{\mathrm{S}}}{r}\right)\frac{r_{\mathrm{S}}}{r^2}\mathrm{c}^2 = 0\,, \tag{31}$$

which becomes

$$\vec{a}_r = -\frac{\mathrm{G}M}{r^2}\left(1-\frac{r_{\mathrm{S}}}{r}\right)\hat{r}\,. \tag{32}$$

We observe that if we leave a particle somewhere, then it moves radially towards the horizon.
Besides, the *speed of a photon in radial motion* around a *Schwarzschild black hole* [4] (p. 547):

$$c_p = \left(1-\frac{r_S}{r}\right)\mathrm{c} \leq \mathrm{c} \tag{33}$$

is only *subluminal* and also the *photon is unmoved on the horizon* (r=r_{S}). So, this *black hole* is *non-traversable*.
Finally, the *acceleration of a photon in radial motion* around a *Schwarzschild black hole* is:

$$\ddot{r} = \frac{r_{\mathrm{S}}\,\mathrm{c}^2}{r^2}\left(1-\frac{r_{\mathrm{S}}}{r}\right), \tag{34}$$

or equivalently,

$$\vec{a}_p = \frac{2\mathrm{G}M}{r^2}\left(1-\frac{r_{\mathrm{S}}}{r}\right)\hat{r}\,. \tag{35}$$

This means that the *Schwarzschild black hole repels the photon*!

Schwarzschild Repulsive Hole

The substitution

$$c_1 = \frac{r_S}{2} = \frac{GM}{c^2} \quad ; \quad c_2 = 0 \quad ; \quad c_3 = -\frac{r_S}{2} = -\frac{GM}{c^2} \tag{36}$$

to the general *wormhole with spherical symmetry*, implies the *Schwarzschild repulsive hole*. It is noted that this substitution is not equivalent to the substitution γ =-1 to the *Janis-Newman-Winicour wormhole*. We also remind that the *general Schwarzschild solution* gives a metric which contains two arbitrary constants [7]. The *weak field approximation* to the *Newtonian gravitational field* determines the constants (the *Schwarzschild radius* (15) is one of them) and we obtain the metric (22). Furthermore, the substitution (36) is equivalent to the substitution

$$r_S \to -r_S = -\frac{2GM}{c^2} \tag{37}$$

to the *Schwarzschild black hole*. Then, we have

$$dS^2 = -c^2 d\tau_{GR}{}^2 = -\left(1+\frac{r_S}{r}\right)c^2 dt^2 + \frac{1}{1+\frac{r_S}{r}}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right); \tag{38}$$

$$\dot{t}_{GR} = \left[1+\frac{r_S}{r} - \frac{1}{1+\frac{r_S}{r}}\frac{\dot{r}^2}{c^2} - \frac{r^2}{c^2}\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right)\right]^{-\frac{1}{2}} ; \tag{39}$$

$$L_{SR} = -mc^2\left[1+\frac{r_S}{r} - \frac{1}{1+\frac{r_S}{r}}\frac{\dot{r}^2}{c^2} - \frac{r^2}{c^2}\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right)\right]^{\frac{1}{2}} ; \tag{40}$$

$$V_{SR} = c^2\left(\left[1+\frac{r_S}{r} - \frac{1}{1+\frac{r_S}{r}}\frac{\dot{r}^2}{c^2} - \frac{r^2}{c^2}\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right)\right]^{\frac{1}{2}} - \frac{1}{\gamma_{(\beta_P)}}\right). \tag{41}$$

The results of the above *repulsive hole* can be extracted by the results of *black hole* [4] with the substitution (37). For instance the *radial velocity of a particle* is [4] (p. 543):

$$\dot{r}^2 = c^2\left(1+\frac{r_S}{r}\right)^2\left[1 - \frac{m^2c^4}{E_{GR}^2}\left(1+\frac{r_S}{r}\right)\left(1+\frac{1}{c^2}\frac{h_{GR}^2}{r^2}\right)\right], \tag{42}$$

where the *integral of motions* are the *GR total energy* [4] (p. 544):

$$E_{GR} = \frac{1+\frac{r_S}{r}}{\sqrt{1+\frac{r_S}{r} - \frac{1}{1+\frac{r_S}{r}}\frac{\dot{r}^2}{c^2} - \frac{r^2\dot{\phi}^2}{c^2}}}mc^2 \geq 0 \tag{43}$$

and the *GR angular momentum per mass unit* [4] (p. 546):

$$h_{GR} = r^2\dot{\phi}\left[1+\frac{r_S}{r} - \frac{1}{1+\frac{r_S}{r}}\frac{\dot{r}^2}{c^2} - \frac{r^2\dot{\phi}^2}{c^2}\right]^{-\frac{1}{2}} . \tag{44}$$

The differentiation of (42) gives

$$\ddot{r}=\frac{c^2}{2}\left(1+\frac{r_S}{r}\right)\left(-\frac{2r_S}{r^2}\left[1-\frac{m^2c^4}{E_{GR}^2}\left(1+\frac{r_S}{r}\right)\left(1+\frac{1}{c^2}\frac{h_{GR}^2}{r^2}\right)\right]-\frac{m^2c^4}{E_{GR}^2}\left(1+\frac{r_S}{r}\right)\left[-\frac{r_S}{r^2}\left(1+\frac{h_{GR}^2}{c^2r^2}\right)-\frac{2h_{GR}^2}{c^2r^3}\left(1+\frac{r_S}{r}\right)\right]\right). \quad (45)$$

Thus, a *momentarily unmoved particle* ($\dot{r}=0$; $\dot{\phi}=0$) has

$$E_{GR}=\sqrt{1+\frac{r_S}{r}}mc^2\geq 0\,;\; h_{GR}=0\,. \quad (46)$$

So, equation (29) gives the *initial acceleration of a momentarily unmoved particle* [4] (p. 563):

$$2\ddot{r}-\left(1+\frac{r_S}{r}\right)\frac{r_S}{r^2}c^2=0\,, \quad (47)$$

which becomes

$$\vec{a}_r=\frac{GM}{r^2}\left(1+\frac{r_S}{r}\right)\hat{r}\,. \quad (48)$$

We observe that if we leave a particle somewhere, then it is repelled radially away the horizon. Besides, the *speed of a photon in radial motion* around a *Schwarzschild repulsive hole* [4] (p. 547):

$$c_p=\left(1+\frac{r_S}{r}\right)c\geq c \quad (49)$$

is only *superluminal* and also the photon has *infinite speed at the zero-point*. So, this *repulsive hole* is *traversable*. Finally, the *acceleration of a photon in radial motion* around a *Schwarzschild repulsive hole* is:

$$\ddot{r}=-\frac{r_Sc^2}{r^2}\left(1+\frac{r_S}{r}\right), \quad (50)$$

or equivalently,

$$\vec{a}_r=-\frac{2GM}{r^2}\left(1+\frac{r_S}{r}\right)\hat{r}\,. \quad (51)$$

This means that the *Schwarzschild repulsive hole attracts the photon*!

Black Hole according to Teleparallel gravity

Another application is the *teleparallel gravity*, which is based on *Weitzenböck spacetime* (T^4) [8]. The corresponding isotropic metric by a static, spherical body may take the form

$$dS^2=-c^2d\tau_{GR}{}^2=-\left(1-\frac{r_S}{r}\right)c^2dt^2+\frac{1}{1-\frac{r_S}{r}}\left(1-\frac{8\varepsilon}{1-5\varepsilon}\frac{r_S}{4r}\right)dr^2+r^2d\theta^2+r^2\sin^2\theta\, d\phi^2 \quad (52)$$

that looks like the *Schwarzschild metric* [4] (p. 567). The experimental value of the arbitrary parameter ε is

$$\varepsilon=-0.004\pm 0.004 \quad (53)$$

according to observations in the solar system (*gravitational deflection of light* and *precession of Mercury's perihelion*) [8] (p. 3538). From the metric (52) we obtain

$$\dot{t}_{GR}=\left[1-\frac{r_S}{r}-\left(1-\frac{8\varepsilon}{1-5\varepsilon}\frac{r_S}{4r}\right)\frac{1}{1-\frac{r_S}{r}}\frac{\dot{r}^2}{c^2}-\frac{r^2}{c^2}\left(\dot{\theta}^2+\sin^2\theta\,\dot{\phi}^2\right)\right]^{-\frac{1}{2}}\;; \quad (54)$$

$$L_{SR}=-mc^2\left[1-\frac{r_S}{r}-\left(1-\frac{8\varepsilon}{1-5\varepsilon}\frac{r_S}{4r}\right)\frac{1}{1-\frac{r_S}{r}}\frac{\dot{r}^2}{c^2}-\frac{r^2}{c^2}\left(\dot{\theta}^2+\sin^2\theta\,\dot{\phi}^2\right)\right]^{\frac{1}{2}}\;; \quad (55)$$

$$V_{\mathrm{SR}(r,\dot r,\dot\phi)} = \mathrm{c}^2\left(\left[1-\frac{r_{\mathrm S}}{r}-\left(1-\frac{8\varepsilon}{1-5\varepsilon}\frac{r_{\mathrm S}}{4r}\right)\frac{1}{1-\frac{r_{\mathrm S}}{r}}\frac{\dot r^2}{\mathrm{c}^2}-\frac{r^2}{\mathrm{c}^2}\left(\dot\theta^2+\sin^2\theta\,\dot\phi^2\right)\right]^{\frac{1}{2}}-\frac{1}{\gamma_{(\beta_P)}}\right). \qquad (56)$$

THE REVERSE PROCEDURE: GR METRICS FROM THE SR GRAVITATIONAL SCALAR GENERALIZED POTENTIAL

Finally, we show the *reverse procedure*: the way to obtain *GR metrics* from the SR GSGP. This can be done via the formula (8ii) which correlates them. Thus, the substitution of the *Newtonian Gravitational Potential*

$$V_{\mathrm N} = -\frac{\mathrm G M}{r} \qquad (57)$$

gives the corresponding *GR time dilation*

$$\dot t_{\mathrm{GR}} = \frac{1}{-\frac{\mathrm G M}{\mathrm c^2 r}+\frac{1}{\gamma_{(\beta_P)}}} = \frac{1}{-\frac{r_{\mathrm S}}{2r}+\sqrt{1-\frac{\dot r^2+r^2\dot\theta^2+r^2\sin^2\theta\,\dot\phi^2}{\mathrm c^2}}}. \qquad (58)$$

This implies

$$\mathrm d S^2 = -\mathrm c^2\mathrm d\tau_{\mathrm{GR}}{}^2 = -\left(1-\frac{r_{\mathrm S}}{r}\sqrt{1-\frac{1}{\mathrm c^2}\left[\left(\frac{\mathrm d r}{\mathrm d t}\right)^2+r^2\left(\frac{\mathrm d\theta}{\mathrm d t}\right)^2+r^2\sin^2\theta\left(\frac{\mathrm d\phi}{\mathrm d t}\right)^2\right]}+\frac{r_{\mathrm S}{}^2}{4r^2}\right)\mathrm c^2\mathrm d t^2 + \\ +\mathrm d r^2+r^2\,\mathrm d\theta^2+r^2\sin^2\theta\,\mathrm d\phi^2, \qquad (59)$$

which is *non-Riemannian metric* and we have spacetime with *Finsler geometry* [4] (pp. 572-573). This complicated metric corresponds to a very *simple SR gravitational Lagrangian*. Indeed, (4) combined with (57) gives

$$L_{\mathrm{SR}} = -\frac{1}{\gamma_{(\beta_P)}}m\mathrm c^2+\frac{\mathrm G M}{r}m. \qquad (60)$$

This proves that there exist problems with *SR solution* easier the than the corresponding *GR solution*.

CONCLUSION

The correlation of curved spacetime endowed with variable metric [e.g. the General Relativity (GR)] with the spacetime endowed with *Lorentz Metric* [e.g. the Special Relativity (SR)] becomes via the SR *Lagrangian*, where the *Lorentz γ-factor* is substituted from the *GR time dilation*.

(1) This implies the SR Gravitational Scalar Generalized Potential (GSGP), which corresponds to a *GR metric*.

(2) The presented procedure can be applied to any kind of GR, based on *Riemannian* and *non-Riemannian spacetime*, such as the *classical GR* and *teleparallel gravity*, and on other types of gravitation such as the Modified Newtonian Dynamics (MoND) which is relativized.

(3) The *reverse procedure*: *GR Metrics* from the SR GSGP is also valid. The case study of *Newtonian Gravitational Potential* leads to the corresponding *non-Riemannian metric* of spacetime (*Finsler geometry*).

(4) The *Precession of Mercury's perihelion*, *Gravitational deflection of light*, *Shapiro time delay* and *Gravitational red shift* are explained not only by using the GR (e.g. *Schwarzschild metric*), but also with the usage of SR and *Newtonian Physics* (NPs) [5].

(5) The *wormholes* can be studied not only in the frame of GR, but also in the frame of SR.

(6) There exist gravitational fields (i.e. around a *Schwarzschild repulsive hole*), where the particles have *superluminal speeds* according to GR and SR.

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