# New Central Scalar Gravitational Potential according to Special Relativity and Newtonian Physics, explains the Precession of Mercury's Perihelion, the Gravitational Red Shift and the Rotation Curves in Galaxies, eliminating Dark Matter

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*Keywords*: Cavendish experiment, dark matter, Einstein Relativity Theory, Euclidean closed linear transformations, Euclidean complex relativistic mechanics, Euclidean metric, Galaxies: kinimatics and dynamics of, Galaxies: rotation curve, Galilean transformation, General Relativity, geodesics, gravitation, gravitational red shift, Lagrangian, linear spacetime transformation, Lorentz boost, Lorentz metric, Lorentz transformation, Minkowski space, Modified Newtonian Dynamics, MOND, Newtonian Physics, precession of Earth's perihelion, precession of Mercury's perihelion, relativistic Doppler shift, Schwarzschild metric, Solar System: kinimatics and dynamics of, Special Relativity.

*PACS*: 02.10.Ud, 02.40.Dr, 03.30.+p, 04.20.-q, 04.50.Kd, 04.80.Cc, 95.35.+d, 96.12.De, 96.15.De, 98.20.+d

**Abstract**. The mainstream approach of gravitational field is the development of Geometric theories of gravitation and the application of the Dynamics of General Relativity (GR). Besides, the Generalized Special Relativity (GSR) contains the fundamental parameter ($\xi_I$) of Theories of Physics (TPs). Thus, it expresses at the same time Newtonian Physics (NPs) for $\xi_I \to 0$ and Einstein Relativity Theory (ERT) for $\xi_I=1$. Moreover, the Equivalence Principle (EP) in the context of GSR, has two possible interpretations: $m_G=m$ (1), or $m_G=\gamma(\xi_I,\beta)m$ (2), where $\beta=\upsilon/c$ and $m_G$, $m$, $\gamma$ are the gravitational mass, inertial rest mass and *Lorentz γ-factor*, respectively. In this paper we initially present a new central scalar potential $V=V_{(k,r)}$, where $k=k(\xi_I)$ and $r$ is the distance from the center of gravity. We demand that 'this new GSR gravitational field in accordance with EP (1), gives the same *precession of Mercury's orbit* as Schwarzschild Metric (SM) does' and we obtain $k=6-\xi_I^2$. This emerges *Einsteinian SR-horizon* at $r=5r_S$, while NPs extends the horizon at six *Schwarzschild radius* ($6r_S$).

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We can also explain the Gravitational Red Shift (GRS), if only the proposed GSR Gravitational field strength $g=g(k,r)$ is combined with EP (2). We modify the aforementioned central scalar potential as $V=V(h,k,r)$, where $h=h(r)$. The combination of the above with MOND interpolating functions, or distributions of Dark Matter (DM) in galaxies, provides six different functions $h=h(r)$. Thus, we obtain a new GSR central Gravitational field strength $g=g(h,k,r)$, which not only explains the Precession of Mercury's Perihelion, but also the Rotation Curves in Galaxies, eliminating Dark Matter.

## 1. Introduction

The Equivalence Principle (EP) in the context of Special Relativity (SR), has two possible interpretations [1] (p.245). According to the *mainstream approach (weak EP)*, the *gravitational mass* ($m_G$) is equal to the *inertial rest mass* ($m$):

$$m_G = m. \tag{1}$$

On the other hand, the *alternative approach* is

$$m_G = \gamma\, m. \tag{2}$$

Besides, we have the *gravitational potential energy*

$$U = m_G V, \tag{3}$$

where $V$ is *scalar gravitational potential*.

The consideration of *Newtonian scalar gravitational potential*

$$V_N = -\frac{GM}{r}, \tag{4}$$

according to SR, gives *precession of Mercury's perihelion* only 7.16′′ per century, in case that we follow the *mainstream approach* (1) [2] (p. 355), [3] (p. 338), while the *alternative approach* (2) gives 21.49′′ per century [4] (p. 758), [5] (p. 758). Both the above theoretical results are far away from the experimental value:

$$\Omega_{exp} = 42.9799(9)''\mathrm{cy}^{-1}. \tag{5}$$

This is the contribution of the Sun due to *Schwarzschild Gravitoelectric effect* to the *total precession of Mercury's perihelion* [6] (p. 6). Therefore, when dealing with the gravitational field, we usually apply the Dynamics of General Relativity (GR) and we develop *Geometric theories of gravitation* [7]. The EP in GR is: accelerated motions caused by the gravitational field only (free fall) take place along *geodesics* of the metric, which corresponds to the particular gravitational field [2] (p. 248).

In this paper, we use generalized Relativity Theory (RT), which contains *Einstein Relativity Theory* (ERT) and *Newtonian Physics* (NPs), keeping the formalism of ERT. Thus, the differences between these two Theories of Physics (TPs) are limited to their different value of *metric coefficients of spacetime* for the corresponding Relativistic Inertial observers (RIOs) and the fundamental parameter of TPs: $\xi_I$. NPs has $\xi_I \to 0$, while ERT has $\xi_I = 1$ [8].

The case of observers with variable metric of spacetime, leads to the corresponding GR. For being this clear, we produce the 1st Generalized Schwarzschild Metric (1GSM) and the 3rd Generalized Schwarzschild Metric (3GSM), which are in accordance with any SR based on isotropic Generalized metrics ($g_I$) and *Einstein field equations*.

In case of 1GSM, we compute the corresponding *Lagrangian*, *geodesics*, *equations of motion*, *precession of planets' orbits* etc, resulting formulas which are referred to any TPs. We also present the results of the original *Schwarzschild metric* (SM), by adopting a new separation of *total energy* into *potential energy* (which depends only on distance) and *generalized kinetic energy* (which depends not only on velocity, but also on distance). This emerges a new *central potential*, which gives the well-known *Schwarzschild gravitational field strength*. The next step is the modification of the above potential (by introducing a real parameter $k$), because is going to be used according to Generalized Special Relativity (GSR) (as a pure GSR field in the spacetime of RIOs). The condition: 'this new

GSR gravitational field strength gives the same *precession of Mercury's orbit* as the original SM does', emerges the value of parameter $k$=6-$\xi_{\mathrm{I}}^2$. Thus, we obtain the new GSR central scalar gravitational potential $V$=[sqrt(1-$kr_{\mathrm{S}}/r$)-1]$\mathrm{c}^2/k$. NPs (with $\xi_{\mathrm{I}}\rightarrow 0$) gives $k$=6, while ERT (with $\xi_{\mathrm{I}}$=1) emerges $k$=5. Finally, we compare the SR and GR approaches of gravity and conclude no significant variation.

In case of 3GSM, the combination of its *Newtonian version* with Modified Newtonian Dynamics (MOND), leads to MOND relativization. After, we pass to RIOs of ordinary flat spacetime (*Minkowski space*) with *Lorentz metric*, extending MOND methods to ERT. We use *Simple* and *Standard* Interpolating Function ($\mu$) to the *Lorentzian version* of 3GSM, for the explanation of the Rotation Curves in Galaxies as well as the Solar system, eliminating Dark Matter. Generally, this approach, in non rotating black hole, planetary and star system scale, coincides to the original Schwarzschild metric, while in galactic scale, it gives MONDian results. We also modify the aforementioned central scalar potential as $V$=[sqrt(1-$hkr_{\mathrm{S}}/r$)-1]$\mathrm{c}^2/k$, where $h$=$h(r)$. The combination of the above with MOND interpolating functions, or distributions of Dark Matter (DM) in galaxies, provides six different functions $h$=$h(r)$. Thus, we obtain a new GSR central Gravitational field strength $g$=$g(h,k,r)$, which not only explains the *Precession of Mercury's Perihelion*, but also the *Rotation Curves in Galaxies*, eliminating Dark Matter.

## 2. Isometric Euclidean Closed Linear Transformations of Complex Spacetime endowed with the Corresponding Metrics

In this paper, the *metric coefficients of time and space have different signs*. Moreover, 3D-space is isotropic, in case of Isometric Euclidean Closed Linear Transformations of Complex Spacetime (IECLSTTs) [9]. Thus, for RIOs, the representation of the non-degenerate inner product in holonomic basis $\{\mathbf{e}_{ct}, \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ of 'flat' spacetime is the real matrix

$$g_{\mathrm{I}} = \mathrm{diag}\left(g_{\mathrm{I}00},\ g_{\mathrm{I}11},\ g_{\mathrm{I}22},\ g_{\mathrm{I}33}\right) = g_{\mathrm{I}11}\,\mathrm{diag}\left(-\frac{1}{\xi_{\mathrm{I}}^{2}},\ 1,\ 1,\ 1\right) = g_{\mathrm{I}00}\,\mathrm{diag}\left(1,\ -\xi_{\mathrm{I}}^{2},\ -\xi_{\mathrm{I}}^{2},\ -\xi_{\mathrm{I}}^{2}\right), \tag{6}$$

where

$$\xi_I = \sqrt{\frac{g_{\mathrm{I}11}}{-g_{\mathrm{I}00}}} \tag{7}$$

The index I remind us that we are referred to the spacetime of the RIOs of each specific TPs. This GSR has real Universal Speed ($c_{\mathrm{I}}$):

$$c_{\mathrm{I}} = \frac{1}{\xi_{\mathrm{I}}}\mathrm{c} \tag{8}$$

and the transformation of a contravariant four-vector is

$$\mathrm{d}X' = \Lambda_{\mathrm{I}(\xi_{\mathrm{I}},\vec{\beta})}\,\mathrm{d}X, \tag{9}$$

where

$$\Lambda_{\mathrm{I}(\xi_{\mathrm{I}},\vec{\beta})} = \gamma_{(\xi_{\mathrm{I}}\vec{\beta})}\begin{bmatrix} 1 & -\xi_{\mathrm{I}}^2\beta_x & -\xi_{\mathrm{I}}^2\beta_y & -\xi_{\mathrm{I}}^2\beta_z \\ -\beta_x & 1 & \mathrm{i}\,\xi_{\mathrm{I}}\beta_z & -\mathrm{i}\,\xi_{\mathrm{I}}\beta_y \\ -\beta_y & -\mathrm{i}\,\xi_{\mathrm{I}}\beta_z & 1 & \mathrm{i}\,\xi_{\mathrm{I}}\beta_x \\ -\beta_z & \mathrm{i}\,\xi_{\mathrm{I}}\beta_y & -\mathrm{i}\,\xi_{\mathrm{I}}\beta_x & 1 \end{bmatrix} = \gamma_{(\xi_{\mathrm{I}}\vec{\beta})}\begin{bmatrix} 1 & -\xi_{\mathrm{I}}^2\beta^T \\ -\beta & \mathrm{I}_3 + \mathrm{i}\,\xi_{\mathrm{I}}\,\mathrm{A}_{(\beta)} \end{bmatrix}, \tag{10}$$

$$\beta^i = \frac{\mathrm{d}x^i}{\mathrm{d}x^0} \quad ; \quad \beta = \begin{bmatrix} \beta_x \\ \beta_y \\ \beta_z \end{bmatrix} \quad ; \quad \mathrm{A}_{(\beta)} = \begin{bmatrix} 0 & \beta_z & -\beta_y \\ -\beta_z & 0 & \beta_x \\ \beta_y & -\beta_x & 0 \end{bmatrix} \tag{11}$$

and

$$\gamma_{(\vec{\delta})} = \frac{1}{\sqrt{1-\vec{\delta}^2}} \tag{12}$$

is Lorentz $\gamma$-factor.

The specific value $\xi_I \to 0$ ($g_{I11} \to 0$, $g_{I00} \neq 0$) gives Galilean Transformation (GT) with Infinite Universal Speed ($c_I \to +\infty$) and the corresponding metric of the spacetime (let us call *Galilean metric*)

$$g_\Gamma = \lim_{g_{I11}\to 0} \mathrm{diag}\left(g_{I00},\ g_{I11},\ g_{I11},\ g_{I11}\right) = g_{I00} \lim_{\xi_I \to 0} \mathrm{diag}\left(1,\ -\xi_I^{\,2},\ -\xi_I^{\,2},\ -\xi_I^{\,2}\right). \tag{13}$$

The corresponding spacetime (let us call *Galilean spacetime*) has infinite curvature ($K \to +\infty$) in any orientation $\kappa \mathbf{e_x} + \lambda \mathbf{e_y} + \mu \mathbf{e_z}$ of 3D-space. This is the reason that time is absolute for any type of observers as well as the Universal speed is infinite ($c_I \to +\infty$).

The specific value $\xi_I = 1$ ($g_{I11} = -g_{I00}$) gives *Vossos Transformation* (VT) with $c_I = c$ (the universal speed is the speed of light in vacuum) and the corresponding metric of spacetime (let us call *Vossos metric*)

$$g_B = g_{I11}\, \mathrm{diag}\left(-1,\ 1,\ 1,\ 1\right) = g_{I11}\, \eta\,, \tag{11}$$

which for $g_{I1}=1$ becomes the *Lorentz metric* (η). Thus, we have the *Lorentzian case* of Euclidean Complex Relativistic Mechanics (ECRMs) [10], which is associated with ERT.

We now make the option that observer O measures *real spacetime*. As matrix $\Lambda_I$ contains some elements which are imaginary numbers, we conclude that *the spacetime of one moving observer is complex*. Thus, we put an index C to the complex natural sizes and the real natural sizes have no index. The typical matrix of IECLSTTs along x-axis (Generalized; Galilean-Newtonian; Lorentzian-Einsteinian) is

$$\Lambda_{Ityp} = \gamma_{(\xi_I \beta)} \begin{bmatrix} 1 & -\xi_I^{\,2}\beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & \mathrm{i}\,\xi_I \beta \\ 0 & 0 & -\mathrm{i}\,\xi_I \beta & 1 \end{bmatrix};\ \Lambda_{\Gamma typ} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};\ \Lambda_{Btyp} = \gamma_{(\beta)} \begin{bmatrix} 1 & -\beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & \mathrm{i}\,\beta \\ 0 & 0 & -\mathrm{i}\,\beta & 1 \end{bmatrix}. \tag{12}$$

In addition, any complex *Cartesian Coordinates* (CCs) of the theory may be turned to the corresponding real CCs, in order to be perceived by human senses. This is achieved, if the moving Observer O′ considers as Real CCs the corresponding lengths of rods [8] (p. 6). Thus, it emerges the (Generalized; Galilean-Newtonian; Lorentzian-Einsteinian) Real Boost (RB)

$$\mathrm{d}X' = \Lambda_{IR(\beta)}\, \mathrm{d}X\ ;\ \mathrm{d}X' = \Lambda_{\Gamma(\beta)}\, \mathrm{d}X\ ;\ \mathrm{d}X' = \Lambda_{L(\beta)}\, \mathrm{d}X, \tag{13}$$

where

$$\Lambda_{IR(\vec\beta)} = \begin{bmatrix} \gamma_{(\xi_I\vec\beta)} & -\gamma_{(\xi_I\vec\beta)}\xi_I^{\,2}\beta^T \\ -\gamma_{(\xi_I\vec\beta)}\beta & I_3 + \dfrac{\gamma_{(\xi_I\vec\beta)}-1}{\beta^T\beta}\beta\beta^T \end{bmatrix};\ \Lambda_{\Gamma(\vec\beta)} = \begin{bmatrix} 1 & 0 \\ -\beta & I_3 \end{bmatrix};\ \Lambda_{L(\vec\beta)} = \begin{bmatrix} \gamma_{(\vec\beta)} & -\gamma_{(\vec\beta)}\beta^T \\ -\gamma_{(\vec\beta)}\beta & I_3 + \dfrac{\gamma_{(\vec\beta)}-1}{\beta^T\beta}\beta\beta^T \end{bmatrix}. \tag{14}$$

The typical matrix of (Generalized; Galilean-Newtonian; Lorentzian-Einsteinian) RB along x-axis is

$$\Lambda_{IR\,typ(\beta)} = \begin{bmatrix} \gamma_{(\xi_I\beta)} & -\xi_I^{\,2}\gamma_{(\xi_I\beta)}\beta & 0 & 0 \\ -\gamma_{(\xi_I\beta)}\beta & \gamma_{(\xi_I\beta)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};\ \Lambda_{\Gamma typ(\beta)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};\ \Lambda_{L typ(\beta)} = \begin{bmatrix} \gamma_{(\beta)} & -\gamma_{(\beta)}\beta & 0 & 0 \\ -\gamma_{(\beta)}\beta & \gamma_{(\beta)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{15}$$

We observe that for $\xi_I=1$, we have the original typical proper *Lorentz Boost* (LB) (see e.g. [2] p. 21, eq. 1.38) and the corresponding general proper LB (see e.g. [2] p. 24, eq. 1.47).

Supposing one Particle ($P$) with real mass $m$ moving with velocity $\vec{\upsilon}_P = \vec{\beta}_P\, c$ wrt observer O, we calculate the *Generalized relativistic kinetic energy*; *Generalized relativistic energy*; *Generalized energy of Rest mass* [8] (p. 10):

$$K = \frac{\gamma_{(\xi_I \vec\beta_P)} - 1}{\xi_I^{\,2}} m c^2\ ;\ E = \frac{\gamma_{(\xi_I \vec\beta_P)}}{\xi_I^{\,2}} m c^2\ ;\ E_{rest} = \frac{1}{\xi_I^{\,2}} m c^2 \tag{16}$$

## 3. GR: Generalized Schwarzschild metrics

*3.1. The metric of a static and centrally symmetric gravitational field*

*Einstein field equations* in vacuum [11] (pp. 303, 396) are reduced to the *single tensor equation* $R_{\mu\nu}=0$. This emerges the *metric of a static and centrally symmetric gravitational field*

$$\mathrm{d}S^2 = g_{100} f_{(r)}\,\mathrm{c}^2\mathrm{d}t^2 + g_{\mathrm{I}11} g_{(r)}\,\mathrm{d}r^2 + g_{\mathrm{I}11} h_{(r)}\,\mathrm{d}\theta^2 + g_{\mathrm{I}11} h_{(r)} \sin^2\theta\,\mathrm{d}\phi^2 , \tag{17}$$

with the following conditions [12] (p. 2):

$$g_{(r)} = \frac{\mu}{f_{(r)}\left(1-f_{(r)}\right)^4}\left(\frac{\mathrm{d}f}{\mathrm{d}r}\right)^2 \; ; \; h_{(r)} = \frac{\mu}{\left(1-f_{(r)}\right)^2} , \tag{18}$$

where $\mu$ is an arbitrary constant and $f$ is an arbitrary function of $r$ (not constant).

*3.2. The 3rd Generalized Schwarzschild Metric, Relativistic potential and Field strength*

We define a new relativistic potential $\Phi$ around a center of gravity with mass $M$ (let us call *3rd Generalized Schwarzschild Relativistic Potential*-3GSRP) as

$$\Phi = \frac{\mathrm{c}^2}{2\xi_{\mathrm{I}}^{\,2}} \ln\left(1 - a_{(r)} \frac{\xi_{\mathrm{I}}^{\,2} r_{\mathrm{S}}}{r}\right) , \tag{19}$$

where

$$r_{\mathrm{S}} = \frac{2\mathrm{G}M}{\mathrm{c}^2} \tag{20}$$

is *Schwarzschild radius* and $a_{(r)}$ is unspecified function, in accordance with any TPs. The 3GSP is connected with $\Phi$, via the formula

$$\ln f_{(r)} = \frac{2}{c_{\mathrm{I}}^{\,2}}\Phi = \frac{2\xi_{\mathrm{I}}^{\,2}}{\mathrm{c}^2}\Phi . \tag{21}$$

Thus, we obtain

$$f_{(r)} = 1 - a_{(r)} \frac{\xi_{\mathrm{I}}^{\,2} r_{\mathrm{S}}}{r} . \tag{22}$$

After replacing the above equation and $\mu=\xi_{\mathrm{I}}^4 r_{\mathrm{S}}^2$ to (18), we also have

$$g_{(r)} = \frac{\left(r\dfrac{\mathrm{d}a}{\mathrm{d}r} - a_{(r)}\right)^2}{a_{(r)}^{\,4}\left(1 - a_{(r)}\dfrac{\xi_{\mathrm{I}}^{\,2} r_{\mathrm{S}}}{r}\right)} \; ; \; h_{(r)} = \frac{r^2}{a_{(r)}^{\,2}} . \tag{23}$$

So, we obtain the *3rd Generalized Schwarzschild Metric* (3GSM)

$$\mathrm{d}S^2 = g_{100}\left(1 - a_{(r)}\frac{\xi_{\mathrm{I}}^{\,2} r_{\mathrm{S}}}{r}\right)\mathrm{c}^2\mathrm{d}t^2 + \frac{g_{\mathrm{I}11}\left(r\dfrac{\mathrm{d}a}{\mathrm{d}r} - a_{(r)}\right)^2}{a_{(r)}^{\,4}\left(1 - a_{(r)}\dfrac{\xi_{\mathrm{I}}^{\,2} r_{\mathrm{S}}}{r}\right)}\mathrm{d}r^2 + \frac{g_{\mathrm{I}11} r^2}{a_{(r)}^{\,2}}\mathrm{d}\theta^2 + \frac{g_{\mathrm{I}11} r^2}{a_{(r)}^{\,2}}\sin^2\theta\,\mathrm{d}\phi^2 , \tag{24}$$

with spatial part

$$\mathrm{d}l^2 = \frac{g_{\mathrm{I}11}\left(r\dfrac{\mathrm{d}a}{\mathrm{d}r} - a_{(r)}\right)^2}{a_{(r)}^{\,4}\left(1 - a_{(r)}\dfrac{\xi_{\mathrm{I}}^{\,2} r_{\mathrm{S}}}{r}\right)}\mathrm{d}r^2 + \frac{g_{\mathrm{I}11} r^2}{a_{(r)}^{\,2}}\mathrm{d}\theta^2 + \frac{g_{\mathrm{I}11} r^2}{a_{(r)}^{\,2}}\sin^2\theta\,\mathrm{d}\phi^2 , \tag{25}$$

where *a* is an arbitrary function of *r* (or constant). Now, we can calculate this *radial field strength*, by defining

$$\vec{g} = -\sqrt{g_{\mathrm{I}11}}\nabla\Phi = -\sqrt{g_{\mathrm{I}11}}\frac{d\Phi}{dl}\hat{r} = -\sqrt{g_{\mathrm{I}11}}\frac{d\Phi}{dr}\frac{dr}{dl}\hat{r}\,, \tag{26}$$

and

$$g = \sqrt{g_{\mathrm{I}11}}\frac{d\Phi}{dr}\frac{dr}{dl}\,. \tag{27}$$

The positive value of field strength means gravity, while negative value means antigravity. So, it is

$$g = \frac{GM}{r^2}\left(1 - a_{(r)}\frac{\xi_{\mathrm{I}}^2 r_{\mathrm{S}}}{r}\right)^{-\frac{1}{2}} a_{(r)}{}^2 > 0\,. \tag{28}$$

We also prefer $a>0$, in order to ensure *Gravitational Red Shift* (GRS).

*3.3. The 1st Generalized Schwarzschild Metric, Relativistic potential, Field strength, Lagrangian, Geodesics, Equations of motion and Precession of planets' orbits*

In case that $a_{(r)}=1$, (19) gives the 1st Generalized Schwarzschild Relativistic Potential (1GSRP) [9] (p.11):

$$\Phi = \frac{\mathrm{c}^2}{2\xi_{\mathrm{I}}^2}\ln\left(1 - \frac{\xi_{\mathrm{I}}^2 r_{\mathrm{S}}}{r}\right) = -\frac{\mathrm{c}^2}{2}\frac{r_{\mathrm{S}}}{r} + \ldots = -\frac{\mathrm{G}M}{r} + \ldots \tag{29}$$

Thus, (24) emerges the 1st Generalized Schwarzschild metric (1GSM):

$$\mathrm{d}S^2 = g_{100}\left(1 - \xi_{\mathrm{I}}^2\frac{r_{\mathrm{S}}}{r}\right)\mathrm{c}^2\mathrm{d}t^2 + \frac{g_{\mathrm{I}11}}{1 - \xi_{\mathrm{I}}^2\frac{r_{\mathrm{S}}}{r}}\mathrm{d}r^2 + g_{\mathrm{I}11}r^2\,\mathrm{d}\theta^2 + g_{\mathrm{I}11}r^2\sin^2\theta\,\mathrm{d}\phi^2\,. \tag{30}$$

Besides, the *1st Generalized Schwarzschild field strength* (*g*) is

$$\vec{g} = -\frac{GM}{r^2}\left(1 - \xi_I^2\frac{r_s}{r}\right)^{-\frac{1}{2}}\hat{r}\,. \tag{31}$$

The usual definition of *Lagrangian* of gravitational system (*M*, *m*) [11] (p. 205)

$$L = m\dot{x}^{\mu} g_{\mu\nu}\dot{x}^{\nu}\,, \tag{32}$$

for orbit on the 'plane' $\theta=\pi/2$, gives the *1st Generalized Schwarzschild Lagrangian* (1GSL) [8] (p. 15):

$$L = mg_{100}\left[\left(1 - \xi_{\mathrm{I}}^2\frac{r_{\mathrm{S}}}{r}\right)\mathrm{c}^2\,\dot{t}^2 - \frac{\xi_{\mathrm{I}}^2}{1 - \xi_{\mathrm{I}}^2\frac{r_{\mathrm{S}}}{r}}\dot{r}^2 - \xi_{\mathrm{I}}^2 r^2\dot{\phi}^2\right] \;;\; {}^{\cdot} = \frac{\mathrm{d}}{\mathrm{d}\tau}\,. \tag{33}$$

The well-known *Euler-Lagrange equations*

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial\dot{x}^{\mu}}\right) - \frac{\partial L}{\partial x^{\mu}} = 0 \;;\; \mu=0,\,1,\,3 \tag{34}$$

give us

$$E = \left(1 - \xi_{\mathrm{I}}^2\frac{r_{\mathrm{S}}}{r}\right)\frac{m\mathrm{c}^2}{\xi_{\mathrm{I}}^2}\dot{t} \;;\; {}^{\cdot} = \frac{\mathrm{d}}{\mathrm{d}\tau} \;; \tag{35}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{2\dot{r}}{1 - \xi_{\mathrm{I}}^2\frac{r_{\mathrm{S}}}{r}}\right) - \left[-\frac{r_{\mathrm{S}}}{r^2}\mathrm{c}^2\,\dot{t}^2 + \frac{\partial}{\partial r}\left(\frac{1}{1 - \xi_{\mathrm{I}}^2\frac{r_{\mathrm{S}}}{r}}\right)\dot{r}^2 + 2r\dot{\phi}^2\right] = 0 \;; \tag{36}$$

$$J = mr^2\dot{\phi} \quad ; \quad \dot{} = \frac{\mathrm{d}}{\mathrm{d}\tau} \ , \tag{37}$$

where $E$ is the total energy and $J$ is the total angular momentum of the system (the *integrals of motion*). The solutions of the above *equations of motion* satisfy the condition

$$L = mg_{100}\,\mathrm{c}^2 \,. \tag{38}$$

So, they can also be used for the practical determination of *geodesics* [11] (p. 205).

Now, we study the motion of particle $P$ around the center of gravity of mass $M$. The case of Uniform Circular Motion (UCM) is obtained, by putting $r$=$R$=constant to (36). The orbit of *non-circular motion* comes with similar way to the original *Schwarzschild space* [11] (pp. 238-45). Thus, the *exact differential equation of motion* is

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = \frac{\mathrm{G}M}{h^2} + 3\xi_{\mathrm{I}}^{\,2}\frac{\mathrm{G}M}{\mathrm{c}^2}u^2 \quad ; \quad u = \frac{1}{r} \quad ; \quad h = r^2\dot{\phi} \quad ; \quad \dot{} = \frac{\mathrm{d}}{\mathrm{d}\tau} \,, \tag{39}$$

where $h$=$J/m$ is the angular momentum per mass unit.

In case of *small velocities* relative to $c_{\mathrm{I}}$ ($\upsilon$<<c/$\xi_{\mathrm{I}}$), we replace the solution of the *simplified differential equation*

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = \frac{\mathrm{G}M}{h^2} \quad ; \quad u = \frac{\mathrm{G}M}{h^2}\left(1 + e\cos\phi\right) \quad ; \quad \frac{\mathrm{G}M}{h^2} = \frac{1}{a\left(1-e^2\right)} \tag{40}$$

to the last term of the exact differential equation of motion ($e$ is the eccentricity of the *conic section*, $\alpha$ is the semimajor axis in case of *ellipse*). Thus, we have the *approximate differential equation of motion* (which also validates UCM):

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = \frac{\mathrm{G}M}{h^2} + 3\xi_{\mathrm{I}}^{\,2}\frac{\mathrm{G}^3M^3}{\mathrm{c}^2h^4}\left(1 + e\cos\phi\right)^2 \quad ; \quad u = \frac{1}{r} \quad ; \quad h = r^2\dot{\phi} \quad ; \quad \dot{} = \frac{\mathrm{d}}{\mathrm{d}\tau} \tag{41}$$

with *exact* and *approximate solution*, correspondingly

$$u = \frac{\mathrm{G}M}{h^2}\left(1 + e\cos\phi + 3\xi_{\mathrm{I}}^{\,2}\frac{\mathrm{G}^2M^2}{\mathrm{c}^2h^2}e\phi\sin\phi\right); \quad u = \frac{1}{r} \quad ; \quad h = r^2\dot{\phi} \quad ; \quad \dot{} = \frac{\mathrm{d}}{\mathrm{d}t} \quad ; \quad \frac{\mathrm{G}M}{h^2} = \frac{1}{a\left(1-e^2\right)}, \tag{42}$$

$$u \approx \frac{\mathrm{G}M}{h^2}\left(1 + e\cos\left[\left(1 - 3\xi_{\mathrm{I}}^{\,2}\frac{\mathrm{G}^2M^2}{\mathrm{c}^2h^2}\right)\phi\right]\right) \quad ; \quad 0 < \frac{6\pi\xi_{\mathrm{I}}^{\,2}\mathrm{G}^2M^2}{\mathrm{c}^2h^2} << 1\,. \tag{43}$$

The last equation can be written as

$$u = \frac{1}{r} \approx \frac{\mathrm{G}M}{h^2}\left[1 + e\cos\left(\lambda_{\mathrm{GR}}\phi\right)\right] \quad ; \quad \lambda_{\mathrm{GR}} = 1 - 3\xi_{\mathrm{I}}^{\,2}\frac{\mathrm{G}^2M^2}{\mathrm{c}^2h^2} \quad ; \quad 0 < \frac{6\pi\xi_{\mathrm{I}}^{\,2}\mathrm{G}^2M^2}{\mathrm{c}^2h^2} << 1\,. \tag{44}$$

Hence the orbit can be regarded as an *ellipse* that rotates ('precesses') about one of its foci by an amount

$$\Delta = \frac{2\pi}{1 - 3\xi_{\mathrm{I}}^{\,2}\dfrac{\mathrm{G}^2M^2}{\mathrm{c}^2h^2}} - 2\pi \approx \frac{6\pi\xi_{\mathrm{I}}^{\,2}\mathrm{G}^2M^2}{\mathrm{c}^2h^2} = \frac{6\pi\xi_{\mathrm{I}}^{\,2}\mathrm{G}M}{a\left(1-e^2\right)\mathrm{c}^2} \quad ; \quad h = r^2\dot{\phi} \quad ; \quad \dot{\phi} = \frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \frac{\mathrm{d}\phi}{\mathrm{d}t}\dot{t} \tag{45}$$

rad per revolution. Finally the angular velocity of ellipse rotation is given by the formula

$$\Omega\left(\frac{''}{cy}\right) = \Delta\left(\frac{\mathrm{rad}}{\mathrm{rev}}\right)\left(\frac{360^o}{2\,\pi \ \ rad}\right)\left(\frac{3600''}{1^o}\right)\frac{1}{T}\left(\frac{\mathrm{rev}}{\mathrm{day}}\right)\left(\frac{365.242\,\mathrm{day}}{\mathrm{year}}\right)\left(\frac{100\,\mathrm{year}}{\mathrm{cy}}\right), \tag{46}$$

or equivalently

$$\Omega\left(\frac{''}{\mathrm{cy}}\right) = \Delta\left(\frac{\mathrm{rad}}{\mathrm{rev}}\right)\left(\frac{7533657\times10^3\,''\cdot\mathrm{day}}{\mathrm{rad}\cdot\mathrm{cy}}\right)\frac{1}{T}\left(\frac{\mathrm{rev}}{\mathrm{day}}\right). \tag{47}$$

Accordingly to the mainstream approach in textbooks, the further study is based on the *superposition principle*. This emerges the *relation of time to proper time.* Replacing this to (35), they obtain the final formula of the *total relativistic energy*. Finally, the *generalized potential energy* is calculated, by reducing the kinetic energy (which is considered equal to this of SR) from the total relativistic energy. But SM is a static and stationary metric of non-rotating mass. So, there is no gravitomagnetism and we expect that the gravitational force is independent from the velocity of the particle. Thus we adapt the following approach which gives simple *central potential* which describes Gravitoelectric Effect (GEE).

The isometry of spacetime relieves us the *relation of time to proper time* [8] (p. 16):

$$\dot{t} = \frac{\mathrm{d}t}{\mathrm{d}\tau} = \left[1 - \xi_{\mathrm{I}}^2\left(\frac{r_{\mathrm{S}}}{r} + \frac{1}{1-\xi_{\mathrm{I}}^2\frac{r_{\mathrm{S}}}{r}}\beta_{Pr}^{\ 2} + \beta_{P\phi}^{\ 2}\right)\right]^{-\frac{1}{2}} \geq 1 \quad ; \quad \theta = \frac{\pi}{2} . \tag{48}$$

Replacing the above equation to (35), we obtain the final formula of the *total relativistic energy*

$$E = \frac{1-\xi_{\mathrm{I}}^2\frac{r_{\mathrm{S}}}{r}}{\sqrt{1 - \xi_{\mathrm{I}}^2\left(\frac{r_{\mathrm{S}}}{r} + \frac{1}{1-\xi_{\mathrm{I}}^2\frac{r_{\mathrm{S}}}{r}}\beta_{Pr}^{\ 2} + \beta_{P\phi}^{\ 2}\right)}}\frac{m\mathrm{c}^2}{\xi_{\mathrm{I}}^2} \geq 0 \quad ; \quad \theta = \frac{\pi}{2} . \tag{49}$$

We observe the different contribution of the radial and orbital velocity to the total energy! Now, we demand zero kinetic energy ($K$=0), in case that the particle is static $\left(\vec{\beta}_P = 0\right)$. Then $E_{\left(\vec{\beta}_P=0\right)} = E_{\mathrm{rest}} + U$ , where $U$ is the *potential energy*. Replacing (16iii) and (49) to the above equation, we have

$$U = \left(\sqrt{1-\xi_{\mathrm{I}}^2\frac{r_{\mathrm{S}}}{r}} - 1\right)\frac{m\mathrm{c}^2}{\xi_{\mathrm{I}}^2} \leq 0 \ ; \tag{50}$$

$$V = \left(\sqrt{1-\xi_{\mathrm{I}}^2\frac{r_{\mathrm{S}}}{r}} - 1\right)\frac{\mathrm{c}^2}{\xi_{\mathrm{I}}^2} \leq 0 \tag{51}$$

where $V$ is the 1$^{\mathrm{st}}$ Generalized Schwarzschild Potential (1GSP). This is a central potential:

$$\vec{g} = -\frac{dV}{dr}\hat{r} = -\frac{GM}{r^2}\left(1-\xi_I^{\ 2}\frac{r_s}{r}\right)^{-\frac{1}{2}}\hat{r} . \tag{52}$$

We observe that the result is the same as (31). The *generalized Relativistic Kinetic energy* is defined as $K_{\mathrm{g}}=E-E_{\mathrm{rest}}-U$. So,

$$K_{\mathrm{g}} = \left(\frac{1-\xi_{\mathrm{I}}^2\frac{r_{\mathrm{S}}}{r}}{\sqrt{1 - \xi_{\mathrm{I}}^2\left(\frac{r_{\mathrm{S}}}{r} + \frac{1}{1-\xi_{\mathrm{I}}^2\frac{r_{\mathrm{S}}}{r}}\beta_{Pr}^{\ 2} + \beta_{P\phi}^{\ 2}\right)}} - \sqrt{1-\xi_{\mathrm{I}}^2\frac{r_{\mathrm{S}}}{r}}\right)\frac{m\mathrm{c}^2}{\xi_{\mathrm{I}}^2} \geq 0 \quad ; \quad \theta = \frac{\pi}{2} . \tag{53}$$

We also observe that if $r \rightarrow +\infty$, the above equation becomes the *Relativistic Kinetic energy* of GSR: (16i). Finally the *Relativistic mechanic energy* $E_{\mathrm{m}}=E-E_{\mathrm{rest}}=K_{\mathrm{g}}+U$ is

$$E_{\mathrm{m}} = \left( \frac{1-\xi_{\mathrm{I}}^{2}\frac{r_{\mathrm{S}}}{r}}{\sqrt{1-\xi_{\mathrm{I}}^{2}\left(\frac{r_{\mathrm{S}}}{r}+\frac{1}{1-\xi_{\mathrm{I}}^{2}\frac{r_{\mathrm{S}}}{r}}\beta_{Pr}^{2}+\beta_{P\phi}^{2}\right)}} - 1 \right)\frac{m\mathrm{c}^{2}}{\xi_{\mathrm{I}}^{2}} \quad ; \quad \theta=\frac{\pi}{2} . \tag{54}$$

In case that $\xi_{\mathrm{I}}\to 0^{+}$ (*Galilean metric*), (48) gives $\dot{t}=1$. Thus, we obtain the *Newtonian results*:

$$\Phi_{\mathrm{N}} = \lim_{\xi_{\mathrm{I}}\to 0}\Phi = \frac{\mathrm{c}^{2}}{2}\lim_{\xi_{\mathrm{I}}\to 0}\left[\frac{1}{\xi_{\mathrm{I}}^{2}}\ln\left(1-\frac{\xi_{\mathrm{I}}^{2}r_{\mathrm{S}}}{r}\right)\right] = \frac{\mathrm{c}^{2}}{4}\lim_{\xi_{\mathrm{I}}\to 0}\left[\frac{1}{\xi_{\mathrm{I}}}\frac{\frac{-2\xi_{\mathrm{I}}r_{\mathrm{S}}}{r}}{1-\frac{\xi_{\mathrm{I}}^{2}r_{\mathrm{S}}}{r}}\right] = -\frac{\mathrm{c}^{2}}{2}\frac{r_{\mathrm{S}}}{r} = -\frac{\mathrm{G}M}{r} \quad ; \tag{55}$$

$$\mathrm{d}S_{\mathrm{N}}^{2} = g_{\mathrm{I}00}\lim_{\xi_{\mathrm{I}}\to 0}\left[\left(1-\frac{\xi_{\mathrm{I}}^{2}r_{\mathrm{S}}}{r}\right)\mathrm{c}^{2}\mathrm{d}t^{2} - \frac{\xi_{\mathrm{I}}^{2}}{1-\frac{\xi_{\mathrm{I}}^{2}r_{\mathrm{S}}}{r}}\mathrm{d}r^{2} - \xi_{\mathrm{I}}^{2}r^{2}\,\mathrm{d}\theta^{2} - \xi_{\mathrm{I}}^{2}r^{2}\sin^{2}\theta\,\mathrm{d}\phi^{2}\right] \quad ; \tag{56}$$

$$\vec{g}_{\mathrm{N}} = -\frac{\mathrm{G}M}{r^{2}}\hat{r} \quad ; \tag{57}$$

$$L_{\mathrm{N}} = m g_{\mathrm{I}00}\mathrm{c}^{2} \;\; ; \;\; E_{\mathrm{N}} = +\infty \;\; ; \;\; \ddot{r} + \frac{\mathrm{G}M}{r^{2}} - r\dot{\phi}^{2} = 0 \;\; ; \;\; J_{\mathrm{N}} = m r^{2}\dot{\phi} \;\; ; \;\; \dot{} = \frac{\mathrm{d}}{\mathrm{d}t} \;\; ; \;\; \theta = \frac{\pi}{2} . \tag{58}$$

The *Newtonian differential equation* of motion and the corresponding solution are

$$\frac{\mathrm{d}^{2}u_{\mathrm{N}}}{\mathrm{d}\phi^{2}} + u_{\mathrm{N}} = \frac{\mathrm{G}M}{h_{\mathrm{N}}^{2}} \;\; ; \;\; u_{\mathrm{N}} = \frac{\mathrm{G}M}{h_{\mathrm{N}}^{2}}\left(1+e_{\mathrm{N}}\cos\phi\right) \;\; ; \;\; u = \frac{1}{r} \;\; ; \;\; h_{\mathrm{N}} = r^{2}\dot{\phi} \;\; ; \;\; \dot{} = \frac{\mathrm{d}}{\mathrm{d}t} \;\; ; \tag{59}$$

$$e_{\mathrm{N}} = \sqrt{1+\frac{2E_{\mathrm{mN}}h_{\mathrm{N}}^{2}}{\mathrm{G}^{2}M^{2}m}} \;\; ; \;\; E_{\mathrm{mN}} = -\frac{\mathrm{G}Mm}{2a_{\mathrm{N}}} , \tag{60}$$

where $\alpha_{\mathrm{N}}$ is the semimajor axis of *Newtonian ellipse* which do not rotate ($\Delta_{\mathrm{N}}$=0). Besides

$$U_{\mathrm{N}} = -\frac{\mathrm{G}Mm}{r} \;\; ; \;\; V_{\mathrm{N}} = -\frac{\mathrm{G}M}{r} \;\; ; \;\; K_{\mathrm{N}} = \frac{1}{2}\left|\vec{\beta}_{P}\right|^{2}m\mathrm{c}^{2} = \frac{1}{2}m\left|\vec{\upsilon}\right|^{2} \quad E_{\mathrm{mN}} = \frac{1}{2}m\left|\vec{\upsilon}\right|^{2} - \frac{\mathrm{G}M}{r} . \tag{61}$$

In case that $\xi_{\mathrm{I}}$=1, it emerges the well-known results of the original *Schwarzschild metric* in ERT (see e.g. [11] pp. 228-45):

$$\Phi_{E} = \frac{\mathrm{c}^{2}}{2}\ln\left(1-\frac{r_{\mathrm{S}}}{r}\right) \quad ; \tag{62}$$

$$\mathrm{d}S_{\mathrm{E}}^{2} = g_{\mathrm{I}00}\left[\left(1-\frac{r_{\mathrm{S}}}{r}\right)\mathrm{c}^{2}\mathrm{d}t^{2} - \frac{1}{1-\frac{r_{\mathrm{S}}}{r}}\mathrm{d}r^{2} - r^{2}\,\mathrm{d}\theta^{2} - r^{2}\sin^{2}\theta\,\mathrm{d}\phi^{2}\right] \quad ; \tag{63}$$

$$\vec{g}_{\mathrm{EGR}} = -\frac{GM}{r^{2}}\left(1-\frac{r_{s}}{r}\right)^{-\frac{1}{2}}\hat{r} \quad ; \tag{64}$$

$$L_{\mathrm{EGR}} = mg_{100}\left[\left(1-\frac{r_{\mathrm{S}}}{r}\right)\mathrm{c}^2\dot{t}^2 - \frac{1}{1-\frac{r_{\mathrm{S}}}{r}}\dot{r}^2 - r^2\dot{\phi}^2\right] \;;\; E_{\mathrm{EGR}} = \left(1-\frac{r_{\mathrm{S}}}{r}\right)m\mathrm{c}^2\dot{t} \;;\; \dot{} = \frac{\mathrm{d}}{\mathrm{d}\tau_{\mathrm{EGR}}} \;;\; \theta = \frac{\pi}{2}. \quad (65)$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau_{\mathrm{EGR}}}\left(\frac{2\dot{r}}{1-\frac{r_{\mathrm{S}}}{r}}\right) - \left[-\frac{r_{\mathrm{S}}}{r^2}\mathrm{c}^2\dot{t}^2 + \frac{\partial}{\partial r}\left(\frac{1}{1-\frac{r_{\mathrm{S}}}{r}}\right)\dot{r}^2 + 2r\dot{\phi}^2\right] = 0 \;;\; J_{\mathrm{EGR}} = mr^2\dot{\phi} \;;\; \dot{} = \frac{\mathrm{d}}{\mathrm{d}\tau_{\mathrm{EGR}}}. \quad (66)$$

The *differential equation of non-UCMs* of the original *Schwarzschild metric* has come from (39):

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = \frac{\mathrm{G}M}{h_{\mathrm{E}}^2} + 3\frac{\mathrm{G}M}{\mathrm{c}^2}u^2 \;;\; u = \frac{1}{r} \;;\; h_{\mathrm{EGR}} = r^2\dot{\phi} \;;\; \dot{} = \frac{\mathrm{d}}{\mathrm{d}\tau_{\mathrm{EGR}}}. \quad (67)$$

The corresponding ERT *approximate differential equation of motion* (which also validates UCM) is:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = \frac{\mathrm{G}M}{h_{\mathrm{EGR}}^2} + 3\frac{\mathrm{G}^3M^3}{\mathrm{c}^2 h_{\mathrm{EGR}}^4}(1+e\cos\phi)^2 \;;\; u = \frac{1}{r} \;;\; h_{\mathrm{EGR}} = r^2\dot{\phi} \;;\; \dot{} = \frac{\mathrm{d}}{\mathrm{d}\tau_{\mathrm{EGR}}} \quad (68)$$

with *exact* and *approximate solution*, correspondingly

$$u = \frac{\mathrm{G}M}{h_{\mathrm{EGR}}^2}\left(1 + e_{\mathrm{EGR}}\cos\phi + 3\frac{\mathrm{G}^2M^2}{\mathrm{c}^2 h_{\mathrm{EGR}}^2}e_{\mathrm{EGR}}\phi\sin\phi\right) \;;\; \frac{\mathrm{G}M}{h_{\mathrm{EGR}}^2} = \frac{1}{a_{\mathrm{EGR}}\left(1-e_{\mathrm{EGR}}^2\right)} \;; \quad (69)$$

$$u \approx \frac{\mathrm{G}M}{h_{\mathrm{EGR}}^2}\left(1 + e_{\mathrm{EGR}}\cos\left[\left(1-3\frac{\mathrm{G}^2M^2}{\mathrm{c}^2 h_{\mathrm{EGR}}^2}\right)\phi\right]\right) \;;\; 0 < \frac{6\pi\mathrm{G}^2M^2}{\mathrm{c}^2 h_{\mathrm{EGR}}^2} << 1. \quad (70)$$

The last equation can be written as

$$u = \frac{1}{r} \approx \frac{\mathrm{G}M}{h^2}\left[1 + e\cos(\lambda_{\mathrm{EGR}}\phi)\right] \;;\; \lambda_{\mathrm{EGR}} = 1 - 3\frac{\mathrm{G}^2M^2}{\mathrm{c}^2h^2} \;;\; 0 < \frac{6\pi\mathrm{G}^2M^2}{\mathrm{c}^2h^2} << 1. \quad (71)$$

Hence the ERT orbit can be regarded as an *Einsteinian ellipse* (with $\alpha_{\mathrm{EGR}}$ semimajor axis) which rotates (‘precesses) about one of its foci by an amount

$$\Delta_{\mathrm{EGR}} = \frac{2\pi}{1-3\frac{\mathrm{G}^2M^2}{\mathrm{c}^2h_{\mathrm{EGR}}^2}} - 2\pi \approx \frac{6\pi\mathrm{G}^2M^2}{\mathrm{c}^2h_{\mathrm{EGR}}^2} = \frac{6\pi\mathrm{G}M}{a_{\mathrm{EGR}}\left(1-e_{\mathrm{EGR}}^2\right)\mathrm{c}^2} \;;\; h_{\mathrm{EGR}} = r^2\dot{\phi} \;;\; \dot{\phi} = \frac{\mathrm{d}\phi}{\mathrm{d}\tau_{\mathrm{EGR}}} = \frac{\mathrm{d}\phi}{\mathrm{d}t}\dot{t} \quad (72)$$

rad per revolution. Accordingly to our *non-mainstream approach*, we have

$$\dot{t} = \frac{\mathrm{d}t}{\mathrm{d}\tau_{\mathrm{EGR}}} = \left[1-\left(\frac{r_{\mathrm{S}}}{r} + \frac{1}{1-\frac{r_{\mathrm{S}}}{r}}\beta_{Pr}^2 + \beta_{P\phi}^2\right)\right]^{-\frac{1}{2}} \geq 1 \;;\; E_{\mathrm{EGR}} = \frac{1-\frac{r_{\mathrm{S}}}{r}}{\sqrt{1-\left(\frac{r_{\mathrm{S}}}{r} + \frac{1}{1-\frac{r_{\mathrm{S}}}{r}}\beta_{Pr}^2 + \beta_{P\phi}^2\right)}}m\mathrm{c}^2 \geq 0 \quad (73)$$

$$U_{\mathrm{EGR}} = \left(\sqrt{1-\frac{r_{\mathrm{S}}}{r}} - 1\right)m\mathrm{c}^2 \leq 0 \;;\; V_{\mathrm{EGR}} = \left(\sqrt{1-\frac{r_{\mathrm{S}}}{r}} - 1\right)\mathrm{c}^2 \leq 0 \;;\; K_{\mathrm{g\,EGR}} = \left(\frac{1-\frac{r_{\mathrm{S}}}{r}}{\sqrt{1-\left(\frac{r_{\mathrm{S}}}{r} + \frac{1}{1-\frac{r_{\mathrm{S}}}{r}}\beta_{Pr}^2 + \beta_{P\phi}^2\right)}} - \sqrt{1-\frac{r_{\mathrm{S}}}{r}}\right)m\mathrm{c}^2 \geq 0 \quad (74)$$

$$E_{\mathrm{m\,EGR}}=\left(\frac{1-\frac{r_{\mathrm{S}}}{r}}{\sqrt{1-\left(\frac{r_{\mathrm{S}}}{r}+\frac{1}{1-\frac{r_{\mathrm{S}}}{r}}\beta_{Pr}{}^{2}+\beta_{P\phi}{}^{2}\right)}}-1\right)m\mathrm{c}^{2}\quad;\quad\theta=\frac{\pi}{2}\,. \tag{75}$$

## 4. GSR: Metric of RIOs, Gravitational Potential, Field strength, Lagrangian, Equations of motion and Precession of planets' orbits

In case of GSR, the geometry of spacetime has steady metric (6). So, gravity is studied as a field, which comes from *GSR gravitational potential* $(V_{\mathrm{GSR}},\vec{w}_{\mathrm{GSR}})$. This adds extra terms to the *GSR Lagrangian of a free particle P*. In this paper, we examine the case that $\vec{w}_{SR}=0$, according to the mainstream approach of EP (1). Thus, the *GSR Lagrangian* in the frame of mass *M*, is [2] (p. 351):

$$L_{\mathrm{GSR}}=-g_{\mathrm{I}00}\left(-\frac{1}{\gamma_{\left(\xi_{\mathrm{I}}\vec{\beta}_{P}\right)}}m\mathrm{c}^{2}-\xi_{\mathrm{I}}{}^{2}mV_{\mathrm{GSR}}\right), \tag{76}$$

where $V_{\mathrm{GSR}}$ is *central gravitational potential*. Besides, the orbit of particle *P* is on the 'plane' θ=π/2 and we have:

$$\upsilon^{2}=\dot{r}^{2}+r^{2}\dot{\phi}^{2}\quad;\quad\gamma_{\left(\xi_{\mathrm{I}}\vec{\beta}_{P}\right)}=\frac{1}{\sqrt{1-\xi_{\mathrm{I}}{}^{2}\frac{\dot{r}^{2}+r^{2}\dot{\phi}^{2}}{\mathrm{c}^{2}}}}\quad;\quad\dot{}=\frac{\mathrm{d}}{\mathrm{d}t}, \tag{77}$$

$$L_{\mathrm{GSR}}=-g_{\mathrm{I}00}\left(-\sqrt{1-\xi_{\mathrm{I}}{}^{2}\frac{\dot{r}^{2}+r^{2}\dot{\phi}^{2}}{\mathrm{c}^{2}}}m\mathrm{c}^{2}-\xi_{\mathrm{I}}{}^{2}mV_{\mathrm{GSR}}\right). \tag{78}$$

The *GSR total energy definition* (16ii) and *Euler-Lagrange equations* (34) give us the *equations of motion*:

$$E_{\mathrm{tot\,GSR}}=\frac{\gamma_{\left(\xi_{\mathrm{I}}\vec{\beta}_{P}\right)}m\mathrm{c}^{2}}{\xi_{\mathrm{I}}{}^{2}}+mV_{\mathrm{GSR}}\quad; \tag{79}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma_{\left(\xi_{\mathrm{I}}\vec{\beta}_{P}\right)}\dot{r}\right)-\gamma_{\left(\xi_{\mathrm{I}}\vec{\beta}_{P}\right)}r\dot{\phi}^{2}+\frac{\partial V_{\mathrm{GSR}}}{\partial r}=0\quad; \tag{80}$$

$$J=mh=\gamma_{\left(\xi_{\mathrm{I}}\vec{\beta}_{P}\right)}mr^{2}\dot{\phi}\quad;\quad\dot{}=\frac{\mathrm{d}}{\mathrm{d}t}, \tag{81}$$

where the *integrals of motion* are: the *GSR total energy* ($E_{\mathrm{totGSR}}$) and the *GSR total angular momentum* (*J*). Besides, *h*=*J*/m is the *GSR angular momentum per mass unit*. Solving (79) in terms of *γ*, we find

$$\gamma_{\left(\xi_{\mathrm{I}}\vec{\beta}_{P}\right)}=\xi_{\mathrm{I}}{}^{2}\frac{E_{\mathrm{tot\,GSR}}-mV_{\mathrm{GSR}}}{m\mathrm{c}^{2}}\,. \tag{82}$$

Moreover, we have

$$\upsilon^{2}=\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^{2}\dot{\phi}^{2}+r^{2}\dot{\phi}^{2}=\left[\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^{2}+r^{2}\right]\frac{h^{2}}{\gamma_{\left(\xi_{\mathrm{I}}\vec{\beta}_{P}\right)}{}^{2}r^{4}}\quad;\quad\dot{}=\frac{\mathrm{d}}{\mathrm{d}t}. \tag{83}$$

Replacing the above in the identity $1+\xi_{\mathrm{I}}{}^{2}\frac{\upsilon^{2}}{\mathrm{c}^{2}}\gamma_{\left(\xi_{\mathrm{I}}\vec{\beta}_{P}\right)}{}^{2}=\gamma_{\left(\xi_{\mathrm{I}}\vec{\beta}_{P}\right)}{}^{2}$, we obtain the equation of trajectory:

$$1+\xi_{\mathrm{I}}^{2}\frac{h^{2}}{\mathrm{c}^{2}}\left[\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^{2}+u^{2}\right]=\xi_{\mathrm{I}}^{4}\left(\frac{E_{\mathrm{tot\,GSR}}-mV_{\mathrm{GSR}}}{m\mathrm{c}^{2}}\right)^{2} \;;\; u=\frac{1}{r} \;;\; h=r^{2}\dot{\phi} \;;\; \dot{}=\frac{\mathrm{d}}{\mathrm{d}t}, \tag{84}$$

Differentiation wrt $\varphi$, emerges the *equation of trajectory for a central gravitational potential*:

$$\frac{\mathrm{d}^{2}u}{\mathrm{d}\phi^{2}}+u=-\frac{\xi_{\mathrm{I}}^{2}}{h^{2}}\frac{E_{\mathrm{tot\,GSR}}-mV_{\mathrm{GSR}}}{m\mathrm{c}^{2}}\frac{\mathrm{d}V_{\mathrm{GSR}}}{\mathrm{d}u} \;;\; u=\frac{1}{r} \;;\; h=r^{2}\dot{\phi} \;;\; \dot{}=\frac{\mathrm{d}}{\mathrm{d}t}, \tag{85}$$

where the points with extreme values and circular motion are excluded [2] (p. 352).

Now, we propose the formula of *GSR gravitational central potential*:

$$V_{\mathrm{GSR}}=\left(\sqrt{1-k\frac{r_{\mathrm{S}}}{r}}-1\right)\frac{\mathrm{c}^{2}}{k}\leq 0 \;;\; k=k(\xi_{\mathrm{I}}). \tag{86}$$

The idea comes from 1GSP (51). This is modified (by introducing a real number $k$), because it is used according to GSR (in the space of RIOs). We also observe that the *GSR horizon* is located at distance $k$ *Schwarzschild radius* ($kr_{\mathrm{S}}$) from the center of gravity. Besides, we obtain the following *GSR gravitational field strength*:

$$\vec{g}=-\frac{dV_{\mathrm{GSR}}}{dr}\hat{r}=-\frac{\mathrm{G}M}{r^{2}}\left(1-k\frac{r_{s}}{r}\right)^{-\frac{1}{2}}\hat{r}. \tag{87}$$

The replacement of (86) to (85), gives the *exact equation of trajectory for GSR Gravitational field*:

$$\frac{\mathrm{d}^{2}u}{\mathrm{d}\phi^{2}}+u=-\frac{\xi_{\mathrm{I}}^{2}}{h^{2}}\left[-\mathrm{G}M\frac{E_{\mathrm{tot\,GSR}}}{m\mathrm{c}^{2}}(1-kr_{s}u)^{-\frac{1}{2}}+\frac{\mathrm{G}M}{k}-\frac{\mathrm{G}M}{k}(1-kr_{s}u)^{-\frac{1}{2}}\right] \;;\; u=\frac{1}{r}. \tag{88}$$

In order to make the above equation similar to the corresponding of *Newtonian scalar gravitational potential* at large distance ($kr_{\mathrm{S}}u$<<1 or equivalently $r$>>$kr_{\mathrm{S}}$), we apply *Taylor theorem* to the quantity:

$$(1-kr_{s}u)^{-\frac{1}{2}}\approx 1+\frac{k}{2}r_{s}u \;;\; u=\frac{1}{r} \;;\; r>>kr_{\mathrm{S}}. \tag{89}$$

So, we obtain the *approximate equation of trajectory for GSR Gravitational field*:

$$\frac{\mathrm{d}^{2}u}{\mathrm{d}\phi^{2}}+\left[1-\xi_{\mathrm{I}}^{2}\frac{\mathrm{G}M}{h^{2}}\frac{E_{\mathrm{tot\,GSR}}}{m\mathrm{c}^{2}}\frac{k\mathrm{G}M}{\mathrm{c}^{2}}-\xi_{\mathrm{I}}^{2}\frac{\mathrm{G}M}{h^{2}}\frac{\mathrm{G}M}{\mathrm{c}^{2}}\right]u=\xi_{\mathrm{I}}^{2}\frac{\mathrm{G}M}{h^{2}}\frac{E_{\mathrm{tot\,GSR}}}{m\mathrm{c}^{2}} \;;\; u=\frac{1}{r} \;;\; r>>kr_{\mathrm{S}}, \tag{90}$$

which after replacing

$$\lambda_{\mathrm{GSR}}^{2}=1-\xi_{\mathrm{I}}^{2}\frac{\mathrm{G}M}{h^{2}}\frac{E_{\mathrm{tot\,GSR}}}{m\mathrm{c}^{2}}\frac{k\mathrm{G}M}{\mathrm{c}^{2}}-\xi_{\mathrm{I}}^{2}\frac{\mathrm{G}M}{h^{2}}\frac{\mathrm{G}M}{\mathrm{c}^{2}}, \tag{91}$$

gives

$$\frac{\mathrm{d}^{2}u}{\mathrm{d}\phi^{2}}+\lambda_{\mathrm{GSR}}^{2}u=\xi_{\mathrm{I}}^{2}\frac{\mathrm{G}M}{h^{2}}\frac{E_{\mathrm{tot\,GSR}}}{m\mathrm{c}^{2}} \;;\; u=\frac{1}{r} \;;\; r>>kr_{\mathrm{S}}. \tag{92}$$

The above *equation of trajectory for GSR Gravitational field at large distance*, has the following solution:

$$u=\frac{1}{r}=\frac{\xi_{\mathrm{I}}^{2}\mathrm{G}ME_{\mathrm{tot\,GSR}}}{\lambda_{\mathrm{GSR}}^{2}m\mathrm{c}^{2}h^{2}}\left[1+e\cos(\lambda_{\mathrm{GSR}}\phi)\right] \;;\; u=\frac{1}{r} \;;\; \frac{\mathrm{G}M}{h^{2}}=\frac{1}{a(1-e^{2})} \;;\; r>>kr_{\mathrm{S}}. \tag{93}$$

Hence we have obtained again, the *ellipse* which rotates (‘precesses’) about one of its foci, by using only GSR.

The system of equations (91) and (93) contains the variables $\lambda_{\mathrm{GSR}}$ and $E_{\mathrm{totGSR}}$. So, we calculate them, by working at the perihelion, where $\varphi$=0, $r$=$\alpha$(1-$e$):

$$\frac{1}{a(1-e)}=\frac{\xi_{\mathrm{I}}^{2}\mathrm{G}ME_{\mathrm{tot\,GSR}}}{\lambda_{\mathrm{GSR}}^{2}m\mathrm{c}^{2}h^{2}}(1+e) \;;\; \frac{\mathrm{G}M}{h^{2}}=\frac{1}{a(1-e^{2})} \;;\; r>>kr_{\mathrm{S}}. \tag{94}$$

This emerges

$$E_{\text{tot GSR}} = \lambda_{\text{GSR}}{}^2 \frac{m\text{c}^2}{\xi_{\text{I}}{}^2} \quad ; \; r>>kr_{\text{S}}. \tag{95}$$

So, the combination of the last equation with (91) gives the SR total energy at large distance:

$$E_{\text{tot GSR}} = \frac{\dfrac{1}{\xi_{\text{I}}{}^2} - \dfrac{\text{G}M}{\text{c}^2 a(1-e^2)}}{1 + \dfrac{k\text{G}M}{\text{c}^2 a(1-e^2)}} m\text{c}^2 \le \frac{m\text{c}^2}{\xi_{\text{I}}{}^2} = E_{\text{rest}} \quad ; \; r>>kr_{\text{S}}. \tag{96}$$

Besides, the replacement of the above equation to (91) gives

$$\lambda_{\text{GSR}}{}^2 = 1 - \frac{(k+\xi_{\text{I}}{}^2)\text{G}M}{\text{c}^2 a(1-e^2) + k\text{G}M} = \frac{\text{c}^2 a(1-e^2) - \xi_{\text{I}}{}^2 \text{G}M}{\text{c}^2 a(1-e^2) + k\text{G}M} \ge 0 \quad ; \; r>>kr_{\text{S}}. \tag{97}$$

The above leads to the final conditions

$$0 \le e \le \sqrt{1 - \xi_{\text{I}}{}^2 \frac{r_{\text{S}}}{2a}} \quad ; \; a>>kr_{\text{S}}. \tag{98}$$

Moreover, the combination of (95) and (97) with (93) gives

$$u = \frac{1}{r} = \frac{1}{a(1-e^2)}\left[1 + e\cos(\lambda_{\text{GSR}}\phi)\right] \quad ; \; \lambda_{\text{GSR}} = \sqrt{\frac{\text{c}^2 a(1-e^2) - \xi_{\text{I}}{}^2 \text{G}M}{\text{c}^2 a(1-e^2) + k\text{G}M}} \quad ; \; r>>kr_{\text{S}}. \tag{99}$$

Thus, the precession of ellipse is

$$\Delta_{\text{GSR}} = \left(\frac{1}{\lambda_{\text{GSR}}} - 1\right)2\pi = \left(\sqrt{\frac{\text{c}^2 a(1-e^2) + k\text{G}M}{\text{c}^2 a(1-e^2) - \xi_{\text{I}}{}^2 \text{G}M}} - 1\right)2\pi = \left(\sqrt{\frac{1 + \dfrac{kr_{\text{S}}}{2a(1-e^2)}}{1 - \dfrac{\xi_{\text{I}}{}^2 r_{\text{S}}}{2a(1-e^2)}}} - 1\right)2\pi \quad ; \; a>>kr_{\text{S}} \tag{100}$$

rad per revolution.

The condition: 'this new *GSR gravitational field* gives the same *precession of Mercury's orbit* as does the original *Schwarzschild metric*', is equivalent to

$$\lambda_{\text{GSR}}{}^2 = \lambda_{\text{EGR}}{}^2. \tag{101}$$

This combined with (71ii) and (97), gives

$$k = \frac{6 - \xi_{\text{I}}{}^2 - \dfrac{9}{2}\dfrac{r_{\text{S}}}{a(1-e^2)}}{\left(1 - \dfrac{3}{2}\dfrac{r_{\text{S}}}{a(1-e^2)}\right)^2}. \tag{102}$$

*Parameter k* (reason) must be independent from *a* and *e* (results). So, we prefer to adopt the integer value (Generalized; Newtonian; Einsteinian):

$$k = 6 - \xi_{\text{I}}{}^2 \; ; \; k = 6 \; ; \; k = 5. \tag{103}$$

According to (87), the force is

$$\vec{F} = m\vec{g} = -m\frac{dV}{dr}\hat{r} = -\frac{\text{G}Mm}{r^2}\left(1 - k\frac{r_s}{r}\right)^{-\frac{1}{2}}\hat{r}. \tag{104}$$

At large distance ($r>>kr_{\text{S}}$), we apply *Taylor theorem* to the quantity:

$$\left(1 - k\frac{r_s}{r}\right)^{-\frac{1}{2}} \approx 1 + \frac{k}{2}\frac{r_s}{r} = 1 + k\frac{\text{G}M}{\text{c}^2 r} \quad ; \; r>>kr_{\text{S}}. \tag{105}$$

Thus, we obtain

$$\vec{F}=\left(-\frac{\mathrm{G}Mm}{r^2}-k\frac{\mathrm{G}^2M^2m}{\mathrm{c}^2r^3}\right)\hat{r}\;\;;\;\; r>>kr_\mathrm{S}. \tag{106}$$

According to NPs ($\xi_\mathrm{I}\to 0$; $k\to 6$), we obtain

$$\vec{F}=\left(-\frac{\mathrm{G}Mm}{r^2}-6\frac{\mathrm{G}^2M^2m}{\mathrm{c}^2r^3}\right)\hat{r}\;\;;\;\; r>>kr_\mathrm{S}, \tag{107}$$

which is the force that predicts the *precession of Mercury's perihelion*, by using *Perturbation Theory* [1] (p. 246), [3] (p.512), [13] (p. 536-539).

In case of *planet Mercury*, it is $\alpha$=0.38709893 AU, $e$=0.20563069 and $T$=87.968 days [14]. The values: AU= $1.4959787066\times10^{11}$ m, G=$6.67428(67)\times10^{-11}\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}$, c=299792458 $\mathrm{ms}^{-1}$ (exact) [15] (pp. 1-1, 1-20, 14-2) and $M$=$1{,}988{,}500\times10^{24}$ kg [16], give

$$\frac{r_\mathrm{S}}{a\left(1-e^2\right)}=\frac{2\mathrm{G}M}{\mathrm{c}^2a\left(1-e^2\right)}=5.32518(53)\times10^{-8}<<1. \tag{108}$$

The case of *Earth*, with $\alpha$= 1.00000011 AU, $e$= 0.01671022 and $T$=365.242 days [17], emerges

$$\frac{r_\mathrm{S}}{a\left(1-e^2\right)}=\frac{2\mathrm{G}M}{\mathrm{c}^2a\left(1-e^2\right)}=1.97476(20)\times10^{-8}<<1. \tag{109}$$

Now, we can return to all the previous formulas and replace the above values. Thus (100) and (47) give the results, which are summarized in Table 1. We observe that both ESR and NPs give the same precessions.

**Table 1.** Angular velocity of ellipse perihelion rotation ('precession') for *Mercury* and *Earth*, according to *GSR Gravitational field* ($\Omega_\mathrm{GSR}$) for *Newtonian Physics* ($\xi_\mathrm{I}$=0) and *Einsteinian Special Relativity* ($\xi_\mathrm{I}$=1) and according to the original *Schwarzschild Gravitoelectric Effect* ($\Omega_\mathrm{EGR}$). $\Delta\Omega_\mathrm{GSRr}$ (%) is the percentile relative change.

| | | **Mercury** | | | **Earth** | | |
|---|---|---|---|---|---|---|---|
| $\xi_\mathrm{I}$ | $k$ | $\Omega_\mathrm{GSR}$ / ′′cy$^{-1}$ | $\Omega_\mathrm{EGR}$ / ′′cy$^{-1}$ | $\Delta\Omega_\mathrm{GSRr}$ (%) | $\Omega_\mathrm{GSR}$ / ′′cy$^{-1}$ | $\Omega_\mathrm{EGR}$ / ′′cy$^{-1}$ | $\Delta\Omega_\mathrm{GSRr}$ (%) |
| 0 | 6 | 42.9820(43) | 42.9799(9) (1) | 0.005(10) | 3.83893(38) | 3.8401(4) | -0.030(14) |
| 1 | 5 | 42.9820(43) | 42.9799(9) (1) | 0.005(10) | 3.83893(38) | 3.8401(4) | -0.030(14) |

(1) [6] (p. 6)

## 5. GSR: Gravitational Red Shift

The proposed *GSR Gravitational field* was combined with the *mainstream approach of EP* (1). This combination cannot produce *Gravitational Red Shift* (GRS) as SM does. On the other hand, GRS is achieved, if only the proposed GSR Gravitational field is combined with the *alternative approach of EP* (2)

$$m_G=\gamma_{(\xi_\mathrm{I}\vec{\beta}_P)}m. \tag{110}$$

More specifically in GSR, the *photon* (the particle which is associated with the E/M radiation) [8] (p. 13) has

$$m{=}0\;\;;\;\;\beta_P=\frac{1}{\xi_\mathrm{I}}\;\;;\;\;\gamma_{(\xi_\mathrm{I}\vec{\beta}_P)}\to+\infty\;\;;\;\;E=\frac{\gamma_{(\xi_\mathrm{I}\vec{\beta}_P)}}{\xi_\mathrm{I}^{\,2}}m\mathrm{c}^2=hf\;\;;\;\;m_G=\gamma_{(\xi_\mathrm{I}\vec{\beta}_P)}m=\xi_\mathrm{I}^{\,2}\frac{hf}{\mathrm{c}^2}. \tag{111}$$

Thus, the energy conservation gives

$$E_\mathrm{tot\,GSR}=E+m_\mathrm{G}\mathrm{V}_\mathrm{GSR(r)}=\frac{\gamma_{(\xi_\mathrm{I}\vec{\beta}_P)}}{\xi_\mathrm{I}^{\,2}}m\mathrm{c}^2+\gamma_{(\xi_\mathrm{I}\vec{\beta}_P)}m\mathrm{V}_\mathrm{GSR(r)}=hf_{(r)}+\xi_\mathrm{I}^{\,2}hf_{(r)}\frac{\mathrm{V}_\mathrm{GSR(r)}}{\mathrm{c}^2}=hf_\infty, \tag{112}$$

where h is *Plank constant* and $f(r)$ ; $f_\infty$ are the frequencies of E/M radiation at distance $r$ from the center of gravity and at infinite distance, respectively. So, we obtain GRS:

$$f_\infty=f_{(r)}(1+\xi_I^2\, V_{GSR(r)}/c^2)=f(r)(1+\xi_I^2/k[\mathrm{sqrt}(1-kr_S/r)-1])\approx f_{(r)}(1-\xi_I^2/c^2GM/r)])<f_{(r)}, \tag{113}$$

where (113iii) is referred to light which was emitted from atoms located far away from the GSR horizon. We observe that NPs (with $\xi_I\rightarrow 0$) has no GRS, in contrast to ESR (with $\xi_I=1$) which gives the well-known GRS of ERT. For instance, the replacement of $f_{(r)}$ with the *Earth laboratory value* of line $D_1$ at the spectrum of Sodium (Na) to the above formula, emerges $f_\infty$ equal to the data from astronomical observation [18].

## 6. Modification of GSR Gravitational Field in order to also explain the Rotation Curves in Galaxies

The next step is the Modification of GSR Gravitational Field (86) in order to also explain the Rotation Curves in Galaxies

$$V_{GSR}=\left(\sqrt{1-h_{(r)}k\frac{r_S}{r}}-1\right)\frac{c^2}{k}\le 0 \;\; ; \;\; k=6-\xi_I^2 \;\; ; \;\; h(kr_S)\approx 1, \tag{114}$$

where $h$ is an unspecified function of the distance with slow evolution, in accordance with any TPs. The condition (114iii) simplifies the modified GSR Gravitational Field to (86), near to the GSR horizon. Besides, we obtain the following *GSR gravitational field strength*:

$$\vec{g}=-\frac{dV_{GSR}}{dr}\hat{r}=\left[-\frac{GM}{r^2}\left(h_{(r)}-r\frac{dh}{dr}\right)\left(1-kh_{(r)}\frac{r_s}{r}\right)^{-\frac{1}{2}}\right]\hat{r}=-g\hat{r} \;\; ; \tag{115}$$

$$g=\frac{GM}{r^2}\left(h_{(r)}-r\frac{dh}{dr}\right)\left(1-kh_{(r)}\frac{r_s}{r}\right)^{-\frac{1}{2}} \;\; ; \;\; k=6-\xi_I^2. \tag{116}$$

The positive value of field strength $g$ means gravity, while negative value means antigravity. In case of UCM, it is $g=\upsilon^2/r$. So,

$$\upsilon^2=\frac{GM}{r}\left(h_{(r)}-r\frac{dh}{dr}\right)\left(1-kh_{(r)}\frac{r_s}{r}\right)^{-\frac{1}{2}} \; ; \; k=6-\xi_I^2. \tag{117}$$

The above (116) reminds us the corresponding of 3GSM (28), which express only gravity. Far away from the horizon, the *gravitational field strengths* (28) and (116) become the same if only

$$a_{(r)}{}^2=h_{(r)}-r\frac{dh}{dr}. \tag{118}$$

We extend the above condition at any distance.

### *6.1. The Combination of Modified GSR Gravitational Field strength with MOND*

Modified Newtonian Dynamics (MOND) explains the rotation curves in many galaxies, by using suitable *Interpolating Function* ($\mu$) in *Milgrom's Law* [19]. In case of a spherical or cylindrical distribution of mass, the *Modified Newtonian field strength* is

$$g=\frac{1}{\mu_{(r)}}\frac{GM}{r^2} \tag{119}$$

The combination of the 3GSM field strength (28) with MOND (188) and condition (118) emerges

$$a_{(r)}{}^2=\frac{1}{\mu_{(r)}}=h_{(r)}-r\frac{dh}{dr} \; . \tag{120}$$

Two common choices are the *Simple* and *Standard interpolating function*, correspondingly

$$\frac{1}{\mu}=1+\frac{a_0}{g}=\frac{1}{2}\left(1+\sqrt{1+\left(\frac{r}{r_0}\right)^2}\right)\ ;\ \frac{1}{\mu}=\sqrt{1+\left(\frac{a_0}{g}\right)^2}=\frac{1}{\sqrt{2}}\sqrt{1+\sqrt{1+\frac{1}{4}\left(\frac{r}{r_0}\right)^4}}\ ;\ r_0=\sqrt{\frac{GM}{4a_0}}\ ,\quad (121)$$

where $r_0$ is called *Milgrom radius* [20] (p. 3) and $a_0=1.2(\pm0.1)\times10^{-10}\,\mathrm{ms}^{-2}$ [19] (p. 1) is a new (acceleration-dimensional) physical constant. Both the *Interpolating functions* give the same velocity at infinite distance from the center of gravity

$$\upsilon_\infty{}^2=\sqrt{\mathrm{G}Ma_0} \quad (122)$$

From (120), we calculate that

$$h_{(r)}=-r\int\frac{\mathrm{d}r}{r^2\mu_{(r)}}=-\frac{r}{r_0}\int\frac{1}{\left(\frac{r}{r_0}\right)^2\mu_{(r)}}\mathrm{d}\left(\frac{r}{r_0}\right)=-\frac{r}{r_0}I\,. \quad (123)$$

Besides, the integrals of *Simple* and *Standard interpolating functions* are correspondingly:

$$I_{\text{Simpl}}=\int\frac{1}{2x^2}\left(1+\sqrt{1+x^2}\right)\mathrm{d}x=\frac{1}{2}\left(-\frac{1}{x}-\frac{\sqrt{1+x^2}}{x}+\operatorname{arcsinh}x\right)+C_{\text{Simpl}}=\frac{1}{2}\left(-\frac{1}{x}-\frac{\sqrt{1+x^2}}{x}+\ln\left(x+\sqrt{1+x^2}\right)\right)+C_{\text{Simpl}} \quad (124)$$

$$I_{\text{Stand}}=\int\frac{1}{\sqrt{2}x^2}\sqrt{1+\sqrt{1+\frac{x^4}{4}}}\,\mathrm{d}x=\frac{1}{2}\frac{\left(\sqrt{1+\frac{x^4}{4}}+1\right)^{\frac{1}{4}}}{\left(\sqrt{1+\frac{x^4}{4}}-1\right)^{\frac{1}{4}}}\left(-1+2^{\frac{3}{4}}\left(1-\sqrt{1+\frac{x^4}{4}}\right)^{\frac{1}{4}}{}^2\mathrm{F}_1\left(\frac{1}{4},\frac{1}{4};\frac{5}{4};\frac{1}{2}\left(\sqrt{1+\frac{x^4}{4}}+1\right)\right)\right)+C_{\text{Stand}} \quad (125)$$

The last solution contains *Gauss hypergeometric function* and has steady imaginary part

$$\mathrm{Im}[\mathrm{I}_{\text{Stand}}-\mathrm{C}_{\text{Stand}}]=\frac{1}{\sqrt{2}}{}^2\mathrm{F}_1\left(\frac{1}{4},\frac{1}{4};\frac{5}{4};1\right)=\frac{1}{\sqrt{2}}\Gamma_{\left(\frac{3}{4}\right)}\Gamma_{\left(\frac{5}{4}\right)}=0.7853981633974484. \quad (126)$$

Thus, we have correspondingly

$$h_{(r)}=-xI_{\text{Simpl}}=\frac{1}{2}\left(1+\sqrt{1+x^2}-x\operatorname{arcsinh}x\right)-xC_{\text{Simpl}}=\frac{1}{2}\left(1+\sqrt{1+x^2}-x\ln\left(x+\sqrt{1+x^2}\right)\right)-xC_{\text{Simpl}}\ ;\ x=\frac{r}{r_0}\ ; \quad (127)$$

$$h_{(r)}=-xI_{\text{Stand}}=\frac{1}{\sqrt{2}}\sqrt{1+\sqrt{1+\frac{x^4}{4}}}\left(1-2^{\frac{3}{4}}\left(1-\sqrt{1+\frac{x^4}{4}}\right)^{\frac{1}{4}}{}^2\mathrm{F}_1\left(\frac{1}{4},\frac{1}{4};\frac{5}{4};\frac{1}{2}\left(\sqrt{1+\frac{x^4}{4}}+1\right)\right)\right)-xC_{\text{Stand}}\,. \quad (128)$$

We observe that in case of *Simple Interpolating function*: $h_{(0)}$=1. So, we prefer $C_{\text{Simpl}}$=0 and we have

$$h_{(r)}=\frac{1}{2}\left(1+\sqrt{1+x^2}-x\operatorname{ArcSinh}x\right)=\frac{1}{2}\left(1+\sqrt{1+x^2}-x\ln\left(x+\sqrt{1+x^2}\right)\right)\ ;\ x=\frac{r}{r_0} \quad (129)$$

We also observe that the *Standard Interpolating function* has $h_{(0)}$=1, but we now prefer

$$C_{\text{Stand}}=-\frac{1}{\sqrt{2}}{}^2\mathrm{F}_1\left(\frac{1}{4},\frac{1}{4};\frac{5}{4};1\right)i=-\frac{1}{\sqrt{2}}\Gamma_{\left(\frac{3}{4}\right)}\Gamma_{\left(\frac{5}{4}\right)}i=-0.7853981633974484\ \mathrm{i}. \quad (130)$$

in order to get rid of the imaginary part. Thus, we obtain

$$h_{(r)}=-xI_{\text{Stand}}=\frac{1}{\sqrt{2}}\sqrt{1+\sqrt{1+\frac{x^4}{4}}}\left(1-2^{\frac{3}{4}}\left(1-\sqrt{1+\frac{x^4}{4}}\right)^{\frac{1}{4}}{}^2\mathrm{F}_1\left(\frac{1}{4},\frac{1}{4};\frac{5}{4};\frac{1}{2}\left(\sqrt{1+\frac{x^4}{4}}+1\right)\right)\right)+x\frac{1}{\sqrt{2}}\Gamma_{\left(\frac{3}{4}\right)}\Gamma_{\left(\frac{5}{4}\right)}i \quad (131)$$

This kind of *'old' MONDian field strength*:

i. is efficient to explain the *rotation curves in galaxies* as well as the *precession of Mercury's orbit* (because $a\approx\mu\approx h\approx1$ in the Solar system), but

ii. gives extra larges values of the gravitational field strength around bodies with small mass. For instance a body of $M$=1 Kg at distance $r$=1 m, produces $\mu$=0.518 ($1/\mu$=1.93) according to the *Simple interpolating function*. This means *twice value* of the *Newtonian field strength* and contradicts to the *Cavendish experiment*.

In this paper, we also make changes to MOND resolving the above contradiction. Thus, we define the *New Simple* and *New Standard interpolating function* ($\mu$) respectively

$$\frac{1}{\mu}=1+\frac{M}{M_0}\frac{a_0}{g} \;\; ; \;\; \frac{1}{\mu}=\sqrt{1+\left(\frac{M}{M_0}\frac{a_0}{g}\right)^2} \,. \tag{132}$$

Let us also define the following *characteristic radii of α system* with mass $M$:

$$r_\infty=\frac{2r_0}{\sqrt{\beta_\infty}}=\sqrt[8]{\frac{G^3c^4}{a_0^5}}\sqrt[8]{M^3}=\mathrm{C}\sqrt[8]{M^3} \;\; ; \;\; R_0=\sqrt{\frac{GM_0}{4a_0}} \,, \tag{133}$$

where $M_0$ is the mass that is contained in a sphere of radius $r_\infty$ with the same center as the center of the system with mass $M$. We also calculate **C=4.2(0.2)×10$^6$ mKg$^{-3/8}$**, by using $a_0$=1.2(±0.1)×10$^{-10}$ ms$^{-2}$ [19] (p.1), G=6.67428(67)×10$^{-11}$m$^3$kg$^{-1}$s$^{-2}$ and c=299792458 ms$^{-1}$ (exact) [15] (pp. 1-1, 1-20, 14-2). For instance:

(i) In case that we examine the gravitational field at distance $r$=1 m from *a body* with $M$=1 Kg *on planet Earth* (at high $h$=2 m), we calculate $r_\infty$=1.4(0.1)×10$^{-13}$ Kpc=2.8(0.1)×10$^{-5}$ AU= 4.2(0.2)×10$^6$ m. Taking into account that the Earth has radius $R_{\text{Earth}}$=6,378,140 m and mass $M_{\text{Earth}}$=5.9742 10$^{24}$ kg [15] (pp.1-1,14-2), the accurate calculation of $M_0$ needs the part of the mass of Earth that is contained into the sphere with center the body. Of course, this mass has not spherical or cylindrical symmetry. So, the following calculation gives us only order of magnitude. We can easily find that $M_0$= ($V_0/V_{\text{Earth}}$) $M_{\text{Earth}}$, where $V_0$ is the common part of these spheres. Besides, we approximately consider that we have one sphere with center on the surface of another equal sphere (Figure 1). Thus the common volume can be calculated, by using the *second Pappus's centroid theorem* $V_0$=$d$·$S$=2π$CS$ [21], where $C$=(4$r$sin3$a$)/[2$a$-sin(2$a$)]-$r$cos$a$ is the distance of the center of Circular segment of central angle 2$a$=120°=2π/3 rad from the rotation axis and $S$=($r^2$/2)[2$a$-sin(2$a$)] is the area of the aforementioned Circular segment. So, it emerges $V_0$=π$r^3$[(4/3)sin3(π/3)-2(π/3)cos(π/3)+2sin(π/3)cos2(π/3)]= 0.251840554π$r^3$, where $r$=CubicRoot[($R_{\text{Earth}}^3$+$r_\infty^3$)/2]=5.5(0.3)×10$^6$ m and the mass of interest is $M_0$=($V_0/V_{\text{Earth}}$)$M_{\text{Earth}}$=[3$V_0$/(4π$R_{\text{Earth}}^3$)]$M_{\text{Earth}}$=[3·0.251840554π$r^3$/(4π$R_{\text{Earth}}^3$)]$M_{\text{Earth}}$= [3·0.251840554 (5.5×10$^6$)$^3$/(4·6378140$^3$)]·5.9742×10$^{24}$ kg=7.2(±1.2)×10$^{23}$ kg. This results $R_0$=1.0(0.2)×10$^{-8}$ Kpc=2.1(0.3) AU=3.2(0.5)×10$^{11}$ m.

(ii) The calculation of the gravitational field of *planet Earth* at high $h$=2 m, gives $r_\infty$=2.7(0.1)×10$^{-4}$Kpc=5.5(0.3)×10$^4$ AU=8.2(0.4)×10$^{15}$ m. Taking into account that our Solar system has radius $r_{\text{SolarSys}}$=2.4×10$^{-7}$Kpc=50 AU=7.5×10$^{12}$ m and the closest star to the Earth is *Alpha Centauri* at distance $r$=4.37 ly=1.34 pc=276×10$^3$ AU=4.13×10$^{16}$ m [22] (pp. 219–236), we understand that $M_0$=$M_{\text{SolSys}}$=1.9918×10$^{30}$ kg (the total mass of the Solar System). This gives $R_0$=1.7055×10$^{-5}$ Kpc=3,517.8 AU=5.2626×10$^{14}$ m.

(iii) The study of the gravitational field of the *Sun* at *planet Earth*, emerges $r_\infty$=0.0314(0.0014) Kpc= 6.5(0.3)×10$^6$ AU=9.7(0.5)×10$^{17}$ m. Taking into account that our galaxy (*Milky Way*) has mass $m_{\text{G}}$=1.3(±0.3)×10$^{12}$ M$_\odot$ =2.5(±0.6)×1042 Kg [23], diameter $d_{\text{G}}$=2$r_{\text{G}}$=175(±25)×10$^3$ ly=53.6(±7.7) Kpc= 1.103(±0.158)×10$^{10}$ AU=1.65(±0.24)×10$^{21}$ m [24] and our Solar System is located at distance $r$=2.65(±0.1)×10$^3$ ly=0.812(±0.003) Kpc=1.67×10$^8$ AU=2.51×10$^{19}$ m from the Galactic Center, it is obvious that the sphere of radius $r_\infty$ does not enclose the supermassive black hole of *Sagittarius A** with mass $m$=4.31(06)×10$^6$ M$_\odot$ =8.57(12)×10$^{36}$ Kg [25]. Thus, it is efficient to use as $M_0$=($r_\infty^2/r_{\text{G}}^2$)$m_{\text{G}}$=3.5×10$^{36}$ Kg (about 0.00014% of the mass of *Milky Way*) and we obtain $R_0$=0.023 Kpc=4.7×10$^6$ AU=7.0×10$^{17}$ m.

(iv) The calculation of the gravitational field of *Galaxy NGC 3198* on a *star* at distance $r$=8 Kpc from its center, emerges $r_\infty$=279 Kpc=5.7×10$^{10}$ AU=8.6×10$^{21}$ m. Thus, the sphere of radius $r_\infty$, encloses the whole galaxy of radius $r_G$=50 Kpc= $1.03\times10^{10}$ AU=$1.54\times10^{21}$ m and we put $M_0=m_G=6.76294\times10^{40}$ Kg [26] (p. 56). This gives $R_0$=3.14 Kpc=$6.5\times10^8$ AU=$9.7\times10^{19}$ m.

(v) The study of the gravitational field of the *Observable Universe* at the *Limit of observation*, emerges $r_\infty=1.02\times10^7$ Kpc= $2.10\times10^{15}$ AU=$3.14\times10^{26}$ m. Taking into account that the Observable Universe has mass $m_U=10^{53}$ Kg [27] (p. 43) and radius $r_U=4.5\times10^{10}$ ly=$1.4\times10^7$ Kpc= $2.9\times10^{15}$ AU=$4.3\times10^{26}$ m [28] (p. 27), it is efficient to use as $M_0=(r_\infty^3/r_U^3)m_U=3.9\times10^{52}$ Kg (about 39% of mass of the Observable Universe). Thus, we obtain $R_0=7.8\times10^9$ ly= $2.4\times10^6$ Kpc=$4.9\times10^{14}$ AU=$7.4\times10^{25}$ m.

The above manifest that *the gravitational field of a system* which is enclosed within a circular orbit of radius $r$, is not only affected by the *internal mass* ($M$), but it is also affected by the part of *total mass* ($M_0$) (internal & external) of the *hyper-system* (that is enclosed within radius $r_\infty$), where the system belongs!

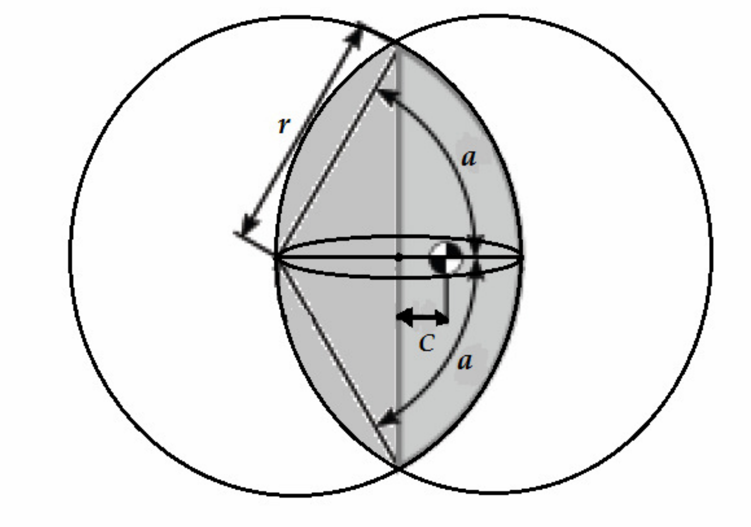


**Figure 1.** One sphere with center on the surface of another equal sphere. The common volume is calculated, by using the *second Pappus's centroid theorem*: $V_0=d\cdot S=2\pi CS$, where $C=(4r\sin^3 a)/[2a-\sin(2a)]-r\cos a$ is the distance of the center of Circular segment of central angle $2a=120^o=2\pi/3$ rad from the rotation axis and $S=(r^2/2)[2a-\sin(2a)]$ is the area of the aforementioned Circular segment.

Thus, we obtain the *New Simple* & *New Standard interpolating function* ($\mu$), respectively

$$\frac{1}{\mu}=\frac{1}{2}\left(1+\sqrt{1+\left(\frac{r}{R_0}\right)^2}\right)\ ;\ \frac{1}{\mu}=\frac{1}{\sqrt{2}}\sqrt{1+\sqrt{1+\frac{1}{4}\left(\frac{r}{R_0}\right)^4}}\ . \tag{134}$$

Both the *New Interpolating functions* give the same *new velocity at infinite distance* from the center of gravity (e.g. Galaxy)

$$\upsilon_{0\infty}{}^2=\sqrt{\mathrm{G}\frac{M^2}{M_0}a_0}=M\sqrt{\frac{\mathrm{G}\,a_0}{M_0}}\ ;\ \beta_{0\infty}=\frac{\upsilon_{0\infty}}{c}=\frac{1}{c}\sqrt[4]{G\frac{M^2}{M_0}a_0}=\frac{\sqrt{M}}{c}\sqrt[4]{\frac{\mathrm{G}\,a_0}{M_0}} \tag{135}$$

The above referred examples [System (Location) – Hyper-system] give, respectively the following results (data from Table 1):

(a) *New Simple interpolating function*: *(i)* $M$=1Kg (located at high $h$=2 m from Earth at $r$=1m) – Earth: $\mu$=0.9999999999999999999999998 ($1/\mu$=1.0000000000000000000000002=$1+2\times10^{-24}$), *(ii)* Earth (at $h$=2m from Earth surface) – Solar System: $\mu$=0.99999999999999996 ($1/\mu$=1.00000000000000004=$1+4\times10^{-17}$), *(iii)* Sun (at Earth) - 0.00014% Milky Way:

$\mu$=0.999999999999989 (1/$\mu$=1.00000000000001=1+1×$10^{-14}$), *(iv)* NGC 3198 (at Star 8 Kpc) - NGC 3198: $\mu$=0.53528 (1/$\mu$=1.8682), *(v)* Observable Universe (at the Limit of observation) - 39% Observable Universe: $\mu$=0.32 (1/$\mu$=3.2).

(b) *New Standard interpolating function*: *(i)* $M$=1Kg (located at high $h$=2 m from Earth at $r$=1m) – Earth: $\mu$=0.999999999999999999999999999999999999999999999997 (1/$\mu$=1.000000000000000000000000000000000000000000000003=1+3×$10^{-48}$), *(ii)* Earth (at $h$=2m from Earth surface) – Solar System: $\mu$=0.9999999999999999999999999999999993 (1/$\mu$=1.0000000000000000000000000000000007=1+7×$10^{-34}$), *(iii)* Sun (at Earth) – 0.00014% Milky Way: $\mu$=0.99999999999999999999999999993 (1/$\mu$=1.00000000000000000000000000007=1+7×$10^{-29}$), *(iv)* NGC 3198 (at Star 8 Kpc) - NGC 3198: $\mu$=0.67462 (1/$\mu$=1.48232), *(v)* Observable Universe (at the Limit of observation) - 39% Observable Universe: $\mu$=0.37 (1/$\mu$=2.7).

The results are in accordance with the *Cavendish experiment*, observations in our solar system and also explain the rotation curves in Galaxies, especially in case of *Standard* $\mu$. Besides, they are summarized in Table 1.

### *6.2. The Combination of Modified GSR Gravitational Field strength with the concept of phantom Dark Matter and the Velocity at Infinite Distance of MOND*

Below, we shall find the metric of spacetime that corresponds to the concept of phantom DM [1] (p. 356). We consider a *very simple distribution of phantom DM*:

$$\rho_{\text{dark}} = \frac{C_{\text{dark}}}{r^2} \;\; ; \;\; M_{\text{dark}} = \int_0^r 4\pi r^2 \rho_{\text{dark}} dr = 4\pi C_{\text{dark}} r \tag{136}$$

and also *all the luminous-baryonic mass at the center of gravity*. In case of a *spherical* or *cylindrical distribution of mass*, the *Modified Newtonian field strength* is

$$g = \frac{\mathrm{G}(M + M_{\text{dark}})}{r^2} = \left( \frac{\mathrm{G} M}{r^2} + \frac{4\pi \mathrm{G}\, C_{\text{dark}}}{r} \right). \tag{137}$$

Thus, the velocity in UCM is given by the formula

$$\upsilon^2 = \frac{\mathrm{G}(M + M_{\text{dark}})}{r} = \left( \frac{\mathrm{G} M}{r} + 4\pi \mathrm{G}\, C_{\text{dark}} \right), \tag{138}$$

which at infinite distance from the center of gravity, gives

$$\upsilon_\infty{}^2 = 4\pi \mathrm{G}\, C_{\text{dark}} . \tag{139}$$

The combination of the above equation with the *'old'* (122) or *New* (135i) *MONDian formula* gives correspondingly

$$C_{\text{dark}} = \frac{1}{4\pi}\sqrt{\frac{M a_0}{\mathrm{G}}} \;\; ; \;\; C_{\text{dark}} = \frac{M}{4\pi}\sqrt{\frac{a_0}{\mathrm{G} M_0}} , \tag{140}$$

where $M_0$ is the *part of mass of the hyper-system* that is enclosed within radius $r_\infty$.

*i)* According to (140i), the initial equations can be written as

$$\rho_{\text{dark}} = \frac{1}{4\pi}\sqrt{\frac{a_0}{\mathrm{G}}}\frac{\sqrt{M}}{r^2} = \frac{1}{8\pi}\frac{M}{r_0 r^2} \;\; ; \;\; M_{\text{dark}} = \sqrt{\frac{M a_0}{\mathrm{G}}} r = \frac{M}{2}\frac{r}{r_0} . \tag{141}$$

The replacement of *'old' MONDian* (140i) to (137) and the combination with (119) and (120) give

$$a_{(r)}{}^2 = \frac{1}{\mu_{(r)}} = h_{(r)} - r\frac{\mathrm{d}h}{\mathrm{d}r} = 1 + \frac{1}{2}\frac{r}{r_0} . \tag{142}$$

*ii)* According to *'new' MONDian* (140ii), the initial equations can be written as

$$\rho_{\text{dark}} = \frac{M}{4\pi}\sqrt{\frac{a_0}{\mathrm{G} M_0}}\frac{1}{r^2} = \frac{1}{8\pi}\frac{M}{R_0 r^2} \;\; ; \;\; M_{\text{dark}} = M\sqrt{\frac{a_0}{\mathrm{G} M_0}} r = \frac{M}{2}\frac{r}{R_0} . \tag{143}$$

The replacement of (140ii) to (137) and the combination with (119) and (120) give

$$a_{(r)}{}^{2}=\frac{1}{\mu_{(r)}}=h_{(r)}-r\frac{\mathrm{d}h}{\mathrm{d}r}=1+\frac{1}{2}\frac{r}{R_0}. \tag{144}$$

Thus, the integral of (123) in case of DM is

$$I_{DM}=\int\frac{1}{x^2}\left(1+\frac{1}{2}x\right)\mathrm{d}x=-\frac{1}{x}+\frac{1}{2}\ln x+C_{\mathrm{DM}}, \tag{145}$$

where

$$x=\frac{r}{r_0}\quad;\quad x=\frac{r}{R_0} \tag{146}$$

for *'old'* or *'new'* distribution of DM, correspondingly. Thus, we have

$$h_{(r)}=-xI_{\mathrm{Simpl}}=1-\frac{x}{2}\ln x-xC_{\mathrm{DM}} \tag{147}$$

We observe that $h_{(0)}$=1. So, we prefer $C_{\mathrm{Simpl}}$=0 and we also have

$$h_{(r)}=-xI_{\mathrm{Simpl}}=1-\frac{x}{2}\ln x. \tag{148}$$

The *'new'* concept, for $M\to 0$ gives $a\approx\mu\approx h\approx 1$. This turns metric (24) approximately equal to the metric of RIOs (6). This is a general property, because we have used (135i). Besides, the examination of the dark matter around a body with $M$=1 Kg, *near to planet Earth*, within radius $r$=1 m (at high $h$=2 m), gives $M_{\mathrm{dark}}=5.488\times 10^{-13}$ Kg. This corresponds to the *Newtonian field strength*, in accordance with the *Cavendish experiment*.

Finally, it is proven that the corresponding values of function $a_{(r)}$ have the properties: *Standard Interpolating function* < *Simple Interpolating function* < *Absorption of DM* and also *'New'* < *'old'*.

## 7. Experimental Validation - Discussion

In Table 2, we show the values of *'old'* and *'new'* characteristic parameters for the original 1Kg, the Earth, the Sun (see [9] (p. 8) and data from [15] (pp. 1-1, 14-2)), Galaxy NGC 3198 (data from [26] (p. 56) and [29] (p. 3)) and the Observable Universe (data from [27] (p. 43) and [28] (p. 27)). Besides, $M_0$ is the mass that is enclosed in a sphere of radius $r_\infty$ and the radii $r_\infty$, $R_0$, the new velocity at infinite distance $\upsilon_{0\infty}$, $\beta_{0\infty}$, the inverse of the *New Interpolating functions* $1/\mu_{\mathrm{Nsimpl}}$ and $1/\mu_{\mathrm{Nstand}}$ on a sphere of radius $R$ and they have been obtained from (133i), (133ii), (135i), (135ii), (134i) & (134ii), respectively. Besides, we have used the following values of physical constants: $a_0=1.2(0.1)\times 10^{-10}$ ms$^{-2}$ [19] (p.1), AU=$1.4959787066\times 10^{11}$ m, G=$6.67428(67)\times 10^{-11}$m$^3$kg$^{-1}$s$^{-2}$, c=299792458 ms$^{-1}$ (exact) [15] (pp. 1-1, 1-20, 14-2).

### *7.1. The Combination of Lorentzian-Einsteinian 3rd Generalized Schwarzschild Metric or Modified GSR Gravitational Field with 'old' or 'New' MOND Simple & Standard Interpolating Function and 'old' or 'New' Absorption of the Dark Matter into the field in Galaxy NGC 3198*

In order to find out what is the effect of the modification at large mass and size systems, we analytically examine Galaxy NGC 3198.

The values of Circular Velocities [experimental ($V_{\mathrm{exp}}$) and calculated by the Combination of *Lorentzian-Einsteinian 3GSM* or *Modified GSR Gravitational Field* with the corresponding *Simple* $\mu$ ($V_{\mathrm{simp,Lor}}$) or *New Simple* $\mu$ ($V_{\mathrm{Nsimp,Lor}}$) or *Standard* $\mu$ ($V_{\mathrm{stand,Lor}}$) or *New Standard* $\mu$ ($V_{\mathrm{Nstand,Lor}}$) or *Absorption of DM into the Metric by using distribution (141)* ($V_{\mathrm{DM,Lor}}$) or *Absorption of DM into the Metric by using distribution (143)* ($V_{\mathrm{NDM,Lor}}$)], the Luminous Mass of the galaxy that is enclosed within the circular orbit ($M_{\mathrm{d}}$), the corresponding *Schwarzschild radius* ($r_{\mathrm{S}}$), *Milgrom radius* ($r_0$), the corresponding values of the function $a_{(r)}$ ($a_{\mathrm{simp,Lor}}$ or $a_{\mathrm{Nsimp,Lor}}$ or $a_{\mathrm{stand,Lor}}$ or $a_{\mathrm{Nstand,Lor}}$ or $a_{\mathrm{DM,Lor}}$ or $a_{\mathrm{NDM,Lor}}$), function $h_{(r)}$ ($h_{\mathrm{simp,Lor}}$ or $h_{\mathrm{Nsimp,Lor}}$ or $h_{\mathrm{stand,Lor}}$ or $h_{\mathrm{Nstand,Lor}}$ or $h_{\mathrm{DM,Lor}}$ or $h_{\mathrm{NDM,Lor}}$) and time coefficient of metric $g_{00}$ ($g_{00,\mathrm{simp,Lor}}$ or $g_{00,\mathrm{Nsimp,Lor}}$ or $g_{00,\mathrm{stand,Lor}}$ or $g_{00,\mathrm{Nstand,Lor}}$ or $g_{00,\mathrm{DM,Lor}}$ or $g_{00,\mathrm{NDM,Lor}}$) wrt the distance from the center of Galaxy NGC 3198, are contained in Table 3 (data from [30] (p. 2)). The Circular Velocities ($V_{\mathrm{simp,Lor}}$, $V_{\mathrm{stand,Lor}}$, $V_{\mathrm{Nsimp,Lor}}$, $V_{\mathrm{Nstand,Lor}}$, $V_{\mathrm{DM,Lor}}$, and $V_{\mathrm{NDM,Lor}}$) have been calculated by using (119) or (28) or (116), the values of function $a_{(r)}$ ($a_{\text{simp,Lor}}$ or $a_{\text{stand,Lor}}$ or $a_{\text{Nsimp,Lor}}$ or $a_{\text{Nstand,Lor}}$ or $a_{\text{DM,Lor}}$ or $a_{\text{NDM,Lor}}$), by using (120) combined with (121) or (134) or (142) or (144), the values of function $h_{(r)}$ ($h_{\text{simp,Lor}}$ or $h_{\text{stand,Lor}}$ or $h_{\text{Nsimp,Lor}}$ or $h_{\text{Nstand,Lor}}$ or $h_{\text{DM,Lor}}$ or $h_{\text{NDM,Lor}}$), by using and (129) or (131) or (148) combined with (146), respectively. Finally the values of time coefficient of metric ($g_{00}$) have been calculated from (24) for $\xi_I$=1.

The calculation of the gravitational field of Galaxy NGC 3198 on a star at distance $r$=2 Kpc from its center, emerges $r_\infty$=118 Kpc=2.4×10$^{10}$ AU=3.6×10$^{21}$ m. Thus, the sphere of radius $r_\infty$, encloses the whole galaxy of radius $r_G$=50 Kpc= 1.03×10$^{10}$ AU=1.54×10$^{21}$ m and we put $M_0$=$m_G$=6.76294×10$^{40}$ Kg [4] (p. 56). This gives steady *new Milgrom radius* $R_0$=3.14 Kpc=6.5×10$^8$ AU=9.7×10$^{19}$ m.

**Table 2.** Characteristic parameters (mass $M$, distance or size radius $R$, *Schwarzschild radius* $r_S$, *Milgrom radius* $r_0$, $r_0/r_S$, velocity at infinite distance $\upsilon_\infty$, $\beta_\infty$, *new Milgrom radius* $R_0$, new velocity at infinite distance $\upsilon_{0\infty}$, $\beta_{0\infty}$, New Interpolating functions $1/\mu_{\text{NSimpl}}$ and $1/\mu_{\text{NStand}}$ on a sphere of radius $R$) for 1 Kg, the Earth, the Sun, galaxy NGC 3198 and the Observable Universe.

| | **1 Kg (original)** | **Earth** | **Sun** | **NGC 3198** | **Observable Universe** |
|---|---|---|---|---|---|
| **M /** Kg | 1 | 5.9742×10$^{24}$ (1) | 1.9891×10$^{30}$ (1) | 6.76294×10$^{40}$ (2) | 10$^{53}$ (4) |
| ***R*** **/** m | 1 | 6378140 (1) | 6.9599×10$^8$ (1) | 2.47×10$^{20}$ | 4.3×10$^{26}$ |
| /AU | 6.68×10$^{-12}$ | 4.263523×10$^{-5}$ | 4.6524×10$^{-3}$ | 1.65×10$^9$ | 2.9×10$^{15}$ |
| **/** Kpc | 3.24×10$^{-20}$ | 2.066999×10$^{-13}$ | 2.2555×10$^{-11}$ | 8 (3) | 14×10$^6$ (5) |
| $r_s$ **/** m | 2.96×10$^{-27}$ | 8.8736×10$^{-3}$ | 2,954.4 | 1.004451×10$^{14}$ | 1.48×10$^{26}$ |
| /AU | 1.98×10$^{-38}$ | 5.9316×10$^{-14}$ | 1.9749×10$^{-8}$ | 671.434 | 9.9×10$^{14}$ |
| **/** Kpc | 9.61×10$^{-47}$ | 2.8757×10$^{-22}$ | 9.5746×10$^{-17}$ | 0.0000680703 | 4.80×10$^6$ |
| $r_0$ **/** m | 0.373 | 9.1143×10$^{11}$ | 5.2591×10$^{14}$ | 9.6972671×10$^{19}$ | 1.18×10$^{26}$ |
| /AU | 2.49×10$^{-12}$ | 6.0925 | 3,515.5 | 6.45222×10$^8$ | 7.9×10$^{14}$ |
| **/** Kpc | 1.21×10$^{-20}$ | 2.9537×10$^{-8}$ | 0.000017043 | 3.14265 | 3.8×10$^6$ |
| $r_0/r_s$ | 1.26×10$^{26}$ | 1.02712×10$^{14}$ | 1.7801×10$^{11}$ | 965,430 | 0.80 |
| $\upsilon_\infty$ **/** m s$^{-1}$ | 9.45×10$^{-6}$ | 14.7899 | 355.27 | 152,556 | 1.68×10$^8$ |
| $\beta_\infty$ | 3.15×10$^{-14}$ | 4.93339×10$^{-8}$ | 1.1851×10$^{-6}$ | 0.000508873 | 0.56 |
| $r_\infty$ / m | 4.2(0.2)×10$^6$ | 8.2(0.4)×10$^{15}$ | 9.7(0.5)×10$^{17}$ | 8.6×10$^{21}$ | 3.14×10$^{26}$ |
| / AU | 2.8(0.1)×10$^{-5}$ | 5.5(0.3)×10$^4$ | 6.5(0.3)×10$^6$ | 5.7×10$^{10}$ | 2.10×10$^{15}$ |
| / Kpc | 1.4(0.1)×10$^{-13}$ | 2.7 (0.1)×10$^{-4}$ | 0.0314(0.0014) | 279 | 1.02×10$^7$ |
| $M_0$ / Kg | 7.2(1.2)×10$^{23}$ | 1.9918× 10$^{30}$ | 3.5×10$^{36}$ | 6.76294×10$^{40}$ | 3.9×10$^{52}$ |
| $R_0$ / m | 3.2(0.5)×10$^{11}$ | 5.2626×10$^{14}$ | 9.7×10$^{17}$ | 9.7×10$^{19}$ | 7.4×10$^{25}$ |
| / AU | 2.1(0.3) | 3,517.8 | 4.7×10$^6$ | 6.5×10$^8$ | 4.9×10$^{14}$ |
| / Kpc | 1.0(0.2)×10$^{-8}$ | 1.7055×10$^{-5}$ | 0.023 | 3.14 | 2.4×10$^6$ |
| $\upsilon_{0\infty}$ / m s$^{-1}$ | 1.0×10$^{-11}$ | 0.615 | 9.75 | 152,556 | 2.1×10$^8$ |
| $\beta_{0\infty}$ | 3.3×10$^{-20}$ | 2.05×10$^{-9}$ | 3.25×10$^{-8}$ | 0.000508873 | 0.70 |
| $1/\mu_{\text{Nsimpl}}$ | 1+2×10$^{-24}$ | 1+4×10$^{-17}$ | 1+1×10$^{-14}$ | 1.8682 | 3.2 |
| $1/\mu_{\text{Nstand}}$ | 1+3×10$^{-48}$ | 1+7×10$^{-34}$ | 1+7×10$^{-29}$ | 1.48232 | 2.7 |

[1][15] (pp. 1-1, 14-2), [2][26] (p. 56), [3][29] (p. 3), [4] [27] (p. 43), [5] [28] (p. 27).

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**Table 3.** Circular Velocities [experimental ($V_{exp}$) and calculated by the Combination of *Lorentzian-Einsteinian 3rd Generalized Schwarzschild metric* or *Modified GSR Gravitational Field* with the corresponding *Simple μ* ($V_{simp,Lor}$) or *New Simple μ* ($V_{Nsimp,Lor}$) or *Standard μ* ($V_{stand,Lor}$) or *New Standard μ* ($V_{Nstand,Lor}$) or *Absorption of DM into the Metric by using distribution (141)* ($V_{DM,Lor}$) or *Absorption of DM into the Metric by using distribution (143)* ($V_{NDM,Lor}$)], the Luminous Mass of the galaxy that is enclosed within the circular orbit ($M_d$), the corresponding values of function $a_{(r)}$, function $h_{(r)}$ and time coefficient of metric ($g_{00}$) wrt the distance from the center of Galaxy NGC 3198. The relative errors of the experimental Velocities are $(\Delta V_{exp})_r \approx 8\%$. [1][30] (p. 2)

| ***r*** / Kpc<br>/$10^{20}$m | $M_d$<br>/$10^{40}$ kg | $V_{exp}$ [(1)]<br>/ Km s$^{-1}$ | $a_{simp,Lor}$<br>$a_{stand,Lor}$<br>$a_{Nsimp,Lor}$<br>$a_{Nstand,Lor}$<br>$a_{DM,Lor}$<br>$a_{NDM,Lor}$ | $h_{simp,Lor}$<br>$h_{stand,Lor}$<br>$h_{Nsimp,Lor}$<br>$h_{Nstand,Lor}$<br>$h_{DM,Lor}$<br>$h_{NDM,Lor}$ | $g_{00,simp,Lor}$<br>$g_{00,stand,Lor}$<br>$g_{00,Nsimp,Lor}$<br>$g_{00,Nstand,Lor}$<br>$g_{00,DM,Lor}$<br>$g_{00,NDM,Lor}$ | $V_{simp,Lor}$<br>$V_{stand,Lor}$<br>$V_{Nsimp,Lor}$<br>$V_{Nstand,Lor}$<br>$V_{DM,Lor}$<br>$V_{NDM,Lor}$ / Km s$^{-1}$ | $(\Delta V)_r$<br>% |
|---|---|---|---|---|---|---|---|
| 4.0 | 1.620 | 118.0 | 1.3759 | -0.2964 | -0.9999997318 | 128.783 | 9 |
| 1.23 | | | 1.2265 | -1.3149 | -0.9999997609 | 114.801 | -3 |
| | | | 1.0341 | 0.6330 | -0.9999997769 | 107.103 | -9 |
| | | | 1.3437 | -0.0247 | -0.9999997984 | 96.790 | -18 |
| | | | 1.5167 | -0.2426 | -0.9999997043 | 141.959 | 20 |
| | | | 1.2792 | 0.8407 | -0.9999997506 | 119.735 | 1 |
| 8.0 | 5.825 | 150.3 | 1.3999 | -0.4181 | -0.9999995093 | 175.687 | 17 |
| 2.47 | | | 1.2506 | -1.4709 | -0.9999995617 | 156.950 | 4 |
| | | | 1.3666 | -0.2523 | -0.9999995210 | 171.504 | 14 |
| | | | 1.2173 | -1.2549 | -0.9999995734 | 152.765 | 2 |
| | | | 1.5399 | -0.3837 | -0.9999994603 | 193.262 | 29 |
| | | | 1.5076 | -0.2073 | -0.9999994716 | 189.202 | 26 |
| 16.1 | 7.237 | 155.3 | 1.7396 | -2.6769 | -0.9999996236 | 171.526 | 10 |
| 4.97 | | | 1.6060 | -4.2377 | -0.9999996526 | 158.351 | -2 |
| | | | 1.7635 | -2.8797 | -0.9999996185 | 173.880 | 12 |
| | | | 1.6312 | -4.4717 | -0.9999996471 | 160.839 | 4 |
| | | | 1.8645 | -2.9619 | -0.9999995966 | 183.838 | 18 |
| | | | 1.8872 | -3.2344 | -0.9999995917 | 186.078 | 20 |
| 32.2 | 6.544 | 148.4 | 2.3942 | -10.0944 | -0.9999997658 | 158.734 | 7 |
| 9.94 | | | 2.2927 | -12.8678 | -0.9999997757 | 152.004 | 2 |
| | | | 2.3764 | -9.8497 | -0.9999997675 | 157.557 | 6 |
| | | | 2.2742 | -12.5675 | -0.9999997775 | 150.781 | 2 |
| | | | 2.4916 | -11.2046 | -0.9999997444 | 165.194 | 11 |
| | | | 2.4745 | -11.0458 | -0.9999997580 | 164.058 | 11 |
| 48.2 | 6.072 | 151.9 | 2.9340 | -19.5417 | -0.9999998221 | 153.157 | 1 |
| 14.87 | | | 2.8503 | -23.5855 | -0.9999998272 | 148.785 | -2 |
| | | | 2.8609 | -18.0986 | -0.9999998265 | 149.341 | -2 |
| | | | 2.7751 | -21.9252 | -0.9999998317 | 144.862 | -5 |
| | | | 3.0155 | -21.5329 | -0.9999998172 | 157.409 | 4 |
| | | | 2.9443 | -20.1474 | -0.9999998215 | 153.691 | 1 |
| 13800 | 6.763 | - | 46.8625 | -17,741.1 | -0.9999999889 | 152.574 | - |
| 4,258.3 | | | 46.8571 | -18703. 6 | -0.9999999889 | 152.557 | - |

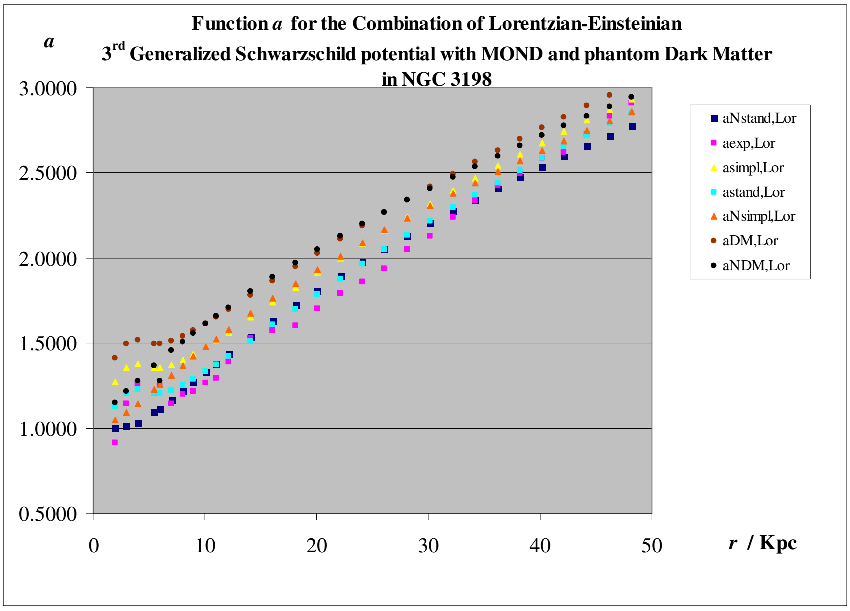


**Figure 2.** Plot of function $a_{(r)}$ wrt the distance ($r$) from the center of Galaxy NGC 3198 for the Combination of *Lorentzian-Einsteinian 3rd Generalized Schwarzschild metric* or *Modified GSR Gravitational Field* with *Simple interpolating function* ($a_{\mathrm{simp,Lor}}$), or *Standard interpolating function* ($a_{\mathrm{stand,Lor}}$), or *New Simple Interpolating Function* ($a_{\mathrm{Nsimp,Lor}}$), or *New Standard Interpolating Function* ($a_{\mathrm{Nstand,Lor}}$), or *Absorption of phantom Dark Matter into the Metric by using distribution (141)* ($a_{\mathrm{DM,Lor}}$), or *Absorption of phantom Dark Matter into the Metric by using distribution (165)* ($a_{\mathrm{NDM,Lor}}$). The experimental values ($a_{\mathrm{exp,Lor}}$) have been obtained, by replacing the experimental acceleration ($g_{\mathrm{exp}}=V_{\mathrm{exp}}^2/r$) in (28).

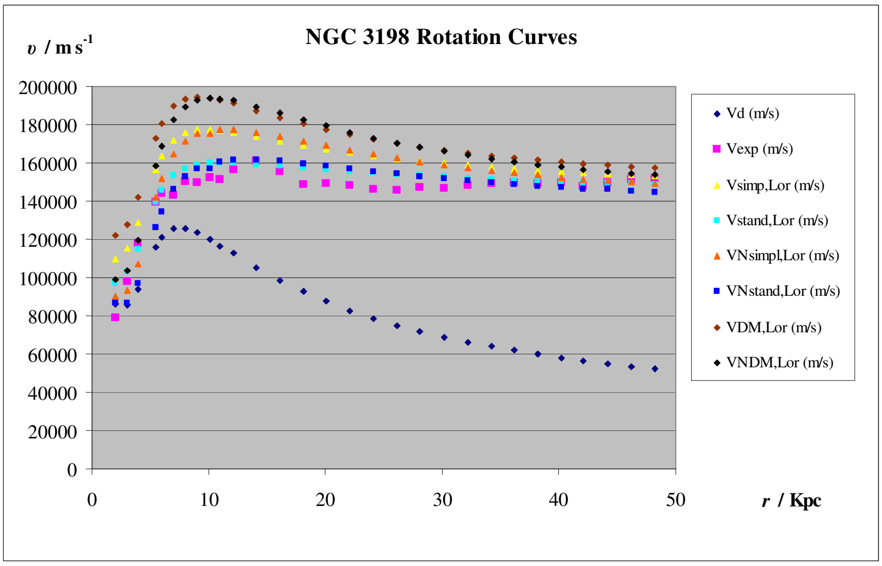


**Figure 3.** Rotation Curves in Galaxy NGC 3198. Rotational Velocities [experimental ($V_{\mathrm{exp}}$), calculated by *Schwarzschild* or *Newtonian field strength* ($V_{\mathrm{d}}$) and the Combination of *Lorentzian-Einsteinian 3rd Generalized Schwarzschild metric* or *Modified GSR Gravitational Field* with *Simple interpolating function* ($V_{\mathrm{simp,Lor}}$), or *Standard interpolating function* ($V_{\mathrm{stand,Lor}}$), or *New Simple Interpolating Function* ($V_{\mathrm{Nsimp,Lor}}$), or *New Standard Interpolating Function* ($V_{\mathrm{Nstand,Lor}}$), or *Absorption of phantom Dark Matter into the Metric by using distribution (141)* ($V_{\mathrm{DM,Lor}}$), or *Absorption of phantom Dark Matter into the Metric by using distribution (165)* ($V_{\mathrm{NDM,Lor}}$)] wrt the distance ($r$) from the center of Galaxy NGC 3198.

In Figure 2, we show the plot of function $a_{(r)}$ wrt the distance from the center of Galaxy NGC 3198 for the Combination of *Lorentzian-Einsteinian 3GSM* or *Modified GSR Gravitational Field* with *Simple μ* ($a_{\text{simp,Lor}}$), or *Standard μ* ($a_{\text{stand,Lor}}$), or *New Simple μ* ($a_{\text{Nsimp,Lor}}$) or *New Standard μ* ($a_{\text{Nstand,Lor}}$), or *Absorption of phantom Dark Matter into the Metric by using distribution (141)* ($a_{\text{DM,Lor}}$), or *Absorption of phantom Dark Matter into the Metric by using distribution (143)* ($a_{\text{NDM,Lor}}$). The experimental values ($a_{\text{exp,Lor}}$) have been obtained, by replacing the experimental acceleration ($g_{\text{exp}}=V_{\text{exp}}^2/r$) in (28). In addition, the corresponding Rotation Curves in Galaxy NGC 3198 are shown in Figure 3.

We observe that in case of Galaxy NGC 3198, *Schwarzschild* or *Newtonian field strength* produces maximum relative error about 66% at extra large distances. The *Simple μ* gives better results, producing maximum relative error 39% near to the galactic center and it is improved as *New Simple μ* with corresponding maximum relative error 18% at 11.0 Kpc. The *Standard μ* in (119) gives even better results, producing maximum relative error about 23% at the center of the galaxy is also improved as *New Standard μ* with corresponding maximum relative error -18% at 4.0 Kpc. The *Absorption of phantom DM into the Metric by using distribution (158)* ($V_{\text{DM,Lor}}$) has maximum relative error 54% near to the galactic center and it is improved with *DM distribution (165)* ($V_{\text{NDM,Lor}}$) with corresponding maximum relative error 28% at 9.0 Kpc. It is noted that the relative error of experimental Circular Velocities is $(\Delta V_{\text{exp}})_r \approx 8\%$ related to the uncertainty of the Hubble constant $H_0$ [11] (p.356-357). Finally, the values at distance 13.8 Mpc=$2.846\times10^{12}$ AU=$4.258\times10^{23}$ m, which is the distance of Galaxy NGC 3198 from Earth, give us the image of what happens at extremely large distances: $g_{00}\to -1$.

The same procedure can be followed in any galaxy, by using only the mass of the visible disk. Thus, it explains the rotation curves of many galaxies, eliminating the corresponding DM (see Figure 4 [31]).

**Galaxies well fit by MOND**

*84 listed at present*

UGC 2885 NGC 5533 NGC 6674 NGC 7331 NGC 5907 NGC 2998
NGC 801 NGC 5371 NGC 5033 NGC 2903 NGC 3521 NGC 2683 NGC 3198
NGC 6946 NGC 2403 NGC 6503 NGC 1003 NGC 247 NGC 7739 NGC 300
NGC 5585 NGC 55 NGC 1560 NGC 3109 UGC 128 UGC 2259 M 33
IC 2574 DDO 170 DDO 168 NGC 3726 NGC 3769 NGC 3877 NGC 3893
NGC 3917 NGC 3949 NGC 3953 NGC 3972 NGC 3992 NGC 4010
NGC 4013 NGC 4051 NGC 4085 NGC 4088 NGC 4100 NGC 4138
NGC 4157 NGC 4183 NGC 4217 NGC 4389 UGC 6399 UGC 6446
UGC 6667 UGC 6818 UGC 6917 UGC 6923 UGC 6930 UGC 6973
UGC 6983 UGC 7089 NGC 1024 NGC 3593 NGC 4698 NGC 5879 IC 724
F563-1 F563-V2 F568-1 F568-3 F568-V1 F571-V1 F574-1 F583-1
F583-4 UGC 1230 UGC 5005 UGC 5999 Carina Fornax
Leo I Leo II Sculptor Sextans Sgr

**Figure 4.** Galaxies with rotation curves well fit by MOND [31].

### *7.2. The Combination of Lorentzian-Einsteinian 3rd Generalized Schwarzschild Metric or Modified GSR Gravitational Field with 'old' or 'New' MOND Simple & Standard Interpolating Function and 'old' or 'New' Absorption of Dark Matter into the field in the Solar System*

In order to find out what is the effect of the modification at medium mass and size systems, we now examine our Solar System.

**Table 4.** Rotational Velocities [experimental ($V_{exp}$) and calculated by the Combination of *Lorentzian-Einsteinian 3rd Generalized Schwarzschild metric* or *Modified GSR Gravitational Field* with MOND *Simple & Standard Interpolating Function* ($V_{simp,Lor}$ & $V_{stand,,Lor}$)], the Luminous Mass of the Solar System that is enclosed within the circular orbit ($M_d$), the value of function $h_{(r)}$ and the value of time coefficient of metric ($g_{00}$) wrt the mean distance from the Sun. Data from [15] (p. 14-3).

| ***Name*** | ***r*** /AU /$10^{11}$ m | ***M***$_d$ /$10^{24}$ kg | $a_{Schwar}$ $a_{simp,Lor}$ $a_{stand,Lor}$ | $h_{Schwar}=1$ $h_{simp,Lor}$ $h_{stand,Lor}$ | $g_{00,Schwar}$ $g_{00,simp,Lor}$ $g_{00,stand,Lor}$ | $V_{Schwar}$ $V_{simp,Lor}$ $V_{stand,Lor}$ / Km s$^{-1}$ |
|---|---|---|---|---|---|---|
| Sun | 0.00465 | 1,989,100 | 1 | 1 | -0.9999957553 | 436.747 |
| Surface | 0.00696 | | 1.000000000000 | 1.000000000000 | -0,9999957553 | 436.747 |
| | | | 1.000000000000 | 1.000000000000 | -0.9999957553 | 436.747 |
| Mercury | 0.38710 | 1,989,100.0000 | 1 | 1 | -0.9999999490 | 47.880 |
| | 0.57909 | | 1.000000001516 | 0.999999996969 | -0.9999999490 | 47.880 |
| | | | 1.000000000000 | 1.000000000000 | -0.9999999490 | 47.880 |
| Venus | 0.72333 | 1,989,100.3302 | 1 | 1 | -0.9999999727 | 35.027 |
| | 1.08209 | | 1.000000005292 | 0.999999989416 | -0.9999999727 | 35.027 |
| | | | 1.000000000000 | 0.999838760132 | -0.9999999727 | 35.027 |
| Earth | 1.00000 | 1,989,105.1992 | 1 | 1 | -0.9999999803 | 29.790 |
| | 1.49598 | | 1.000000010114 | 0.999999979771 | -0.9999999803 | 29.790 |
| | | | 1.000000000000 | 0.999771972392 | -0.9999999803 | 29.790 |
| Mars | 1.52369 | 1,989,111.1715 | 1 | 1 | -0.9999999870 | 24.134 |
| | 2.27941 | | 1.000000023482 | 0.999999953036 | -0.9999999870 | 24.134 |
| | | | 1.000000000000 | 0.999659019194 | -0.9999999870 | 24.134 |
| Jupiter | 5.20283 | 1,989,111.8134 | 1 | 1 | -0.9999999962 | 13.060 |
| | 7.78332 | | 1.000000273790 | 0.999999452420 | -0.9999999962 | 13.060 |
| | | | 1.000000000000 | 0.998837605409 | -0.9999999962 | 13.060 |
| Saturn | 9.53876 | 1,991,010.6134 | 1 | 1 | -0.9999999979 | 9.650 |
| | 14.26978 | | 1.000000919405 | 0.999998161187 | -0.9999999979 | 9.650 |
| | | | 1.000000000001 | 0.997869945819 | -0.9999999979 | 9.650 |
| Uranus | 19.19139 | 1,991,579.1134 | 1 | 1 | -0.9999999990 | 6.804 |
| | 28.70991 | | 1.000003720565 | 0.999992558819 | -0.9999999990 | 6.804 |
| | | | 1.000000000012 | 0.995715092796 | -0.9999999990 | 6.804 |
| Neptune | 30.06107 | 1,991,665.7384 | 1 | 1 | -0.9999999993 | 5.437 |
| | 44.97072 | | 1.000009128095 | 0.999981743504 | -0.9999999993 | 5.437 |
| | | | 1.000000000083 | 0.993288340427 | -0.9999999993 | 5.437 |
| Pluto | 39.52940 | 1,991,768.5184 | 1 | 1 | -0.9999999995 | 4.741 |
| | 59.13514 | | 1.000015782729 | 0.999968433629 | -0.9999999995 | 4.741 |
| | | | 1.000000000249 | 0.991174598083 | -0.9999999995 | 4.741 |
| NGC | $2.846\times10^{12}$ | 1,991,768.5334 | 1 | 1 | -1,0000000000 | $3.12\times10^{-7}$ |
| 3198 | $4.258\times10^{12}$ | | 20,114.1129122 | $-8.17434687\times10^{9}$ | -1,0000000000 | 355.39 |
| | | | 20,114.1128998 | $-8.35188544\times10^{9}$ | -1,0000000000 | 355.39 |

The mean values of Rotational Velocities, the Mass of the Solar System that is enclosed within the orbit wrt the mean distance the planet from the Sun, are contained in Table 4 [data from [15] (p. 14-3)]. The Circular Velocities ($V_{Schwar}$, $V_{simp,Lor}$, $V_{stand,Lor}$) have been calculated by using (31) or (119) or (28) or (116), the values of function $a_{(r)}$ ($a_{simp,Lor}$ or $a_{stand,Lor}$) by using (120) combined with (121) and the values of function $h_{(r)}$ ($h_{\mathrm{simp,Lor}}$ or $h_{\mathrm{stand,Lor}}$) by using (129) or (131), respectively. Finally, the values of time coefficient of metric ($g_{00}$) have been calculated from (24) for $\xi_{\mathrm{I}}$=1. In addition, the corresponding Rotation Curves and Mass Distribution in the Solar System are shown in Figure 10 and Figure 11 of [20], respectively.

We observe that in case of Solar System, the Combination of *Lorentzian-Einsteinian 3GSM* or *Modified GSR Gravitational Field* with MOND *Simple* or *Standard* $\mu$, gives almost the same Rotational Velocities ($V_{\mathrm{simp,Lor}}$, $V_{\mathrm{stand,Lor}}$) and coefficients of metric ($g_{00}$) as those calculated by the original *Schwarzschild field strength* ($V_{\mathrm{Schwar}}$). Thus, there are not significant changes to the *Relativistic Doppler Shift*, the *gravitational red shift* as well as the *precession of Mercury's orbit* ($g_{00}$=0.9999999490). Finally, the values at distance 13.8 Mpc=$2.846\times10^{12}$ AU=$4.258\times10^{23}$ m, which is the distance of Galaxy NGC 3198 from Earth give us the image of what happens at extra large distances: $g_{00}\to-1$ (see also the corresponding velocities at infinite distance in Table 1).

Finally, we conclude that the Combination of *Lorentzian-Einsteinian* 3GSM or *Modified GSR Gravitational Field* with *'old'* or *'New'* MOND or *Absorption of phantom DM into the Metric-Field*, in scale of black hole, planetary and star system, coincides to the original *Schwarzschild metric* ($a\approx\mu\approx h\approx1$), while in galactic scale, it gives MONDian results ($a>1$, $\mu>1$), eliminating the corresponding DM.

**Abbreviations**

1GSL: 1st Generalized Schwarzschild Lagrangian
1GSM: 1st Generalized Schwarzschild Metric
1GSP: 1st Generalized Schwarzschild Potential
1GSRP: 1st Generalized Schwarzschild Relativistic Potential
3GSM: 3rd Generalized Schwarzschild Metric
3GSRP: 3rd Generalized Schwarzschild Relativistic Potential
CCs: Cartesian Coordinates
$c_{\mathrm{I}}$: Universal Speed
DM: Dark Matter
ECRMs: Euclidean Complex Relativistic Mechanics
EGR: Einsteinian General Relativity
EP: Equivalence Principle
ERT: Einstein Relativity Theory
ESR: Einsteinian Special Relativity
GEE: Gravitoelectric Effect
GR: General Relativity
GRS: Gravitational Red Shift
GSR: Generalized Special Relativity
GT: Galilean Transformation
IECLSTTs: Isometric Euclidean Closed Linear Transformations of Complex Spacetime
LSTT: Linear Spacetime Transformation
LB: Lorentz Boost
MOND: Modified Newtonian Dynamics
NPs: Newtonian Physics
$r_0$: Milgrom radius
RB: Real Boost
RIOs: Relativistic Inertial observers
$r_{\mathrm{S}}$: Schwarzschild radius
$r_{\mathrm{SI}}$: 1st Generalized Schwarzschild radius
RT: Relativity Theory
SM: Schwarzschild Metric
SR: Special Relativity
TPs: Theory of Physics
UCM: Uniform Circular Motion
VT: Vossos Transformation
$\mu$: Interpolating function

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