Explanation of Light Deflection, Precession of Mercury's Perihelion, Gravitational Red Shift and Rotation Curves in Galaxies, by using General Relativity or equivalent Generalized Scalar Gravitational Potential, according to Special Relativity and Newtonian Physics

Spyridon Vossos[1,2], Elias Vossos[1,3] and Christos G. Massouros[1,4]

[1]Core department, National and Kapodistrian University of Athens, Euripus Campus GR 34400, Greece

Keywords: Cavendish experiment, dark matter, Einstein relativity theory, closed linear transformation, Euclidean metric, galaxy NGC 3198, Galilean transformation, General relativity, gravitation, gravitational deflection of light, gravitational red shift, kinimatics and dynamics of galaxies, Lagrangian, linear spacetime transformation, Lorentz boost, Lorentz metric, Lorentz transformation, Minkowski space, Modified Newtonian Dynamics, MOND, Newtonian Physics, precession of Earth's perihelion, precession of Mercury's perihelion, rotation curve of Galaxies, Schwarzschild metric, kinimatics and dynamics of Solar system, Special relativity.

PACS: 02.10.Ud, 02.40.Dr, 03.30.+p, 04.20.-q, 04.50.Kd, 04.80.Cc, 95.35.+d, 96.12.De, 96.15.De, 98.20.+d

Abstract. The development of Geometric theories of gravitation and the application of the Dynamics of General Relativity (GR) is the mainstream approach of gravitational field. Besides, the Generalized Special Relativity (GSR) contains the fundamental parameter (ξ_I) of Theories of Physics (TPs). Thus, it expresses at the same time Newtonian Physics (NPs) for $\xi_I \rightarrow 0$ and Special Relativity (SR) for $\xi_I=1$. Moreover, the weak Equivalence Principle (EP) in the context of GSR, has the interpretation: $m_G=m$, where m_G and m are the gravitational mass and the inertial rest mass, respectively. In this paper, we bridge GR with GSR. This is achieved, by using a GSR-Lagrangian, which contains the corresponding GR-proper time. Thus, we obtain a new central scalar GSR-gravitational generalized potential $V=V(k,l,r,r\_dot,\varphi\_dot)$, where $k=k(\xi_I)$, $l=l(\xi_I)$, r is the distance from the center of gravity and r_dot, φ_dot are the radial and angular velocity, respectively. The replacement $k=1$ and $l=\xi_I^2$ makes the above GSR-potential equivalent to the original Schwarzschild Metric (SM). Thus, it explains the Precession of Mercury's Perihelion (PMP), Gravitational Deflection of Light (GDL),

[2] e-mail: svossos@uoa.gr.
[3] e-mail: evossos@uoa.gr.
[4] e-mail: ChrMas@uoa.gr.

Gravitational Red Shift (GRS) etc, by using SR and/or NPs. The procedure described in this paper can be applied to any other GR-spacetime metric, in order to find out the corresponding GSR-gravitational potential. So, we also use the GR-proper time of the 3rd Generalized Schwarzschild Metric (3GSM) and we obtain the central scalar GSR-gravitational potential *V*=*V*(*a*,*k*,*l*,*r_dot*,*φ_dot*), where *a*=*a*(*r*). The combination of the above with MOND interpolating functions, or distributions of Dark Matter (DM) in galaxies, provides the functions corresponding *a*=*a*(*r*). Thus, we obtain a new GSR-Gravitational field, which explains the PMP, GDL, GRS as well as the *Rotation Curves in Galaxies*, eliminating the corresponding DM.

1. Introduction

The Equivalence Principle (EP) in the context of Special Relativity (SR), has many possible interpretations [1] (p. 245). In this paper, we follow the *weak EP,* where the *gravitational mass* (m_G) is equal to the *inertial rest mass* (m):

$$m_G = m. \tag{1}$$

This SR-interpretation coincides to the case of Newtonian Physics (NPs). Besides, the *gravitational potential energy* is usually

$$U=m_G V=mV, \tag{2}$$

where V is *scalar gravitational potential*. The above equation is valid, if the *scalar gravitational potential* depends only on the distance: $V_{GSR}=V_{GSR(r)}$. In case that *generalized scalar gravitational potential* is used, as we do in this paper, (2) is valid only for unmoved particle. Below, we shall explain the most significant gravitational phenomena:

(i) Precession of Mercury's Perihelion (PMP)
(ii) Gravitational Deflection of Light (GDL)
(iii) Gravitational Red Shift (GRS), and
(iv) *rotation curv*es in galaxies,

by using initially General Relativity (GR) and after SR and/or NPs.

The EP (1) according to SR, combined with *Newtonian scalar gravitational potential*

$$V_N = -\frac{GM}{r}, \tag{3}$$

gives GR-PMP (Figure 1a): $\Omega = 7''.16$ per century, [2] (p. 355), [3] (p. 338). This theoretical result is far away from the experimental value: $\Omega_{exp} = 42''.9799(9)$ cy^{-1}, which is the contribution of the Sun due to *Schwarzschild* Gravito-Electric effect (GEE) to the total PMP [4] (p. 6), [5] (p. 152). Moreover, we have already presented the scalar gravitational potential

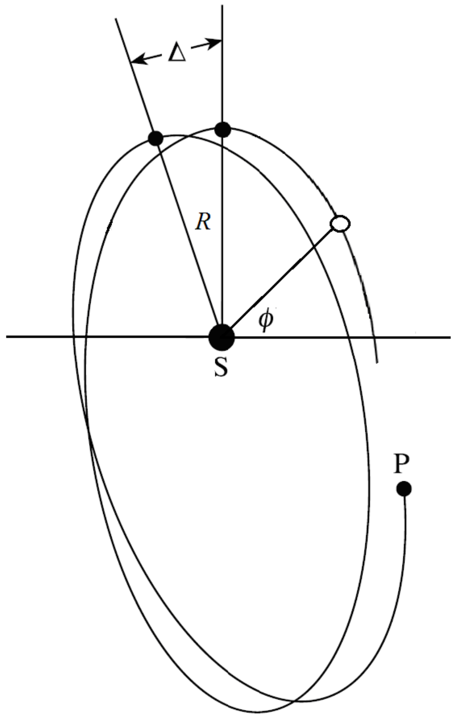
$$V = \left(\sqrt{1-k\frac{r_S}{r}}-1\right)\frac{c^2}{k} \le 0, \tag{4}$$

where

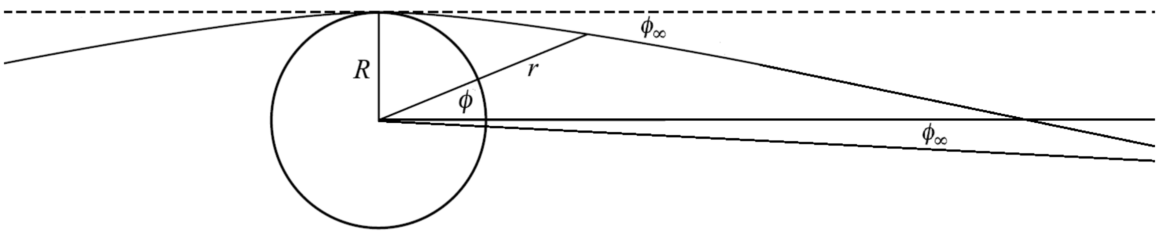
$$r_S = \frac{2GM}{c^2} \tag{5}$$

is *Schwarzschild radius.* [6]. The combination of EP (1) with potential (4) and k=5 according to SR, or the combination of EP (1) with potential (4) and k=6 according NPs, give the same *precession of Mercury's perihelion*: $42''.9820(43)$ cy^{-1}, [6] (p. 14). This is in accordance with the experimental value.

On the other hand, the GDL (Figure 1b) is an effect that was firstly predicted by Johann von Soldner, in 1801. He supposed that a ray grazing the Earth (or the Moon, or the Sun) contains particles (photons) moving with steady speed $\upsilon = c$ and he solved the problem, by using NPs and



(a)



(b)

Figure 1. (a) Precession of the pericenter / perihelion of the orbit of a particle / planet by a spherical mass / Sun. (b) Deflection of a ray by a spherical mass / Sun. R is the (minimum) distance of the perihelion / pericenter from the center of gravity and Δ is the angle that the perihelion precesses per revolution.

Newtonian scalar gravitational potential (3) [7] (p.169). The result of the half deflection (φ_∞) has

$$\tan\phi_\infty = \frac{\mathrm{G}M}{\mathrm{c}\sqrt{\mathrm{c}^2 R^2 - 2\mathrm{G}MR}} \approx \frac{\mathrm{G}M}{\mathrm{c}^2 R} = \frac{r_\mathrm{S}}{2R}, \tag{6}$$

which gives the magnitude of the total deflection of a ray

$$\Theta \approx \frac{2\mathrm{G}M}{\mathrm{c}\sqrt{\mathrm{c}^2 R^2 - 2\mathrm{G}MR}} \approx \frac{2\mathrm{G}M}{\mathrm{c}^2 R} = \frac{r_\mathrm{S}}{R}, \tag{7}$$

where R is the minimum distance from the center of gravity. In 1911, a similar result was obtained by Albert Einstein, before the development of GR. He solved the problem, by using SR, the EP & *Newtonian scalar gravitational potential* (3) and he calculated exactly [8] (p. 904):

$$\Theta = \frac{2\mathrm{G}M}{\mathrm{c}^2 R} = \frac{r_{\mathrm{S}}}{R}. \tag{8}$$

For a ray grazing the Sun, they calculated $\Theta = 0''.84$ and $\Theta = 0''.83$, respectively. These results are about the half the observed value $\Theta_{\exp} = 1''.75$ [9] (p. 249), which is also calculated by Schwarzschild Metric (SM) formula [5] (p. 153):

$$\Theta = \frac{4\mathrm{G}M}{\mathrm{c}^2 R} = \frac{2r_{\mathrm{S}}}{R}. \tag{9}$$

The same result can be obtained, by using A. Einstein 1911-method and scalar gravitational potential (4) for

$$k = \frac{4\mathrm{c}^2 R}{\pi \mathrm{G} M} = \frac{8R}{\pi\, r_{\mathrm{S}}}. \tag{10}$$

This means variable k>>5 (k=5 is the value which predicts the Precession of Mercury's perihelion). So, potential (4) is also inefficient to explain the GDL according to SR, in contrast to *scalar gravitational generalized potential* (236) (see below).

The above analysis explains why the gravitational field is usually studied, by using the Dynamics of GR and the development of *Geometric theories of gravitation* [10]. The EP in GR is: accelerated motions caused by the gravitational field only (free fall) take place along *geodesics* of the metric, which corresponds to the particular gravitational field [2] (p. 248).

In this paper, we use generalized Relativity Theory (RT), which contains *Einstein Relativity Theory* (ERT) and *Newtonian Physics* (NPs), keeping the formalism of ERT. Thus, the differences between these two Theories of Physics (TPs) are limited to their different value of *metric coefficients of spacetime* for the corresponding Relativistic Inertial observers (RIOs) and the fundamental parameter of TPs: ξ_{I}. NPs has $\xi_{\mathrm{I}}\to 0$, while ERT has $\xi_{\mathrm{I}}=1$ [11]. The case of observers with variable metric of spacetime, leads to the corresponding GR. For being this clear, we present the 1$^{\text{st}}$ Generalized Schwarzschild Metric (1GSM) and the 3$^{\text{rd}}$ Generalized Schwarzschild Metric (3GSM), which are in accordance with any SR based on isotropic Generalized metrics (g_{I}) and *Einstein field equations*.

In case of 1GSM, we compute the corresponding *Lagrangian*, *Equations of motion*, *Precession of planets' orbits*, *Deflection of light* etc, resulting formulas which are referred to any TPs. We also present the results of the original *Schwarzschild metric* (SM), by adopting *no-superposition principle,* in contrast with many textbooks, and we obtain the *total GR-energy*. Finally, the *generalized potential energy* is calculated, by reducing the kinetic energy (which is considered equal to this of GSR) from the total GR-energy. Thus, we conclude that although SM is a static and stationary metric of non-rotating mass, it produces Gravito-Magnetic Effect (GME), because the GSR-gravitational potential and the GSR-gravitational force depend on the velocity of the particle.

The next step is the invention of a method which bridges GR with GSR. This is achieved, by using a GSR-Lagrangian, which contains the time dilation of the corresponding GR-Lagrangian. Thus, we obtain a new central scalar GSR-Gravitational generalized potential $V=V(k,l,r,\dot{r},\dot{\phi})$, where $k=k(\xi_{\mathrm{I}})$, $l=l(\xi_{\mathrm{I}})$, r is the distance from the center of gravity and $\dot{r}$, $\dot{\phi}$ are the radial and angular velocity, respectively. We demand that 'this new GSR-gravitational field in accordance with EP (1), gives the same *equation of orbit* as SM does' and we obtain k=1 and $l=\xi_{\mathrm{I}}^2$.

In case of 3GSM, we apply the above procedure and we obtain a new central scalar GSR-gravitational potential $V=V(a,k,l,\dot{r},\dot{\phi})$, where $a=a(r)$. The combination of the above with Modified Newtonian Dynamics (MOND) and/or distributions of phantom Dark Matter (DM) in galaxies, provides the corresponding functions $a=a(r)$. Thus, we have also achieved the relativization of MOND. More specifically, we use a new generalized interpolating function (μ) (which expresses both the *Simple* and the *Standard* μ) and/or a very simple distribution of DM, for the explanation of the Rotation Curves in Galaxies (e.g. NGC-3198) as well as the Solar system, eliminating DM. Generally, this approach, in non rotating black hole, planetary and star system-scale, coincides to the original SM, while in galactic scale, it gives MONDian or DM-results. Finally, we have obtained a new Gravitational field, which not only explains the PMP, GDL, GRS etc, but also the *Rotation Curves in Galaxies*, eliminating the corresponding DM.

2. Isometric Closed Linear Transformations of Complex Spacetime endowed with the Corresponding Metrics

In this paper, the *metric coefficients of time and space have different signs*. Moreover, 3D-space is isotropic, in case of Isometric Closed Linear Transformations of Complex Spacetime (ICLSTTs) [12]. Thus, for RIOs, the representation of the non-degenerate inner product in holonomic basis [$\mathbf{e}_\mu$]=[$\mathbf{e}_0$, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$]=[$\mathbf{e}_{ct}$, $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$] is the real matrix of metric:

$$g_{\mathrm{I}} = \mathrm{diag}\left(g_{\mathrm{I}00},\ g_{\mathrm{I}11},\ g_{\mathrm{I}22},\ g_{\mathrm{I}33}\right) = g_{\mathrm{I}11}\,\mathrm{diag}\left(-\frac{1}{\xi_{\mathrm{I}}^{2}},\ 1,\ 1,\ 1\right) = g_{\mathrm{I}00}\,\mathrm{diag}\left(1,\ -\xi_{\mathrm{I}}^{2},\ -\xi_{\mathrm{I}}^{2},\ -\xi_{\mathrm{I}}^{2}\right), \quad (11)$$

where

$$\xi_{\mathrm{I}} = \sqrt{\frac{g_{\mathrm{I}11}}{-g_{\mathrm{I}00}}} \quad (12)$$

The index I remind us that we are referred to the spacetime of the RIOs of each specific TP. Besides the GSR has real Universal Speed (c_{I}):

$$c_{\mathrm{I}} = \frac{1}{\xi_{\mathrm{I}}}\mathrm{c} \quad (13)$$

and the transformation of a contravariant four-vector is

$$\mathrm{d}X' = \Lambda_{\mathrm{I}(\xi_{\mathrm{I}},\beta)}\,\mathrm{d}X, \quad (14)$$

where

$$\Lambda_{\mathrm{I}(\xi_{\mathrm{I}},\beta)} = \gamma_{(\xi_{\mathrm{I}}\beta)}\begin{bmatrix} 1 & -\xi_{\mathrm{I}}^{2}\beta_x & -\xi_{\mathrm{I}}^{2}\beta_y & -\xi_{\mathrm{I}}^{2}\beta_z \\ -\beta_x & 1 & \mathrm{i}\,\xi_{\mathrm{I}}\beta_z & -\mathrm{i}\,\xi_{\mathrm{I}}\beta_y \\ -\beta_y & -\mathrm{i}\,\xi_{\mathrm{I}}\beta_z & 1 & \mathrm{i}\,\xi_{\mathrm{I}}\beta_x \\ -\beta_z & \mathrm{i}\,\xi_{\mathrm{I}}\beta_y & -\mathrm{i}\,\xi_{\mathrm{I}}\beta_x & 1 \end{bmatrix} = \gamma_{(\xi_{\mathrm{I}}\beta)}\begin{bmatrix} 1 & -\xi_{\mathrm{I}}^{2}\beta^{T} \\ -\beta & \mathrm{I}_3 + \mathrm{i}\,\xi_{\mathrm{I}}\,\mathrm{A}_{(\beta)} \end{bmatrix}, \quad (15)$$

$$\beta^{i} = \frac{\mathrm{d}x^{i}}{\mathrm{d}x^{0}} \quad ; \quad \beta = \begin{bmatrix} \beta_x \\ \beta_y \\ \beta_z \end{bmatrix} \quad ; \quad \mathrm{A}_{(\beta)} = \begin{bmatrix} 0 & \beta_z & -\beta_y \\ -\beta_z & 0 & \beta_x \\ \beta_y & -\beta_x & 0 \end{bmatrix} \quad (16)$$

and

$$\gamma_{(\delta)} = \frac{1}{\sqrt{1-\delta^{\mathrm{T}}\delta}} \quad (17)$$

is *Lorentz γ-factor*. The typical matrix of IECLSTTs along x-axis (Generalized; Galilean-Newtonian; Lorentzian-Einsteinian) is

$$\Lambda_{\mathrm{I}typ} = \gamma_{(\xi_{\mathrm{I}}\beta)}\begin{bmatrix} 1 & -\xi_{\mathrm{I}}^{2}\beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & \mathrm{i}\,\xi_{\mathrm{I}}\beta \\ 0 & 0 & -\mathrm{i}\,\xi_{\mathrm{I}}\beta & 1 \end{bmatrix} ; \ \Lambda_{\Gamma typ} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; \ \Lambda_{\mathrm{B}typ} = \gamma_{(\beta)}\begin{bmatrix} 1 & -\beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & \mathrm{i}\,\beta \\ 0 & 0 & -\mathrm{i}\,\beta & 1 \end{bmatrix}. \quad (18)$$

The specific value $\xi_{\mathrm{I}}\to 0$ ($g_{\mathrm{I}11}\to 0$, $g_{\mathrm{I}00}\neq 0$) gives Galilean Transformation (GT) with infinite Universal Speed ($c_{\mathrm{I}}\to +\infty$) and the corresponding metric of the spacetime (let us call *Galilean metric*):

$$g_\Gamma = \lim_{g_{I11}\to 0} \mathrm{diag}\left(g_{I00},\ g_{I11},\ g_{I11},\ g_{I11}\right) = g_{I00} \lim_{\xi_I \to 0} \mathrm{diag}\left(1,\ -\xi_I^{\,2},\ -\xi_I^{\,2},\ -\xi_I^{\,2}\right). \tag{19}$$

The corresponding spacetime (let us call *Galilean spacetime*) has infinite curvature ($K \to +\infty$) in any orientation $\kappa\mathbf{e_x}+\lambda\mathbf{e_y}+\mu\mathbf{e_z}$ of 3D-space. This is the reason that time is absolute for any type of observers as well as the Universal speed is infinite ($c_I \to +\infty$).

The specific value ξ_I=1 (g_{I11}= - g_{I00}) gives transformation with c_I=c (the universal speed is the well-known present speed of light in vacuum) and the corresponding metric of spacetime

$$g_B = g_{I11}\ \mathrm{diag}\left(-1,\ 1,\ 1,\ 1\right) = g_{I11}\ \eta, \tag{20}$$

which for g_{I11}=1 becomes the *Lorentz metric* (η). Thus, we have the *Lorentzian case* of GSR [13], [14], which is associated with ERT.

We now make the option that observer O measures *real spacetime*. As some elements of matrix Λ_I are imaginary numbers, we conclude that *the spacetime of one moving observer is complex*. Thus, we put an index C to the complex natural sizes and the real natural sizes have no index. In addition, any complex *Cartesian Coordinates* (CCs) of the theory may be turned to the corresponding real CCs, in order to be perceived by human senses. This is achieved, if the moving Observer O′ considers as Real CCs the corresponding lengths of rods [11] (p. 6). Thus, it emerges the (Generalized; Galilean-Newtonian; Lorentzian-Einsteinian) Real Boost (RB)

$$\mathrm{d}X' = \Lambda_{IR(\beta)}\,\mathrm{d}X \ ;\ \mathrm{d}X' = \Lambda_{\Gamma(\beta)}\,\mathrm{d}X \ ;\ \mathrm{d}X' = \Lambda_{L(\beta)}\,\mathrm{d}X, \tag{21}$$

where

$$\Lambda_{IR(\beta)} = \begin{bmatrix} \gamma_{(\xi_I\beta)} & -\gamma_{(\xi_I\beta)}\xi_I^{\,2}\beta^T \\ -\gamma_{(\xi_I\beta)}\beta & I_3 + \dfrac{\gamma_{(\xi_I\beta)}-1}{\beta^T\beta}\beta\beta^T \end{bmatrix} ;\ \Lambda_{\Gamma(\beta)} = \begin{bmatrix} 1 & 0 \\ -\beta & I_3 \end{bmatrix} ;\ \Lambda_{L(\beta)} = \begin{bmatrix} \gamma_{(\beta)} & -\gamma_{(\beta)}\beta^T \\ -\gamma_{(\beta)}\beta & I_3 + \dfrac{\gamma_{(\beta)}-1}{\beta^T\beta}\beta\beta^T \end{bmatrix}. \tag{22}$$

The typical matrix of (Generalized; Galilean-Newtonian; Lorentzian-Einsteinian) RB along x-axis is

$$\Lambda_{IR\,typ(\beta)} = \begin{bmatrix} \gamma_{(\xi_I\beta)} & -\xi_I^{\,2}\gamma_{(\xi_I\beta)}\beta & 0 & 0 \\ -\gamma_{(\xi_I\beta)}\beta & \gamma_{(\xi_I\beta)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ;\ \Lambda_{\Gamma typ(\beta)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ;\ \Lambda_{L\,typ(\beta)} = \begin{bmatrix} \gamma_{(\beta)} & -\gamma_{(\beta)}\beta & 0 & 0 \\ -\gamma_{(\beta)}\beta & \gamma_{(\beta)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{23}$$

We observe that for ξ_I=1, we have the original typical proper *Lorentz Boost* (LB) (see e.g. [2] p. 21, eq. 1.38) and the corresponding general proper LB (see e.g. [2] p. 24, eq. 1.47).

Supposing one Particle (*P*) with real mass *m* moving with velocity $\vec{\upsilon}_P = \vec{\beta}_P\,\mathrm{c}$ wrt observer O, we calculate the *Generalized kinetic energy*; *Generalized relativistic energy*; *Generalized energy of Rest mass* [11] (p. 10):

$$K = \frac{\gamma_{(\xi_I\vec{\beta}_P)}-1}{\xi_I^{\,2}} m\mathrm{c}^2 \ ;\ E = \frac{\gamma_{(\xi_I\vec{\beta}_P)}}{\xi_I^{\,2}} m\mathrm{c}^2 \ ;\ E_{rest} = \frac{1}{\xi_I^{\,2}} m\mathrm{c}^2 \tag{24}$$

3. GR: Generalized Schwarzschild metrics

3.1. The metric of a static and centrally symmetric gravitational field

Einstein field equations in vacuum [9] (pp. 303, 396) are reduced to the *single tensor equation* $R_{\mu\nu}$=0. This emerges the *metric of a static and centrally symmetric gravitational field*

$$\mathrm{d}S^2 = g_{I00} f_{(r)}\,\mathrm{c}^2\mathrm{d}t^2 + g_{I11} g_{(r)}\,\mathrm{d}r^2 + g_{I11} h_{(r)}\,\mathrm{d}\theta^2 + g_{I11} h_{(r)} \sin^2\theta\,\mathrm{d}\phi^2\,, \tag{25}$$

with the following conditions [15] (p. 2):

$$g_{(r)} = \frac{\mu}{f_{(r)}\left(1-f_{(r)}\right)^4}\left(\frac{\mathrm{d}f}{\mathrm{d}r}\right)^2 \ ;\ h_{(r)} = \frac{\mu}{\left(1-f_{(r)}\right)^2}\,, \tag{26}$$

where μ is an arbitrary constant and f is an arbitrary function of r (not constant).

3.2. The 3rd Generalized Schwarzschild Metric, Relativistic potential and Field strength

We define the *3rd Generalized Schwarzschild Relativistic Potential* (3GSRP) around a center of gravity with mass M as

$$\Phi = \frac{c^2}{2\xi_I^{\,2}} \ln\left(1 - a_{(r)} \frac{\xi_I^{\,2} r_S}{r}\right), \tag{27}$$

where $a_{(r)}$ is unspecified function, in accordance with any TPs. The 3GSP is connected with Φ, via the formula

$$\ln f_{(r)} = \frac{2}{c_I^{\,2}} \Phi = \frac{2\xi_I^{\,2}}{c^2} \Phi, \tag{28}$$

which emerges

$$f_{(r)} = 1 - a_{(r)} \frac{\xi_I^{\,2} r_S}{r}. \tag{29}$$

After replacing the above equation and $\mu = \xi_I^4 r_S^2$ to (26), we also have

$$g_{(r)} = \frac{\left(r\dfrac{\mathrm{d}a}{\mathrm{d}r} - a_{(r)}\right)^2}{a_{(r)}^{\,4}\left(1 - a_{(r)} \dfrac{\xi_I^{\,2} r_S}{r}\right)} \;;\; h_{(r)} = \frac{r^2}{a_{(r)}^{\,2}}. \tag{30}$$

So, we obtain the *3rd Generalized Schwarzschild Metric* (3GSM)

$$\mathrm{d}S^2 = g_{I00}\left(1 - a_{(r)} \frac{\xi_I^{\,2} r_S}{r}\right) c^2 \mathrm{d}t^2 + \frac{g_{I11}\left(r\dfrac{\mathrm{d}a}{\mathrm{d}r} - a_{(r)}\right)^2}{a_{(r)}^{\,4}\left(1 - a_{(r)} \dfrac{\xi_I^{\,2} r_S}{r}\right)} \mathrm{d}r^2 + \frac{g_{I11} r^2}{a_{(r)}^{\,2}} \mathrm{d}\theta^2 + \frac{g_{I11} r^2}{a_{(r)}^{\,2}} \sin^2\theta \,\mathrm{d}\phi^2, \tag{31}$$

with spatial part

$$\mathrm{d}l^2 = \frac{g_{I11}\left(r\dfrac{\mathrm{d}a}{\mathrm{d}r} - a_{(r)}\right)^2}{a_{(r)}^{\,4}\left(1 - a_{(r)} \dfrac{\xi_I^{\,2} r_S}{r}\right)} \mathrm{d}r^2 + \frac{g_{I11} r^2}{a_{(r)}^{\,2}} \mathrm{d}\theta^2 + \frac{g_{I11} r^2}{a_{(r)}^{\,2}} \sin^2\theta \,\mathrm{d}\phi^2, \tag{32}$$

where $a_{(r)}$ is an arbitrary function of the distance r (or constant). Now, we can calculate the following quantity [which *usually is considered as the radial field strength in textbooks* [9] (p. 230)], by defining

$$\vec{g} = -\sqrt{g_{I11}} \nabla\Phi = -\sqrt{g_{I11}} \frac{d\Phi}{dl} \hat{r} = -\sqrt{g_{I11}} \frac{d\Phi}{dr} \frac{dr}{dl} \hat{r}, \tag{33}$$

and

$$g = \sqrt{g_{I11}} \frac{d\Phi}{dr} \frac{dr}{dl}. \tag{34}$$

The positive value ($g>0$) means gravity, while negative value ($g<0$) means antigravity. So, it is

$$g = \frac{GM}{r^2}\left(1 - a_{(r)} \frac{\xi_I^{\,2} r_S}{r}\right)^{-\frac{1}{2}} a_{(r)}^{\,2} > 0. \tag{35}$$

We also prefer $a_{(r)}>0$, in order to ensure *Gravitational Red Shift* (GRS). We shall see that the *field strength on moving particle* is given by a different formula, which also contains the velocity of the particle and also the field strength of unmoved particle is given by (35), if only $a_{(r)}=1$.

3.3. The 1st Generalized Schwarzschild Metric, Relativistic potential, Field strength, Lagrangian, Geodesics, Equations of motion, Precession of planets' orbits and Deflection of light

In case that $a_{(r)}$=1, (27) gives the 1st Generalized Schwarzschild Relativistic Potential (1GSRP) [12] (p. 11):

$$\Phi = \frac{c^2}{2\xi_I^{\,2}} \ln\left(1 - \frac{\xi_I^{\,2} r_S}{r}\right) = -\frac{c^2}{2}\frac{r_S}{r} + \ldots = -\frac{GM}{r} + \ldots \tag{36}$$

Thus, (31) emerges the 1st Generalized Schwarzschild metric (1GSM):

$$dS^2 = g_{I00}\left(1 - \xi_I^{\,2}\frac{r_S}{r}\right)c^2 dt^2 + \frac{g_{I11}}{1 - \xi_I^{\,2}\frac{r_S}{r}} dr^2 + g_{I11} r^2 d\theta^2 + g_{I11} r^2 \sin^2\theta\, d\phi^2 . \tag{37}$$

Besides, (35) gives

$$\vec{g}_{(r)} = -\frac{GM}{r^2}\left(1 - \xi_I^{\,2}\frac{r_s}{r}\right)^{-\frac{1}{2}} \hat{r} , \tag{38}$$

which is the *1st Generalized Schwarzschild field strength* (g) for *unmoved particle*.

The usual definition of *Lagrangian* of gravitational system (M, m) [9] (p. 205)

$$L = m\dot{x}^{\mu} g_{\mu\nu} \dot{x}^{\nu} , \tag{39}$$

for orbit on the 'plane' θ=π/2, emerges the *1st Generalized Schwarzschild Lagrangian* (1GSL) [11] (p. 15):

$$L = mg_{I00}\left[\left(1 - \xi_I^{\,2}\frac{r_S}{r}\right)c^2 \dot{t}^2 - \frac{\xi_I^{\,2}}{1 - \xi_I^{\,2}\frac{r_S}{r}} \dot{r}^2 - \xi_I^{\,2} r^2 \dot{\phi}^2\right] \;\; ; \;\; \dot{} = \frac{d}{d\tau} . \tag{40}$$

The well-known *Euler-Lagrange equations*

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^{\mu}}\right) - \frac{\partial L}{\partial x^{\mu}} = 0 \;\; ; \;\; \mu\text{=0, 1, 2} \tag{41}$$

give us the *equations of motion*:

$$E_{GR} = \left(1 - \xi_I^{\,2}\frac{r_S}{r}\right)\frac{mc^2}{\xi_I^{\,2}}\dot{t} \;\; ; \;\; \dot{} = \frac{d}{d\tau} \;\; ; \tag{42}$$

$$\frac{d}{d\tau}\left(\frac{2\dot{r}}{1 - \xi_I^{\,2}\frac{r_S}{r}}\right) - \left[-\frac{r_S}{r^2}c^2 \dot{t}^2 + \frac{\partial}{\partial r}\left(\frac{1}{1 - \xi_I^{\,2}\frac{r_S}{r}}\right)\dot{r}^2 + 2r\dot{\phi}^2\right] = 0 \;\; ; \tag{43}$$

$$J_{GR} = mh_{GR} = mr^2\dot{\phi} \;\; ; \;\; \dot{} = \frac{d}{d\tau} , \tag{44}$$

where the *integrals of motion* are: the total GR-energy (E_{GR}) and the total GR-angular momentum (J_{GR}) of the system (h_{GR}=J_{GR}/m is the *GR-angular momentum per mass unit*). The solutions of the above *equations of motion* satisfy the condition

$$L = mg_{I00}\, c^2 . \tag{45}$$

So, they can also be used for the practical determination of *geodesics* [9] (p. 205). It is noted that

$$h_{GR} = r^2\dot{\phi} = r^2 \frac{d\phi}{d\tau}\frac{dt}{dt} = r^2 \frac{d\phi}{dt}\frac{dt}{d\tau} = h_N \dot{t} \;\; ; \;\; h_N = r^2\frac{d\phi}{dt} \;\; ; \;\; \dot{} = \frac{d}{d\tau} \tag{46}$$

where h_N=J_N/m is the corresponding *Newtonian-angular momentum per mass unit*. Besides (43) is also written as

$$2\ddot{r}\left(1-\xi_{\rm I}^{2}\frac{r_{\rm S}}{r}\right)-\xi_{\rm I}^{2}\frac{r_{\rm S}}{r^{2}}\dot{r}^{2}+\left(1-\xi_{\rm I}^{2}\frac{r_{\rm S}}{r}\right)^{2}\frac{r_{\rm S}}{r^{2}}{\rm c}^{2}\dot{t}^{2}-2\left(1-\xi_{\rm I}^{2}\frac{r_{\rm S}}{r}\right)^{2}r\dot{\phi}^{2}=0\;\;;\;\;\dot{}=\frac{\rm d}{{\rm d}\tau}, \tag{47}$$

or equivalently,

$$2\frac{{\rm d}^{2}r}{{\rm d}t^{2}}\left(1-\xi_{\rm I}^{2}\frac{r_{\rm S}}{r}\right)-\xi_{\rm I}^{2}\frac{r_{\rm S}}{r^{2}}\frac{\dot{r}^{2}}{\dot{t}^{2}}+\left(1-\xi_{\rm I}^{2}\frac{r_{\rm S}}{r}\right)^{2}\frac{r_{\rm S}}{r^{2}}{\rm c}^{2}-2\left(1-\xi_{\rm I}^{2}\frac{r_{\rm S}}{r}\right)^{2}r\frac{\dot{\phi}^{2}}{\dot{t}^{2}}=0\;\;;\;\;\dot{}=\frac{\rm d}{{\rm d}\tau}. \tag{48}$$

Thus, we obtain

$$2\ddot{r}\left(1-\xi_{\rm I}^{2}\frac{r_{\rm S}}{r}\right)-\xi_{\rm I}^{2}\frac{r_{\rm S}}{r^{2}}\dot{r}^{2}+\left(1-\xi_{\rm I}^{2}\frac{r_{\rm S}}{r}\right)^{2}\frac{r_{\rm S}}{r^{2}}{\rm c}^{2}-2\left(1-\xi_{\rm I}^{2}\frac{r_{\rm S}}{r}\right)^{2}r\dot{\phi}^{2}=0\;\;;\;\;\dot{}=\frac{\rm d}{{\rm d}t}. \tag{49}$$

Now, we study the motion of particle *P* around the center of gravity of mass *M*. In case that $\dot{r}=0$, we have motion at the *perihelion* and/or *aphelion* or Uniform Circular Motion (UCM). Thus,

$$2\ddot{r}+\left(1-\xi_{\rm I}^{2}\frac{r_{\rm S}}{r}\right)\frac{r_{\rm S}}{r^{2}}{\rm c}^{2}-2\left(1-\xi_{\rm I}^{2}\frac{r_{\rm S}}{r}\right)r\dot{\phi}^{2}=0\;\;;\;\;\dot{}=\frac{\rm d}{{\rm d}t}\;\;;\;\;r\to R. \tag{50}$$

The UCM (with r=R=const) has the extra condition $\ddot{r}=0$. Thus, the above eqn gives the same *angular velocity* and the same *centripetal acceleration* for any TPs

$$\dot{\phi}=\frac{{\rm d}\phi}{{\rm d}t}=\omega=\sqrt{\frac{GM}{R^{3}}}\;\;;\;\;a=\frac{\upsilon^{2}}{R}=\omega^{2}R=\frac{GM}{R^{2}}=g_{\rm N}. \tag{51}$$

Let us remind that a solution of the system of (N −1) Euler–Lagrange equations automatically satisfies the Nth equation, except for the solution $x_{\rm N}$ = const [9] (p. 213). Since we have already dealt with r = const, we can now forget about eqn (43). Instead, we use Lagrangian (40) combined with (45) [9] (p. 239). Thus, we obtain

$$\dot{r}^{2}=-\frac{{\rm c}^{2}}{\xi_{\rm I}^{2}}\left(1-\xi_{\rm I}^{2}\frac{r_{\rm S}}{r}\right)+\frac{{\rm c}^{2}}{\xi_{\rm I}^{2}}\left(1-\xi_{\rm I}^{2}\frac{r_{\rm S}}{r}\right)^{2}\dot{t}^{2}-\left(1-\xi_{\rm I}^{2}\frac{r_{\rm S}}{r}\right)r^{2}\dot{\phi}^{2}\;\;;\;\;\dot{}=\frac{\rm d}{{\rm d}\tau}, \tag{52}$$

or equivalently,

$$\left(\frac{{\rm d}r}{{\rm d}t}\right)^{2}=-\frac{{\rm c}^{2}}{\xi_{\rm I}^{2}}\left(1-\xi_{\rm I}^{2}\frac{r_{\rm S}}{r}\right)\frac{1}{\dot{t}^{2}}+\frac{{\rm c}^{2}}{\xi_{\rm I}^{2}}\left(1-\xi_{\rm I}^{2}\frac{r_{\rm S}}{r}\right)^{2}-\left(1-\xi_{\rm I}^{2}\frac{r_{\rm S}}{r}\right)\frac{r^{2}\dot{\phi}^{2}}{\dot{t}^{2}}\;\;;\;\;\dot{}=\frac{\rm d}{{\rm d}\tau}. \tag{53}$$

The above eqns by using (42) and (44) become, respectively:

$$\dot{r}^{2}=-\frac{{\rm c}^{2}}{\xi_{\rm I}^{2}}\left(1-\xi_{\rm I}^{2}\frac{r_{\rm S}}{r}\right)+\frac{\xi_{\rm I}^{2}E_{\rm GR}^{2}}{m^{2}{\rm c}^{2}}-\left(1-\xi_{\rm I}^{2}\frac{r_{\rm S}}{r}\right)\frac{h_{\rm GR}^{2}}{r^{2}}\;\;;\;\;\dot{}=\frac{\rm d}{{\rm d}\tau}, \tag{54}$$

$$\left(\frac{{\rm d}r}{{\rm d}t}\right)^{2}=\frac{{\rm c}^{2}}{\xi_{\rm I}^{2}}\left(1-\xi_{\rm I}^{2}\frac{r_{\rm S}}{r}\right)^{2}\left[1-\frac{m^{2}{\rm c}^{4}}{\xi_{\rm I}^{4}E_{\rm GR}^{2}}\left(1-\xi_{\rm I}^{2}\frac{r_{\rm S}}{r}\right)\left(1+\frac{\xi_{\rm I}^{2}}{{\rm c}^{2}}\frac{h_{\rm GR}^{2}}{r^{2}}\right)\right]. \tag{55}$$

Accordingly to the *mainstream approach* in textbooks, the further study is based on the *superposition principle*. This emerges the relation of time to proper time (GR-time dilation). Replacing this to (42), they obtain the final formula of the *total GR-energy*. Finally, the *generalized potential energy* is calculated, by reducing the kinetic energy (which is considered equal to this of SR) from the total relativistic energy. In this paper, we follow a similar approach with *no-superposition principle*. Thus, we obtain simple *central potential* which describes GEE in case of unmoved particle, while moving particle has also GME. So, we conclude that even SM is a static and stationary metric of non-rotating mass, there exists gravitomagnetism, because the GSR-gravitational potential and the GSR-gravitational force depend on the velocity of the particle. This is not obvious in case of GR, because the motion on the curved geodesics is considered as inertial motion. But, a space endowed with steady metric (like Minkowski spacetime) makes clearest the above consideration.

The isometry of spacetime relieves us the *relation of time to proper time* [11] (p. 16):

$$\mathrm{d}S^2 = g_{\mathrm{I}00}\,\mathrm{c}^2\mathrm{d}\tau^2 = g_{\mathrm{I}00}\left(1-\xi_{\mathrm{I}}^{\,2}\frac{r_{\mathrm{S}}}{r}\right)\mathrm{c}^2\mathrm{d}t^2 + \frac{g_{\mathrm{I}11}}{1-\xi_{\mathrm{I}}^{\,2}\frac{r_{\mathrm{S}}}{r}}\mathrm{d}r^2 + g_{\mathrm{I}11}r^2\,\mathrm{d}\theta^2 + g_{\mathrm{I}11}r^2\sin^2\theta\,\mathrm{d}\phi^2 \;\; ; \;\; \theta=\frac{\pi}{2}, \quad (56)$$

or equivalently,

$$\left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^2 = 1-\frac{\xi_{\mathrm{I}}^{\,2}r_{\mathrm{S}}}{r}-\frac{\xi_{\mathrm{I}}^{\,2}}{1-\xi_{\mathrm{I}}^{\,2}\frac{r_{\mathrm{S}}}{r}}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2\frac{1}{\mathrm{c}^2}-\xi_{\mathrm{I}}^{\,2}r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2\frac{1}{\mathrm{c}^2} \;\; ; \;\; \theta=\frac{\pi}{2}. \quad (57)$$

This gives the GR-time dilation

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \left[1-\xi_{\mathrm{I}}^{\,2}\left(\frac{r_{\mathrm{S}}}{r}+\frac{1}{1-\xi_{\mathrm{I}}^{\,2}\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^2}{\mathrm{c}^2}+\frac{r^2\dot{\phi}^2}{\mathrm{c}^2}\right)\right]^{-\frac{1}{2}} \geq 1 \;\; ; \;\; \dot{} = \frac{\mathrm{d}}{\mathrm{d}t}. \quad (58)$$

Replacing the above equation to (42), we obtain the final formula of the *total GR-energy*

$$E_{\mathrm{GR}} = \frac{1-\xi_{\mathrm{I}}^{\,2}\frac{r_{\mathrm{S}}}{r}}{\sqrt{1-\xi_{\mathrm{I}}^{\,2}\left(\frac{r_{\mathrm{S}}}{r}+\frac{1}{1-\xi_{\mathrm{I}}^{\,2}\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^2}{\mathrm{c}^2}+\frac{r^2\dot{\phi}^2}{\mathrm{c}^2}\right)}}\frac{m\mathrm{c}^2}{\xi_{\mathrm{I}}^{\,2}} \geq 0 \;\; ; \;\; \dot{} = \frac{\mathrm{d}}{\mathrm{d}t}. \quad (59)$$

We observe the different contribution of the radial and orbital velocity to the total energy! Now, we demand zero kinetic energy (K=0), in case that the particle is static $\left(\vec{\beta}_P = 0\right)$. Thus, $E_{GR\left(\vec{\beta}_P=0\right)} = E_{\mathrm{rest}} + U_{(r)}$, where $U_{(r)}$ is the *potential energy of unmoved particle* . Replacing (24iii) and (59) to the above equation, we have

$$U_{(r)} = \left(\sqrt{1-\xi_{\mathrm{I}}^{\,2}\frac{r_{\mathrm{S}}}{r}}-1\right)\frac{m\mathrm{c}^2}{\xi_{\mathrm{I}}^{\,2}} \leq 0 \;\; ; \quad (60)$$

$$V_{(r)} = \left(\sqrt{1-\xi_{\mathrm{I}}^{\,2}\frac{r_{\mathrm{S}}}{r}}-1\right)\frac{\mathrm{c}^2}{\xi_{\mathrm{I}}^{\,2}} \leq 0, \quad (61)$$

where V is the 1st Generalized Schwarzschild Potential (1GSP) of unmoved particle(where (2) has been used, too). This is a central potential with field strength:

$$\vec{g}_{(r)} = -\frac{dV}{dr}\hat{r} = -\frac{GM}{r^2}\left(1-\xi_I^{\,2}\frac{r_s}{r}\right)^{-\frac{1}{2}}\hat{r}. \quad (62)$$

We observe that the result is the same as (38). Besides, the *mechanic energy* E_{m}=E_{GR}-E_{rest}=K+U_{g} is

$$E_{\mathrm{m}} = \left(\frac{1-\xi_{\mathrm{I}}^{\,2}\frac{r_{\mathrm{S}}}{r}}{\sqrt{1-\xi_{\mathrm{I}}^{\,2}\left(\frac{r_{\mathrm{S}}}{r}+\frac{1}{1-\xi_{\mathrm{I}}^{\,2}\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^2}{\mathrm{c}^2}+\frac{r^2\dot{\phi}^2}{\mathrm{c}^2}\right)}}-1\right)\frac{m\mathrm{c}^2}{\xi_{\mathrm{I}}^{\,2}} \;\; ; \;\; \dot{} = \frac{\mathrm{d}}{\mathrm{d}t}. \quad (63)$$

The *generalized Potential energy* is defined as U_{g}=E_{GR}-E_{rest}-K=E_{GR}-E. The consideration of the Generalized relativistic energy as equal to this of SR (24ii), gives

$$U_g = \left(\left(1-\xi_{\mathrm{I}}^2 \frac{r_{\mathrm{S}}}{r}\right) \left[1-\xi_{\mathrm{I}}^2 \left(\frac{r_{\mathrm{S}}}{r} + \frac{1}{1-\xi_{\mathrm{I}}^2 \frac{r_{\mathrm{S}}}{r}} \frac{\dot{r}^2}{\mathrm{c}^2} + \frac{r^2 \dot{\phi}^2}{\mathrm{c}^2} \right)\right]^{-\frac{1}{2}} - \gamma_{\left(\xi_{\mathrm{I}} \vec{\beta}_P\right)} \right) \frac{m\mathrm{c}^2}{\xi_{\mathrm{I}}^2} . \quad (64)$$

We also observe that if $\vec{\beta}_P \to 0$, the above equation becomes equal to (61). Finally, the replacement of (58) to (46i) gives

$$h_{\mathrm{GR}} = h_{\mathrm{N}} \frac{\mathrm{d}t}{\mathrm{d}\tau} = h_{\mathrm{N}} \left[1-\xi_{\mathrm{I}}^2 \left(\frac{r_{\mathrm{S}}}{r} + \frac{1}{1-\xi_{\mathrm{I}}^2 \frac{r_{\mathrm{S}}}{r}} \frac{\dot{r}^2}{\mathrm{c}^2} + \frac{r^2 \dot{\phi}^2}{\mathrm{c}^2} \right)\right]^{-\frac{1}{2}} \geq h_{\mathrm{N}} \;\; ; \;\; h_{\mathrm{N}} = r^2 \frac{\mathrm{d}\phi}{\mathrm{d}t} \;\; ; \;\; {}^{\cdot} = \frac{\mathrm{d}}{\mathrm{d}t} . \quad (65)$$

Besides, for a *particle* or *planet* at the *perihelion* and/or *aphelion*, or in UCM (where r=R; $\dot{r}=0$), the above equation becomes

$$h_{\mathrm{GR}} = h_{\mathrm{N}} \left[1-\xi_{\mathrm{I}}^2 \left(\frac{r_{\mathrm{S}}}{R} + \frac{R^2 \dot{\phi}_{(R)}^{\;2}}{\mathrm{c}^2} \right)\right]^{-\frac{1}{2}} \geq h_{\mathrm{N}} \;\; ; \;\; h_{\mathrm{N}} = r^2 \frac{\mathrm{d}\phi}{\mathrm{d}t} \;\; ; \;\; {}^{\cdot} = \frac{\mathrm{d}}{\mathrm{d}t} . \quad (66)$$

Morever, for a *particle* or *planet* in UCM, we obtain

$$h = h_{\mathrm{N}} \left[1-\xi_{\mathrm{I}}^2 \left(\frac{r_{\mathrm{S}}}{R} + \frac{\mathrm{G}M}{\mathrm{c}^2 R} \right)\right]^{-\frac{1}{2}} = h_{\mathrm{N}} \left[1-\xi_{\mathrm{I}}^2 \frac{3r_{\mathrm{S}}}{2R}\right]^{-\frac{1}{2}} \geq h_{\mathrm{N}} \;\; ; \;\; h_{\mathrm{N}} = r^2 \frac{\mathrm{d}\phi}{\mathrm{d}t} \;\; ; \;\; {}^{\cdot} = \frac{\mathrm{d}}{\mathrm{d}t} , \quad (67)$$

where (51i) has been also used.

In case of *Generalized photon*, it is m=0 and the velocity at infinite distance from the center of gravity is $c_{\mathrm{I}} = \mathrm{c}/\xi_{\mathrm{I}}$. But, the total angular momentum of the system $J_{\mathrm{GR}} = mr^2\dot{\phi} = mh_{\mathrm{GR}}$ is generally finite $\neq 0$ (except for radial motion). Thus, must be

$$h_{\mathrm{GR}} = +\infty , \quad (68)$$

and (65) demands

$$1-\xi_{\mathrm{I}}^2 \left(\frac{r_{\mathrm{S}}}{r} + \frac{1}{1-\xi_{\mathrm{I}}^2 \frac{r_{\mathrm{S}}}{r}} \frac{\dot{r}^2}{\mathrm{c}^2} + \frac{r^2 \dot{\phi}^2}{\mathrm{c}^2} \right) = 0 \;\; ; \;\; {}^{\cdot} = \frac{\mathrm{d}}{\mathrm{d}t} . \quad (69)$$

This can also be concluded, by using the energy formula (59) and demanding E$\neq$0. So, eqn (69) correlates the radial and the angular velocity of *Generalized photon*. Besides, the velocity of the Generalized photon (c_P) at random position is given by the formula

$$c_P^{\;2} = \dot{r}^2 + r^2 \dot{\phi}^2 . \quad (70)$$

Thus, we have

$$1-\xi_{\mathrm{I}}^2 \left(\frac{r_{\mathrm{S}}}{r} + \frac{1}{1-\xi_{\mathrm{I}}^2 \frac{r_{\mathrm{S}}}{r}} \frac{\dot{r}^2}{\mathrm{c}^2} + \frac{c_P^{\;2} - \dot{r}^2}{\mathrm{c}^2} \right) = 0 . \quad (71)$$

In case of *Generalized photon in radial motion*, the above eqn gives

$$c_p = \left(1-\xi_{\mathrm{I}}^2 \frac{r_{\mathrm{S}}}{r}\right) \frac{\mathrm{c}}{\xi_{\mathrm{I}}} \;\; ; \;\; \gamma_{\left(\xi_{\mathrm{I}} \beta_p\right)} = \frac{1}{\xi_{\mathrm{I}} \sqrt{\frac{r_{\mathrm{S}}}{r} \left(2-\xi_{\mathrm{I}}^2 \frac{r_{\mathrm{S}}}{r}\right)}} . \quad (72)$$

We observe that the photon is unmoved on the 1st *generalized Schwarzschild radius* ($r_{SI}=\xi_I^2 r_S$) as well as *Lorentz γ-factor* is infinite only at infinite distance (except for NPs where it is infinite everywhere). Besides, eqn (71) is transformed to

$$1-\xi_I^2\left(\frac{r_S}{r}+\frac{\xi_I^2\frac{r_S}{r}}{1-\xi_I^2\frac{r_S}{r}}\frac{\dot{r}^2}{c^2}+\frac{c_P^2}{c^2}\right)=0\,. \quad (73)$$

In case of UCM, or motion at the *perihelion/aphelion*, where $r=R$; $\dot{r}=0$, the velocity of the Generalized photon is denoted as c_R and the (71) becomes

$$1-\xi_I^2\left(\frac{r_S}{R}+\frac{c_R^2}{c^2}\right)=0\,, \quad (74)$$

or equivalently,

$$c_R=c\sqrt{\frac{1}{\xi_I^2}-\frac{r_S}{R}}\,. \quad (75)$$

Besides the combination of (67) with (68) gives the radius of UCM for a photon

$$R=\frac{3}{2}\xi_I^2 r_S\,. \quad (76)$$

The above result has accordance with ERT, by replacing $\xi_I=1$ [9] (p. 239). Moreover, the replacement of (76) to (75) emerges

$$c_R=\frac{c}{\xi_I}\sqrt{1-\frac{2}{3}}=\frac{1}{\sqrt{3}}\frac{c}{\xi_I}\,. \quad (77)$$

The *orbit of motion* comes with similar way to the original *Schwarzschild space* [9] (pp. 238-45). Thus, the *exact differential equation of motion* is [11] (p. 15):

$$\frac{d^2u}{d\phi^2}+u=\frac{GM}{h_{GR}^2}+3\xi_I^2\frac{GM}{c^2}u^2 \;\;;\;\; u=\frac{1}{r} \;\;;\;\; h_{GR}=r^2\dot{\phi} \;\;;\;\; \dot{}=\frac{d}{d\tau}\,. \quad (78)$$

This reminds us the orbit of *conic section* with differential eqn and solution, respectively:

$$\frac{d^2u}{d\phi^2}+u=\frac{1}{R(1+e)}=\frac{1}{a(1-e^2)}=\frac{GM}{h_{GR}^2} \;\;;\;\; u=\frac{1}{r}=\frac{1+e\sin\phi}{R(1+e)}=\frac{1+e\sin\phi}{a(1-e^2)}=\frac{GM}{h_{GR}^2}(1+e\sin\phi), \quad (79)$$

where R is the (minimum) distance of the perihelion / pericenter from the center of gravity, e is the eccentricity of the *conic section*, α is the semimajor axis in case of *ellipse* and angle ϕ is measured from axis which passes the center of gravity and it is perpendicular to the radius of perihelion R as it is shown in Figure 1a. It noted that

$$R=a(1-e) \;\;;\;\; \frac{h_{GR}^2}{GM}=R(1+e)=a(1-e^2)\,. \quad (80)$$

In case of *small velocities* relative to c_I (υ<<c/ξ_I, or equivalently r>> $\xi_I^2 r_S$), we replace the solution of the *simplified differential equation* (79) to the last term of the exact differential equation of motion (46). Thus, we have the *approximate differential equation of motion* (which only approximately validates UCM):

$$\frac{d^2u}{d\phi^2}+u=\frac{GM}{h_{GR}^2}+3\xi_I^2\frac{G^3M^3}{c^2h_{GR}^4}(1+e\sin\phi)^2 \;;\; u=\frac{1}{r} \;;\; h_{GR}=r^2\dot{\phi} \;;\; \dot{}=\frac{d}{d\tau}\,, \quad (81)$$

with *exact solution*:

$$u=\frac{GM}{h_{GR}^2}\left(1+e\sin\phi+3\xi_I^2\frac{G^2M^2}{c^2h_{GR}^2}e\left(\frac{\pi}{2}-\phi\right)\cos\phi\right) \;;\; h_{GR}=r^2\dot{\phi} \;;\; \dot{}=\frac{d}{dt} \;;\; \frac{GM}{h_{GR}^2}=\frac{1}{R(1+e)}=\frac{1}{a(1-e^2)}\,. \quad (82)$$

The *approximate solution* is obtained as following. We rewrite (82i) as

$$u = \frac{\mathrm{G}M}{h_{\mathrm{GR}}^{2}}\left[1 + e\left(\sin\phi + 3\xi_{\mathrm{I}}^{2}\frac{\mathrm{G}^{2}M^{2}}{\mathrm{c}^{2}h_{\mathrm{GR}}^{2}}\left(\frac{\pi}{2} - \phi\right)\cos\phi\right)\right]. \quad (83)$$

and we remember the identity

$$\sin(\phi + d) = \sin\phi\cos d + \cos\phi\sin d\,. \quad (84)$$

These are associated, by using

$$d = \frac{3\xi_{\mathrm{I}}^{2}\mathrm{G}^{2}M^{2}}{\mathrm{c}^{2}h_{\mathrm{GR}}^{2}}\left(\frac{\pi}{2} - \phi\right) = \frac{3\xi_{\mathrm{I}}^{2}\mathrm{G}M}{\mathrm{c}^{2}R(1+e)}\left(\frac{\pi}{2} - \phi\right) = \frac{3\xi_{\mathrm{I}}^{2}\mathrm{G}M}{\mathrm{c}^{2}a(1-e^{2})}\left(\frac{\pi}{2} - \phi\right) << 1 \;\; ; \;\; \cos d \approx 1 \;\; ; \;\; \sin d \approx d\,. \quad (85)$$

Thus, we obtain

$$u = \frac{\mathrm{G}M}{h_{\mathrm{GR}}^{2}}\left[1 + e\sin\left(\phi + \frac{3\xi_{\mathrm{I}}^{2}\mathrm{G}^{2}M^{2}}{\mathrm{c}^{2}h_{\mathrm{GR}}^{2}}\left(\frac{\pi}{2} - \phi\right)\right)\right] = \frac{\mathrm{G}M}{h_{\mathrm{GR}}^{2}}\left[1 + e\sin\left(\phi\left(1 - 3\xi_{\mathrm{I}}^{2}\frac{\mathrm{G}^{2}M^{2}}{\mathrm{c}^{2}h_{\mathrm{GR}}^{2}}\right) + \frac{3\pi\,\xi_{\mathrm{I}}^{2}\mathrm{G}^{2}M^{2}}{2\mathrm{c}^{2}h_{\mathrm{GR}}^{2}}\right)\right]. \quad (86)$$

The above eqn can be written as

$$u \approx \frac{\mathrm{G}M}{h_{\mathrm{GR}}^{2}}\left[1 + e\sin\left(\lambda_{\mathrm{GR}}\phi + (1-\lambda_{\mathrm{GR}})\frac{\pi}{2}\right)\right] \;;\; \lambda_{\mathrm{GR}} = 1 - \frac{3\xi_{\mathrm{I}}^{2}\mathrm{G}^{2}M^{2}}{\mathrm{c}^{2}h_{\mathrm{GR}}^{2}} = 1 - \frac{3\xi_{\mathrm{I}}^{2}\mathrm{G}M}{\mathrm{c}^{2}R(1+e)} = 1 - \frac{3\xi_{\mathrm{I}}^{2}\mathrm{G}M}{\mathrm{c}^{2}a(1-e^{2})}, \quad (87)$$

or equivalently,

$$u = \frac{1}{r} \approx \frac{\mathrm{G}M}{h_{\mathrm{GR}}^{2}}\left[1 + e\sin\left(\lambda_{\mathrm{GR}}\left(\phi - \frac{\pi}{2}\right) + \frac{\pi}{2}\right)\right]. \quad (88)$$

Thus, we also obtain

$$u = \frac{1}{r} \approx \frac{\mathrm{G}M}{h_{\mathrm{GR}}^{2}}\left[1 + e\cos\left(\lambda_{\mathrm{GR}}\left(\phi - \frac{\pi}{2}\right)\right)\right] = \frac{1}{R(1+e)}\left[1 + e\cos\left(\lambda_{\mathrm{GR}}\left(\phi - \frac{\pi}{2}\right)\right)\right] = \frac{1}{a(1-e^{2})}\left[1 + e\cos\left(\lambda_{\mathrm{GR}}\left(\phi - \frac{\pi}{2}\right)\right)\right], \quad (89)$$

where

$$\lambda_{\mathrm{GR}} = 1 - \frac{3\xi_{\mathrm{I}}^{2}\mathrm{G}^{2}M^{2}}{\mathrm{c}^{2}h_{\mathrm{GR}}^{2}} = 1 - \frac{3\xi_{\mathrm{I}}^{2}\mathrm{G}M}{\mathrm{c}^{2}R(1+e)} = 1 - \frac{3\xi_{\mathrm{I}}^{2}\mathrm{G}M}{\mathrm{c}^{2}a(1-e^{2})} \quad (90)$$

with condition

$$0 < \frac{6\pi\,\xi_{\mathrm{I}}^{2}\mathrm{G}^{2}M^{2}}{\mathrm{c}^{2}h_{\mathrm{GR}}^{2}} = \frac{6\pi\,\xi_{\mathrm{I}}^{2}\mathrm{G}M}{\mathrm{c}^{2}R(1+e)} = \frac{6\pi\,\xi_{\mathrm{I}}^{2}\mathrm{G}M}{\mathrm{c}^{2}a(1-e^{2})} << 1\,. \quad (91)$$

For every perihelion, we have

$$\cos\left(\lambda_{\mathrm{GR}}\left(\phi - \frac{\pi}{2}\right)\right) = 1\,. \quad (92)$$

The first, the second and the *n*-th perihelion correspond to $\phi = \frac{\pi}{2}$, $\phi = \frac{2\pi}{\lambda_{\mathrm{GR}}} + \frac{\pi}{2}$ and $\phi = \frac{2n\pi}{\lambda_{\mathrm{GR}}} + \frac{\pi}{2}$, respectively (Figure 1a). Hence, the orbit can be regarded as an *ellipse* that rotates ('precesses') about one of its foci by an amount

$$\Delta = \frac{2\pi}{\lambda_{\mathrm{GR}}} - 2\pi = \left(\frac{1}{\lambda_{\mathrm{GR}}} - 1\right)2\pi \approx 2\pi\,\lambda_{\mathrm{GR}} = \frac{6\pi\,\xi_{\mathrm{I}}^{2}\mathrm{G}^{2}M^{2}}{\mathrm{c}^{2}h^{2}} = \frac{6\pi\,\xi_{\mathrm{I}}^{2}\mathrm{G}M}{R(1+e)\mathrm{c}^{2}} = \frac{6\pi\,\xi_{\mathrm{I}}^{2}\mathrm{G}M}{a(1-e^{2})\mathrm{c}^{2}} \quad (93)$$

rad per revolution.

We observe that the above eqn predicts precession of cycle (e=0) for $\xi_{\mathrm{I}} \neq 0$, because it comes form the *approximate solution* (83) of the *approximate differential equation of motion* (81). Finally, the angular velocity of ellipse rotation is given by the formula

$$\Omega\left(\frac{''}{cy}\right)=\Delta\left(\frac{\text{rad}}{\text{rev}}\right)\left(\frac{360^o}{2\pi \quad rad}\right)\left(\frac{3600''}{1^o}\right)\frac{1}{T}\left(\frac{\text{rev}}{\text{day}}\right)\left(\frac{365.242\,\text{day}}{\text{year}}\right)\left(\frac{100\,\text{year}}{\text{cy}}\right), \tag{94}$$

or equivalently

$$\Omega\left(\frac{''}{\text{cy}}\right)=\Delta\left(\frac{\text{rad}}{\text{rev}}\right)\left(\frac{7533657\times 10^3\,''\cdot\text{day}}{\text{rad}\cdot\text{cy}}\right)\frac{1}{T}\left(\frac{\text{rev}}{\text{day}}\right). \tag{95}$$

The corresponding *angular* and *radial velocities* are obtained as following. We initially calculate E_{GR} and h_{GR}, by working at the 1st perihelion, where $\phi=\frac{\pi}{2}$; $R=\alpha$ (1-e); $\dot{r}=0$; $\ddot{\phi}=0$. Thus, (59) and (65) become

$$E_{\text{GR}}=\frac{1-\xi_{\text{I}}^2\frac{r_{\text{S}}}{R}}{\sqrt{1-\xi_{\text{I}}^2\left(\frac{r_{\text{S}}}{R}+\frac{R^2\dot{\phi}_{(R)}^2}{\text{c}^2}\right)}}\frac{m\text{c}^2}{\xi_{\text{I}}^2}\geq 0 \quad ; \quad h_{\text{GR}}=R^2\dot{\phi}_{(R)}\left[1-\xi_{\text{I}}^2\left(\frac{r_{\text{S}}}{R}+\frac{R^2\dot{\phi}_{(R)}^2}{\text{c}^2}\right)\right]^{-\frac{1}{2}} \quad ; \quad \dot{}=\frac{\text{d}}{\text{d}t}. \tag{96}$$

The null *radial velocity* at the perihelion turns (55) to

$$1-\frac{m^2\text{c}^4}{\xi_{\text{I}}^4E_{\text{GR}}^2}\left(1-\xi_{\text{I}}^2\frac{r_{\text{S}}}{R}\right)\left(1+\frac{\xi_{\text{I}}^2}{\text{c}^2}\frac{h_{\text{GR}}^2}{R^2}\right)=0. \tag{97}$$

Moreover, the replacement of (96) to the above eqn gives

$$1-\frac{1-\xi_{\text{I}}^2\left(\frac{r_{\text{S}}}{R}+\frac{R^2\dot{\phi}_{(R)}^2}{\text{c}^2}\right)}{\left(1-\xi_{\text{I}}^2\frac{r_{\text{S}}}{R}\right)}\left(1+\frac{\xi_{\text{I}}^2}{\text{c}^2}\frac{R^2\dot{\phi}_{(R)}^2}{1-\xi_{\text{I}}^2\left(\frac{r_{\text{S}}}{R}+\frac{R^2\dot{\phi}_{(R)}^2}{\text{c}^2}\right)}\right)=0, \tag{98}$$

or equivalently,

$$1-\frac{1}{\left(1-\xi_{\text{I}}^2\frac{r_{\text{S}}}{R}\right)}\left(1-\xi_{\text{I}}^2\left(\frac{r_{\text{S}}}{R}+\frac{R^2\dot{\phi}_{(R)}^2}{\text{c}^2}\right)+\frac{\xi_{\text{I}}^2}{\text{c}^2}R^2\dot{\phi}_{(R)}^2\right)=0, \tag{99}$$

which is valid for any value of $\dot{\phi}_{(R)}$ and R. Thus, (96) combined with (55) gives the *radial velocity* at any position.

Alternatively, we differentiate (89) wrt time and we obtain

$$\frac{\dot{r}}{r^2}=\frac{\text{G}Me}{h_{\text{GR}}^2}\lambda_{\text{GR}}\dot{\phi}\sin\left(\lambda_{\text{GR}}\left(\phi-\frac{\pi}{2}\right)\right)=\frac{e}{R(1+e)}\lambda_{\text{GR}}\dot{\phi}\sin\left(\lambda_{\text{GR}}\left(\phi-\frac{\pi}{2}\right)\right)=\frac{e}{a\left(1-e^2\right)}\lambda_{\text{GR}}\dot{\phi}\sin\left(\lambda_{\text{GR}}\left(\phi-\frac{\pi}{2}\right)\right) \quad ; \tag{100}$$

$$\frac{\ddot{r}r^2-2r\dot{r}^2}{r^4}=\frac{\text{G}Me}{h^2}\lambda_{\text{GR}}\left[\lambda_{\text{GR}}\dot{\phi}^2\cos\left(\lambda_{\text{GR}}\left(\phi-\frac{\pi}{2}\right)\right)+\ddot{\phi}\sin\left(\lambda_{\text{GR}}\left(\phi-\frac{\pi}{2}\right)\right)\right] \quad ; \quad \dot{}=\frac{\text{d}}{\text{d}t}. \tag{101}$$

At the perihelion ($\dot{r}=0$) the above eqn becomes

$$\frac{\ddot{r}_{(R)}}{R^2}=\frac{\text{G}Me}{h_{\text{GR}}^2}\lambda_{\text{GR}}^2\dot{\phi}_{(R)}^2. \tag{102}$$

Besides, the combination of (80ii) with (96ii) gives

$$h_{\mathrm{GR}}{}^{2} = \mathrm{G}\,MR(1+e) = \mathrm{G}\,Ma\left(1-e^{2}\right) = \frac{R^{4}\dot{\phi}_{(R)}{}^{2}}{1-\xi_{\mathrm{I}}{}^{2}\left(\dfrac{r_{\mathrm{S}}}{R}+\dfrac{R^{2}\dot{\phi}_{(R)}{}^{2}}{\mathrm{c}^{2}}\right)} \quad ; \quad {}^{\cdot} = \frac{\mathrm{d}}{\mathrm{d}\,t}\,. \tag{103}$$

The above emerges

$$\left[1-\xi_{\mathrm{I}}{}^{2}\left(\frac{r_{\mathrm{S}}}{R}+\frac{R^{2}\dot{\phi}_{(R)}{}^{2}}{\mathrm{c}^{2}}\right)\right]\mathrm{G}\,MR(1+e) = \left[1-\xi_{\mathrm{I}}{}^{2}\left(\frac{r_{\mathrm{S}}}{R}+\frac{R^{2}\dot{\phi}_{(R)}{}^{2}}{\mathrm{c}^{2}}\right)\right]\mathrm{G}\,Ma\left(1-e^{2}\right) = R^{4}\dot{\phi}_{(R)}{}^{2}\,, \tag{104}$$

or equivalently,

$$\dot{\phi}_{(R)} = \sqrt{\frac{\left(1-\xi_{\mathrm{I}}{}^{2}\dfrac{r_{\mathrm{S}}}{R}\right)\mathrm{G}\,MR(1+e)}{R^{4}+\xi_{\mathrm{I}}{}^{2}\dfrac{r_{\mathrm{S}}R^{3}}{2}(1+e)}} \quad ; \quad R = a(1-e) >> r_{\mathrm{S}}\,. \tag{105}$$

Moreover, the *total GR-energy* can be calculated by replacing (105) to (96i):

$$E_{\mathrm{GR}} = \frac{1-\xi_{\mathrm{I}}{}^{2}\dfrac{r_{\mathrm{S}}}{R}}{\sqrt{1-\xi_{\mathrm{I}}{}^{2}\left(\dfrac{r_{\mathrm{S}}}{R}+\dfrac{r_{\mathrm{S}}}{2R}\dfrac{\left(1-\xi_{\mathrm{I}}{}^{2}\dfrac{r_{\mathrm{S}}}{R}\right)(1+e)}{1+\xi_{\mathrm{I}}{}^{2}\dfrac{r_{\mathrm{S}}}{2R}(1+e)}\right)}}\frac{m\mathrm{c}^{2}}{\xi_{\mathrm{I}}{}^{2}} \geq 0 \quad ; \quad R = a(1-e) >> r_{\mathrm{S}}\,. \tag{106}$$

or equivalently,

$$E_{\mathrm{GR}} = \frac{1-\xi_{\mathrm{I}}{}^{2}\dfrac{r_{\mathrm{S}}}{R}}{\sqrt{1-\xi_{\mathrm{I}}{}^{2}\dfrac{r_{\mathrm{S}}}{2R}\left(\dfrac{2+\xi_{\mathrm{I}}{}^{2}\dfrac{r_{\mathrm{S}}}{R}(1+e)+\left(1-\xi_{\mathrm{I}}{}^{2}\dfrac{r_{\mathrm{S}}}{R}\right)(1+e)}{1+\xi_{\mathrm{I}}{}^{2}\dfrac{r_{\mathrm{S}}}{2R}(1+e)}\right)}}\frac{m\mathrm{c}^{2}}{\xi_{\mathrm{I}}{}^{2}} \geq 0 \quad ; \quad R = a(1-e) >> r_{\mathrm{S}}\,. \tag{107}$$

Thus, we obtain

$$E_{\mathrm{GR}} = \frac{1-\xi_{\mathrm{I}}{}^{2}\dfrac{r_{\mathrm{S}}}{R}}{\sqrt{1-\xi_{\mathrm{I}}{}^{2}\dfrac{r_{\mathrm{S}}}{2R}\left(\dfrac{3+e}{1+\xi_{\mathrm{I}}{}^{2}\dfrac{r_{\mathrm{S}}}{2R}(1+e)}\right)}}\frac{m\mathrm{c}^{2}}{\xi_{\mathrm{I}}{}^{2}} \geq 0 \quad ; \quad R = a(1-e) >> r_{\mathrm{S}}\,. \tag{108}$$

In this way, the mechanic energy (63) becomes

$$E_{\mathrm{m}} = \left[\left(1-\xi_{\mathrm{I}}{}^{2}\frac{r_{\mathrm{S}}}{R}\right)\left(1-\xi_{\mathrm{I}}{}^{2}\frac{r_{\mathrm{S}}}{2R}\left(\frac{3+e}{1+\xi_{\mathrm{I}}{}^{2}\dfrac{r_{\mathrm{S}}}{2R}(1+e)}\right)\right)^{-\frac{1}{2}} - 1\right]\frac{m\mathrm{c}^{2}}{\xi_{\mathrm{I}}{}^{2}} \quad ; \quad R = a(1-e) >> r_{\mathrm{S}}\,. \tag{109}$$

In case of UCM ($e \to 0$, $a \to R$), (105) becomes

$$\dot{\phi}_{(R)} = \sqrt{\frac{\left(1-\xi_{\mathrm{I}}^{2}\dfrac{r_{\mathrm{S}}}{R}\right)\mathrm{G}\,MR}{R^{4}+\xi_{\mathrm{I}}^{2}\dfrac{r_{\mathrm{S}}R^{3}}{2}}}\,, \tag{110}$$

which is slightly smaller than the valid (51i), because it comes form the *approximate solution* (83) of the *approximate differential equation of motion* (81).

Moreover, the *Generalized Gravitational Deflection of light* can be obtained in a similar way to the original SM [9] (pp. 248-49). The combination of (78) with (68) gives

$$\frac{\mathrm{d}^{2}u}{\mathrm{d}\phi^{2}}+u=3\xi_{\mathrm{I}}^{2}\frac{\mathrm{G}M}{\mathrm{c}^{2}}u^{2}\ \ ;\ \ u=\frac{1}{r}\,. \tag{111}$$

In case of *large distances* from the center of gravity relative to r_{S} ($r>>r_{\mathrm{S}}$; $u<< 1/r_{\mathrm{S}}$), we replace the solution (straight line) of the *simplified equation of orbit*

$$\frac{\mathrm{d}^{2}u}{\mathrm{d}\phi^{2}}+u=0\ \ ;\ \ u=\frac{\sin\phi}{R}\,. \tag{112}$$

to the last term of the exact differential equation of orbit (Figure 1b). Thus, we have the *approximate differential equation of orbit*

$$\frac{\mathrm{d}^{2}u}{\mathrm{d}\phi^{2}}+u=3\xi_{\mathrm{I}}^{2}\frac{\mathrm{G}M}{\mathrm{c}^{2}}\frac{\sin^{2}\phi}{R^{2}}=3\xi_{\mathrm{I}}^{2}\frac{\mathrm{G}M}{\mathrm{c}^{2}R^{2}}\left(1-\cos^{2}\phi\right) \tag{113}$$

with solution

$$u=\frac{\sin\phi}{R}+3\xi_{\mathrm{I}}^{2}\frac{\mathrm{G}M}{2\mathrm{c}^{2}R^{2}}\left(1+\frac{1}{3}\cos 2\phi\right). \tag{114}$$

Here, angle ϕ is measured from axis which passes the center of gravity and it is perpendicular to the radius of perihelion R (Figure 1b).
For $r\rightarrow+\infty$:

$$u\rightarrow 0\ \ ;\ \ \phi\rightarrow\phi_{\infty}\ \ ;\ \ \sin\phi_{\infty}\rightarrow\phi_{\infty}\ \ ;\ \ \cos 2\phi_{\infty}\rightarrow 1\,. \tag{115}$$

Thus, it emerges

$$\phi_{\infty}=-2\xi_{\mathrm{I}}^{2}\frac{\mathrm{G}M}{\mathrm{c}^{2}R}\,, \tag{116}$$

which is only the right hand deflection. There also exists the left hand deflection with

$$\phi_{\infty l}=\pi+2\xi_{\mathrm{I}}^{2}\frac{\mathrm{G}M}{\mathrm{c}^{2}R}\,. \tag{117}$$

So, we obtain the magnitude of the total deflection of a ray

$$\Theta=4\xi_{\mathrm{I}}^{2}\frac{\mathrm{G}M}{\mathrm{c}^{2}R}=2\xi_{\mathrm{I}}^{2}\frac{r_{\mathrm{S}}}{R}\,. \tag{118}$$

In case that $\xi_{\mathrm{I}}\rightarrow 0^{+}$ (*Galilean metric*), (58) gives $\dot{t}=1$ for $m\neq 0$, or $\dot{t}=+\infty$ (for generalized photons m=0). Thus, we obtain the *Newtonian results*:

$$\Phi_{\mathrm{N}}=\lim_{\xi_{\mathrm{I}}\to 0}\Phi=\frac{\mathrm{c}^{2}}{2}\lim_{\xi_{\mathrm{I}}\to 0}\left[\frac{1}{\xi_{\mathrm{I}}^{2}}\ln\left(1-\frac{\xi_{\mathrm{I}}^{2}r_{\mathrm{S}}}{r}\right)\right]=\frac{\mathrm{c}^{2}}{4}\lim_{\xi_{\mathrm{I}}\to 0}\left[\frac{1}{\xi_{\mathrm{I}}}\frac{\dfrac{-2\xi_{\mathrm{I}}r_{\mathrm{S}}}{r}}{1-\dfrac{\xi_{\mathrm{I}}^{2}r_{\mathrm{S}}}{r}}\right]=-\frac{\mathrm{c}^{2}}{2}\frac{r_{\mathrm{S}}}{r}=-\frac{\mathrm{G}M}{r}\ \ ; \tag{119}$$

$$\mathrm{d}S_{\mathrm{N}}^{2}=g_{\mathrm{I}00}\lim_{\xi_{\mathrm{I}}\to 0}\left[\left(1-\frac{\xi_{\mathrm{I}}^{2}r_{\mathrm{S}}}{r}\right)\mathrm{c}^{2}\mathrm{d}t^{2}-\frac{\xi_{\mathrm{I}}^{2}}{1-\dfrac{\xi_{\mathrm{I}}^{2}r_{\mathrm{S}}}{r}}\mathrm{d}r^{2}-\xi_{\mathrm{I}}^{2}r^{2}\,\mathrm{d}\theta^{2}-\xi_{\mathrm{I}}^{2}r^{2}\sin^{2}\theta\,\mathrm{d}\phi^{2}\right]\ \ ; \tag{120}$$

$$\vec{g}_{\mathrm{N}(r)}=-\frac{\mathrm{G}M}{r^2}\hat{r} \quad ; \tag{121}$$

$$L_{\mathrm{N}}=mg_{\mathrm{I}00}\lim_{\xi_{\mathrm{I}}\to 0}\left[\left(1-\xi_{\mathrm{I}}{}^2\frac{r_{\mathrm{S}}}{r}\right)\mathrm{c}^2\dot{t}^2-\frac{\xi_{\mathrm{I}}{}^2}{1-\xi_{\mathrm{I}}{}^2\frac{r_{\mathrm{S}}}{r}}\dot{r}^2-\xi_{\mathrm{I}}{}^2r^2\dot{\phi}^2\right] \; ; \; E_{\mathrm{N}}=+\infty \quad ; \tag{122}$$

$$\ddot{r}+\frac{\mathrm{G}M}{r^2}-r\dot{\phi}^2=0 \quad ; J_{\mathrm{N}}=mr^2\dot{\phi} \quad ; \quad h_{\mathrm{N}}=r^2\dot{\phi} \quad ; \quad \dot{}=\frac{\mathrm{d}}{\mathrm{d}t} \quad ; \quad \theta=\frac{\pi}{2}\,. \tag{123}$$

The *Newtonian differential equation* of motion and the corresponding solution are

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2}+u=\frac{\mathrm{G}M}{h_{\mathrm{N}}{}^2} \quad ; \quad u=\frac{\mathrm{G}M}{h_{\mathrm{N}}{}^2}\left(1+e_{\mathrm{N}}\cos\phi\right) \quad ; \quad u=\frac{1}{r} \quad ; \quad h_{\mathrm{N}}=r^2\dot{\phi} \quad ; \quad \dot{}=\frac{\mathrm{d}}{\mathrm{d}t} \quad ; \tag{124}$$

$$e_{\mathrm{N}}=\sqrt{1+\frac{2E_{\mathrm{mN}}h_{\mathrm{N}}{}^2}{\mathrm{G}^2M^2m}} \quad ; \quad E_{\mathrm{mN}}=-\frac{\mathrm{G}Mm}{2a_{\mathrm{N}}} \quad ; \quad \frac{h_{\mathrm{N}}{}^2}{\mathrm{G}M}=R_{\mathrm{N}}\left(1+e_{\mathrm{N}}\right)=a_{\mathrm{N}}\left(1-e_{\mathrm{N}}{}^2\right), \tag{125}$$

where α_{N} is the semimajor axis of *Newtonian ellipse* which do not rotate (Δ_{N}=0). Besides

$$U_{\mathrm{N}}=-\frac{\mathrm{G}Mm}{r} \quad ; \quad V_{\mathrm{N}}=-\frac{\mathrm{G}M}{r} \quad ; \quad K_{\mathrm{N}}=\frac{1}{2}\left|\vec{\beta}_P\right|^2m\mathrm{c}^2=\frac{1}{2}m\left|\vec{\upsilon}\right|^2 \qquad E_{\mathrm{mN}}=\frac{1}{2}m\left|\vec{\upsilon}\right|^2-\frac{\mathrm{G}M}{r}\,. \tag{126}$$

The Generalized Newtonian photon has

$$\dot{t}=+\infty \; ; \; c_R=\mathrm{c}\lim_{\xi_{\mathrm{I}}\to 0}\sqrt{\frac{1}{\xi_{\mathrm{I}}{}^2}-\frac{r_{\mathrm{S}}}{R}}=+\infty \quad ; \quad \Theta_{\mathrm{N}}=0\,. \tag{127}$$

We observe that the speed of light is infinite $\left(c_R=+\infty\right)$ at the perihelion as well as at infinite distance from the center of gravity and also there is no-deflection of light.

In case that ξ_{I}=1, it emerges the well-known results of the original *Schwarzschild metric* in ERT (see e.g. [8] pp. 228-45):

$$\Phi_E=\frac{\mathrm{c}^2}{2}\ln\left(1-\frac{r_{\mathrm{S}}}{r}\right) \quad ; \tag{128}$$

$$\mathrm{d}S_{\mathrm{E}}{}^2=g_{\mathrm{I}00}\left[\left(1-\frac{r_{\mathrm{S}}}{r}\right)\mathrm{c}^2\mathrm{d}t^2-\frac{1}{1-\frac{r_{\mathrm{S}}}{r}}\mathrm{d}r^2-r^2\,\mathrm{d}\theta^2-r^2\sin^2\theta\,\mathrm{d}\phi^2\right] \quad ; \tag{129}$$

$$\vec{g}_{\mathrm{E}(r)}=-\frac{GM}{r^2}\left(1-\frac{r_s}{r}\right)^{-\frac{1}{2}}\hat{r} \quad ; \tag{130}$$

$$L_{\mathrm{E}}=mg_{\mathrm{I}00}\left[\left(1-\frac{r_{\mathrm{S}}}{r}\right)\mathrm{c}^2\dot{t}^2-\frac{1}{1-\frac{r_{\mathrm{S}}}{r}}\dot{r}^2-r^2\dot{\phi}^2\right] \quad ; \quad E_{\mathrm{E}}=\left(1-\frac{r_{\mathrm{S}}}{r}\right)m\mathrm{c}^2\dot{t} \quad ; \quad \dot{}=\frac{\mathrm{d}}{\mathrm{d}\tau_{\mathrm{E}}} \quad ; \tag{131}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau_{\mathrm{E}}}\left(\frac{2\dot{r}}{1-\frac{r_{\mathrm{S}}}{r}}\right)-\left[-\frac{r_{\mathrm{S}}}{r^2}\mathrm{c}^2\dot{t}^2+\frac{\partial}{\partial r}\left(\frac{1}{1-\frac{r_{\mathrm{S}}}{r}}\right)\dot{r}^2+2r\dot{\phi}^2\right]=0 \quad ; \quad J_{\mathrm{E}}=mr^2\dot{\phi} \quad ; \quad \dot{}=\frac{\mathrm{d}}{\mathrm{d}\tau_{\mathrm{E}}}\,. \tag{132}$$

The *differential equation of motion* of the original *Schwarzschild metric* has come from (39):

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2}+u=\frac{\mathrm{G}M}{h_{\mathrm{E}}{}^2}+3\frac{\mathrm{G}M}{\mathrm{c}^2}u^2 \quad ; \quad u=\frac{1}{r} \quad ; \quad h_{\mathrm{E}}=r^2\dot{\phi} \quad ; \quad \dot{}=\frac{\mathrm{d}}{\mathrm{d}\tau_{\mathrm{E}}}\,. \tag{133}$$

The corresponding ERT *approximate differential equation of motion* (which also approximately validates UCM) is:

$$\frac{d^2 u}{d\phi^2}+u=\frac{GM}{h_E^2}+3\frac{G^3 M^3}{c^2 h_E^4}(1+e_E\cos\phi)^2 \;;\; u=\frac{1}{r} \;;\; h_E=r^2\dot{\phi} \;;\; \dot{}=\frac{d}{d\tau_E} \tag{134}$$

with *exact* and *approximate solution*, correspondingly

$$u=\frac{GM}{h_E^2}\left(1+e_E\cos\phi+3\frac{G^2M^2}{c^2 h_E^2}e_E\,\phi\sin\phi\right) \;;\; \frac{h_E^2}{GM}=R_E(1+e_E)=a_E\left(1-e_E^2\right) \;; \tag{135}$$

$$u\approx\frac{GM}{h_E^2}\left(1+e_E\cos\left[\left(1-3\frac{G^2M^2}{c^2h_E^2}\right)\phi\right]\right) \;;\; 0<\frac{6\pi G^2M^2}{c^2h_E^2}<<1. \tag{136}$$

The last eqn can be written as

$$u=\frac{1}{r}\approx\frac{GM}{h_E^2}\left[1+e\cos(\lambda_E\phi)\right] \;;\; \lambda_E=1-3\frac{G^2M^2}{c^2h_E^2} \;;\; 0<\frac{6\pi G^2M^2}{c^2h_E^2}<<1. \tag{137}$$

Hence the *Einsteinian-orbit* can be regarded as an *Einsteinian ellipse* (with α_E semimajor axis) which rotates ('precesses') about one of its foci by an amount

$$\Delta_E=\frac{2\pi}{1-3\frac{G^2M^2}{c^2h_E^2}}-2\pi\approx\frac{6\pi G^2M^2}{c^2h_E^2}=\frac{6\pi GM}{a_E\left(1-e_E^2\right)c^2} \;;\; h_E=r^2\dot{\phi} \;;\; \dot{\phi}=\frac{d\phi}{d\tau_E}=\frac{d\phi}{dt}\dot{t} \tag{138}$$

rad per revolution. Accordingly to our *no-superposition approach*, we have

$$\dot{t}=\frac{dt}{d\tau_E}=\left[1-\left(\frac{r_S}{r}+\frac{1}{1-\frac{r_S}{r}}\frac{\dot{r}^2}{c^2}+\frac{r^2\dot{\phi}^2}{c^2}\right)\right]^{-\frac{1}{2}}\geq1 \;;\; E_E=\frac{1-\frac{r_S}{r}}{\sqrt{1-\left(\frac{r_S}{r}+\frac{1}{1-\frac{r_S}{r}}\frac{\dot{r}^2}{c^2}+\frac{r^2\dot{\phi}^2}{c^2}\right)}}mc^2\geq0 \;; \tag{139}$$

$$U_{gE}=\left(\left(1-\frac{r_S}{r}\right)\left[1-\left(\frac{r_S}{r}+\frac{1}{1-\frac{r_S}{r}}\frac{\dot{r}^2}{c^2}+\frac{r^2\dot{\phi}^2}{c^2}\right)\right]^{-\frac{1}{2}}-\gamma_{\left(\xi_I\vec{\beta}_P\right)}\right)\frac{mc^2}{\xi_I^2}\leq0 \;;\; V_{E(r)}=\left(\sqrt{1-\frac{r_S}{r}}-1\right)c^2\leq0 \;; \tag{140}$$

$$K_E=\left(\gamma_{\left(\vec{\beta}_P\right)}-1\right)mc^2\geq0 \;;\; E_{mE}=\left(\left(1-\frac{r_S}{r}\right)\left[1-\left(\frac{r_S}{r}+\frac{1}{1-\frac{r_S}{r}}\frac{\dot{r}^2}{c^2}+\frac{r^2\dot{\phi}^2}{c^2}\right)\right]^{-\frac{1}{2}}-1\right)mc^2. \tag{141}$$

The Generalized Einsteinian photon has

$$\dot{t}=+\infty \;;\; c_R=c\sqrt{1-\frac{r_S}{R}} \;;\; \Theta=\frac{4GM}{c^2R}. \tag{142}$$

We observe that the speed of light is zero $(c_R=0)$ on the horizon (R=r_S), while as at infinite distance from the center of gravity is $c_I=c$ and also the well-known deflection of light.

4. Generalized SR: Gravitational Field from Generalized Central Potential

4.1. GSR-Gravitational Potential, Lagrangian, Equations of motion and correlation to GR

We study the *motion of particle P* with mass *m,* around a center of gravity with mass *M*. The usual definition of *Lagrangian* of gravitational system (*M*, *m*) [9] (p. 205) gives

$$L = m\dot{x}^{\mu} g_{\mu\nu} \dot{x}^{\nu} = \frac{m\,\mathrm{d}S^2}{\mathrm{d}\tau^2} = \frac{mg_{100}\,\mathrm{c}^2\mathrm{d}\tau^2}{\mathrm{d}\tau^2} = mg_{100}\,\mathrm{c}^2 \;\; ; \;\; \dot{} = \frac{\mathrm{d}}{\mathrm{d}\tau}. \tag{143}$$

This is valid in both the GR and SR [2] (p. 345). In case of GSR, the geometry of spacetime has steady metric (11). So, gravity is studied as a field, which comes from *GSR-gravitational potential* $\left(V_{\mathrm{GSR}}, \vec{w}_{\mathrm{GSR}}\right)$. This adds extra terms to the *GSR-Lagrangian of a free particle P*. In this paper, we examine the case that $\vec{w}_{\mathrm{GSR}} = 0$, according to the weak approach of EP (1). Thus, the *GSR-Lagrangian* in the frame of mass *M*, is [2] (p. 351):

$$L_{\mathrm{GSR}} = -g_{100}\left(-\frac{1}{\gamma_{(\xi_1\beta_P)}} m\mathrm{c}^2 - \xi_{\mathrm{I}}^{\,2} mV_{\mathrm{GSR}(r,\dot{r},\dot{\phi})}\right) = g_{\mathrm{I}11}\left(-\frac{1}{\gamma_{(\xi_1\beta_P)}}\frac{m\mathrm{c}^2}{\xi_{\mathrm{I}}^{\,2}} - mV_{\mathrm{GSR}(r,\dot{r},\dot{\phi})}\right), \tag{144}$$

where V_{GSR} is generalized *central scalar gravitational potential*. Besides, the orbit of particle *P* is on the 'plane' $\theta=\pi/2$ and we have:

$$\upsilon_P^{\,2} = \dot{r}^2 + r^2\dot{\phi}^2 \;\; ; \;\; \gamma_{(\xi_1\beta_P)} = \frac{1}{\sqrt{1 - \xi_{\mathrm{I}}^{\,2}\dfrac{\dot{r}^2 + r^2\dot{\phi}^2}{\mathrm{c}^2}}}, \tag{145}$$

$$L_{\mathrm{GSR}} = -g_{100}\left(-\sqrt{1 - \xi_{\mathrm{I}}^{\,2}\frac{\dot{r}^2 + r^2\dot{\phi}^2}{\mathrm{c}^2}}\, m\mathrm{c}^2 - \xi_{\mathrm{I}}^{\,2} mV_{\mathrm{GSR}(r,\dot{r},\dot{\phi})}\right) \;\; ; \;\; \dot{} = \frac{\mathrm{d}}{\mathrm{d}t}. \tag{146}$$

Let us find the *first integral of motion* for the above *GSR-Lagrangian*

$$C_1 = \sum_{\mu=1}^{n=2}\left(\frac{\partial L_{\mathrm{GSR}}}{\partial \dot{x}^{\mu}}\right)\dot{x}^{\mu} - L_{\mathrm{GSR}} \;\; ; \;\; \mu=1, 2, \tag{147}$$

which gives

$$E^* = \frac{C_1}{g_{\mathrm{I}11}} = \gamma_{(\xi_1\beta_P)}\frac{m\mathrm{c}^2}{\xi_{\mathrm{I}}^{\,2}} + mV_{\mathrm{GSR}(r,\dot{r},\dot{\phi})} - m\mathrm{c}^2\left(\frac{\partial V_{\mathrm{GSR}}}{\partial\dot{\phi}}\frac{\dot{\phi}}{\mathrm{c}^2} + \frac{\partial V_{\mathrm{GSR}}}{\partial\dot{r}}\frac{\dot{r}}{\mathrm{c}^2}\right). \tag{148}$$

The *GSR-relativistic energy* definition (24ii) plus the potential energy (2) give the quantity

$$E_{\mathrm{tGSR}} = \frac{\gamma_{(\xi_1\beta_P)} m\mathrm{c}^2}{\xi_{\mathrm{I}}^{\,2}} + mV_{\mathrm{GSR}(r,\dot{r},\dot{\phi})}, \tag{149}$$

which is maintained if only

$$\frac{\partial V_{\mathrm{GSR}}}{\partial\dot{\phi}}\frac{\dot{\phi}}{\mathrm{c}^2} + \frac{\partial V_{\mathrm{GSR}}}{\partial\dot{r}}\frac{\dot{r}}{\mathrm{c}^2} = 0. \tag{150}$$

In any other case there exists a non-null quantity

$$E_{\mathrm{dGSR}} = -m\mathrm{c}^2\left(\frac{\partial V_{\mathrm{GSR}}}{\partial\dot{\phi}}\frac{\dot{\phi}}{\mathrm{c}^2} + \frac{\partial V_{\mathrm{GSR}}}{\partial\dot{r}}\frac{\dot{r}}{\mathrm{c}^2}\right). \tag{151}$$

For instance, if $V_{\mathrm{GSR}}=V_{\mathrm{GSR}(r)}$, then there is no d*GSR- energy* and the *tGSR-energy* is maintained [6] (pp. 11-12). Generally, the *first integral of motion* gives the *total energy*

$$E^* = \frac{C_1}{g_{\mathrm{I}11}} = E_{\mathrm{tGSR}} + E_{\mathrm{dGSR}}. \tag{152}$$

Thus, we obtain the *generalized potential energy*:

$$U^* = E^* - E = mV_{\mathrm{GSR}(r,\dot{r},\dot{\phi})} - m\mathrm{c}^2\left(\frac{\partial V_{\mathrm{GSR}}}{\partial\dot{\phi}}\frac{\dot{\phi}}{\mathrm{c}^2} + \frac{\partial V_{\mathrm{GSR}}}{\partial\dot{r}}\frac{\dot{r}}{\mathrm{c}^2}\right). \tag{153}$$

We observe that if only condition (150) is valid (e.g. $V_{\mathrm{GSR}}=V_{\mathrm{GSR}(r)}$), then the potential energy is given by the formula

$$U = mV_{\mathrm{GSR}(r,\dot{r},\dot{\phi})}. \tag{154}$$

We also observe that the coordinate φ is ignored in *GSR-Lagrangian* (146). So, the *second integral of motion* is

$$C_2 = \frac{\partial L_{\mathrm{GSR}}}{\partial\dot{\phi}}, \tag{155}$$

which gives

$$C_2 = -g_{100}\left(\xi_1^{\,2} m\gamma_{(\xi_1\beta_P)} r^2\dot{\phi} - \xi_1^{\,2} m\frac{\partial V_{\mathrm{GSR}}}{\partial\dot{\phi}}\right) = g_{111}\left(m\gamma_{(\xi_1\beta_P)} r^2\dot{\phi} - m\frac{\partial V_{\mathrm{GSR}}}{\partial\dot{\phi}}\right). \tag{156}$$

The *tGSR-angular momentum* (J) is defined as

$$J = mh = m\gamma_{(\xi_1\beta_P)} r^2\dot{\phi} \quad ; \quad \dot{} = \frac{\mathrm{d}}{\mathrm{d}t}, \tag{157}$$

where $h=J/m$ is the *tGSR-angular momentum per rest mass unit*. So, *tGSR-angular momentum* (J) is maintained only if

$$\frac{\partial V_{\mathrm{GSR}(r,\dot{r},\dot{\phi})}}{\partial\dot{\phi}} = 0. \tag{158}$$

In any other case, there exists a quantity

$$J_{\mathrm{dGSR}} = mh_{\mathrm{dGSR}} = -m\frac{\partial V_{\mathrm{GSR}}}{\partial\dot{\phi}}. \tag{159}$$

that we call *dGSR-angular momentum*. In case that $V_{\mathrm{GSR}}=V_{\mathrm{GSR(r)}}$, there is no *dGSR-angular momentum* and the *tGSR-angular momentum* (J) is maintained [6] (pp. 11-12). Generally, the *second integral of motion* gives

$$J^* = mh^* = J + J_{\mathrm{dGSR}} = m\left(h + h_{\mathrm{dGSR}}\right) = \frac{C_2}{g_{111}}. \tag{160}$$

Now, let us pass to *Euler-Lagrange equations*

$$\frac{d}{dt}\left(\frac{\partial L_{\mathrm{GSR}}}{\partial\dot{x}^\mu}\right) - \frac{\partial L_{\mathrm{GSR}}}{\partial x^\mu} = 0 \quad ; \; \mu=1,\,2, \tag{161}$$

which give us the *equations of motion*:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma_{(\xi_1\beta_P)}\dot{r} - \frac{\partial V_{\mathrm{GSR}}}{\partial\dot{r}}\right) - \gamma_{(\xi_1\beta_P)} r\dot{\phi}^2 + \frac{\partial V_{\mathrm{GSR}}}{\partial r} = 0 \quad ; \tag{162}$$

$$J^* = mh^* = m\gamma_{(\xi_1\beta_P)} r^2\dot{\phi} - m\frac{\partial V_{\mathrm{GSR}}}{\partial\dot{\phi}} \quad ; \quad \dot{} = \frac{\mathrm{d}}{\mathrm{d}t}. \tag{163}$$

The case of *circular motion* is obtained by putting $r=R$=constant to (123).

The only thing that we have to do, is the proposition of function V_{GSR}. Fortunately, GR can help by reminding us that the EP in GR is: ‘accelerated motions caused by the gravitational field only (free fall) take place along *geodesics* of the metric, which corresponds to the particular gravitational field’ [2] (p. 248). So, the curved spacetime of GR demands *no force* and also *Lorentz γ-factor* is replaced by the GR-time dilation $\dot{t}$:

$$\gamma_{\left(\xi_1\vec{\beta}_P\right)} \rightarrow \dot{t} = \frac{\mathrm{d}t}{\mathrm{d}\tau_{\mathrm{GR}}}. \tag{164}$$

Moreover, it is

$$\mathrm{d}S^2 = g_{\mathrm{I}00}\,\mathrm{c}^2\mathrm{d}\tau_{\mathrm{GR}}{}^2 \tag{165}$$

or equivalently,

$$g_{\mathrm{I}00}\,\mathrm{c}^2\,\frac{\mathrm{d}\tau_{\mathrm{GR}}{}^2}{\mathrm{d}t^2} = \frac{\mathrm{d}S^2}{\mathrm{d}t^2}\,, \tag{166}$$

which gives

$$\dot{t} = \frac{\mathrm{d}t}{\mathrm{d}\tau_{\mathrm{GR}}} = \left(\frac{\mathrm{d}S^2}{g_{\mathrm{I}00}\,\mathrm{c}^2\mathrm{d}t^2}\right)^{-\frac{1}{2}} \geq 1\,. \tag{167}$$

The *GSR-Lagrangian of a free particle P* [2] (p. 351)

$$L_{\mathrm{GSR}} = -g_{\mathrm{I}00}\left(-\frac{1}{\gamma_{(\xi_{\mathrm{I}}\beta_P)}}m\mathrm{c}^2\right) = g_{\mathrm{I}11}\left(-\frac{1}{\gamma_{(\xi_{\mathrm{I}}\beta_P)}}\frac{m\mathrm{c}^2}{\xi_{\mathrm{I}}{}^2}\right), \tag{168}$$

by using (127) becomes

$$L_{\mathrm{GSR}} = -g_{\mathrm{I}00}\left(-\frac{1}{\dot{t}}m\mathrm{c}^2\right) = g_{\mathrm{I}11}\left(-\frac{1}{\dot{t}}\frac{m\mathrm{c}^2}{\xi_{\mathrm{I}}{}^2}\right) \;\;;\;\; \dot{t} = \frac{\mathrm{d}t}{\mathrm{d}\tau_{\mathrm{GR}}}\,, \tag{169}$$

We observe that *GSR-Lagrangian* (169i) is not the same as the corresponding of GR (39) (because GR is referred to spacetime with variable curvature, while GSR is valid in spacetime with steady curvature), but we shall see that they give exactly the same results. Besides, (169) combined with (144ii) gives

$$\frac{1}{\dot{t}}\frac{m\mathrm{c}^2}{\xi_{\mathrm{I}}{}^2} = \frac{1}{\gamma_{(\xi_{\mathrm{I}}\beta_P)}}\frac{m\mathrm{c}^2}{\xi_{\mathrm{I}}{}^2} + mV_{\mathrm{GSR}(r,\dot{r},\dot{\phi})} \;\;;\;\; \dot{t} = \frac{\mathrm{d}t}{\mathrm{d}\tau_{\mathrm{GR}}}\,. \tag{170}$$

Finally, we obtain the potential

$$V_{\mathrm{GSR}(r,\dot{r},\dot{\phi})} = \frac{\mathrm{c}^2}{\xi_{\mathrm{I}}{}^2}\left(\frac{1}{\dot{t}} - \frac{1}{\gamma_{(\xi_{\mathrm{I}}\beta_P)}}\right) \;\;;\;\; \dot{t} = \frac{\mathrm{d}t}{\mathrm{d}\tau_{\mathrm{GR}}}\,. \tag{171}$$

Thus, we need the formula of GR-time dilation $\dot{t} = \dot{t}_{(r,\dot{r},\dot{\phi})}$. Furthermore, the replacement of the potential to (105), give us the *GSR-Lagrangian*

$$L_{\mathrm{GSR}} = g_{\mathrm{I}00}m\mathrm{c}^2\frac{1}{\dot{t}} = -g_{\mathrm{I}11}\frac{m\mathrm{c}^2}{\xi_{\mathrm{I}}{}^2}\frac{1}{\dot{t}} \;\;;\;\; \dot{t} = \frac{\mathrm{d}t}{\mathrm{d}\tau_{\mathrm{GR}}}\,. \tag{172}$$

Finally, the weak EP (1) combined with central potential gives:

$$\vec{F} = m\vec{g} \;\;;\;\; \vec{g} = -\frac{\partial V_{\mathrm{GSR}(r,\dot{r},\dot{\phi})}}{\partial r}\hat{r} \;\;;\;\; g = \frac{\partial V_{\mathrm{GSR}(r,\dot{r},\dot{\phi})}}{\partial r}\,, \tag{173}$$

where $\vec{g}$ is the *field strength*. The positive value of field strength g means gravity, while negative value means antigravity.

4.2. GSR combined with 1GSM: Gravitational Potential, Field strength, Lagrangian, Equations of motion, Precession of planets' orbits and Deflection of Light

Now, it is time to specify the above procedure, by combining the GSR with the 1GSM. The replacement of (58) to (171), gives the GSR-1st Generalized Schwarzschild Potential (GSR-1GSP):

$$V_{\mathrm{GSR}(r,\dot{r},\dot{\phi})} = \frac{\mathrm{c}^2}{\xi_{\mathrm{I}}{}^2}\left(\left[1 - \xi_{\mathrm{I}}{}^2\left(\frac{r_{\mathrm{S}}}{r} + \frac{1}{1 - \xi_{\mathrm{I}}{}^2\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^2}{\mathrm{c}^2} + \frac{r^2\dot{\phi}^2}{\mathrm{c}^2}\right)\right]^{\frac{1}{2}} - \frac{1}{\gamma_{(\xi_{\mathrm{I}}\beta_P)}}\right) \;\;\cdot = \frac{\mathrm{d}}{\mathrm{d}t}\,. \tag{174}$$

Journal of Physics: Conference Series **1730** (2021) 012080 doi:10.1088/1742-6596/1730/1/012080

We make the above potential more flexible, by adapting

$$V_{\mathrm{GSR}(r,\dot{r},\dot{\phi})}=\frac{\mathrm{c}^2}{\xi_{\mathrm{I}}^{\,2}}\left(l\left[1-k\left(\frac{r_{\mathrm{S}}}{r}+\frac{1}{1-k\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^2}{\mathrm{c}^2}+\frac{r^2\dot{\phi}^2}{\mathrm{c}^2}\right)\right]^{\frac{1}{2}}-\frac{1}{\gamma_{(\xi_{\mathrm{I}}\beta_P)}}\right)\ ;\ k=k(\xi_{\mathrm{I}}).\ ;\ l=l(\xi_{\mathrm{I}})\ ;\ \dot{}=\frac{\mathrm{d}}{\mathrm{d}t},\quad(175)$$

which is called as Modified GSR-1st Generalized Schwarzschild Potential (M-GSR-1GSP). This modification makes the *GSR-generalized potential* and the *GSR-Lagrangian* more flexible, in order to obtain results in accordance to the experimental data, by using different TPs. Of course, the values

$$k=\xi_{\mathrm{I}}^{\,2}\ ;\ l=1\quad(176)$$

makes the above potential equal to (174): the *GSR-gravitational generalized potential* which corresponds to the 1GSM. Moreover, we calculate

$$\frac{\partial V_{\mathrm{GSR}}}{\partial\dot{\phi}}=-kl\frac{r^2\dot{\phi}}{\xi_{\mathrm{I}}^{\,2}}\left[1-k\left(\frac{r_{\mathrm{S}}}{r}+\frac{1}{1-k\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^2}{\mathrm{c}^2}+\frac{r^2\dot{\phi}^2}{\mathrm{c}^2}\right)\right]^{-\frac{1}{2}}+r^2\dot{\phi}\gamma_{(\xi_{\mathrm{I}}\beta_P)}\ ;\quad(177)$$

$$\frac{\partial V_{\mathrm{GSR}}}{\partial\dot{r}}=-\frac{kl}{1-k\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}}{\xi_{\mathrm{I}}^{\,2}}\left[1-k\left(\frac{r_{\mathrm{S}}}{r}+\frac{1}{1-k\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^2}{\mathrm{c}^2}+\frac{r^2\dot{\phi}^2}{\mathrm{c}^2}\right)\right]^{-\frac{1}{2}}+\dot{r}\gamma_{(\xi_{\mathrm{I}}\beta_P)}\ ;\quad(178)$$

$$g=\frac{\partial V_{\mathrm{GSR}}}{\partial r}=\frac{\mathrm{c}^2}{\xi_{\mathrm{I}}^{\,2}}\left[\frac{l}{2}\left(k\frac{r_{\mathrm{S}}}{r^2}+\frac{k^2\frac{r_{\mathrm{S}}}{r^2}}{\left(1-k\frac{r_{\mathrm{S}}}{r}\right)^2}\frac{\dot{r}^2}{\mathrm{c}^2}-\frac{2kr\dot{\phi}^2}{\mathrm{c}^2}\right)\cdot\left[1-k\left(\frac{r_{\mathrm{S}}}{r}+\frac{1}{1-k\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^2}{\mathrm{c}^2}+\frac{r^2\dot{\phi}^2}{\mathrm{c}^2}\right)\right]^{-\frac{1}{2}}+\xi_{\mathrm{I}}^{\,2}\frac{r\dot{\phi}^2}{\mathrm{c}^2}\gamma_{(\xi_{\mathrm{I}}\beta_P)}\right].\quad(179)$$

Besides the *GSR-Lagrangian* (146) becomes

$$L_{\mathrm{GSR}}=g_{100}lm\mathrm{c}^2\left[1-k\left(\frac{r_{\mathrm{S}}}{r}+\frac{1}{1-k\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^2}{\mathrm{c}^2}+\frac{r^2\dot{\phi}^2}{\mathrm{c}^2}\right)\right]^{\frac{1}{2}}=-g_{111}\frac{lm\mathrm{c}^2}{\xi_{\mathrm{I}}^{\,2}}\left[1-k\left(\frac{r_{\mathrm{S}}}{r}+\frac{1}{1-k\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^2}{\mathrm{c}^2}+\frac{r^2\dot{\phi}^2}{\mathrm{c}^2}\right)\right]^{\frac{1}{2}}\ ;\ \dot{}=\frac{\mathrm{d}}{\mathrm{d}t},\quad(180)$$

which is called as Modified GSR-1st Generalized Schwarzschild Lagrangian (M-GSR-1GSL). Moreover, the replacement of (177) and (178) to (148) gives

$$E^*=\gamma_{(\xi_{\mathrm{I}}\vec{\beta}_P)}\frac{m\mathrm{c}^2}{\xi_{\mathrm{I}}^{\,2}}+mV_{\mathrm{GSR}(r,\dot{r},\dot{\phi})}-m\mathrm{c}^2\left(\begin{array}{c}-kl\frac{r^2\dot{\phi}^2}{\xi_{\mathrm{I}}^{\,2}\mathrm{c}^2}\left[1-k\left(\frac{r_{\mathrm{S}}}{r}+\frac{1}{1-k\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^2}{\mathrm{c}^2}+\frac{r^2\dot{\phi}^2}{\mathrm{c}^2}\right)\right]^{-\frac{1}{2}}+r^2\frac{\dot{\phi}^2}{\mathrm{c}^2}\gamma_{(\xi_{\mathrm{I}}\beta_P)}\\-\frac{kl}{1-k\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^2}{\xi_{\mathrm{I}}^{\,2}\mathrm{c}^2}\left[1-k\left(\frac{r_{\mathrm{S}}}{r}+\frac{1}{1-k\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^2}{\mathrm{c}^2}+\frac{r^2\dot{\phi}^2}{\mathrm{c}^2}\right)\right]^{-\frac{1}{2}}+\frac{\dot{r}^2}{\mathrm{c}^2}\gamma_{(\xi_{\mathrm{I}}\beta_P)}\end{array}\right),\quad(181)$$

or equivalently,

$$E^{*}=\gamma_{\left(\xi_{\mathrm{I}}\vec{\beta}_{P}\right)}\frac{m\mathrm{c}^{2}}{\xi_{\mathrm{I}}{}^{2}}+mV_{\mathrm{GSR}\left(r,\dot{r},\dot{\phi}\right)}-m\mathrm{c}^{2}\left(\left[1-k\left(\frac{r_{\mathrm{S}}}{r}+\frac{1}{1-k\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^{2}}{\mathrm{c}^{2}}+\frac{r^{2}\dot{\phi}^{2}}{\mathrm{c}^{2}}\right)\right]^{-\frac{1}{2}}\left(-kl\frac{r^{2}\dot{\phi}^{2}}{\xi_{\mathrm{I}}{}^{2}\mathrm{c}^{2}}-\frac{kl}{1-k\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^{2}}{\xi_{\mathrm{I}}{}^{2}\mathrm{c}^{2}}\right)+\frac{\upsilon^{2}}{\mathrm{c}^{2}}\gamma_{\left(\xi_{\mathrm{I}}\vec{\beta}_{P}\right)}\right). \tag{182}$$

This is also written as

$$E^{*}=\gamma_{\left(\xi_{\mathrm{I}}\beta_{P}\right)}\frac{m\mathrm{c}^{2}}{\xi_{\mathrm{I}}{}^{2}}+mV_{\mathrm{GSR}\left(r,\dot{r},\dot{\phi}\right)}-m\mathrm{c}^{2}\left(\left[1-k\left(\frac{r_{\mathrm{S}}}{r}+\frac{1}{1-k\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^{2}}{\mathrm{c}^{2}}+\frac{r^{2}\dot{\phi}^{2}}{\mathrm{c}^{2}}\right)\right]^{-\frac{1}{2}}\left(-kl\frac{r^{2}\dot{\phi}^{2}}{\xi_{\mathrm{I}}{}^{2}\mathrm{c}^{2}}-\frac{kl}{1-k\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^{2}}{\xi_{\mathrm{I}}{}^{2}\mathrm{c}^{2}}\right)+\frac{\upsilon^{2}}{\mathrm{c}^{2}}\frac{\gamma_{\left(\xi_{\mathrm{I}}\vec{\beta}_{P}\right)}{}^{2}}{\gamma_{\left(\xi_{\mathrm{I}}\vec{\beta}_{P}\right)}}\right). \tag{183}$$

in order to use the identity

$$1+\xi_{\mathrm{I}}{}^{2}\frac{\upsilon^{2}}{\mathrm{c}^{2}}\gamma_{\left(\xi_{\mathrm{I}}\beta_{P}\right)}{}^{2}=\gamma_{\left(\xi_{\mathrm{I}}\beta_{P}\right)}{}^{2}. \tag{184}$$

Thus, it emerges

$$E^{*}=\gamma_{\left(\xi_{\mathrm{I}}\beta_{P}\right)}\frac{m\mathrm{c}^{2}}{\xi_{\mathrm{I}}{}^{2}}+mV_{\mathrm{GSR}\left(r,\dot{r},\dot{\phi}\right)}-m\mathrm{c}^{2}\left(\left[1-k\left(\frac{r_{\mathrm{S}}}{r}+\frac{1}{1-k\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^{2}}{\mathrm{c}^{2}}+\frac{r^{2}\dot{\phi}^{2}}{\mathrm{c}^{2}}\right)\right]^{-\frac{1}{2}}\left(-kl\frac{r^{2}\dot{\phi}^{2}}{\xi_{\mathrm{I}}{}^{2}\mathrm{c}^{2}}-\frac{kl}{1-k\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^{2}}{\xi_{\mathrm{I}}{}^{2}\mathrm{c}^{2}}\right)+\frac{\gamma_{\left(\xi_{\mathrm{I}}\beta_{P}\right)}{}^{2}-1}{\xi_{\mathrm{I}}{}^{2}\gamma_{\left(\xi_{\mathrm{I}}\beta_{P}\right)}}\right). \tag{185}$$

The above eqn is further simplified to

$$E^{*}=mV_{\mathrm{GSR}\left(r,\dot{r},\dot{\phi}\right)}-m\mathrm{c}^{2}\left(\left[1-k\left(\frac{r_{\mathrm{S}}}{r}+\frac{1}{1-k\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^{2}}{\mathrm{c}^{2}}+\frac{r^{2}\dot{\phi}^{2}}{\mathrm{c}^{2}}\right)\right]^{-\frac{1}{2}}\left(-kl\frac{r^{2}\dot{\phi}^{2}}{\xi_{\mathrm{I}}{}^{2}\mathrm{c}^{2}}-\frac{kl}{1-k\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^{2}}{\xi_{\mathrm{I}}{}^{2}\mathrm{c}^{2}}\right)-\frac{1}{\xi_{\mathrm{I}}{}^{2}\gamma_{\left(\xi_{\mathrm{I}}\beta_{P}\right)}}\right). \tag{186}$$

The replacement of (175) to the above, gives

$$E^{*}=\frac{m\mathrm{c}^{2}}{\xi_{\mathrm{I}}{}^{2}}\left(l\left[1-k\left(\frac{r_{\mathrm{S}}}{r}+\frac{1}{1-k\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^{2}}{\mathrm{c}^{2}}+\frac{r^{2}\dot{\phi}^{2}}{\mathrm{c}^{2}}\right)\right]^{\frac{1}{2}}+\left[1-k\left(\frac{r_{\mathrm{S}}}{r}+\frac{1}{1-k\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^{2}}{\mathrm{c}^{2}}+\frac{r^{2}\dot{\phi}^{2}}{\mathrm{c}^{2}}\right)\right]^{-\frac{1}{2}}\left(kl\frac{r^{2}\dot{\phi}^{2}}{\mathrm{c}^{2}}+\frac{kl}{1-k\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^{2}}{\mathrm{c}^{2}}\right)\right), \tag{187}$$

which can also be written as

$$E^{*}=\frac{lm\mathrm{c}^{2}}{\xi_{\mathrm{I}}{}^{2}}\left[1-k\left(\frac{r_{\mathrm{S}}}{r}+\frac{1}{1-k\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^{2}}{\mathrm{c}^{2}}+\frac{r^{2}\dot{\phi}^{2}}{\mathrm{c}^{2}}\right)\right]^{-\frac{1}{2}}\left(1-k\left(\frac{r_{\mathrm{S}}}{r}+\frac{1}{1-k\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^{2}}{\mathrm{c}^{2}}+\frac{r^{2}\dot{\phi}^{2}}{\mathrm{c}^{2}}\right)+k\frac{r^{2}\dot{\phi}^{2}}{\mathrm{c}^{2}}+\frac{k}{1-k\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^{2}}{\mathrm{c}^{2}}\right). \tag{188}$$

So, the *first integral of motion* gave us

$$E^{*}=\frac{lm\mathrm{c}^{2}}{\xi_{\mathrm{I}}{}^{2}}\left[1-k\left(\frac{r_{\mathrm{S}}}{r}+\frac{1}{1-k\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^{2}}{\mathrm{c}^{2}}+\frac{r^{2}\dot{\phi}^{2}}{\mathrm{c}^{2}}\right)\right]^{-\frac{1}{2}}\left(1-k\frac{r_{\mathrm{S}}}{r}\right). \tag{189}$$

This is exactly the *total GR-energy* (59), in case that $k=\xi_{\mathrm{I}}{}^{2}$; l=1 . Now, we demand zero kinetic energy (K=0), in case that the particle is static $\left(\vec{\beta}_{P}=0\right)$. Thus, we have

$$E^{*}_{\left(\vec{\beta}_{P}=0\right)}=E_{\mathrm{rest}}+U\,, \tag{190}$$

where U is the *GSR-gravitational potential energy of a rest body* and

$$E^*_{(\vec{\beta}_P=0)} = \frac{lm\mathrm{c}^2}{\xi_\mathrm{I}^{\ 2}}\left(1-k\frac{r_\mathrm{S}}{r}\right)^{\frac{1}{2}}, \tag{191}$$

is the *total GSR-energy of a rest body*. Replacing the above eqn and (24iii) to (190), we have

$$U_{(r)} = \left(l\sqrt{1-k\frac{r_\mathrm{S}}{r}}-1\right)\frac{m\mathrm{c}^2}{\xi_\mathrm{I}^{\ 2}} \le 0\ ; \tag{192}$$

$$V_{(r)} = \left(l\sqrt{1-k\frac{r_\mathrm{S}}{r}}-1\right)\frac{\mathrm{c}^2}{\xi_\mathrm{I}^{\ 2}} \le 0\ ; \tag{193}$$

where $V_{(r)}$ is the *1st Generalized Schwarzschild Potential (1GSP) of a rest body*. This is a central potential with field strength:

$$\vec{g}_{(r)} = -\frac{dV}{dr}\hat{r} = -\frac{kl}{\xi_I^{\ 2}}\frac{GM}{r^2}\left(1-k\frac{r_s}{r}\right)^{-\frac{1}{2}}\hat{r}\,. \tag{194}$$

We observe that this result is the same as the corresponding GR-formula (62), in case that $k=\xi_\mathrm{I}^{\ 2}$; l=1. Finally, the *GSR-mechanic energy* is

$$E_\mathrm{m} = E^* - E_\mathrm{rest}. \tag{195}$$

Thus we obtain

$$E_\mathrm{m} = \left(\left(1-k\frac{r_\mathrm{S}}{r}\right)\left[1-k\left(\frac{r_\mathrm{S}}{r}+\frac{1}{1-k\frac{r_\mathrm{S}}{r}}\frac{\dot{r}^2}{\mathrm{c}^2}+\frac{r^2\dot{\phi}^2}{\mathrm{c}^2}\right)\right]^{-\frac{1}{2}}-\frac{1}{l}\right)\frac{lm\mathrm{c}^2}{\xi_\mathrm{I}^{\ 2}}\quad ;\ \theta=\frac{\pi}{2}. \tag{196}$$

A part of the above energy is the *dGSR- energy*. From (185) we obtain

$$E_\mathrm{dGSR} = \frac{m\mathrm{c}^2}{\xi_\mathrm{I}^{\ 2}}\left(\frac{kl}{\mathrm{c}^2}\left(r^2\dot{\phi}^2+\frac{1}{1-k\frac{r_\mathrm{S}}{r}}\dot{r}^2\right)\left[1-k\left(\frac{r_\mathrm{S}}{r}+\frac{1}{1-k\frac{r_\mathrm{S}}{r}}\frac{\dot{r}^2}{\mathrm{c}^2}+\frac{r^2\dot{\phi}^2}{\mathrm{c}^2}\right)\right]^{-\frac{1}{2}}-\frac{\gamma_{(\xi_\mathrm{I}\beta_P)}^{\ \ 2}-1}{\gamma_{(\xi_\mathrm{I}\beta_P)}}\right). \tag{197}$$

Besides, the *lGSR-mechanic energy* is defined as

$$E_\mathrm{ml} = K + mV_{\mathrm{GSR}(r,\dot{r},\dot{\phi})} = E^* - E_\mathrm{rest} - E_\mathrm{d}. \tag{198}$$

Thus, we calculate, by using (24i) and (175):

$$E_\mathrm{ml} = \frac{m\mathrm{c}^2}{\xi_\mathrm{I}^{\ 2}}\left(l\left[1-k\left(\frac{r_\mathrm{S}}{r}+\frac{1}{1-k\frac{r_\mathrm{S}}{r}}\frac{\dot{r}^2}{\mathrm{c}^2}+\frac{r^2\dot{\phi}^2}{\mathrm{c}^2}\right)\right]^{-\frac{1}{2}}+\frac{\gamma_{(\xi_\mathrm{I}\vec{\beta}_P)}^{\ \ 2}-\gamma_{(\xi_\mathrm{I}\beta_P)}-1}{\gamma_{(\xi_\mathrm{I}\beta_P)}}\right). \tag{199}$$

Finally, we obtain the generalized *GSR-potential energy*:

$$U^* = E^* - E = \left(\left(1-k\frac{r_\mathrm{S}}{r}\right)\left[1-k\left(\frac{r_\mathrm{S}}{r}+\frac{1}{1-k\frac{r_\mathrm{S}}{r}}\frac{\dot{r}^2}{\mathrm{c}^2}+\frac{r^2\dot{\phi}^2}{\mathrm{c}^2}\right)\right]^{-\frac{1}{2}}-\frac{\gamma_{(\xi_\mathrm{I}\beta_P)}}{l}\right)\frac{lm\mathrm{c}^2}{\xi_\mathrm{I}^{\ 2}}. \tag{200}$$

We observe that the above formula does not associated with eqn (154), because condition (150) is invalid for generalized potential (175).

The case of *circular motion* is obtained by replacing (178) and (179) to *equation of motion* (162)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{kl}{1-k\frac{r_\mathrm{S}}{r}}\frac{\dot{r}}{\xi_\mathrm{I}^{\,2}}\left[1-k\left(\frac{r_\mathrm{S}}{r}+\frac{1}{1-k\frac{r_\mathrm{S}}{r}}\frac{\dot{r}^2}{\mathrm{c}^2}+\frac{r^2\dot{\phi}^2}{\mathrm{c}^2}\right)\right]^{-\frac{1}{2}}\right)+$$
$$\frac{\mathrm{c}^2}{\xi_\mathrm{I}^{\,2}}\frac{l}{2}\left(k\frac{r_\mathrm{S}}{r^2}+\frac{k^2\frac{r_\mathrm{S}}{r^2}}{\left(1-k\frac{r_\mathrm{S}}{r}\right)^2}\frac{\dot{r}^2}{\mathrm{c}^2}-\frac{2kr\dot{\phi}^2}{\mathrm{c}^2}\right)\cdot\left[1-k\left(\frac{r_\mathrm{S}}{r}+\frac{1}{1-k\frac{r_\mathrm{S}}{r}}\frac{\dot{r}^2}{\mathrm{c}^2}+\frac{r^2\dot{\phi}^2}{\mathrm{c}^2}\right)\right]^{-\frac{1}{2}}=0. \tag{201}$$

We then put r=R=constant and we obtain

$$\frac{r_\mathrm{S}}{R^2}-\frac{2R\dot{\phi}^2}{\mathrm{c}^2}=0\,. \tag{202}$$

This gives Uniform Circular Motion (UCM), with the same *angular velocity* and the same *centripetal acceleration* for any TPs

$$\omega=\dot{\phi}=\frac{\mathrm{d}\phi}{\mathrm{d}t}=\sqrt{\frac{\mathrm{G}M}{R^3}}\quad;\quad a=\frac{\upsilon^2}{R}=\omega^2R=\frac{\mathrm{G}M}{R^2}=g_\mathrm{N}\,, \tag{203}$$

exactly as it happens in case of GR.

The *orbit of motion* comes with similar way to the original *Schwarzschild space* [9] (pp. 238-45) as following. We replace (177) to (163) and we obtain

$$h^*=kl\frac{r^2\dot{\phi}}{\xi_\mathrm{I}^{\,2}}\left[1-k\left(\frac{r_\mathrm{S}}{r}+\frac{1}{1-k\frac{r_\mathrm{S}}{r}}\frac{\dot{r}^2}{\mathrm{c}^2}+\frac{r^2\dot{\phi}^2}{\mathrm{c}^2}\right)\right]^{-\frac{1}{2}}\quad;\quad \dot{}=\frac{\mathrm{d}}{\mathrm{d}t}. \tag{204}$$

Besides, (189) gives

$$\left[1-k\left(\frac{r_\mathrm{S}}{r}+\frac{1}{1-k\frac{r_\mathrm{S}}{r}}\frac{\dot{r}^2}{\mathrm{c}^2}+\frac{r^2\dot{\phi}^2}{\mathrm{c}^2}\right)\right]^{-\frac{1}{2}}=\frac{\xi_\mathrm{I}^{\,2}E^*}{lm\mathrm{c}^2\left(1-k\frac{r_\mathrm{S}}{r}\right)}. \tag{205}$$

The replacement of the above to (204), emerges

$$\dot{\phi}=\frac{m\mathrm{c}^2h^*\left(1-k\frac{r_\mathrm{S}}{r}\right)}{kE^*r^2}\quad;\quad \dot{}=\frac{\mathrm{d}}{\mathrm{d}t}. \tag{206}$$

Moreover, (205) can be written as

$$\left[1-k\left(\frac{r_\mathrm{S}}{r}+\frac{1}{1-k\frac{r_\mathrm{S}}{r}}\frac{1}{\mathrm{c}^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2\dot{\phi}^2+\frac{r^2\dot{\phi}^2}{\mathrm{c}^2}\right)\right]^{\frac{1}{2}}=\frac{lm\mathrm{c}^2\left(1-k\frac{r_\mathrm{S}}{r}\right)}{\xi_\mathrm{I}^{\,2}E^*}. \tag{207}$$

The combination of the above eqn with (206) gives

$$\left[1-k\left(\frac{r_{\mathrm{S}}}{r}+\left(\frac{\mathrm{d}\,r}{\mathrm{d}\,\phi}\right)^2\frac{m^2\,\mathrm{c}^2\,h^{*2}\left(1-k\frac{r_{\mathrm{S}}}{r}\right)}{k^2E^{*2}r^4}+\frac{m^2\,\mathrm{c}^2\,h^{*2}\left(1-k\frac{r_{\mathrm{S}}}{r}\right)^2}{k^2E^{*2}r^2}\right)\right]^{\frac{1}{2}}=\frac{lm\mathrm{c}^2\left(1-k\frac{r_{\mathrm{S}}}{r}\right)}{\xi_{\mathrm{I}}{}^2E^*}. \quad (208)$$

The following definition/property

$$u=\frac{1}{r}\quad;\quad r=\frac{1}{u}\quad;\quad\frac{\mathrm{d}\,r}{\mathrm{d}\,\phi}=-\frac{1}{u^2}\frac{\mathrm{d}\,u}{\mathrm{d}\,\phi}=-r^2\frac{\mathrm{d}\,u}{\mathrm{d}\,\phi}, \quad (209)$$

transforms (208) to

$$\left[1-k\left(r_{\mathrm{S}}u+\left(\frac{\mathrm{d}\,u}{\mathrm{d}\,\phi}\right)^2\frac{m^2\,\mathrm{c}^2\,h^{*2}\left(1-kr_{\mathrm{S}}u\right)}{k^2E^{*2}}+\frac{m^2\,\mathrm{c}^2\,h^{*2}\left(1-kr_{\mathrm{S}}u\right)^2u^2}{k^2E^{*2}}\right)\right]^{\frac{1}{2}}=\frac{lm\mathrm{c}^2\left(1-kr_{\mathrm{S}}u\right)}{\xi_{\mathrm{I}}{}^2E^*}. \quad (210)$$

Thus, the above eqn gives

$$1-kr_{\mathrm{S}}u-\left(\frac{\mathrm{d}\,u}{\mathrm{d}\,\phi}\right)^2\frac{m^2\,\mathrm{c}^2\,h^{*2}\left(1-kr_{\mathrm{S}}u\right)}{kE^{*2}}-\frac{m^2\,\mathrm{c}^2\,h^{*2}\left(1-kr_{\mathrm{S}}u\right)^2u^2}{kE^{*2}}=\frac{l^2m^2\,\mathrm{c}^4\left(1-kr_{\mathrm{S}}u\right)^2}{\xi_{\mathrm{I}}{}^4E^{*2}}, \quad (211)$$

which is equivalent to

$$1-\left(\frac{\mathrm{d}\,u}{\mathrm{d}\,\phi}\right)^2\frac{m^2\,\mathrm{c}^2\,h^{*2}}{kE^{*2}}-\frac{m^2\,\mathrm{c}^2\,h^{*2}\left(1-kr_{\mathrm{S}}u\right)u^2}{kE^{*2}}=\frac{l^2m^2\,\mathrm{c}^4\left(1-kr_{\mathrm{S}}u\right)}{\xi_{\mathrm{I}}{}^4E^{*2}}, \quad (212)$$

and even better to

$$\left(\frac{\mathrm{d}\,u}{\mathrm{d}\,\phi}\right)^2+\left(1-kr_{\mathrm{S}}u\right)u^2=-\frac{kl^2\,\mathrm{c}^2\left(1-kr_{\mathrm{S}}u\right)}{\xi_{\mathrm{I}}{}^4h^{*2}}+\frac{kE^{*2}}{m^2\,\mathrm{c}^2\,h^{*2}}, \quad (213)$$

Differentiation wrt φ emerges

$$2\frac{\mathrm{d}\,u}{\mathrm{d}\,\phi}\frac{\mathrm{d}^2\,u}{\mathrm{d}\,\phi^2}+2u\frac{\mathrm{d}\,u}{\mathrm{d}\,\phi}-3kr_{\mathrm{S}}u^2\frac{\mathrm{d}\,u}{\mathrm{d}\,\phi}=\frac{k^2l^2\,\mathrm{c}^2\,r_{\mathrm{S}}}{\xi_{\mathrm{I}}{}^4h^{*2}}\frac{\mathrm{d}\,u}{\mathrm{d}\,\phi}, \quad (214)$$

which generally gives

$$\frac{\mathrm{d}^2\,u}{\mathrm{d}\,\phi^2}+u-\frac{3}{2}kr_{\mathrm{S}}u^2=\frac{k^2l^2\,\mathrm{c}^2\,r_{\mathrm{S}}}{2\xi_{\mathrm{I}}{}^4h^{*2}}. \quad (215)$$

Thus, we obtain the *equation of trajectory for central GSR-gravitational potential (175)*:

$$\frac{\mathrm{d}^2\,u}{\mathrm{d}\,\phi^2}+u=\frac{k^2l^2}{\xi_{\mathrm{I}}{}^4}\frac{\mathrm{G}M}{h^{*2}}+3k\frac{\mathrm{G}M}{\mathrm{c}^2}u^2\quad;\quad u=\frac{1}{r}, \quad (216)$$

where

$$h^*=\frac{kl}{\xi_{\mathrm{I}}{}^2}h_{\mathrm{N}}\left[1-k\left(\frac{r_{\mathrm{S}}}{r}+\frac{1}{1-k\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^2}{\mathrm{c}^2}+\frac{r^2\dot{\phi}^2}{\mathrm{c}^2}\right)\right]^{-\frac{1}{2}}\quad;\quad h_{\mathrm{N}}=r^2\dot{\phi}\;;\;\dot{}=\frac{\mathrm{d}}{\mathrm{d}\,t}. \quad (217)$$

according to (204).

Here, we can make one of the following options:
(i) the GSR-gravitational potential is equivalent to 1GSM, or
(ii) the GSR-gravitational potential is equivalent to the original SM.
The first option demands the differential eqn (216) be the same as (78), while the second option associate it with (133). Both options lead to

$$l = \frac{\xi_{\mathrm{I}}^{\;2}}{k} \,. \tag{218}$$

Thus, we obtain

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = \frac{\mathrm{G}M}{h^{*2}} + 3k\frac{\mathrm{G}M}{\mathrm{c}^2}u^2 \;\;;\;\; h^* = h_{\mathrm{N}}\left[1 - k\left(\frac{r_{\mathrm{S}}}{r} + \frac{1}{1 - k\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^2}{\mathrm{c}^2} + \frac{r^2\dot{\phi}^2}{\mathrm{c}^2}\right)\right]^{-\frac{1}{2}} \;\;;\;\; h_{\mathrm{N}} = r^2\dot{\phi} \;\;;\;\; \dot{} = \frac{\mathrm{d}}{\mathrm{d}t} \,. \tag{219}$$

The comparison of the above *GSR-equation of orbit* to the corresponding of 1GSM (78), shows us that we can easily obtain the GSR-results, by replacing

$$\xi_{\mathrm{I}}^{\;2} \to k \;\;;\;\; h_{\mathrm{GR}} \to h^* \tag{220}$$

to the 1GSM-results. Thus, it emerges the *precession of ellipse* which rotates about one of its foci by an amount

$$\Delta = \frac{2\pi}{1 - 3k\dfrac{\mathrm{G}^2 M^2}{\mathrm{c}^2 h^{*2}}} - 2\pi \approx \frac{6\pi\, k\,\mathrm{G}^2 M^2}{\mathrm{c}^2 h^{*2}} = \frac{6\pi\, k\,\mathrm{G}M}{R(1+e)\mathrm{c}^2} = \frac{6\pi\, k\,\mathrm{G}M}{a\left(1-e^2\right)\mathrm{c}^2} \tag{221}$$

rad per revolution with condition

$$0 < \frac{6\pi\, k\,\mathrm{G}^2 M^2}{\mathrm{c}^2 h_{\mathrm{GR}}^{\;2}} = \frac{6\pi\, k\,\mathrm{G}M}{\mathrm{c}^2 R(1+e)} = \frac{6\pi\, k\,\mathrm{G}M}{\mathrm{c}^2 a\left(1-e^2\right)} << 1 \,. \tag{222}$$

In case of *Generalized photon in radial motion* (72) is transformed to

$$c_p = \left(1 - k\frac{r_{\mathrm{S}}}{r}\right)\frac{\mathrm{c}}{\xi_{\mathrm{I}}} \;\;;\;\; \gamma_{(\xi_{\mathrm{I}}\beta_p)} = \frac{1}{\sqrt{k\dfrac{r_{\mathrm{S}}}{r}\left(2 - k\dfrac{r_{\mathrm{S}}}{r}\right)}} \,. \tag{223}$$

Moreover, the magnitude of the *total Deflection of light* is

$$\Theta = 4k\frac{\mathrm{G}M}{\mathrm{c}^2 R} = 4k\frac{r_{\mathrm{S}}}{R} \,. \tag{224}$$

The options are further differentiated as following:

(i)

$$k = \xi_{\mathrm{I}}^{\;2} \;\;;\;\; l = 1 \;\;;\;\; \frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = \frac{\mathrm{G}M}{h^{*2}} + 3\xi_{\mathrm{I}}^{\;2}\frac{\mathrm{G}M}{\mathrm{c}^2}u^2 \;\;; \tag{225}$$

$$h^* = h_{\mathrm{N}}\left[1 - \xi_{\mathrm{I}}^{\;2}\left(\frac{r_{\mathrm{S}}}{r} + \frac{1}{1 - \xi_{\mathrm{I}}^{\;2}\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^2}{\mathrm{c}^2} + \frac{r^2\dot{\phi}^2}{\mathrm{c}^2}\right)\right]^{-\frac{1}{2}} = h_{\mathrm{N}}\frac{\mathrm{d}t}{\mathrm{d}\tau_{1GSM}} = h_{1GSM} \;\;; \tag{226}$$

$$V_{\mathrm{GSR}(r,\dot{r},\dot{\phi})} = \frac{\mathrm{c}^2}{\xi_{\mathrm{I}}^{\;2}}\left(\left[1 - \xi_{\mathrm{I}}^{\;2}\left(\frac{r_{\mathrm{S}}}{r} + \frac{1}{1 - \xi_{\mathrm{I}}^{\;2}\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^2}{\mathrm{c}^2} + \frac{r^2\dot{\phi}^2}{\mathrm{c}^2}\right)\right]^{\frac{1}{2}} - \frac{1}{\gamma_{(\xi_{\mathrm{I}}\beta_P)}}\right) \;\;;\;\; h_{\mathrm{N}} = r^2\dot{\phi} \;\;;\;\; \dot{} = \frac{\mathrm{d}}{\mathrm{d}t} \;\;; \tag{227}$$

$$E^* = \frac{m\mathrm{c}^2}{\xi_{\mathrm{I}}^{\;2}}\left[1 - \xi_{\mathrm{I}}^{\;2}\left(\frac{r_{\mathrm{S}}}{r} + \frac{1}{1 - \xi_{\mathrm{I}}^{\;2}\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^2}{\mathrm{c}^2} + \frac{r^2\dot{\phi}^2}{\mathrm{c}^2}\right)\right]^{-\frac{1}{2}}\left(1 - \xi_{\mathrm{I}}^{\;2}\frac{r_{\mathrm{S}}}{r}\right) \;\;; \tag{228}$$

$$E_{\mathrm{m}} = \left(\left(1-\xi_{\mathrm{I}}^{2}\frac{r_{\mathrm{S}}}{r}\right)\left[1-\xi_{\mathrm{I}}^{2}\left(\frac{r_{\mathrm{S}}}{r}+\frac{1}{1-\xi_{\mathrm{I}}^{2}\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^{2}}{\mathrm{c}^{2}}+\frac{r^{2}\dot{\phi}^{2}}{\mathrm{c}^{2}}\right)\right]^{-\frac{1}{2}} -1\right)\frac{m\mathrm{c}^{2}}{\xi_{\mathrm{I}}^{2}} \;; \tag{229}$$

$$U^{*} = E^{*}-E = \left(\left(1-\xi_{\mathrm{I}}^{2}\frac{r_{\mathrm{S}}}{r}\right)\left[1-\xi_{\mathrm{I}}^{2}\left(\frac{r_{\mathrm{S}}}{r}+\frac{1}{1-\xi_{\mathrm{I}}^{2}\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^{2}}{\mathrm{c}^{2}}+\frac{r^{2}\dot{\phi}^{2}}{\mathrm{c}^{2}}\right)\right]^{-\frac{1}{2}} -\gamma_{\left(\xi_{\mathrm{I}}\vec{\beta}_{P}\right)}\right)\frac{m\mathrm{c}^{2}}{\xi_{\mathrm{I}}^{2}} \;; \tag{230}$$

$$g = \frac{\mathrm{c}^{2}}{\xi_{\mathrm{I}}^{2}}\left[\frac{1}{2}\left(\xi_{\mathrm{I}}^{2}\frac{r_{\mathrm{S}}}{r^{2}}+\frac{\xi_{\mathrm{I}}^{4}\frac{r_{\mathrm{S}}}{r^{2}}}{\left(1-\xi_{\mathrm{I}}^{2}\frac{r_{\mathrm{S}}}{r}\right)^{2}}\frac{\dot{r}^{2}}{\mathrm{c}^{2}}-\frac{2\xi_{\mathrm{I}}^{2}r\dot{\phi}^{2}}{\mathrm{c}^{2}}\right)\cdot\left[1-\xi_{\mathrm{I}}^{2}\left(\frac{r_{\mathrm{S}}}{r}+\frac{1}{1-\xi_{\mathrm{I}}^{2}\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^{2}}{\mathrm{c}^{2}}+\frac{r^{2}\dot{\phi}^{2}}{\mathrm{c}^{2}}\right)\right]^{-\frac{1}{2}}+\xi_{\mathrm{I}}^{2}\frac{r\dot{\phi}^{2}}{\mathrm{c}^{2}}\gamma_{\left(\xi_{\mathrm{I}}\beta_{P}\right)}\right] \;; \tag{231}$$

$$\Delta = \frac{2\pi}{1-3\xi_{\mathrm{I}}^{2}\frac{\mathrm{G}^{2}M^{2}}{\mathrm{c}^{2}h^{*2}}}-2\pi \approx \frac{6\pi\,\xi_{\mathrm{I}}^{2}\,\mathrm{G}^{2}M^{2}}{\mathrm{c}^{2}h^{*2}} = \frac{6\pi\,\xi_{\mathrm{I}}^{2}\,\mathrm{G}M}{R(1+e)\mathrm{c}^{2}} = \frac{6\pi\,\xi_{\mathrm{I}}^{2}\,\mathrm{G}M}{a\left(1-e^{2}\right)\mathrm{c}^{2}} = \Delta_{1\mathrm{GSM}} \;; \tag{232}$$

$$c_{p} = \left(1-\xi_{\mathrm{I}}^{2}\frac{r_{\mathrm{S}}}{r}\right)\frac{\mathrm{c}}{\xi_{\mathrm{I}}} \;;\; \gamma_{\left(\xi_{\mathrm{I}}\beta_{p}\right)} = \frac{1}{\xi_{\mathrm{I}}\sqrt{\frac{r_{\mathrm{S}}}{r}\left(2-\xi_{\mathrm{I}}^{2}\frac{r_{\mathrm{S}}}{r}\right)}} \;;\; \Theta = 4\xi_{\mathrm{I}}^{2}\frac{\mathrm{G}M}{\mathrm{c}^{2}R} = 2\xi_{\mathrm{I}}^{2}\frac{r_{\mathrm{S}}}{R}. \tag{233}$$

(ii)

$$k=1 \;;\; l=\xi_{\mathrm{I}}^{2} \;;\; \frac{\mathrm{d}^{2}u}{\mathrm{d}\phi^{2}}+u = \frac{\mathrm{G}M}{h^{*2}}+3\frac{\mathrm{G}M}{\mathrm{c}^{2}}u^{2} \;; \tag{234}$$

$$h^{*} = h_{\mathrm{N}}\left[1-\left(\frac{r_{\mathrm{S}}}{r}+\frac{1}{1-\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^{2}}{\mathrm{c}^{2}}+\frac{r^{2}\dot{\phi}^{2}}{\mathrm{c}^{2}}\right)\right]^{-\frac{1}{2}} = h_{\mathrm{N}}\frac{\mathrm{d}t}{\mathrm{d}\tau_{SM}} = h_{\mathrm{E}} \;; \tag{235}$$

$$V_{\mathrm{GSR}\left(r,\dot{r},\dot{\phi}\right)} = \frac{\mathrm{c}^{2}}{\xi_{\mathrm{I}}^{2}}\left(\xi_{\mathrm{I}}^{2}\left[1-\left(\frac{r_{\mathrm{S}}}{r}+\frac{1}{1-\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^{2}}{\mathrm{c}^{2}}+\frac{r^{2}\dot{\phi}^{2}}{\mathrm{c}^{2}}\right)\right]^{\frac{1}{2}}-\frac{1}{\gamma_{\left(\xi_{\mathrm{I}}\beta_{P}\right)}}\right) \;;\; h_{\mathrm{N}} = r^{2}\dot{\phi} \;;\; \dot{} = \frac{\mathrm{d}}{\mathrm{d}t} \;; \tag{236}$$

$$E^{*} = m\mathrm{c}^{2}\left[1-\left(\frac{r_{\mathrm{S}}}{r}+\frac{1}{1-\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^{2}}{\mathrm{c}^{2}}+\frac{r^{2}\dot{\phi}^{2}}{\mathrm{c}^{2}}\right)\right]^{-\frac{1}{2}}\left(1-\frac{r_{\mathrm{S}}}{r}\right) \;; \tag{237}$$

$$E_{\mathrm{m}} = \left(\left(1-\frac{r_{\mathrm{S}}}{r}\right)\left[1-\left(\frac{r_{\mathrm{S}}}{r}+\frac{1}{1-\frac{r_{\mathrm{S}}}{r}}\frac{\dot{r}^{2}}{\mathrm{c}^{2}}+\frac{r^{2}\dot{\phi}^{2}}{\mathrm{c}^{2}}\right)\right]^{-\frac{1}{2}}-1\right)m\mathrm{c}^{2}. \tag{238}$$

$$U^* = E^* - E = \left(\left(1-\frac{r_\mathrm{S}}{r}\right)\left[1-\left(\frac{r_\mathrm{S}}{r}+\frac{1}{1-\frac{r_\mathrm{S}}{r}}\frac{\dot{r}^2}{\mathrm{c}^2}+\frac{r^2\dot{\phi}^2}{\mathrm{c}^2}\right)\right]^{-\frac{1}{2}} - \frac{\gamma_{(\xi_\mathrm{I}\beta_P)}}{\xi_\mathrm{I}^{\,2}}\right) m\mathrm{c}^2 \quad ; \tag{239}$$

$$g = \frac{\mathrm{c}^2}{\xi_\mathrm{I}^{\,2}}\left[\frac{\xi_\mathrm{I}^{\,2}}{2}\left(\frac{r_\mathrm{S}}{r^2}+\frac{\frac{r_\mathrm{S}}{r^2}}{\left(1-\frac{r_\mathrm{S}}{r}\right)^2}\frac{\dot{r}^2}{\mathrm{c}^2}-\frac{2r\dot{\phi}^2}{\mathrm{c}^2}\right)\cdot\left[1-\left(\frac{r_\mathrm{S}}{r}+\frac{1}{1-\frac{r_\mathrm{S}}{r}}\frac{\dot{r}^2}{\mathrm{c}^2}+\frac{r^2\dot{\phi}^2}{\mathrm{c}^2}\right)\right]^{-\frac{1}{2}} + \xi_\mathrm{I}^{\,2}\frac{r\dot{\phi}^2}{\mathrm{c}^2}\gamma_{(\xi_\mathrm{I}\beta_P)}\right] \quad ; \tag{240}$$

$$\Delta = \frac{2\pi}{1-3\frac{\mathrm{G}^2M^2}{\mathrm{c}^2h^2}} - 2\pi \approx \frac{6\pi\mathrm{G}^2M^2}{\mathrm{c}^2h^2} = \frac{6\pi\mathrm{G}M}{R(1+e)\mathrm{c}^2} = \frac{6\pi\mathrm{G}M}{a(1-e^2)\mathrm{c}^2} = \Delta_\mathrm{E} \quad ; \tag{241}$$

In case of *Generalized photon in radial motion*, (223) and (224) are transformed to

$$c_p = \left(1-\frac{r_\mathrm{S}}{r}\right)\frac{\mathrm{c}}{\xi_\mathrm{I}} \quad ; \quad \gamma_{(\xi_\mathrm{I}\beta_p)} = \frac{1}{\sqrt{\frac{r_\mathrm{S}}{r}\left(2-\frac{r_\mathrm{S}}{r}\right)}} \quad ; \quad \Theta = \frac{4\mathrm{G}M}{\mathrm{c}^2R} = 2\frac{r_\mathrm{S}}{R}. \tag{242}$$

In case of UCM, both the options give:

$$\omega = \dot{\phi} = \frac{\mathrm{d}\phi}{\mathrm{d}t} = \sqrt{\frac{\mathrm{G}M}{R^3}} \quad ; \quad a = \frac{\upsilon^2}{R} = \omega^2 R = \frac{\mathrm{G}M}{R^2} = g_\mathrm{N} \quad ; \quad \upsilon = \sqrt{\frac{\mathrm{G}M}{R}} \quad ; \quad g = \frac{\mathrm{G}M}{R^2}\gamma_{(\xi_\mathrm{I}\beta_P)} = \gamma_{(\xi_\mathrm{I}\beta_P)}g_\mathrm{N}. \tag{243}$$

Besides, we have correspondingly:

(i) $$h^* = \sqrt{\mathrm{G}MR}\left(1-\frac{3\xi_\mathrm{I}^{\,2}}{\mathrm{c}^2}\frac{\mathrm{G}M}{R}\right)^{-\frac{1}{2}} \quad ; \quad E_\mathrm{m} = \left(\left(1-\xi_\mathrm{I}^{\,2}\frac{r_\mathrm{S}}{r}\right)\left(1-\frac{3\xi_\mathrm{I}^{\,2}}{\mathrm{c}^2}\frac{\mathrm{G}M}{R}\right)^{-\frac{1}{2}}-1\right)\frac{m\mathrm{c}^2}{\xi_\mathrm{I}^{\,2}}, \tag{244}$$

(ii) $$h^* = \sqrt{\mathrm{G}MR}\left(1-\frac{3}{\mathrm{c}^2}\frac{\mathrm{G}M}{R}\right)^{-\frac{1}{2}} \quad ; \quad E_{\mathrm{m}\,tot} = \left(\left(1-\frac{r_\mathrm{S}}{r}\right)\left(1-\frac{3}{\mathrm{c}^2}\frac{\mathrm{G}M}{R}\right)^{-\frac{1}{2}}-1\right)m\mathrm{c}^2. \tag{245}$$

Moreover, we study the *gravitational field on unmoved particle*. Thus, (179) is transformed to

$$g = \frac{\partial V_\mathrm{GSR}}{\partial r} = \frac{\mathrm{c}^2}{\xi_\mathrm{I}^{\,2}}\frac{lk}{2}\frac{r_\mathrm{S}}{r^2}\left(1-k\frac{r_\mathrm{S}}{r}\right)^{-\frac{1}{2}}. \tag{246}$$

The replacement of (5) and condition (218) to the above eqn gives

$$g = \frac{\partial V_\mathrm{GSR}}{\partial r} = \frac{\mathrm{G}M}{r^2}\left(1-k\frac{r_\mathrm{S}}{r}\right)^{-\frac{1}{2}}. \tag{247}$$

We observe that this formula is the same to the corresponding of 1GSM (for $k=\xi_\mathrm{I}^2$), but it is very different than the corresponding of UCM (243iv). The corresponding *initial acceleration* is computed as following. Eqn (178) is transformed to

$$\frac{\partial V_\mathrm{GSR}}{\partial \dot{r}} = -\frac{kl}{1-k\frac{r_\mathrm{S}}{r}}\frac{\dot{r}}{\xi_\mathrm{I}^{\,2}}\left[1-k\left(\frac{r_\mathrm{S}}{r}+\frac{1}{1-k\frac{r_\mathrm{S}}{r}}\frac{\dot{r}^2}{\mathrm{c}^2}\right)\right]^{-\frac{1}{2}} + \dot{r}\gamma_{(\xi_\mathrm{I}\beta_P)}, \tag{248}$$

by taking $\dot{\phi}=0$. The above eqn and (179) are replaced in (162) and we have for $\dot{\phi}=0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{kl}{1-k\frac{r_\mathrm{S}}{r}}\frac{\dot r}{\xi_\mathrm{I}^{\,2}}\left[1-k\left(\frac{r_\mathrm{S}}{r}+\frac{1}{1-k\frac{r_\mathrm{S}}{r}}\frac{\dot r^2}{\mathrm{c}^2}\right)\right]^{-\frac{1}{2}}\right)+\frac{\mathrm{G}M}{r^2}\left(1-k\frac{r_\mathrm{S}}{r}\right)^{-\frac{1}{2}}=0\ . \qquad (249)$$

This leads to

$$\frac{1}{\xi_\mathrm{I}^{\,2}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{kl\dot r}{1-k\frac{r_\mathrm{S}}{r}}\right)\left(1-k\frac{r_\mathrm{S}}{r}\right)^{-\frac{1}{2}}+\frac{\mathrm{G}M}{r^2}\left(1-k\frac{r_\mathrm{S}}{r}\right)^{-\frac{1}{2}}=0\ , \qquad (250)$$

by taking also $\dot r = 0$. This is equivalent to

$$\left[\frac{1}{\xi_\mathrm{I}^{\,2}}\left(\frac{kl\ddot r\left(1-k\frac{r_\mathrm{S}}{r}\right)}{\left(1-k\frac{r_\mathrm{S}}{r}\right)^2}\right)+\frac{\mathrm{G}M}{r^2}\right]\left(1-k\frac{r_\mathrm{S}}{r}\right)^{-\frac{1}{2}}=0\ , \qquad (251)$$

by taking once again $\dot r = 0$. The replacement of condition (218) to the above eqn gives

$$\frac{\ddot r}{1-k\frac{r_\mathrm{S}}{r}}+\frac{\mathrm{G}M}{r^2}=0 \qquad (252)$$

and we obtain

$$a_r=\ddot r=-\frac{\mathrm{G}M}{r^2}\left(1-k\frac{r_\mathrm{S}}{r}\right) \qquad (253)$$

We observe that the *acceleration of unmoved particle* generally depends on the used TPs and also it is *different than the corresponding field strength* (except for k=0 that corresponds to the *Newtonian potential*, where it is equal). Besides, the *acceleration of unmoved particle* on the *modified Schwarzschild radius* ($r=kr_\mathrm{S}$) is null!

In case of *planet Mercury*, it is α=0.38709893 AU, e=0.20563069 and T=87.968 days [16]. The values: AU= 1.4959787066×10^{11} m, G=6.67428(67)×10^{-11}m^{3}kg^{-1}s^{-2}, c=299792458 ms^{-1} (exact) [17] (pp. 1-1, 1-20, 14-2) and M=1,988,500×10^{24} kg [18], give

$$\frac{r_\mathrm{S}}{a\left(1-e^2\right)}=\frac{2\mathrm{G}M}{\mathrm{c}^2\,a\left(1-e^2\right)}=5.32518(53)\times10^{-8}<<1. \qquad (254)$$

The case of *Earth*, with α= 1.00000011 AU, e= 0.01671022 and T=365.242 days [19], emerges

$$\frac{r_\mathrm{S}}{a\left(1-e^2\right)}=\frac{2\mathrm{G}M}{\mathrm{c}^2\,a\left(1-e^2\right)}=1.97476(20)\times10^{-8}<<1. \qquad (255)$$

Now, we can return to all the previous formulas and replace the above values. Thus, (95) combined with (138) or (241) give the results, which are summarized in Table 1. We observe that both ESR and NPs give the same precessions.

Table 1. Angular velocity ('precession') of ellipse perihelion rotation for *Mercury* and *Earth*, according to k=1 *GSR-Gravitational field* (Ω_GSR) for *Newtonian Physics* (ξ_I=0) and *Einsteinian Special Relativity* (ξ_I=1) and according to the original *Schwarzschild metric* (Ω_EGR). $\Delta\Omega_\mathrm{GSRr}$ (%) is the percentile relative change.

		Mercury			**Earth**		
ξ_I	k	Ω_GSR / ′′cy^{-1}	Ω_EGR / ′′cy^{-1}	$\Delta\Omega_\mathrm{GSRr}$ (%)	Ω_GSR / ′′cy^{-1}	Ω_EGR / ′′cy^{-1}	$\Delta\Omega_\mathrm{GSRr}$ (%)
0	1	42.9820(43) (1)	42.9820(43) (1)	0	3.83893(38) (1)	3.83893(38) (1)	0
1	1	42.9820(43) (1)	42.9820(43) (1)	0	3.83893(38) (1)	3.83893(38) (1)	0

1 [16], [17] (pp. 1-1, 1-20, 14-2), [18], [19]

4.3. GSR combined with 3GSM: Gravitational Potential, Field strength, Lagrangian, Equations of motion and Rotation curves in Galaxies

Now, we specify again the procedure described in 4.1, by combining the GSR with the 3GSM. Firstly, we have to calculate the corresponding GR-time dilation $\dot{t}$. Thus, (31) for θ=π/2 gives

$$g_{100}\,\mathrm{c}^2\mathrm{d}\,\tau^2 = g_{100}\left(1-a_{(r)}\frac{\xi_\mathrm{I}{}^2 r_\mathrm{S}}{r}\right)\mathrm{c}^2\mathrm{d}\,t^2 + \frac{g_{\mathrm{I}11}\left(r\dfrac{\mathrm{d}\,a}{\mathrm{d}\,r}-a_{(r)}\right)^2}{a_{(r)}{}^4\left(1-a_{(r)}\dfrac{\xi_\mathrm{I}{}^2 r_\mathrm{S}}{r}\right)}\mathrm{d}\,r^2 + \frac{g_{\mathrm{I}11}r^2}{a_{(r)}{}^2}\mathrm{d}\,\phi^2, \qquad (256)$$

or equivalently,

$$\left(\frac{\mathrm{d}\,\tau}{\mathrm{d}\,t}\right)^2 = 1-a_{(r)}\frac{\xi_\mathrm{I}{}^2 r_\mathrm{S}}{r} - \frac{\xi_\mathrm{I}{}^2\left(r\dfrac{\mathrm{d}\,a}{\mathrm{d}\,r}-a_{(r)}\right)^2}{a_{(r)}{}^4\left(1-a_{(r)}\dfrac{\xi_\mathrm{I}{}^2 r_\mathrm{S}}{r}\right)}\left(\frac{\mathrm{d}\,r}{\mathrm{d}\,t}\right)^2\frac{1}{\mathrm{c}^2} - \frac{\xi_\mathrm{I}{}^2 r^2}{a_{(r)}{}^2}\left(\frac{\mathrm{d}\,\phi}{\mathrm{d}\,t}\right)^2\frac{1}{\mathrm{c}^2}. \qquad (257)$$

The above eqn gives

$$\frac{\mathrm{d}\,t}{\mathrm{d}\,\tau} = \left[1-\xi_\mathrm{I}{}^2\left(a_{(r)}\frac{r_\mathrm{S}}{r} + \frac{\left(r\dfrac{\mathrm{d}\,a}{\mathrm{d}\,r}-a_{(r)}\right)^2}{a_{(r)}{}^4\left(1-\xi_\mathrm{I}{}^2 a_{(r)}\dfrac{r_\mathrm{S}}{r}\right)}\frac{\dot{r}^2}{\mathrm{c}^2} + \frac{r^2\dot{\phi}^2}{a_{(r)}{}^2\,\mathrm{c}^2}\right)\right]^{-\frac{1}{2}} \geq 1 \;\; ; \;\; {}^{\cdot} = \frac{\mathrm{d}}{\mathrm{d}\,t}. \qquad (258)$$

Moreover, the replacement of the above eqn to (171), gives the GSR-3rd Generalized Schwarzschild Potential (GSR-3GSP):

$$V_{\mathrm{GSR}\left(r,\dot{r},\dot{\phi}\right)} = \frac{\mathrm{c}^2}{\xi_\mathrm{I}{}^2}\left(\left[1-\xi_\mathrm{I}{}^2\left(a_{(r)}\frac{r_\mathrm{S}}{r} + \frac{\left(r\dfrac{\mathrm{d}\,a}{\mathrm{d}\,r}-a_{(r)}\right)^2}{a_{(r)}{}^4\left(1-\xi_\mathrm{I}{}^2 a_{(r)}\dfrac{r_\mathrm{S}}{r}\right)}\frac{\dot{r}^2}{\mathrm{c}^2} + \frac{r^2\dot{\phi}^2}{a_{(r)}{}^2\,\mathrm{c}^2}\right)\right]^{\frac{1}{2}} - \frac{1}{\gamma_{(\xi_\mathrm{I}\beta_P)}}\right) \;\; {}^{\cdot} = \frac{\mathrm{d}}{\mathrm{d}\,t}. \qquad (259)$$

We make the above potential more flexible, by adapting

$$V_{\mathrm{GSR}\left(r,\dot{r},\dot{\phi}\right)} = \frac{\mathrm{c}^2}{\xi_\mathrm{I}{}^2}\left(l\left[1-k\left(a_{(r)}\frac{r_\mathrm{S}}{r} + \frac{\left(r\dfrac{\mathrm{d}\,a}{\mathrm{d}\,r}-a_{(r)}\right)^2}{a_{(r)}{}^4\left(1-k a_{(r)}\dfrac{r_\mathrm{S}}{r}\right)}\frac{\dot{r}^2}{\mathrm{c}^2} + \frac{r^2\dot{\phi}^2}{a_{(r)}{}^2\,\mathrm{c}^2}\right)\right]^{\frac{1}{2}} - \frac{1}{\gamma_{(\xi_\mathrm{I}\beta_P)}}\right) \;; k= k(\xi_\mathrm{I}) \;; l= l(\xi_\mathrm{I}) \;; {}^{\cdot} = \frac{\mathrm{d}}{\mathrm{d}\,t}, \qquad (260)$$

which is called as Modified GSR-3rd Generalized Schwarzschild Potential (M-GSR-3GSP). This modification makes the *GSR-generalized potential* and the *GSR-Lagrangian* more flexible, in order to obtain results in accordance to the experimental data, by using different TPs. Of course, the values

$$k = \xi_\mathrm{I}{}^2 \;\; ; \;\; l=1 \qquad (261)$$

makes the above potential equal to (259): the *GSR-gravitational generalized potential* which corresponds to the 3GSM. Furthermore, we calculate

$$\frac{\partial V_\mathrm{GSR}}{\partial\dot{\phi}} = -kl\frac{r^2\dot{\phi}}{\xi_\mathrm{I}{}^2 a_{(r)}{}^2}\left[1-k\left(a_{(r)}\frac{r_\mathrm{S}}{r} + \frac{\left(r\dfrac{\mathrm{d}\,a}{\mathrm{d}\,r}-a_{(r)}\right)^2}{a_{(r)}{}^4\left(1-k a_{(r)}\dfrac{r_\mathrm{S}}{r}\right)}\frac{\dot{r}^2}{\mathrm{c}^2} + \frac{r^2\dot{\phi}^2}{a_{(r)}{}^2\,\mathrm{c}^2}\right)\right]^{-\frac{1}{2}} + r^2\dot{\phi}\gamma_{(\xi_\mathrm{I}\beta_P)} \;\; ; \qquad (262)$$

$$\frac{\partial V_{\mathrm{GSR}}}{\partial \dot r}=-\frac{kl\left(r\frac{\mathrm{d}a}{\mathrm{d}r}-a_{(r)}\right)^2}{a_{(r)}{}^4\left(1-ka_{(r)}\frac{r_{\mathrm{S}}}{r}\right)}\frac{\dot r}{\xi_1{}^2}\left[1-k\left(a_{(r)}\frac{r_{\mathrm{S}}}{r}+\frac{\left(r\frac{\mathrm{d}a}{\mathrm{d}r}-a_{(r)}\right)^2}{a_{(r)}{}^4\left(1-ka_{(r)}\frac{r_{\mathrm{S}}}{r}\right)}\frac{\dot r^2}{\mathrm{c}^2}+\frac{r^2\dot\phi^2}{a_{(r)}{}^2\mathrm{c}^2}\right)\right]^{-\frac{1}{2}}+\dot r\gamma_{(\xi_1\beta_P)}\ ; \tag{263}$$

$$g=\frac{\partial V_{\mathrm{GSR}}}{\partial r}=\frac{\mathrm{c}^2}{\xi_1{}^2}\left[\frac{l}{2}\left(\begin{array}{l}ka_{(r)}\frac{r_{\mathrm{S}}}{r^2}-k\frac{\mathrm{d}a}{\mathrm{d}r}\frac{r_{\mathrm{S}}}{r}-\left(\begin{array}{l}2ra_{(r)}{}^4\frac{\mathrm{d}^2a}{\mathrm{d}r^2}\left(1-ka_{(r)}\frac{r_{\mathrm{S}}}{r}\right)\left(r\frac{\mathrm{d}a}{\mathrm{d}r}-a_{(r)}\right)-\\-\left(4a_{(r)}{}^3\frac{\mathrm{d}a}{\mathrm{d}r}-5ka_{(r)}{}^4\frac{\mathrm{d}a}{\mathrm{d}r}\frac{r_{\mathrm{S}}}{r}+ka_{(r)}{}^5\frac{r_{\mathrm{S}}}{r^2}\right)\left(r\frac{\mathrm{d}a}{\mathrm{d}r}-a_{(r)}\right)^2\end{array}\right)\cdot\\ \frac{k\dot r^2}{\mathrm{c}^2a_{(r)}{}^8\left(1-ka_{(r)}\frac{r_{\mathrm{S}}}{r}\right)^2}-\left(a_{(r)}-r\frac{\mathrm{d}a}{\mathrm{d}r}\right)\frac{2kr\dot\phi^2}{\mathrm{c}^2a_{(r)}{}^3}\end{array}\right)\cdot\\ \cdot\left[1-k\left(a_{(r)}\frac{r_{\mathrm{S}}}{r}+\frac{\left(r\frac{\mathrm{d}a}{\mathrm{d}r}-a_{(r)}\right)^2}{a_{(r)}{}^4\left(1-ka_{(r)}\frac{r_{\mathrm{S}}}{r}\right)}\frac{\dot r^2}{\mathrm{c}^2}+\frac{r^2\dot\phi^2}{a_{(r)}{}^2\mathrm{c}^2}\right)\right]^{-\frac{1}{2}}+\xi_1{}^2\frac{r\dot\phi^2}{\mathrm{c}^2}\gamma_{(\xi_1\beta_P)}\right], \tag{264}$$

or equivalently,

$$g=\frac{\partial V_{\mathrm{GSR}}}{\partial r}=\frac{\mathrm{c}^2}{\xi_1{}^2}\left[\frac{l}{2}\left(\begin{array}{l}k\frac{r_{\mathrm{S}}}{r^2}\left(a_{(r)}-r\frac{\mathrm{d}a}{\mathrm{d}r}\right)-\left(\begin{array}{l}2ra_{(r)}{}^4\frac{\mathrm{d}^2a}{\mathrm{d}r^2}\left(1-ka_{(r)}\frac{r_{\mathrm{S}}}{r}\right)-\\-a_{(r)}{}^3\left(4\frac{\mathrm{d}a}{\mathrm{d}r}-5ka_{(r)}\frac{\mathrm{d}a}{\mathrm{d}r}\frac{r_{\mathrm{S}}}{r}+ka_{(r)}{}^2\frac{r_{\mathrm{S}}}{r^2}\right)\left(r\frac{\mathrm{d}a}{\mathrm{d}r}-a_{(r)}\right)\end{array}\right)\cdot\\ \frac{k\left(r\frac{\mathrm{d}a}{\mathrm{d}r}-a_{(r)}\right)\dot r^2}{\mathrm{c}^2a_{(r)}{}^8\left(1-ka_{(r)}\frac{r_{\mathrm{S}}}{r}\right)^2}-\left(a_{(r)}-r\frac{\mathrm{d}a}{\mathrm{d}r}\right)\frac{2kr\dot\phi^2}{\mathrm{c}^2a_{(r)}{}^3}\end{array}\right)\cdot\\ \cdot\left[1-k\left(a_{(r)}\frac{r_{\mathrm{S}}}{r}+\frac{\left(r\frac{\mathrm{d}a}{\mathrm{d}r}-a_{(r)}\right)^2}{a_{(r)}{}^4\left(1-ka_{(r)}\frac{r_{\mathrm{S}}}{r}\right)}\frac{\dot r^2}{\mathrm{c}^2}+\frac{r^2\dot\phi^2}{a_{(r)}{}^2\mathrm{c}^2}\right)\right]^{-\frac{1}{2}}+\xi_1{}^2\frac{r\dot\phi^2}{\mathrm{c}^2}\gamma_{(\xi_1\beta_P)}\right]. \tag{265}$$

The above eqn is further simplified to

$$g=\frac{\partial V_{\mathrm{GSR}}}{\partial r}=\frac{\mathrm{c}^2}{\xi_1{}^2}\left[\frac{l}{2}\left(a_{(r)}-r\frac{\mathrm{d}a}{\mathrm{d}r}\right)\left(\begin{array}{l}k\frac{r_{\mathrm{S}}}{r^2}+\left(\begin{array}{l}2ra_{(r)}{}^4\frac{\mathrm{d}^2a}{\mathrm{d}r^2}\left(1-ka_{(r)}\frac{r_{\mathrm{S}}}{r}\right)-\\+a_{(r)}{}^3\left(4\frac{\mathrm{d}a}{\mathrm{d}r}-5ka_{(r)}\frac{\mathrm{d}a}{\mathrm{d}r}\frac{r_{\mathrm{S}}}{r}+ka_{(r)}{}^2\frac{r_{\mathrm{S}}}{r^2}\right)\left(a_{(r)}-r\frac{\mathrm{d}a}{\mathrm{d}r}\right)\end{array}\right)\cdot\\ \frac{k\dot r^2}{\mathrm{c}^2a_{(r)}{}^8\left(1-ka_{(r)}\frac{r_{\mathrm{S}}}{r}\right)^2}-\frac{2kr\dot\phi^2}{\mathrm{c}^2a_{(r)}{}^3}\end{array}\right)\cdot\\ \cdot\left[1-k\left(a_{(r)}\frac{r_{\mathrm{S}}}{r}+\frac{\left(a_{(r)}-r\frac{\mathrm{d}a}{\mathrm{d}r}\right)^2}{a_{(r)}{}^4\left(1-ka_{(r)}\frac{r_{\mathrm{S}}}{r}\right)}\frac{\dot r^2}{\mathrm{c}^2}+\frac{r^2\dot\phi^2}{a_{(r)}{}^2\mathrm{c}^2}\right)\right]^{-\frac{1}{2}}+\xi_1{}^2\frac{r\dot\phi^2}{\mathrm{c}^2}\gamma_{(\xi_1\beta_P)}\right]. \tag{266}$$

The case of *circular motion* is obtained by replacing (263) and (266) to *equation of motion* (162). We then put r=R=constant and we obtain

$$\frac{c^2}{\xi_I^{\;2}}\left[\frac{l}{2}\left(a_{(r)}-r\frac{\mathrm{d}a}{\mathrm{d}r}\right)\left(k\frac{r_S}{r^2}-\frac{2kr\dot{\phi}^2}{c^2 a_{(r)}^{\;3}}\right)\cdot\left[1-k\left(a_{(r)}\frac{r_S}{r}+\frac{r^2\dot{\phi}^2}{a_{(r)}^{\;2}c^2}\right)\right]^{-\frac{1}{2}}\right]=0\,, \tag{267}$$

or equivalently,

$$\left(a_{(r)}-r\frac{\mathrm{d}a}{\mathrm{d}r}\right)\left(\frac{r_S}{r^2}-\frac{2r\dot{\phi}^2}{c^2 a_{(r)}^{\;3}}\right)=0\,. \tag{268}$$

The physical solution is

$$\frac{r_S}{R^2}-\frac{2R\dot{\phi}^2}{c^2 a_{(R)}^{\;3}}=0\,. \tag{269}$$

This gives Uniform Circular Motion (UCM), with the same *angular velocity* and the same *centripetal acceleration* for any TPs:

$$\omega=\dot{\phi}=\frac{\mathrm{d}\phi}{\mathrm{d}t}=\sqrt{\frac{a_{(R)}^{\;3}\,\mathrm{G}M}{R^3}}\quad;\quad a=\frac{\upsilon^2}{R}=\omega^2R=a_{(R)}^{\;3}\frac{\mathrm{G}M}{R^2}=a_{(R)}^{\;3}g_{\mathrm{N}}\,, \tag{270}$$

Thus, the velocity in UCM is given by the formula

$$\upsilon=a_{(R)}^{\;\frac{3}{2}}\sqrt{\frac{\mathrm{G}M}{R}}\,, \tag{271}$$

Now, let as compare the above *centripetal acceleration* (270ii) to the corresponding *field strength* in UCM. Thus, (266) becomes

$$g=\frac{\partial V_{\mathrm{GSR}}}{\partial r}=\frac{c^2}{\xi_I^{\;2}}\left[\frac{l}{2}\left(a_{(r)}-r\frac{\mathrm{d}a}{\mathrm{d}r}\right)\left(k\frac{r_S}{r^2}-\frac{2kr\dot{\phi}^2}{c^2 a_{(r)}^{\;3}}\right)\cdot\left[1-k\left(a_{(r)}\frac{r_S}{r}+\frac{r^2\dot{\phi}^2}{a_{(r)}^{\;2}c^2}\right)\right]^{-\frac{1}{2}}+\xi_I^{\;2}\frac{r\dot{\phi}^2}{c^2}\gamma_{(\xi_I\beta_P)}\right], \tag{272}$$

By taking $\dot{r}=0$. The replacement of (270i) to the above eqn gives

$$g=\frac{\partial V_{\mathrm{GSR}}}{\partial r}=\frac{c^2}{\xi_I^{\;2}}\left[\frac{lk}{2}\left(a_{(r)}-r\frac{\mathrm{d}a}{\mathrm{d}r}\right)\left(\frac{r_S}{R^2}-\frac{2\mathrm{G}M}{c^2R^2}\right)\cdot\left[1-k\left(a_{(r)}\frac{r_S}{R}+\frac{r^2\dot{\phi}^2}{a_{(r)}^{\;2}c^2}\right)\right]^{-\frac{1}{2}}+\xi_I^{\;2}\frac{r\dot{\phi}^2}{c^2}\gamma_{(\xi_I\beta_P)}\right], \tag{273}$$

or equivalently,

$$g=\frac{\partial V_{\mathrm{GSR}}}{\partial r}=R\dot{\phi}^2\gamma_{(\xi_I\beta_P)}=\gamma_{(\xi_I\beta_P)}a_{(R)}^{\;3}\frac{\mathrm{G}M}{R^2}\,. \tag{274}$$

The *field strength* is also written as

$$g=\frac{1}{\sqrt{1-\xi_I^{\;2}\frac{R^2\dot{\phi}^2}{c^2}}}\frac{\mathrm{G}Ma_{(R)}^{\;3}}{R^2}=\frac{1}{\sqrt{1-\xi_I^{\;2}\frac{\mathrm{G}Ma_{(R)}^{\;3}}{c^2R}}}\frac{\mathrm{G}Ma_{(R)}^{\;3}}{R^2}=\frac{1}{\sqrt{1-\xi_I^{\;2}\frac{a_{(R)}^{\;3}r_S}{2R}}}\frac{\mathrm{G}Ma_{(R)}^{\;3}}{R^2}\,, \tag{275}$$

which depends on the used TPs and also is *larger than the centripetal acceleration* (except for $\xi_I\to0$ that corresponds to NPs, where it is equal).

Finally, we study the *gravitational field on unmoved particle*. Thus, (266) is transformed to

$$g=\frac{\partial V_{\mathrm{GSR}}}{\partial r}=\frac{c^2}{\xi_I^{\;2}}\frac{lk}{2}\frac{r_S}{r^2}\left(a_{(r)}-r\frac{\mathrm{d}a}{\mathrm{d}r}\right)\left(1-ka_{(r)}\frac{r_S}{r}\right)^{-\frac{1}{2}}. \tag{276}$$

The replacement of (5) and condition (218) to the above eqn gives

$$g=\frac{\partial V_{\mathrm{GSR}}}{\partial r}=\frac{\mathrm{G}M}{r^2}\left(a_{(r)}-r\frac{\mathrm{d}a}{\mathrm{d}r}\right)\left(1-ka_{(r)}\frac{r_{\mathrm{S}}}{r}\right)^{-\frac{1}{2}}. \tag{277}$$

We observe that this formula of unmoved particle is very different than the corresponding of UCM (274). We also observe that the field strength is not given by eqn (35). The corresponding initial acceleration is computed as following. Eqn (263) is transformed to

$$\frac{\partial V_{\mathrm{GSR}}}{\partial \dot r}=-\frac{kl\left(a_{(r)}-r\frac{\mathrm{d}a}{\mathrm{d}r}\right)^2}{a_{(r)}{}^4\left(1-ka_{(r)}\frac{r_{\mathrm{S}}}{r}\right)}\frac{\dot r}{\xi_{\mathrm{I}}{}^2}\left[1-k\left(a_{(r)}\frac{r_{\mathrm{S}}}{r}+\frac{\left(a_{(r)}-r\frac{\mathrm{d}a}{\mathrm{d}r}\right)^2}{a_{(r)}{}^4\left(1-ka_{(r)}\frac{r_{\mathrm{S}}}{r}\right)}\frac{\dot r^2}{\mathrm{c}^2}\right)\right]^{-\frac{1}{2}}+\dot r\gamma_{(\xi_{\mathrm{I}}\beta_P)}\ , \tag{278}$$

by taking $\dot\phi=0$. The above eqn and (277) are replaced in (162) and we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{kl\left(a_{(r)}-r\frac{\mathrm{d}a}{\mathrm{d}r}\right)^2}{a_{(r)}{}^4\left(1-ka_{(r)}\frac{r_{\mathrm{S}}}{r}\right)}\frac{\dot r}{\xi_{\mathrm{I}}{}^2}\left[1-k\left(a_{(r)}\frac{r_{\mathrm{S}}}{r}+\frac{\left(a_{(r)}-r\frac{\mathrm{d}a}{\mathrm{d}r}\right)^2}{a_{(r)}{}^4\left(1-ka_{(r)}\frac{r_{\mathrm{S}}}{r}\right)}\frac{\dot r^2}{\mathrm{c}^2}\right)\right]^{-\frac{1}{2}}\right) \tag{279}$$

$$+\frac{\mathrm{G}M}{r^2}\left(a_{(r)}-r\frac{\mathrm{d}a}{\mathrm{d}r}\right)\left(1-ka_{(r)}\frac{r_{\mathrm{S}}}{r}\right)^{-\frac{1}{2}}=0.$$

This leads to

$$\frac{1}{\xi_{\mathrm{I}}{}^2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{kl\left(a_{(r)}-r\frac{\mathrm{d}a}{\mathrm{d}r}\right)^2\dot r}{a_{(r)}{}^4\left(1-ka_{(r)}\frac{r_{\mathrm{S}}}{r}\right)}\right)\left(1-ka_{(r)}\frac{r_{\mathrm{S}}}{r}\right)^{-\frac{1}{2}}+\frac{\mathrm{G}M}{r^2}\left(a_{(r)}-r\frac{\mathrm{d}a}{\mathrm{d}r}\right)\left(1-ka_{(r)}\frac{r_{\mathrm{S}}}{r}\right)^{-\frac{1}{2}}=0\ , \tag{280}$$

by taking also $\dot r=0$. This is equivalent to

$$\left[\frac{1}{\xi_{\mathrm{I}}{}^2}\left(\frac{kl\left(a_{(r)}-r\frac{\mathrm{d}a}{\mathrm{d}r}\right)^2\ddot r a_{(r)}{}^4\left(1-ka_{(r)}\frac{r_{\mathrm{S}}}{r}\right)}{a_{(r)}{}^8\left(1-ka_{(r)}\frac{r_{\mathrm{S}}}{r}\right)^2}\right)+\frac{\mathrm{G}M}{r^2}\left(a_{(r)}-r\frac{\mathrm{d}a}{\mathrm{d}r}\right)\right]\left(1-ka_{(r)}\frac{r_{\mathrm{S}}}{r}\right)^{-\frac{1}{2}}=0\ , \tag{281}$$

by taking once again $\dot r=0$. The above emerges

$$\left(a_{(r)}-r\frac{\mathrm{d}a}{\mathrm{d}r}\right)\left[\frac{1}{\xi_{\mathrm{I}}{}^2}\frac{kl\left(a_{(r)}-r\frac{\mathrm{d}a}{\mathrm{d}r}\right)\ddot r}{a_{(r)}{}^4\left(1-ka_{(r)}\frac{r_{\mathrm{S}}}{r}\right)}+\frac{\mathrm{G}M}{r^2}\right]=0\ . \tag{282}$$

The replacement of condition (218) to the above eqn gives

$$\frac{\left(a_{(r)}-r\frac{\mathrm{d}a}{\mathrm{d}r}\right)\ddot r}{a_{(r)}{}^4\left(1-ka_{(r)}\frac{r_{\mathrm{S}}}{r}\right)}+\frac{\mathrm{G}M}{r^2}=0 \tag{283}$$

and we obtain

$$a_r = \ddot{r} = -\frac{{a_{(r)}}^4}{\left(a_{(r)} - r\frac{\mathrm{d}a}{\mathrm{d}r}\right)}\frac{\mathrm{G}M}{r^2}\left(1 - ka_{(r)}\frac{r_\mathrm{S}}{r}\right). \tag{284}$$

We observe that the *acceleration of unmoved particle* generally depends on the used TPs and also is *different than the corresponding field strength* (except for $a_{(r)}$=1 and k=0 that corresponds to the *Newtonian potential*, where it is equal). Besides, the *acceleration of unmoved particle* on the *modified Schwarzschild radius* ($r=ka_{(r)}r_\mathrm{S}$) is null!

4.4. The Combination of Modified GSR-Gravitational Field (m-GSR-3GSM) with MOND

Modified Newtonian Dynamics (MOND) explains the rotation curves in many galaxies, by using suitable *Interpolating Function* (μ) in *Milgrom's Law* [20]. The spherical or cylindrical distribution of mass, causes *Modified Newtonian acceleration*

$$a = \frac{1}{\mu_{(r)}}\frac{GM}{r^2}\,. \tag{285}$$

In case of UCM, the combination of the above with *M-GSR-3GSM-acceleration* (270ii) emerges

$$\frac{1}{\mu_{(r)}} = {a_{(r)}}^3 \;\;;\; a_{(r)} = \frac{1}{{\mu_{(r)}}^{\frac{1}{3}}}\,. \tag{286}$$

Two common choices are the *Simple* and *Standard interpolating function*, correspondingly

$$\frac{1}{\mu_\mathrm{Simpl}} = 1 + \frac{a_0}{a} = \frac{1}{2}\left(1 + \sqrt{1 + \left(\frac{r}{r_0}\right)^2}\right) \;;\; \frac{1}{\mu_\mathrm{Stand}} = \sqrt{1 + \left(\frac{a_0}{a}\right)^2} = \frac{1}{\sqrt{2}}\sqrt{1 + \sqrt{1 + \frac{1}{4}\left(\frac{r}{r_0}\right)^4}} \;\;;\; r_0 = \sqrt{\frac{\mathrm{G}M}{4a_0}}\,, \tag{287}$$

where r_0 is called *Milgrom radius* [21] (p. 3) and $a_0 = 1.2(\pm 0.1)\times 10^{-10}\,\mathrm{ms}^{-2}$ [20] (p. 1) is an extra (acceleration-dimensional) gravitational constant. The above functions are specifications of the *generalized interpolating function*

$$\frac{1}{\mu_{\lambda,n}} = \left(1 + \left(\lambda\frac{a_0}{a}\right)^n\right)^{\frac{1}{n}} = \frac{1}{2^{\frac{1}{n}}}\left(1 + \sqrt{1 + \frac{\lambda^n}{4^{n-1}}\left(\frac{r}{r_0}\right)^{2n}}\right)^{\frac{1}{n}} \;\;;\; r_0 = \sqrt{\frac{\mathrm{G}M}{4a_0}}\,, \tag{288}$$

for λ=1 and n=1, 2, respectively. Thus we obtain the corresponding acceleration and velocity in UCM:

$$a = \frac{1}{2^{\frac{1}{n}}}\left(1 + \sqrt{1 + \frac{\lambda^n}{4^{n-1}}\left(\frac{r}{r_0}\right)^{2n}}\right)^{\frac{1}{n}}\frac{\mathrm{G}M}{r^2} \;\;;\; \upsilon = \frac{1}{2^{\frac{1}{2n}}}\left(1 + \sqrt{1 + \frac{\lambda^n}{4^{n-1}}\left(\frac{r}{r_0}\right)^{2n}}\right)^{\frac{1}{2n}}\sqrt{\frac{\mathrm{G}M}{r}}\,, \tag{289}$$

which give the same *velocity at infinite distance* from the center of gravity for any value of n:

$$\upsilon_\infty = \sqrt[4]{\lambda\mathrm{G}Ma_0} \;\;;\; \beta_\infty = \frac{1}{\mathrm{c}}\sqrt[4]{\lambda\mathrm{G}Ma_0}\,. \tag{290}$$

Besides, (286) emerges

$$a_{(r)} = \frac{1}{{\mu_{(r)}}^{\frac{1}{3}}} = \frac{1}{2^{\frac{1}{3n}}}\left(1 + \sqrt{1 + \frac{\lambda^n}{4^{n-1}}\left(\frac{r}{r_0}\right)^{2n}}\right)^{\frac{1}{3n}}. \tag{291}$$

The value n=0 gives $1/\mu = a_{(r)} = \infty$, which means *infinite acceleration*. Another interesting value is $n \to \infty$ with

$$\frac{1}{\mu_{\lambda,\infty}}=\lim_{n\to\infty}\left(1+\left(\frac{\lambda a_0}{a}\right)^n\right)^{\frac{1}{n}}=\begin{cases}1 & , \; a\geq\lambda a_0\\ \frac{\lambda a_0}{a}, & a\leq\lambda a_0\end{cases}=\begin{cases}1 & , \; r\leq\frac{2r_0}{\sqrt{\lambda}}\\ \sqrt{\frac{\lambda a_0}{GM}}r, & r\geq\frac{2r_0}{\sqrt{\lambda}}\end{cases}=\begin{cases}1 & , \; r\leq\frac{2r_0}{\sqrt{\lambda}}\\ \frac{\sqrt{\lambda}r}{2r_0}, & r\geq\frac{2r_0}{\sqrt{\lambda}}\end{cases}. \quad (292)$$

Thus, we obtain the corresponding acceleration and velocity in UCM:

$$a=\begin{cases}\frac{GM}{r^2} & , \; r\leq\frac{2r_0}{\sqrt{\lambda}}\\ \frac{\sqrt{\lambda GMa_0}}{r}, & r\geq\frac{2r_0}{\sqrt{\lambda}}\end{cases}=\begin{cases}\frac{GM}{r^2} & , \; r\leq\frac{2r_0}{\sqrt{\lambda}}\\ \frac{\upsilon_\infty{}^2}{r}, & r\geq\frac{2r_0}{\sqrt{\lambda}}\end{cases} \; ; \; \upsilon=\begin{cases}\sqrt{\frac{GM}{r}}, & r\leq\frac{2r_0}{\sqrt{\lambda}}\\ \upsilon_\infty \; , & r\geq\frac{2r_0}{\sqrt{\lambda}}\end{cases}, \quad (293)$$

We observe that $\mu_{\lambda,\infty}$ gives *Newtonian acceleration* near to the center of gravity, while this is inversely proportional to the distance far away the center of gravity. Besides, in UCM the velocity has the well-known formula for $r<2r_0/\text{sqrt}(\lambda)$, while it becomes steady for $r>2r_0/\text{sqrt}(\lambda)$. Thus, $\mu_{\lambda,\infty}$ is inefficient to explain the *rotation curves in galaxies*. Besides, (286) gives

$$a_{\infty(r)}=\frac{1}{\mu_{\infty(r)}{}^{\frac{1}{3}}}=\begin{cases}1 & , \; r\leq\frac{2r_0}{\sqrt{\lambda}}\\ \left(\frac{\sqrt{\lambda}r}{2r_0}\right)^{\frac{1}{3}}, & r\geq\frac{2r_0}{\sqrt{\lambda}}\end{cases}. \quad (294)$$

The specific value $\lambda=1$:

i. gives the well-known original *MONDian acceleration in UCM*, which is also efficient to explain the *rotation curves in galaxies* (for n=1,2,…) as well as the *precession of Mercury's orbit* and the *deflection of light* (because $a_{(r)}\approx\mu\approx 1$ in the Solar system), but

ii. in case of *empty of mass space*: $M\to 0$ ($r_0\to 0$), gives $1/\mu_n\to\infty$ and $a_{(r)}\to\infty$ (even if $n\to\infty$). Thus, the 3GSM (31) gives

$$g_{\theta\theta}=\lim_{M\to 0}\frac{g_{111}r^2}{a_{(r)}{}^2}=0\neq g_{111}r^2 \; ; \; g_{\phi\phi}=\lim_{M\to 0}\frac{g_{111}r^2}{a_{(r)}{}^2}\sin^2\theta=0\neq g_{111}r^2\sin^2\theta . \quad (295)$$

Thus, we do not obtain the metric of RIOs (11), except for the case of *Galilean metric* (19).

iii. gives extra larges values of the acceleration around bodies with small mass [except for $n\to\infty$, where $1/\mu=1$ ($a_{(r)}=1$) for $r<2r_0$]. For instance a body of M=1 Kg (r_0=0.373 m) at distance r=1 m, produces $\mu_{\text{Simp}}=0.518$ ($1/\mu_{\text{Simp}}=1.93$) according to the *Simple interpolating function*. Besides, the above has $\mu_{1,\infty}=0.746$ ($1/\mu_{1,\infty}=1.34$). This means *twice value* and 134% stronger than the *Newtonian acceleration*, respectively. Thus, it contradicts to the *Cavendish experiment*.

In this paper, we make changes to MOND ('New' MOND), resolving the above contradiction (ii). Thus, we define

$$\lambda=\lambda_0=\left(\frac{M}{M+m_0}\right)^2<1 , \quad (296)$$

where m_0 is *unspecified non-zero mass-dimensional constant*. Now, (291) becomes

$$a_{(r)}=\frac{1}{2^{\frac{1}{3n}}}\left(1+\sqrt{1+\frac{1}{4^{n-1}}\left(\frac{M}{M+m_0}\right)^{2n}\left(\frac{r}{r_0}\right)^{2n}}\right)^{\frac{1}{3n}}=\frac{1}{2^{\frac{1}{3n}}}\left(1+\sqrt{1+\frac{1}{4^{n-1}}\left(\frac{M^2}{(M+m_0)^2}\right)^n\left(\frac{4a_0r^2}{GM}\right)^n}\right)^{\frac{1}{3n}} . \quad (297)$$

So, the case of *empty of mass space*: $M\to 0$ emerges

$$\lim_{M\to 0}a_{(r)}=1 . \quad (298)$$

Thus, the 3GSM (31) is transformed to the 1GSM (37), which for $M\to 0$ gives the metric of RIOs (11).

4.5. The Combination of Modified GSR Gravitational Field strength with the concept of phantom Dark Matter and the Velocity at Infinite Distance of MOND

Below, we shall find the metric of spacetime that corresponds to the concept of phantom DM [9] (p. 356). We consider a *very simple distribution of phantom DM*:

$$\rho_{\text{dark}} = \frac{C_{\text{dark}}}{r^2} \;\; ; \;\; M_{\text{dark}} = \int_0^r 4\pi r^2 \rho_{\text{dark}} dr = 4\pi C_{\text{dark}} r \tag{299}$$

and also *all the luminous-baryonic mass at the center of gravity*. In case of a *spherical* or *cylindrical distribution of mass*, the *Modified Newtonian acceleration* is

$$a = \frac{G(M + M_{\text{dark}})}{r^2} = \frac{GM}{r^2}\left(1 + \frac{M_{\text{dark}}}{M}\right) = \frac{GM}{r^2}\left(1 + \frac{4\pi C_{\text{dark}} r}{M}\right) = \frac{GM}{r^2} + \frac{4\pi G C_{\text{dark}}}{r} . \tag{300}$$

The combination of the above to (285) gives

$$\frac{1}{\mu_{\text{DM}}} = 1 + \frac{M_{\text{dark}}}{M} = 1 + \frac{4\pi G C_{\text{dark}} r}{M} . \tag{301}$$

Besides, the velocity in UCM is given by the formula

$$\upsilon^2 = \frac{G(M + M_{\text{dark}})}{r} = \frac{GM}{r} + 4\pi G C_{\text{dark}} , \tag{302}$$

which at infinite distance from the center of gravity, gives

$$\upsilon_\infty^{\;2} = 4\pi G C_{\text{dark}} . \tag{303}$$

The combination of the above equation with the (290) *MONDian formula* gives

$$C_{\text{dark}} = \frac{1}{4\pi}\sqrt{\frac{\lambda M a_0}{G}} = \frac{\sqrt{\lambda} M}{8\pi r_0} . \tag{304}$$

The replacement of the above to the initial eqn (299i) gives

$$\rho_{\text{dark}} = \frac{1}{4\pi}\sqrt{\frac{a_0}{G}} \frac{\sqrt{\lambda}\sqrt{M}}{r^2} = \frac{\sqrt{\lambda}}{8\pi} \frac{M}{r_0 r^2} . \tag{305}$$

Thus, (301) combined to (286) gives

$$\frac{1}{\mu_{(r)}} = a_{(r)}^{\;3} = 1 + \frac{\sqrt{\lambda}}{2}\frac{r}{r_0} \;\; ; \;\; a_{(r)} = \frac{1}{\mu_{(r)}^{\frac{1}{3}}} = \left(1 + \frac{\sqrt{\lambda}}{2}\frac{r}{r_0}\right)^{\frac{1}{3}} . \tag{306}$$

Moreover, (270) and (271) give the corresponding acceleration and velocity in UCM:

$$a = \left(1 + \frac{\sqrt{\lambda}}{2}\frac{r}{r_0}\right)\frac{GM}{r^2} \;\; ; \;\; \upsilon = \sqrt{1 + \frac{\sqrt{\lambda}}{2}\frac{r}{r_0}}\sqrt{\frac{GM}{r}} . \tag{307}$$

Finally, it is proven that the corresponding values of function $a_{(r)}$ have the properties: *Standard Interpolating function < Simple Interpolating function < Absorption of DM into the metric* and also *'New' < 'old'*.

In this paper, we use $m_0 = m_e$ (mass of electron) in (296). Thus, observations in macrocosm has

$$\lambda = \lambda_0 = \left(\frac{M}{M + m_e}\right)^2 \approx 1^- \;\; ; \;\; M >> m_e \tag{308}$$

and we obtain the results of original 'old' MOND.

5. Gravitational Red Shift

We initially present the *Gravitational Red Shift* (GRS) according to GR. Thus, we consider two consecutive wave fronts passing first *A* and then *B* [9] (p.188). Thus, we have four events: $A(t_1)$, $A(t_2)$, $B(t_3)$, $B(t_4)$ and also

$$\mathrm{d}S_A{}^2 = g_{100}\,\mathrm{c}^2\mathrm{d}\tau_A{}^2 = g_{100}\left(1 - a_{(r_A)}\frac{\xi_\mathrm{I}{}^2 r_\mathrm{S}}{r_A}\right)\mathrm{c}^2\mathrm{d}t_A{}^2 \;;\; \mathrm{d}S_B{}^2 = g_{100}\,\mathrm{c}^2\mathrm{d}\tau_B{}^2 = g_{100}\left(1 - a_{(r_A)}\frac{\xi_\mathrm{I}{}^2 r_\mathrm{S}}{r_A}\right)\mathrm{c}^2\mathrm{d}t_B{}^2, \quad (309)$$

by using the 3GSM (31). The square root and integration of the above leads to

$$\mathrm{c}T_A = \sqrt{1 - a_{(r_A)}\frac{\xi_\mathrm{I}{}^2 r_\mathrm{S}}{r_A}}\,\mathrm{c}T \;;\; \mathrm{c}T_B = \sqrt{1 - a_{(r_B)}\frac{\xi_\mathrm{I}{}^2 r_\mathrm{S}}{r_B}}\,\mathrm{c}T, \quad (310)$$

where T_A, T_B and T are the period of the wave for unmoved observers located at A, B and infinite distance, correspondingly. The coordinate time (period of the wave) T is considered the same at A and B (t_2-t_1=t_4-t_3=T). So, we obtain

$$\frac{T_B}{T_A} = \sqrt{\frac{1 - a_{(r_B)}\dfrac{\xi_\mathrm{I}{}^2 r_\mathrm{S}}{r_B}}{1 - a_{(r_A)}\dfrac{\xi_\mathrm{I}{}^2 r_\mathrm{S}}{r_A}}} \;;\; \frac{f_B}{f_A} = \sqrt{\frac{1 - a_{(r_A)}\dfrac{\xi_\mathrm{I}{}^2 r_\mathrm{S}}{r_A}}{1 - a_{(r_B)}\dfrac{\xi_\mathrm{I}{}^2 r_\mathrm{S}}{r_B}}}, \quad (311)$$

where f_A and f_B are the frequencies recognized by observers located at A and B (inversely proportional to the times of passing as measured by standard clocks). The above formula emerges

$$T_{(r)} = T\left(1 - a_{(r)}\frac{\xi_\mathrm{I}{}^2 r_\mathrm{S}}{r}\right)^{\frac{1}{2}} \;;\; f_{(r)} = f_\infty\left(1 - a_{(r)}\frac{\xi_\mathrm{I}{}^2 r_\mathrm{S}}{r}\right)^{-\frac{1}{2}}, \quad (312)$$

where $f_{(r)}$ and f_∞ are the frequencies measured by unmoved observers located at distance r from the center of gravity and at infinite distance, respectively. Besides, we can correlate the corresponding *total GR-energies*, by using E_GR=hf:

$$E_{\mathrm{GR}(r)} = E_\mathrm{GR}\left(1 - a_{(r)}\frac{\xi_\mathrm{I}{}^2 r_\mathrm{S}}{r}\right)^{-\frac{1}{2}}, \quad (313)$$

where $E_{\mathrm{GR}(r)}$ and E_GR are the energies measured by unmoved observers located at distance r from the center of gravity and at infinite distance, respectively.
Now, we define GRS *z-factor*:

$$z = \frac{\lambda_\mathrm{O} - \lambda_\mathrm{EL}}{\lambda_\mathrm{EL}} = \frac{\lambda_\mathrm{O}}{\lambda_\mathrm{EL}} - 1 = \frac{\dfrac{c_\mathrm{E}}{f_\mathrm{O}}}{\dfrac{c_\mathrm{E}}{f_\mathrm{EL}}} - 1 = \frac{f_\mathrm{EL}}{f_\mathrm{O}} - 1. \quad (314)$$

Thus, we calculate

$$z = \frac{f_\mathrm{EL}}{f_\mathrm{O}} - 1 = \frac{f_{(r)}}{f_\infty} - 1 = \left(1 - a_{(r)}\frac{\xi_\mathrm{I}{}^2 r_\mathrm{S}}{r}\right)^{-\frac{1}{2}} - 1 \;;\; z \approx \frac{\xi_\mathrm{I}{}^2 a_{(r)} r_\mathrm{S}}{2r} = \frac{\xi_\mathrm{I}{}^2 a_{(r)}\,\mathrm{G}M}{\mathrm{c}^2\, r}, \quad (315)$$

where λ_O is the observed wavelength of radiation which is produced at distance r from the center of gravity and λ_EL is the wavelength of corresponding radiation that is produced in Earth Laboratory (both of them are measured by unmoved observers on Earth, where the speed of light is c_E). The above exact and approximate formula (in case of large distance from the center of gravity), has come by considering

$$f_{(r)} = f_\mathrm{EL}. \quad (316)$$

More specifically, ξ_I=1 gives the *Einsteinian-Lorentzian* 3GSM-results:

$$\frac{T_B}{T_A}=\sqrt{\frac{1-a_{(r)}\dfrac{r_{\mathrm{S}}}{r_B}}{1-a_{(r)}\dfrac{r_{\mathrm{S}}}{r_A}}}\quad;\quad\frac{f_B}{f_A}=\sqrt{\frac{1-a_{(r)}\dfrac{r_{\mathrm{S}}}{r_A}}{1-a_{(r)}\dfrac{r_{\mathrm{S}}}{r_B}}}\,,\tag{317}$$

$$T_{(r)}=T\left(1-\frac{\xi_{\mathrm{I}}{}^2 r_{\mathrm{S}}}{r}\right)^{\frac{1}{2}}\quad;\quad f_{(r)}=f_{\infty}\left(1-\frac{\xi_{\mathrm{I}}{}^2 r_{\mathrm{S}}}{r}\right)^{-\frac{1}{2}},\tag{318}$$

$$z=\left(1-a_{(r)}\frac{r_{\mathrm{S}}}{r}\right)^{-\frac{1}{2}}-1\quad;\quad z\approx a_{(r)}\frac{r_{\mathrm{S}}}{2r}=a_{(r)}\frac{\mathrm{G}M}{\mathrm{c}^2\,r}\,.\tag{319}$$

The choice $a_{(r)}$=1 leads to the 1GSM-results:

$$\frac{T_B}{T_A}=\sqrt{\frac{1-\dfrac{\xi_{\mathrm{I}}{}^2 r_{\mathrm{S}}}{r_B}}{1-\dfrac{\xi_{\mathrm{I}}{}^2 r_{\mathrm{S}}}{r_A}}}\quad;\quad\frac{f_B}{f_A}=\sqrt{\frac{1-\dfrac{\xi_{\mathrm{I}}{}^2 r_{\mathrm{S}}}{r_A}}{1-\dfrac{\xi_{\mathrm{I}}{}^2 r_{\mathrm{S}}}{r_B}}}\,,\tag{320}$$

$$T_{(r)}=T\left(1-\frac{\xi_{\mathrm{I}}{}^2 r_{\mathrm{S}}}{r}\right)^{\frac{1}{2}}\quad;\quad f_{(r)}=f_{\infty}\left(1-\frac{\xi_{\mathrm{I}}{}^2 r_{\mathrm{S}}}{r}\right)^{-\frac{1}{2}},\tag{321}$$

$$z=\left(1-\frac{\xi_{\mathrm{I}}{}^2 r_{\mathrm{S}}}{r}\right)^{-\frac{1}{2}}-1\quad;\quad z\approx\frac{\xi_{\mathrm{I}}{}^2 r_{\mathrm{S}}}{2r}=\frac{\xi_{\mathrm{I}}{}^2\,\mathrm{G}M}{\mathrm{c}^2\,r}\,.\tag{322}$$

We observe that there is no-GRS, in case that light is emitted from position with $r\rightarrow+\infty$ (for any TPs) or $\xi_{\mathrm{I}}\rightarrow 0$ (NPs). Besides, ERT (with ξ_{I}=1) gives the well-known original *Schwarzschild-GRS*:

$$\frac{T_B}{T_A}=\sqrt{\frac{1-\dfrac{r_{\mathrm{S}}}{r_B}}{1-\dfrac{r_{\mathrm{S}}}{r_A}}}\quad;\quad\frac{f_B}{f_A}=\sqrt{\frac{1-\dfrac{r_{\mathrm{S}}}{r_A}}{1-\dfrac{\xi_{\mathrm{I}}{}^2 r_{\mathrm{S}}}{r_B}}}\,,\tag{323}$$

$$T_{(r)}=T\left(1-\frac{r_{\mathrm{S}}}{r}\right)^{\frac{1}{2}}\quad;\quad f_{(r)}=f_{\infty}\left(1-\frac{r_{\mathrm{S}}}{r}\right)^{-\frac{1}{2}},\tag{324}$$

$$z=\left(1-\frac{r_{\mathrm{S}}}{r}\right)^{-\frac{1}{2}}-1\quad;\quad z\approx\frac{r_{\mathrm{S}}}{2r}=\frac{\mathrm{G}M}{\mathrm{c}^2\,r}\,.\tag{325}$$

The application of formula (325) to the Sun surface {r=6.9599×10^{8} m, M=1,988,500×10^{24} kg [18] and G=6.67428(67)×10^{-11}m^{3}kg^{-1}s^{-2}, c=299792458 ms^{-1} (exact) [17] (pp. 1-1, 1-20, 14-2) } emerges $z_{\mathrm{theoretical}}$=2.12244×10^{-6}. In case that we examine the 74 *strong lines* of the spectrum of iron Fe(I), we obtain R=$z_{\mathrm{observed}}/z_{\mathrm{theoretical}}$=0.97(0.16), while all the 738 (weak, medium and strong) lines have R=$z_{\mathrm{observed}}/z_{\mathrm{theoretical}}$ =0.76(0.24) [22] (p. 247).

In case of GSR, the GRS is explained via a different way. Let us consider a ray of light (E/M wave) emitted from source at distance r from the center of gravity. The corresponding period (frequency) of the wave T (f) is considered the same at *any point* for unmoved observers located anywhere, because the space has steady curvature and there exist no time dilation. Thus, the only way to obtain again the above GR-results, is the consideration that (316) is invalid and the emitted radiation is affected by gravitation via the formula

$$f_{\mathrm{O}} = f_{(r)} = f_{\infty} = \sqrt{1 - a_{(r)} \frac{k r_{\mathrm{S}}}{r}} f_{\mathrm{EL}} \,. \tag{326}$$

(i) The first option of GSR ($k=\xi_{\mathrm{I}}^2$) transforms (326) to

$$f_{\mathrm{O}} = f_{(r)} = f_{\infty} = \sqrt{1 - a_{(r)} \frac{\xi_{\mathrm{I}}^{\,2} r_{\mathrm{S}}}{r}} f_{\mathrm{EL}} \,, \tag{327}$$

which gives the 3GSM-results. More specifically, $\xi_{\mathrm{I}}=1$ transforms (327) to

$$f_{\mathrm{O}} = f_{(r)} = f_{\infty} = \sqrt{1 - a_{(r)} \frac{r_{\mathrm{S}}}{r}} f_{\mathrm{EL}} \tag{328}$$

that leads to the *Einsteinian-Lorentzian* 3GSM-results.
The choice $a_{(r)}=1$ transforms (327) to

$$f_{\mathrm{O}} = f_{(r)} = f_{\infty} = \sqrt{1 - \frac{\xi_{\mathrm{I}}^{\,2} r_{\mathrm{S}}}{r}} f_{\mathrm{EL}} \tag{329}$$

which gives the 1GSM-results. More specifically, $\xi_{\mathrm{I}}=1$ transforms (329) to

$$f_{\mathrm{O}} = f_{(r)} = f_{\infty} = \sqrt{1 - \frac{r_{\mathrm{S}}}{r}} f_{\mathrm{EL}} \,, \tag{330}$$

that leads to the original *Schwarzschild metric*-results.

(ii) The second option of GSR (k=1) transforms (326) to (328), which gives again the *Einsteinian-Lorentzian* 3GSM-results. More specifically, $a_{(r)}=1$ transforms (328) to (330), which leads again to the original *Schwarzschild metric*-results.

6. Experimental Validation - Discussion

In Table 2, we show the values of characteristic parameters for the original 1Kg, the Earth, the Sun [data from [17] (pp. 1-1, 14-2)], Galaxy NGC 3198 (data from [23] (p. 56) and [24] (p. 3)) and the Observable Universe (data from [25] (p. 43) and [26] (p. 27)). Besides, M is the mass that is enclosed in a sphere of radius R, r_0 is Milgrom radius, υ_{∞} is new velocity at infinite distance and $\beta_{0\infty}$ is the corresponding velocity factor. The inverse of the *Interpolating functions* $1/\mu_{\mathrm{simpl}}$, $1/\mu_{\mathrm{stand}}$ and $1/\mu_{\mathrm{DM}}$ as well as *functions* a_{simpl}, a_{stand} and a_{DM} on a sphere of radius R and they have been obtained from (287i), (287ii), (306i), (291) for n=1,2 and (306) for λ=1, respectively. Besides, we have used the following values of physical constants: $a_0=1.2(0.1)\times10^{-10}$ ms^{-2} [20] (p.1), AU=$1.4959787066\times10^{11}$ m, G=$6.67428(67)\times10^{-11}$m^{3}kg^{-1}s^{-2}, c=299792458 ms^{-1} (exact) [17] (pp. 1-1, 1-20, 14-2).

6.1. The Combination 3rd Generalized Schwarzschild Metric or Modified GSR-Gravitational Field with MOND Simple & Standard Interpolating Function and Absorption of the Dark Matter into the field in Galaxy NGC 3198

In order to find out what is the effect of the modification at large mass and size systems, we analytically examine Galaxy NGC 3198.

The values of Circular Velocities [experimental (V_{exp}) and calculated by the Combination of *3GSM* or *Modified GSR-Gravitational Field* with the corresponding *Simple* μ (V_{simp}), or *Standard* μ (V_{stand}), or *Absorption of DM into the Metric by using distribution (305) for* $\lambda=1$ (V_{DM})], the Luminous Mass of the galaxy that is enclosed within the circular orbit (M_{d}), the corresponding values of the function $1/\mu_{(r)}$ ($1/\mu_{\mathrm{simp}}$, $1/\mu_{\mathrm{stand}}$, $1/\mu_{\mathrm{DM}}$), function $a_{(r)}$ (a_{simp}, a_{stand}, a_{DM}) wrt the distance from the center of Galaxy NGC 3198, are contained in Table 3 (data from [27] (p. 2)). The Circular Velocities (V_{simp}, V_{stand}, V_{DM}) have been calculated by using (289ii) for n=1, 2 and (307ii) for λ=1, the values of function $1/\mu_{(r)}$ ($1/\mu_{\mathrm{simp}}$, $1/\mu_{\mathrm{stand}}$, $1/\mu_{\mathrm{DM}}$), by using (287i), (287ii) and (306i) for λ=1, the values of function $a_{(r)}$ (a_{simp}, a_{stand}, a_{DM}), by using (291) for n=1, 2 and (306ii) for λ=1, respectively. The experimental values (a_{exp}) have been obtained, by replacing the experimental velocity (V_{exp}) in (271).

In Figure 2, we show the plot of function $a_{(r)}$ wrt the distance from the center of Galaxy NGC 3198 for the Combination of *3GSM* or *Modified GSR-Gravitational Field* with *Simple* μ (a_{simp}), or *Standard* μ (a_{stand}), or *Absorption of phantom Dark Matter into the Metric by using distribution (306ii) for* $\lambda=1$ (a_{DM}). The experimental values (a_{exp}) have been obtained, by replacing the experimental velocity (V_{exp}) in (271). In addition, the corresponding Rotation Curves in Galaxy NGC 3198 are shown in Figure 3.

We observe that in case of Galaxy NGC 3198, *Schwarzschild* or *Newtonian field strength* produces maximum relative error about 66% at extra large distances. The *Simple* μ gives better results, producing maximum relative error 39% near to the galactic center. The *Standard* μ gives even better results, producing maximum relative error about 23% at the center of the galaxy. The *Absorption of phantom DM into the Metric by using distribution (305) for* $\lambda=1$ (V_{DM}) has maximum relative error 54% near to the galactic center. It is noted that the relative error of experimental Circular Velocities is $(\Delta V_{\text{exp}})_{\text{r}} \approx 8\%$ related to the uncertainty of the Hubble constant H_0 [9] (pp. 356-357). Finally, the values at distance 13.8 Mpc=2.846×10^{12} AU=4.258×10^{23} m, which is the distance of Galaxy NGC 3198 from Earth [28], give us the image of what happens at extremely large distances. The replacement of $a_{(r)}$=12.998 to the 3GSM (31) gives g_{00}=0,9999999969$g_{\text{I}00} \rightarrow g_{\text{I}00}$. This means that if $r\rightarrow\infty$, then 3GSM→ metric of RIOs (11).

The same procedure can be followed in any galaxy, by using only the mass of the visible disk. Thus, it explains the rotation curves of many galaxies, eliminating the corresponding DM (see Figure 4 [28]). Besides, we can obtain even better results, by using *value* of n in (288) and (291): $1< n <2$, or other *distribution of phantom DM* such as in [29] (p. 13) that contains the *core radius* R_0.

Table 2. Characteristic parameters (mass M, distance or size radius R, *Schwarzschild radius* r_S, *Milgrom radius* r_0, r_0/r_S, velocity at infinite distance υ_∞, β_∞, $1/\mu_{\text{simp}}$, $1/\mu_{\text{stand}}$, $1/\mu_{\text{DM}}$, a_{simp}, a_{stand}, a_{DM} on a sphere of radius R) for the original 1 Kg, the Earth, the Sun, galaxy NGC 3198 and the Observable Universe.

	1 Kg (original)	**Earth**	**Sun**	**NGC 3198**	**Observable Universe**
M / Kg	1	5.9742×10^{24} (1)	1.9891×10^{30} (1)	6.76294×10^{40} (2)	10^{53} (4)
R / m	1	6378140 (1)	6.9599×10^{8} (1)	2.47×10^{20}	4.3×10^{26}
/AU	6.68×10^{-12}	4.263523×10^{-5}	4.6524×10^{-3}	1.65×10^{9}	2.9×10^{15}
/ Kpc	3.24×10^{-20}	2.066999×10^{-13}	2.2555×10^{-11}	8 (3)	14×10^{6} (5)
r_s / m	2.96×10^{-27}	8.8736×10^{-3}	2,954.4	1.004451×10^{14}	1.48×10^{26}
/AU	1.98×10^{-38}	5.9316×10^{-14}	1.9749×10^{-8}	671.434	9.9×10^{14}
/ Kpc	9.61×10^{-47}	2.8757×10^{-22}	9.5746×10^{-17}	0.0000680703	4.80×10^{6}
r_0 / m	0.373	9.1143×10^{11}	5.2591×10^{14}	9.6972671×10^{19}	1.18×10^{26}
/AU	2.49×10^{-12}	6.0925	3,515.5	6.45222×10^{8}	7.9×10^{14}
/ Kpc	1.21×10^{-20}	2.9537×10^{-8}	0.000017043	3.14265	3.8×10^{6}
r_0/r_s	1.26×10^{26}	1.02712×10^{14}	1.7801×10^{11}	965,430	0.80
υ_∞ / m s^{-1}	9.45×10^{-6}	14.7899	355.27	152,556	1.68×10^{8}
β_∞	3.15×10^{-14}	4.93339×10^{-8}	1.1851×10^{-6}	0.000508873	0.56
$1/\mu_{\text{simpl}}$	1.93	$1+1.22\times10^{-11}$	$1+4.38\times10^{-13}$	1.86819	2.39
$1/\mu_{\text{stand}}$	1.54	$1+2\times10^{-16}$	$1+2\times10^{-16}$	1.48232	1.97
$1/\mu_{\text{DM}}$	2.34	$1+3.50\times10^{-6}$	$1+6.62\times10^{-7}$	2.27355	2.82
a_{simpl}	1.25	$1+4.08\times10^{-12}$	$1+4.86\times10^{-14}$	1.23161	1.34
a_{stand}	1.15	$1+6.67\times10^{-17}$	$1+6.67\times10^{-17}$	1.14020	1.25
a_{DM}	1.33	$1+3.89\times10^{-7}$	$1+2.21\times10^{-7}$	1.31493	1.41

[1][17] (pp. 1-1, 14-2), [2][23] (p. 56), [3][24] (p. 3), [4] [25] (p. 43), [5] [26] (p. 27).

Table 3. Circular Velocities [experimental (V_{exp}) and calculated by the Combination of *Lorentzian-Einsteinian 3rd Generalized Schwarzschild metric* or *Modified GSR Gravitational Field* with the corresponding *Simple μ* (V_{simp}) or *Standard μ* (V_{stand}) or *Absorption of DM into the Metric by using distribution (305) for λ=1* (V_{DM})], the Luminous Mass of the galaxy that is enclosed within the circular orbit (M_d), the corresponding values of function $1/\mu_{(r)}$ and function $a_{(r)}$ wrt the distance from the center of Galaxy NGC 3198. The relative errors of the experimental Velocities are $(\Delta V_{exp})_r \approx 8\%$ [9] (pp. 356-357).

r / Kpc /10^{20}m	***M*****d** /10^{40} kg	***V*****exp** [(1)] / Km s^{-1}	***a*****exp**	**1/*μ*****simp** **1/*μ*****stand** **1/*μ*****DM**	***a*****simp** ***a*****stand** ***a*****DM**	***V*****simp** ***V*****stand** ***V*****DM** / Km s^{-1}	**(Δ*V*)r** %
4.0	1.620	118.0	1.2607	1,8931	1.2371	128.783	9
1.23				1.5043	1.1458	114.801	-3
				2.3002	1.3201	141.959	20
8.0	5.825	150.3	1.1976	1.9597	1.2514	175.687	17
2.47				1.5640	1.1608	156.950	4
				2.3714	1.3335	193.262	29
16.1	7.237	155.3	1.5751	3.0263	1.4464	171.526	10
4.97				2.5792	1.3714	158.351	-2
				3.4763	1.5149	183.838	18
32.2	6.544	148.4	2.2383	5.7321	1.7897	158.734	7
9.94				5.2564	1.7387	152.004	2
				6.2081	1.8379	165.194	11
48.2	6.072	151.9	2.9100	8.6086	2.0495	153.157	1
14.87				8.1242	2.0103	148.785	-2
				9.0932	2.0872	157.409	4
13800	6.763	-	-	2,196.1	12.998	152.574	-
4,258.3			-	2,195.6	12.997	152.557	-
			-	2,196.6	12.999	152.591	-

[1] [27] (p. 2)

6.2. *The Combination of 3rd Generalized Schwarzschild Metric or Modified GSR-Gravitational Field with MOND Simple & Standard Interpolating Function* or *Absorption of Dark Matter into the field in the Solar System*

In order to find out what is the effect of the modification at medium mass and size systems, we now examine our Solar System.

The mean values of Rotational Velocities, the Mass of the Solar System that is enclosed within the orbit wrt the mean distance the planet from the Sun, are contained in Table 4 [data from [17] (p. 14-3)]. The Circular Velocities (V_{Schwar}, V_{simp}, V_{stand}) have been calculated, by using (243iii), (289ii) for λ=1 and n=1, 2, respectively. The values of function $1/\mu_{(r)}$ ($1/\mu_{simp}$, $1/\mu_{stand}$) have been calculated, by using (287i), (287ii) and (306i) for λ=1, the values of function $a_{(r)}$ (a_{exp}, a_{simp}, a_{stand}), by using (271) and (291) for n=1,2, respectively. The coefficients of metric (g_{00}) have been obtained, by replacing the corresponding values of $a_{(r)}$ and also g_{I00}=-1, g_{I11}=1 in (31). In addition, the corresponding Rotation Curves and Mass Distribution in the Solar System are shown in Figure 10 and Figure 11 of [21], respectively.

We observe that in case of Solar System, the Combination of *3GSM* or *Modified GSR-Gravitational Field* with MOND *Simple* or *Standard μ,* gives almost the same Rotational Velocities (V_{simp}, V_{stand}) and the same coefficients of metric (by taking g_{I00}=-1 and g_{I11}=1) (g_{00}) as those calculated by the original

Schwarzschild metric (V_{Schwar}, $g_{00,\text{Lor}}$), because it is $a_{(r)} \approx 1$. Thus, there are not significant changes to the *Relativistic Doppler Shift*, the *gravitational red shift* as well as the *precession of Mercury's orbit* (g_{00}=0.9999999490). Finally, the values at distance 13.8 Mpc=2.846×10^{12} AU=4.258×10^{23} m, (which is the distance of Galaxy NGC 3198 from Earth [23]) give us the image of what happens at extra large distances. The replacement of $a_{(r)}$=739.61 to the 3GSM (31) gives $g_{00}=(1- 5.13\times10^{-18})g_{I00} \rightarrow g_{I00}$. This means that if $r\rightarrow\infty$, then 3GSM→ metric of RIOs (11). Besides, the corresponding velocity in UCM is $V_{\text{simp}}=V_{\text{stand}}$=355.99 ms$^{-1} \neq 0$.

7. Conclusions

The gravitational field can be described equally well, by using *Metrics* according to General Relativity (GR), or *Generalized Potential* according to Special Relativity (SR) and Newtonian Physics (NPs). In this paper, we also use Generalized Special Relativity (GSR) that unifies SR and NPs. Thus, GR is correlated to SR and NPs via the corresponding *GSR-Lagrangian*. More specifically, the *Rotation Curves in Galaxies* are explained, by using the 3rd Generalized Schwarzschild Metric (3GSM) according to General Relativity, or the Modified GSR-3rd Generalized Schwarzschild Potential (M-GSR-3GSP) according to GSR, eliminating the corresponding Dark Matter (DM). The above contain the unspecified function $a_{(r)}$ that is determined, by using *extra-modified interpolating functions* of Modified Newtonian Dynamics (MOND), or *Distributions of phantom DM*. In scale of non rotating black hole, planetary and star system, it is $a_{(r)} \approx 1$. Thus, the 3GSM, or M-GSR-3GSP are simplified to the 1st Generalized Schwarzschild Metric (1GSM) according to GR, or the Modified GSR-1st Generalized Schwarzschild Potential (M-GSR-1GSP) according to SR and NPs, which explain the *Precession of Mercury's perihelion*, *Deflection of Light* and *Gravitational Red Shift*.

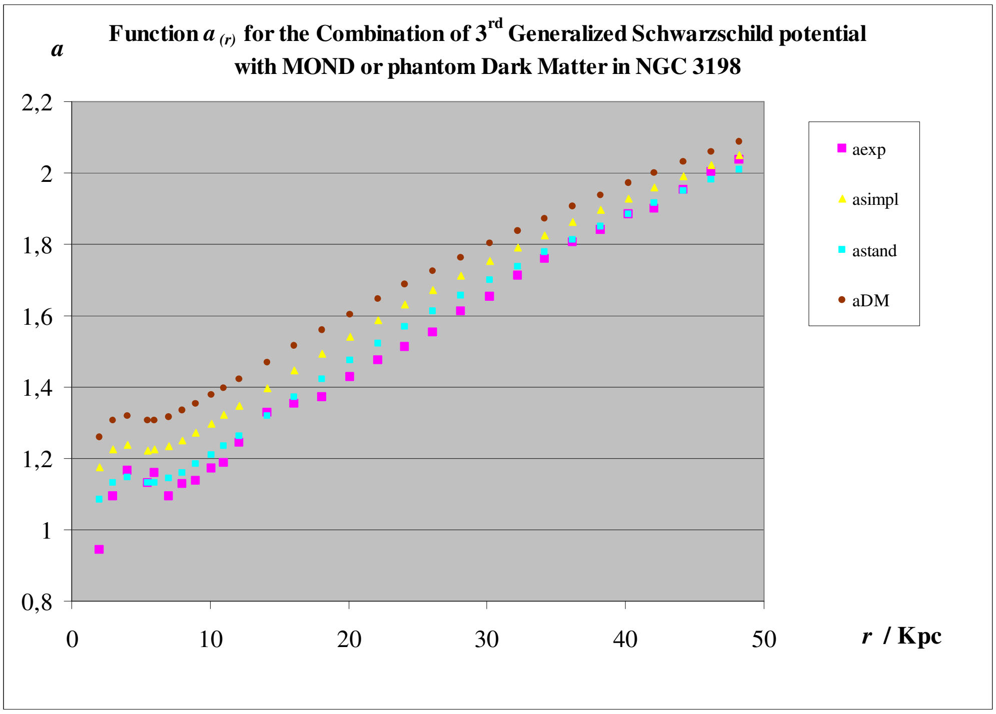


Figure 2. Plot of function $a_{(r)}$ wrt the distance (r) from the center of Galaxy NGC 3198 for the Combination of *3rd Generalized Schwarzschild metric* or *Modified GSR-Gravitational Field* with *Simple interpolating function* (a_{simp}), or *Standard interpolating function* (a_{stand}), or *Absorption of phantom Dark Matter into the Metric by using distribution (305) for λ=1* (a_{DM}). The experimental values (a_{exp}) have been obtained, by replacing the experimental velocity (V_{exp}) in (271).

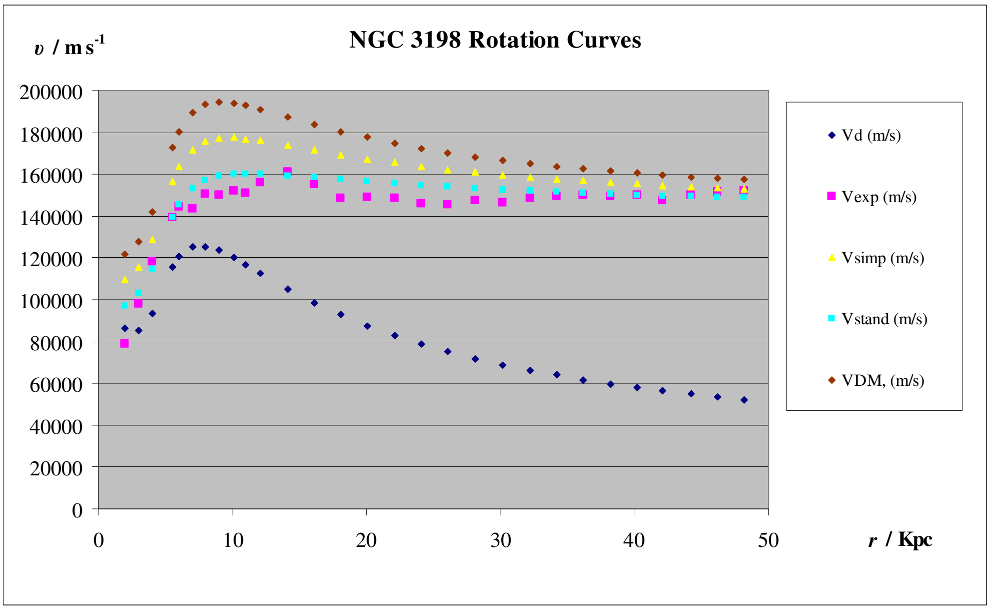


Figure 3. Rotation Curves in Galaxy NGC 3198. Rotational Velocities [experimental (V_{exp}), calculated by *Schwarzschild* or *Newtonian field strength* (V_d) and the Combination of *3rd Generalized Schwarzschild metric* or *Modified GSR-Gravitational Field* with *Simple interpolating function* (V_{simp}), or *Standard interpolating function* (V_{stand}), or *Absorption of phantom Dark Matter into the Metric by using distribution (305) for λ=1* (V_{DM})] wrt the distance (r) from the center of Galaxy NGC 3198.

Galaxies well fit by MOND

84 listed at present

UGC 2885 NGC 5533 NGC 6674 NGC 7331 NGC 5907 NGC 2998
NGC 801 NGC 5371 NGC 5033 NGC 2903 NGC 3521 NGC 2683 NGC 3198
NGC 6946 NGC 2403 NGC 6503 NGC 1003 NGC 247 NGC 7739 NGC 300
NGC 5585 NGC 55 NGC 1560 NGC 3109 UGC 128 UGC 2259 M 33
IC 2574 DDO 170 DDO 168 NGC 3726 NGC 3769 NGC 3877 NGC 3893
NGC 3917 NGC 3949 NGC 3953 NGC 3972 NGC 3992 NGC 4010
NGC 4013 NGC 4051 NGC 4085 NGC 4088 NGC 4100 NGC 4138
NGC 4157 NGC 4183 NGC 4217 NGC 4389 UGC 6399 UGC 6446
UGC 6667 UGC 6818 UGC 6917 UGC 6923 UGC 6930 UGC 6973
UGC 6983 UGC 7089 NGC 1024 NGC 3593 NGC 4698 NGC 5879 IC 724
F563-1 F563-V2 F568-1 F568-3 F568-V1 F571-V1 F574-1 F583-1
F583-4 UGC 1230 UGC 5005 UGC 5999 Carina Fornax
Leo I Leo II Sculptor Sextans Sgr

Figure 4. Galaxies with rotation curves well fit by MOND [31].

Table 4. Rotational Velocities [experimental (V_{exp}) and calculated by the Combination of *3rd Generalized Schwarzschild metric* or *Modified GSR-Gravitational Field* with MOND *Simple or Standard Interpolating Function* (V_{simp}, V_{stand})], the Luminous Mass of the Solar System that is enclosed within the circular orbit (M_d), the corresponding values of function $1/\mu_{(r)}$, function $a_{(r)}$ and time coefficient of metric (by taking g_{I00}=-1 and g_{I11}=1) (g_{00}) wrt the mean distance from the Sun. Data from [17] (p. 14-3).

Name	***r*** /AU /10^{11} m	$\boldsymbol{M_d}$ /10^{24} kg	$1/\mu_{Schwar}=1$ $1/\mu_{simp}$ $1/\mu_{stand}$	$a_{Schwar}=1$ a_{simp} a_{stand}	$g_{00,Schwar}$ $g_{00,simp,Lor}$ $g_{00,stand,Lor}$	V_{Schwar} V_{simp} V_{stand} / Km s^{-1}
Sun	0.00465	1,989,100	1	1	-0.9999957553	436.747
Surface	0.00696		1.00000000000	1.00000000000	-0,9999957553	436.747
			1.00000000000	1.00000000000	-0.9999957553	436.747
Mercury	0.38710	1,989,100.0000	1	1	-0.9999999490	47.880
	0.57909		1.00000000303	1.00000000101	-0.9999999490	47.880
			1.00000000000	1.00000000000	-0.9999999490	47.880
Venus	0.72333	1,989,100.3302	1	1	-0.9999999727	35.027
	1.08209		1.00000001058	1.00000000353	-0.9999999727	35.027
			1.00000000000	1.00000000000	-0.9999999727	35.027
Earth	1.00000	1,989,105.1992	1	1	-0.9999999803	29.790
	1.49598		1.00000002023	1.00000000674	-0.9999999803	29.790
			1.00000000000	1.00000000000	-0.9999999803	29.790
Mars	1.52369	1,989,111.1715	1	1	-0.9999999870	24.134
	2.27941		1.00000004696	1.00000001565	-0.9999999870	24.134
			1.00000000000	1.00000000000	-0.9999999870	24.134
Jupiter	5.20283	1,989,111.8134	1	1	-0.9999999962	13.060
	7.78332		1.00000054758	1.00000018253	-0.9999999962	13.060
			1.00000000000	1.00000000000	-0.9999999962	13.060
Saturn	9.53876	1,991,010.6134	1	1	-0.9999999979	9.650
	14.26978		1.00000183881	1.00000061294	-0.9999999979	9.650
			1.00000000000	1.00000000000	-0.9999999979	9.650
Uranus	19.19139	1,991,579.1134	1	1	-0.9999999990	6.804
	28.70991		1.00000744114	1.00000248038	-0.9999999990	6.804
			1.00000000002	1.00000000001	-0.9999999990	6.804
Neptune	30.06107	1,991,665.7384	1	1	-0.9999999993	5.437
	44.97072		1.00001825627	1.00000608539	-0.9999999993	5.437
			1.00000000017	1.00000000006	-0.9999999993	5.437
Pluto	39.52940	1,991,768.5184	1	1	-0.9999999995	4.741
	59.13514		1.00003156571	1.00001052179	-0.9999999995	4.741
			1.00000000050	1.00000000017	-0.9999999995	4.741
NGC	2.846×10^{12}	1,991,768.5334	1	1	-1,0000000000	3.12×10^{-7}
3198	4.258×10^{12}		404,577,538.2	739.6062784	-1,0000000000	355.39
			404,577,537.7	739.6062781	-1,0000000000	355.39

Abbreviations-Annotations

1GSL: 1st Generalized Schwarzschild Lagrangian
1GSM: 1st Generalized Schwarzschild Metric
1GSP: 1st Generalized Schwarzschild Potential
1GSRP: 1st Generalized Schwarzschild Relativistic Potential
3GSL: 3rd Generalized Schwarzschild Lagrangian
3GSM: 3rd Generalized Schwarzschild Metric
3GSRP: 3rd Generalized Schwarzschild Relativistic Potential
CCs: Cartesian Coordinates
c_I: Universal Speed
DM: Dark Matter

EGR: Einsteinian General Relativity
EP: Equivalence Principle
ERT: Einstein Relativity Theory
ESR: Einsteinian Special Relativity
GDL: Gravitational Deflection of Light
GEE: Gravito-Electric Effect
GME: Gravito-Magnetic Effect
GR: General Relativity
GRS: Gravitational Red Shift
GSR: Generalized Special Relativity
GSR-1GSP: GSR-1st Generalized Schwarzschild Potential
GSR-3GSP: GSR-3rd Generalized Schwarzschild Potential
GT: Galilean Transformation
ICLSTTs: Isometric Closed Linear Transformations of Complex Spacetime
LSTT: Linear Spacetime Transformation
LB: Lorentz Boost
M-GSR-3GSP: Modified GSR-1st Generalized Schwarzschild Potential
M-GSR-3GSP: Modified GSR-3rd Generalized Schwarzschild Potential
MOND: Modified Newtonian Dynamics
NPs: Newtonian Physics
PMP: Precession of Mercury's Perihelion
r_0: Milgrom radius
RB: Real Boost
RIOs: Relativistic Inertial observers
r_S: Schwarzschild radius
r_{SI}: 1st Generalized Schwarzschild radius
r_{SM}: Modified Generalized Schwarzschild radius
RT: Relativity Theory
SM: Schwarzschild Metric
SR: Special Relativity
TPs: Theory of Physics
UCM: Uniform Circular Motion
μ: Interpolating function

References

[1] Phipps T E 1986 Mercury's precession according to special relativity *Am. J. Phys.* **54** (3) 245-247. [DOI: 10.1119/1.14664]
[2] Tsamparlis M 2010 *Special relativity: An introduction with 200 problems and solutions* (Berlin Heidelberg: Springer-Verlag). [ISBN: 978-3-642-03836-5, e-ISBN: 978-3-642-03837-2]
[3] Goldstein H 1980 *Classical Mechanics*, 2nd Edition (Cambridge, London: Addison-Wesley). [ISBN: 0-201-02969-3]
[4] Park R S et al 2017 Precession of Mercury's Perihelion from Ranging to the MESSENGER Spacecraft *Astronomical Journal* **153**, 121, pp 1-7. [DOI: 10.3847/1538-3881/aa5be2]
[5] Einstein, A 1920 *Relativity: The Special and General Theory*; (Holt, New York, USA). Translated by Robert W. Lawson.
[6] Vossos S, Vossos E and Massouros Ch G 2019 New Central Scalar Gravitational Potential according to Special Relativity and Newtonian Physics explains the Precession of Mercury's Perihelion, the Gravitational Red Shift and the Rotation Curves in Galaxies, eliminating Dark Matter *J Phys: Conf Ser* 1391, 012095. [DOI: 10.1088/1742-6596/ 1391/1/012095]
[7] von Soldner J G 1804 Ueber die Ablenkung eines Lichtstrals von seiner geradlinigen Bewegung, durch die Attraktion eines Weltkörpers, an welchem er nahe vorbei geht *Berliner Astronomisches Jahrbuch* **1804**, 161-172. https://de.wikisource.org/w/index.php?title=Ueber_die_Ablenkung_eines_Lichtstrals_von_seiner_geradlinigen_Bewegung&oldid=2052200

[8] Einstein A 1911 On the Influence of Gravitation on the Propagation of Light *Annalen der Physik*, **35**, 898-908. Translated by Michael D. Godfrey. http://gallica.bnf.fr/ark:/12148/bpt6k15338w.image
[9] Rindler W 2006 Relativity: Special, General and Cosmological (New York: Oxford University Press), pp 205, 229-245, 364. [ISBN: 978-0-19-856732-5].
[10] Einstein A 1920 *Relativity: The Special and General Theory* (New York, USA: Holt). Translated by Robert W. Lawson.
[11] Vossos S and Vossos E 2018 Unification of Newtonian Physics with Einstein Relativity Theory, by using Generalized Metrics of Complex Spacetime and application to the Motions of Planets and Stars, eliminating Dark Matter *J Phys.: Conf. Ser.* **1141** 012128. [DOI: 10.1088/1742-6596/1141/1/012128]
[12] Vossos S and Vossos E 2016 Euclidean Closed Linear Transformations of Complex Spacetime and generally of Complex Spaces of dimension four endowed with the Same or Different Metric *J Phys: Conf Ser* **738** 012048. [DOI:10.1088/1742-6596/738/1/012048]
[13] Vossos E, Vossos S and Massouros C. G. 2020 Closed linear transformations of complex space-time endowed with Euclidean or Lorentz metric *IJPAM,* **44-2020**, 1033-1053.
[14] Vossos S and Vossos E 2015 Euclidean Complex Relativistic Mechanics: A New Special Relativity Theory *J. Phys.: Conf. Ser.* **633** 012027. [DOI:10.1088/1742-6596/633/1/012027]
[15] Golovko V A 2019 New metrics for the gravitational field of a point mass *Results in Physics* **13** 102288. [DOI: 10.1016/j.rinp.2019.102288]
[16] Williams D R. Mercury Fact Sheet. [Access: June 27, 2018] https://nssdc.gsfc.nasa.gov/planetary/factsheet/mercuryfact.html
[17] Lide D R ed 2009 *CRC Handbook of Chemistry and Physics* 89th Edition (Internet Version) (FL: CRC Press/Taylor and Francis, Boca Raton).
[18] Williams D R. Sun Fact Sheet. [Access: June 27, 2018] https://nssdc.gsfc.nasa.gov/planetary/factsheet/sunfact.html
[19] Williams D R. Earth Fact Sheet. [Access: June 27, 2018] https://nssdc.gsfc.nasa.gov/planetary/factsheet/earthfact.html
[20] McGaugh S and Milgrom M 2013 Andromeda Dwarfs in Light of MOND *Astrophysical Journal* 766(1), 22, pp 1-7. [DOI: 10.1088/0004-637X/766/1/22]
[21] Vossos S and Vossos E 2017 Explanation of Rotation Curves in Galaxies and Clusters of them, by Generalization of Schwarzschild Metric and Combination with MOND, eliminating Dark Matter, *J. Phys.: Conf Ser.* **936** 012008. [DOI:10.1088/1742-6596/936/1/012008]
[22] Lopresto J C, Chapman R D and Sturgis E A 1980 Solar gravitational redshift *Solar Physics* **66**, 245-249.
[23] Begeman KG 1989 H_I rotation curves of spiral galaxies. I - NGC 3198 *Astron Astrophys* 223(1-2) 47-60. http://articles.adsabs.harvard.edu/cgi-bin/nph-iarticle_query?1989A%26A...223...47B&defaultprint=YES&filetype=.pdf
[24] Heymans C 2017 *The Dark Universe* (IOP Publishing, Bristol, UK). [ISBN: 978-0-7503-1373-5] [DOI:10.1088/978-0-7503-1373-5]
[25] Davies P 2006 *The Goldilocks Enigma* (New York, USA: First Mariner Books). [ISBN: 978-0-618-59226-5]
[26] Bars I and Terning J 2009 Extra Dimensions in Space and Time (New York, USA: Springer). [ISBN: 978-0-387-77637-8].
[27] Karukes EV, Salucci P and Gentile G 2015 The Dark Matter Distribution in the Spiral NGC3198 out of 0.22 R_{vir} *Astron Astrophys* **578**, A13, pp 1-8. [DOI: 10.1051/0004-6361/201425339]
[28] McGaugh S. MOND Rotation Curve Fits–"The MOND pages". [Access: April 19, 2019] http://astroweb.case.edu/ssm/mond/fitroster.html

[29] Corbelli E, Thilker D, Zibetti S, Giovanardi C, Salucci P 2014 Dynamical signatures of a ΛCDM-halo and the distribution of the baryons in M33 *Astron Astrophys* **572**, A23, pp 1-18. [DOI: 10.1051/0004-6361/201424033]