Separation and Relevant Properties in Hypergroups

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Abstract. The hypergroup, being a very general algebraic structure, was enriched with additional axioms, some less and some more powerful. These axioms led to the creation of more specific types of hypergroups such as the transposition, the cambiste and the convexity ones. This paper deals with the notion of the separation and relevant properties in hypergroups, in join spaces and in convexity hypergroups as well.

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Hypercompositional Algebra is the branch of Algebra which deals with structures endowed with multi-valued operations. Multi-valued operations, also called *hyperoperations* or *hypercompositions*, are operations in which the result is multi-valued, rather than a single element. More precisely, a hypercomposition in a non-void set H is a function from the Cartesian product $H \times H$ to the powerset $P(H)$ of H. Hypercompositional structures came into being through the notion of the *hypergroup*. The hypergroup was introduced by F. Marty in 1934, during the 8th congress of the Scandinavian Mathematicians [5]. A hypergroup, satisfies the following axioms:

i. $(ab)c = a(bc)$ for all $a,b,c \in H$ (associativity),

ii. $aH = Ha = H$ for all $a \in H$ (reproduction).

Note that, if «·» is a hypercomposition in a set H and A, B are subsets of H, then $A \cdot B$ signifies the union $\bigcup_{(a,b)\in A\times B} a\cdot b$ ($A = \varnothing \vee B = \varnothing \Leftrightarrow A{\cdot}B = \varnothing$). In both cases, aA and Aa have the same meaning as $\{a\}A$ and $A\{a\}$ respectively. Generally, the singleton $\{a\}$ is identified with its member a. In a hypergroup, the result of the hypercomposition is always a non-empty set (see [14, 16]). In [5], F. Marty also defined the two induced hypercompositions (right and left division) that result from the hypercomposition of the hypergroup, i.e.

$$\frac{a}{|\,b} = \{x \in H \mid a \in xb\} \quad \text{and} \quad \frac{a}{b\,|} = \{x \in H \mid a \in bx\}.$$

It is obvious that the two induced hypercompositions coincide, if the hypergroup is commutative. For the sake of notational simplicity, a/b or $a:b$ is used to denote the right division (as well as the division in commutative hypergroups) and $b\backslash a$ or $a..b$ is used to denote the left division [4, 7, 22, 23].

A non-empty subset K of H is called *semi-subhypergroup* when it is stable under the hypercomposition, i.e. it has the property $xy \subseteq K$ for all $x, y \in K$. K is a subhypergroup of H, if it satisfies the axiom of reproduction, i.e. if the equality $xK = Kx = K$ is valid for all $x \in K$ [15]. This means that when K is a subhypergroup, the relations $a \in bx$ and $a \in yb$ can always be solved in K. The non-void intersection of two subhypergroups, although stable under the hypercomposition, usually is not a subhypergroup, since the reproduction is not always valid. In other words the solutions of the relation $a \in yb$ and $a \in bx$ do not lie in the intersection when a and b are elements of the intersection. This led (from the very early steps of hypergroup theory) to the consideration of more special types of subhypergroups. One of them is the *closed subhypergroup*. A subhypergroup K of H is called *left closed* with respect to H, if for any two elements a and b in K all possible solutions of the relation $a \in yb$ lie in K. This means that K is left closed if and only if $a/b \subseteq K$, for all $a,b \in K$. Similarly, K is *right* *closed* when all possible solutions of the relation $a \in bx$ lie in K or, equivalently, if $b \backslash a \subseteq K$ for all $a,b \in K$. Finally, K is *closed* when it is both right and left closed. The non-void intersection of two closed subhypergroups is a closed subhypergroup.

It is has been proven [7, 8] that the set of the semi-subhypergroups (resp. the set of the closed subhypergroups) which contain a non-void subset E is a complete lattice. Hence, given a non-empty subset E of a hypergroup H, the minimum semi-subhypergroup (in the sense of inclusion) which contains E can be assigned. This semi-subhypergroup is denoted by $[E]$ and it is called the generated by E semi-subhypergroup of H. Similarly, $\langle E \rangle$ is the generated by E closed subhypergroup of H. For notational simplicity, if $E = \{a_1,...,a_n\}$, $[E] = [a_1,...,a_n]$ and $\langle E \rangle = \langle a_1,...,a_n \rangle$ are used instead. A subset B of a hypergroup H is called free or independent if either $B = \varnothing$, or $x \notin \langle B - \{x\} \rangle$ for all $x \in B$, otherwise it is called non-free or dependent. B generates H, if $\langle B \rangle = H$, in which case B is a set of generators of H. If H has a finite set of generators, it is called a finite type hypergroup. A free set of generators is a basis of H.

Proposition 1. *If S, T are semi-subhypergroup of a commutative hypergroup H, then ST is a semi-subhypergroup of H as well.*

An element a of a semi-subhypergroup S is called *interior element* of S if for each $x \in S$ there exists $y \in S$ such that $a \in xy$. An element of S which is not an interior element is called *frontier element* of S.

Proposition 2. *Let a be an interior element and b a frontier element of a semi-subhypergroup S of H. Then ab and ba consists only of interior elements of S.*

P r o o f. Let c be an element of ab and x an arbitrary element of S. Since a is an interior element of S it derives that there exists $z \in S$ such that $a \in xz$. Hence $c \in (xz)b = x(zb)$. Therefore there exists $y \in zb \subseteq S$ such that $c \in xy$. Thus c is an interior element of S. Dually ba consists of interior elements and the Proposition is established.

Proposition 3. *If a semi-subhypergroup S of a hypergroup H consists only of interior elements, then S is a subhypergroup of H.*

P r o o f. Suppose that a is an arbitrary element of S. Since S is a semi-subhypergroup, $aS \subseteq S$ is valid. Next let y be any element of S. Since y is an interior element, there exists $x \in S$ such that $y \in ax$. Hence $S \subseteq aS$. Therefore $S = aS$. Dually, $S = Sa$ holds.

A hypercomposition is called *right open* if $a \notin ba$ for all $a,b \in H$ with $b \neq a$. The definition of *left open* hypercomposition is similar. Obviously, a hypercomposition is *open* if it is both right and left open.

Proposition 4. *Let H be a hypergroup endowed with an open hypercomposition and K a subhypergroup of H. Then any element of K is an interior element.*

Proposition 5. *Let H be a hypergroup endowed with an open hypercomposition, S a semi-subhypergroup of H and I the subset of the interior elements of S. Then I absorbs S, i.e. $IS \subseteq I$*

P r o o f. Suppose that $a \in I$ and $b \in S$. Let r be an element of ab. In order to prove that r is an interior element, we have to show that for any $x \in S$ it exists $y \in S$ such that $r \in xy$. Since a is an interior element, there exists $z \in S$, such that $a \in xz$. Hence, $r \in ab \subseteq (xz)b = x(zb)$. But $zb \subseteq S$. So there exists $y \in S$ such that $r \in xy$, QED.

Proposition 6. *Let H be a hypergroup endowed with an open hypercomposition, S a semi-subhypergroup of H and I the subset of the interior elements of S. Then I is a subhypergroup of H.*

P r o o f. Suppose that $a \in I$. Per Proposition 4.4, $aI \subseteq I$. To prove the reverse inclusion, let $b \in I$. Since b is an interior element, there exists $z \in S$, such that $b \in az$. Per Proposition 1.5, $aa = a$, hence $az = (aa)z = a(az)$. Per Proposition 4.4, $az \subseteq aS \subseteq I$. Thus there exists $w \in I$, such that $b \in aw$, QED.

Let A and B be subhypergroups of H. Suppose $B \subset A$, $A \neq B$ and no subhypergroup lies between B and A. Then we say that A *covers* B, or that B *is covered* by A.

Proposition 7. *If the closed subhypergroup* A *covers the closed subhypergroup* B*, and* $c \in A - B$*, then* $A = \left\langle B \cup \{c\} \right\rangle$.

Proof. $B \subset \left\langle B \cup \{c\} \right\rangle$ and moreover $\left\langle B \cup \{c\} \right\rangle \subseteq A$. Hence by definition of covering $A = \left\langle B \cup \{c\} \right\rangle$.□

The hypergroup (as defined by F. Marty), being a very general algebraic structure, was enriched with additional axioms, some less and some more powerful. These axioms led to the creation of more specific types of hypergroups. One of these axioms is the *transposition axiom* [4, 13]:

$$b \backslash a \cap c / d \neq \varnothing \quad \text{implies} \quad ad \cap bc \neq \varnothing, \ \text{for all} \ a,b,c,d \in H$$

A hypergroup *H* which satisfies the transposition axion is called *transposition hypergroup* [4]. W. Prenowitz called the commutative transposition hypergroup *join space* [22, 23]. This type of hypergroup has been widely utilized in the study of geometry via the use of hypercompositional algebra tools [9, 22, 23] and it has numerous applications in formal languages, the theory of automata and graph theory [12, 18, 19, 20]. A *convexity hypergroup* is a join hypergroup which satisfies the axioms:

i. the hypercomposition is open,
ii. $ab \cap ac \neq \varnothing$ implies $b = c$ or $b \in ac$ or $c \in ab$.

Among the results reached in *convexity hypergroup* are [2, 3, 17]:

Proposition 8. *Let* B *be a non-empty subset of a convexity hypergroup* H. B *is a basis of* H *if and only if:*
(i) B *is a maximal free set and*
(ii) B *is a minimal set of* H *generators*

Proposition 9. *Every convexity hypergroup has at least one basis.*

Proposition 10. *All the bases of a convexity hypergroup have the same cardinality.*

The *dimension* of a convexity hypergroup H (denoted by $\dim H$) is the cardinality of any basis of H.

Proposition 11. *Let* V *be a vector space over an ordered field* F. *Then,* V*, when endowed with the hypercomposition*

$$ab = \left\{ \kappa a + \lambda b \mid \kappa, \lambda > 0, \kappa + \lambda = 1 \right\},$$

becomes a convexity hypergroup.

This hypergroup, which was derived from the vector space and is connected with it, was named *attached hypergroup* of V [7, 21]. A direct consequence of the above proposition is that the convex sets of V are the semi-subhypergroups of the attached hypergroup H_V, while the subspaces of V are the closed subhypergroups of this hypergroup [7, 10, 21].

Let A and B be two non-empty subsets of a hypergroup H. Suppose that a subset S of H has the property $a \in A,\ b \in B$ implies $ab \cap S \neq \varnothing$. Then we say that S *separates* A and B. If $S \cap A = \varnothing$ and $S \cap B = \varnothing$, we say that S *strictly separates* A and B. In addition if S is a closed subhypergroup covered by $\left\langle S \cup \{a\} \right\rangle,\ a \in A$ and $\left\langle S \cup \{b\} \right\rangle,\ b \in B$, we say that S is the *separating closed subhypergroup* of A and B.

Proposition 12. *Let H be a convexity hypergroup and* A *a nonempty closed subhypergroup which strictly separates the closed subhypergroup* B *into sets* M *and* N*. Then* B *covers* A *and* $B = A/M \cup A \cup A/N$.

Proof. Let $p \in M$ and $q \in N$. Since A separates B, for any $x \in M$ we have $qx \cap A \neq \varnothing$, thus $x \in A/q$, therefore $M \subseteq A/q \subseteq B$. Similarly $N \subseteq A/p \subseteq B$. Since the hypercomposition is open the sets $A/M, A/N, A$ are disjoined. Thus $B = M \cup A \cup N = A/q \cup A \cup A/p$, hence $M = A/q$ and $N = A/p$. It remains to prove that B covers A. Suppose that C is a closed subhypergroup such that $A \subset C \subset B$. Then $M \cap C \neq \varnothing$ or $N \cap C \neq \varnothing$. Let $M \cap C \neq \varnothing$. Then $M = A/q \subseteq A/C \subseteq C$. Moreover, since $pq \cap A \neq \varnothing$ we have $N = A/p \subseteq A/C \subseteq C$. But $B = M \cup A \cup N$, therefore $C = B$.

Proposition 13. *Let* S *and* T *be two finite sets of elements in an n-dimensional convexity hypergroup* H. *If any semi-subhypergroup generated by* $k+1,\ k \leq n$ *elements of* S *may be separated from any semi-subhypergroup generated by* $l+1,\ l \leq n$ *elements of* T*, then* $[S] \cap [T] = \varnothing$.

P r o o f. If H be a commutative hypergroup and $\{a_1,...a_n\} \subseteq H$. Then,

$$[a_1,a_2,...,a_n] = ([a_1] \cup [a_2] \cup ... \cup [a_n]) \cup ([a_1][a_2] \cup ... \cup [a_{n-1}][a_n]) \cup ... \cup ([a_1]...[a_n])$$

(Proposition 2.1. [17]). Since H is a convexity hypergroup, the hypercomposition is open, thus $aa = a$ and therefore $[a] = a$. Hence $[a_1,a_2,...,a_n] = \{a_1,a_2,...a_n\} \cup (a_1a_2 \cup ... \cup a_{n-1}a_n) \cup ... \cup a_1...a_n$. Therefore if we suppose that $[S] \cap [T] \neq \varnothing$, then for any element $x \in [S] \cap [T]$ it holds that $x \in s_1...s_i \cap t_1...t_j$, where $\{s_1,...,s_i\} \subseteq S$ and $\{t_1,...,t_j\} \subseteq T$. But, it is known that [17] if an element x of an n-dimensional convexity hypergroup H belongs to a hyperproduct of $n+1$ elements, then there exists a proper subset of these elements which contains x in their hyperproduct. So, there exists proper subsets of $\{s_1,...,s_i\}$ and $\{t_1,...,t_j\}$ not exceeding n elements, which contains x in their hyperproduct, i.e. $x \in s_1...s_p \cap t_1...t_q \;\; p,q \leq n$. The contradiction obtained proves the validity of the Proposition.

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