

On the enumeration of rigid hypercompositional structures

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Abstract. This paper deals with hypercompositional structures endowed with a single hypercomposition (i.e. hypergroupoids, semi-hypergroups, quasi-hypergroups and hypergroups) and studies and categorizes those structures in which the group of automorphisms is of order 1.

Keywords: hypergroups, transposition hypergroups, hypercompositional structures, isomorphism.

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INTRODUCTION

Hypercompositional Algebra is the branch of Algebra which deals with structures endowed with multi-valued operations. Multi-valued operations, also called *hyperoperations* or *hypercompositions*, are operations in which the result is multi-valued, rather than a single element. More precisely, a hypercomposition in a non-void set H is a function from the Cartesian product $H \times H$ to the powerset $P(H)$ of H . Hypercompositional structures came into being through the notion of the *hypergroup*. The hypergroup was introduced by F. Marty in 1934, during the 8th congress of the Scandinavian Mathematicians [10]. A hypergroup, satisfies the following axioms:

- i. $(ab)c = a(bc)$ for all $a, b, c \in H$ (associativity),
- ii. $aH = Ha = H$ for all $a \in H$ (reproduction).

Note that, if « \cdot » is a hypercomposition in a set H and A, B are subsets of H , then $A \cdot B$ signifies the union

$\bigcup_{(a,b) \in A \times B} a \cdot b$ ($A = \emptyset \vee B = \emptyset \Leftrightarrow A \cdot B = \emptyset$). In both cases, aA and Aa have the same meaning as $\{a\}A$ and $A\{a\}$

respectively. Generally, the singleton $\{a\}$ is identified with its member a . In a hypergroup, the result of the hypercomposition is always a non-empty set (see [14, 22]). In [10], F. Marty also defined the two induced hypercompositions (right and left division) that result from the hypercomposition of the hypergroup, i.e.

$$\frac{a}{|b} = \{x \in H \mid a \in xb\} \quad \text{and} \quad \frac{a}{b|} = \{x \in H \mid a \in bx\}.$$

It is obvious that the two induced hypercompositions coincide, if the hypergroup is commutative. For the sake of notational simplicity, W. Prenowitz [29] denoted division in commutative hypergroups by a/b . Later on, J. Jantosciak used the notation a/b for right division and $b \setminus a$ for left division [7].

The hypergroup (as defined by F. Marty), being a very general algebraic structure, was enriched with additional axioms, some less and some more powerful. These axioms led to the creation of more specific types of hypergroups. One of these axioms is the transposition axiom. It was introduced by W. Prenowitz, who used it in commutative hypergroups. W. Prenowitz called the resulting hypergroup *join space* [29]. Thus, *join space* (or *join hypergroup*) is defined as a commutative hypergroup H , in which

$$a/b \cap c/d \neq \emptyset \text{ implies } ad \cap bc \neq \emptyset, \text{ for all } a, b, c, d \in H \text{ (transposition axiom)}$$

is true. This type of hypergroup has been widely utilized in the study of geometry via the use of hypercompositional algebra tools [29, 30] and it has numerous applications in formal languages, the theory of automata and graph theory [18, 24, 25, 26]. Later, J. Jantosciak generalized the transposition axiom in an arbitrary hypergroup as follows:

$$b \setminus a \cap c/d \neq \emptyset \text{ implies } ad \cap bc \neq \emptyset, \text{ for all } a, b, c, d \in H.$$

He named this particular hypergroup *transposition hypergroup* and studied its properties in [7].

Mathematical study of hypercompositional structures followed the reverse course as well. Instead of enriching the hypergroup with more axioms, certain axioms were removed thus leading to the generation of weaker structures.

In this way, the pair (H, \cdot) , where H is a non-empty set and " \cdot " a hypercomposition, was named *partial hypergroupoid*, while it was called *hypergroupoid* if $ab \neq \emptyset$ for all $a, b \in H$ [2]. A hypergroupoid in which the associativity is valid, is called *semi-hypergroup*, while it is called *quasi-hypergroup*, if the reproductivity is valid. The quasi-hypergroups in which the weak associativity is valid, i.e. $(ab)c \cap a(bc) \neq \emptyset$ for all $a, b, c \in H$, were named *H_V-groups* [33]. In [22] the transposition axiom is introduced in hypergroupoids, semi-hypergroups and quasi-hypergroups, thus defining transposition hypergroupoids, transposition semi-hypergroups and transposition quasi-hypergroups. Moreover, in [21], the transposition axiom is introduced in H_V-groups, thus defining transposition H_V-groups and join H_V-groups when the hyperoperation is commutative. Properties and examples of these hypercompositional structures are given in [21] and [22]. However, issues pertaining to the enumeration of hypercompositional structures are of special interest [e.g. 3, 4, 5, 20, 23, 31, 32]. In the following paragraph we will deal with the enumeration of a class of hypercompositional structures that satisfy the transposition axiom.

RIGID TRANSPOSITION HYPERCOMPOSITIONAL STRUCTURES

The enumeration of hypergroups of order 3 reveals that there are 23.192 hypergroups which are partitioned into 3999 equivalence classes [31]. 3739 of the above classes consist of 6 members, 244 consist of 3 members, 10 consist of 2 members and the last 6 are one-member classes. As mentioned in [31], the 6 one-member classes correspondingly contain the following 6 hypergroups:

| | |
|---|--|
| (H ₁) $ab = H$, for all $a, b \in H$ | (H ₂) $ab = \{a, b\}$, for all $a, b \in H$ |
| (H ₃) $ab = \begin{cases} \{a, b\}, & \text{if } a \neq b \\ H - \{a\}, & \text{if } a = b \end{cases}$ | (H ₄) $ab = \begin{cases} H, & \text{if } a \neq b \\ H - \{a\}, & \text{if } a = b \end{cases}$ |
| (H ₅) $ab = \begin{cases} \{a, b\}, & \text{if } a \neq b \\ H, & \text{if } a = b \end{cases}$ | (H ₆) $ab = \begin{cases} H, & \text{if } a \neq b \\ a, & \text{if } a = b \end{cases}$ |

All the above hypergroups are commutative. H₁ is called *total hypergroup*. H₂ is called *B(iset)-hypergroup* and emerged during the study of formal languages and automata theory through the use of hypercompositional algebra [24, 25, 26]. A *B-hypergroup* is a join hypergroup (see [24] for proof). The free monoid of the words generated by an alphabet A can be endowed with the B-hypergroup structure, thus becoming a join hyperring, which is named *linguistic hyperring* [25, 26, 27]. Moreover, hypergroup H₅ was named *monogene*, since it is spanned by a single element (see [13]). If the monogene hypergroup is the additive part of a hyperfield, then the matter of monogene hyperfields isomorphism to quotient hyperfields [9] is a hitherto open problem in the theory of hyperfields. This also leads to open questions in the theory of fields [16, 17, 19]. On the other hand, if H₃, endowed with a scalar neutral element, is the additive part of a hyperfield, then the class of these hyperfields contains elements that do not belong to the class of quotient hyperfields [11, 12, 15]. With regard to the validity (or not) of the transposition axiom in the above hypergroups, we initially observe that, since all these are commutative, the two induced hypercompositions coincide (i.e. $b \setminus a = a / b$). If $\text{card } H > 2$, then the following Lemmas are valid.

Lemma 1. *The transposition axiom is valid in H₁.*

Proof. The induced hypercomposition in H₁ is the following: $a / b = H$, for all $a, b \in H$. Since $ab = H$, for all $a, b \in H$, the intersection $ab \cap cd$ is equal to H , for all $a, b, c, d \in H$.

Lemma 2. *The transposition axiom is valid in H₂ (see [24]).*

Lemma 3. *The transposition axiom is valid in H₃.*

Proof. The induced hypercomposition in H₃ is the following:

$$a / b = \begin{cases} \{a, b\}, & \text{if } a \neq b \\ H - \{a\}, & \text{if } a = b \end{cases}$$

Next, suppose that $a / b \cap c / d \neq \emptyset$. This means that at least one element among a, b, c, d is equal to one of the rest. Hence, the intersection $ad \cap bc$ is always non-void.

Lemma 4. *The transposition axiom is valid in H₄.*

Proof. The induced hypercomposition in H₄ is the following:

$$a / b = \begin{cases} H, & \text{if } a \neq b \\ H - \{a\}, & \text{if } a = b \end{cases}$$

Next, suppose that $a/b \cap c/d \neq \emptyset$. Then, $H - \{a, b\}$ is always a subset of the intersection $ad \cap bc$. Therefore, $ad \cap bc \neq \emptyset$.

Lemma 5. *The transposition axiom is valid in H_5 .*

Proof. The induced hypercomposition in H_5 is the following:

$$a/b = \begin{cases} \{a, b\}, & \text{if } a \neq b \\ H, & \text{if } a = b \end{cases}$$

If $a/b \cap c/d \neq \emptyset$, then at least one element among a, b, c, d must be equal to one of the rest. Hence, the intersection $ad \cap bc$ is always non-void.

Lemma 6. *The transposition axiom is not valid in H_6 .*

Proof. The induced hypercomposition in H_6 is the following:

$$a/b = \begin{cases} H - \{b\}, & \text{if } a \neq b \\ H, & \text{if } a = b \end{cases}$$

Next, if $a \neq b$, we have: $a/b \cap b/a = [H - \{b\}] \cap [H - \{a\}] \neq \emptyset$, while $aa \cap bb = \{a\} \cap \{b\} = \emptyset$.

Hence, the following is valid:

Theorem 1. *The hypergroups H_i , $i=1, \dots, 5$ are join hypergroups, while the transposition axiom is not valid in H_6 .*

Remark. One can observe that in hypergroups H_i , $i=1, \dots, 5$ the equality $ab = a/b$ is valid for all $a, b \in H$.

Definition. A hyperstructure is called *rigid*, if its group of automorphisms is of order 1.

In [1] the rigid quasi-hypergroups are presented and it is proven that, there also exist 8 H_V -groups, in addition to the above 6 hypergroups. Assuming that $\text{card } H > 2$, these are the following:

| | |
|--|--|
| $(H_{V1}) \quad ab = \begin{cases} H - \{a\}, & \text{if } a \neq b \\ H, & \text{if } a = b \end{cases}$ | $(H_{V2}) \quad ab = \begin{cases} H - \{b\}, & \text{if } a \neq b \\ H, & \text{if } a = b \end{cases}$ |
| $(H_{V3}) \quad ab = \begin{cases} H - \{a, b\}, & \text{if } a \neq b \\ H, & \text{if } a = b \end{cases}$ | $(H_{V4}) \quad ab = \begin{cases} H - \{a, b\}, & \text{if } a \neq b \\ a, & \text{if } a = b \end{cases}$ |
| $(H_{V5}) \quad ab = \begin{cases} H - \{a\}, & \text{if } a \neq b \\ a, & \text{if } a = b \end{cases}$ | $(H_{V6}) \quad ab = \begin{cases} H - \{b\}, & \text{if } a \neq b \\ a, & \text{if } a = b \end{cases}$ |
| $(H_{V7}) \quad ab = \begin{cases} a, & \text{if } a \neq b \\ H, & \text{if } a = b \end{cases}$ | $(H_{V8}) \quad ab = \begin{cases} b, & \text{if } a \neq b \\ H, & \text{if } a = b \end{cases}$ |

In [7] and then in [8], a principle of duality is established in the theory of hypergroups. This principle is valid for all hypercompositional structures. More precisely, two statements of the theory of hypercompositional structures are dual statements, if each results from the other by interchanging the order of the hypercomposition, i.e. by interchanging any hypercomposition ab with the hypercomposition ba . One can observe that the transposition axiom, as well as the associativity axiom (whether weak or not), are self-dual. The left and the right division have dual definitions, thus they must be interchanged in the construction of a dual statement. Therefore, the following principle of duality holds for the theory of hypercompositional structures:

Given a theorem, the dual statement resulting from interchanging the order of hypercomposition “ \cdot ” (and, necessarily, interchanging of the left and the right divisions) is also a theorem.

In accordance to that, given a hypercomposition its dual one derives if the result ab of any two elements a, b is interchanged by the result ba . Obviously the commutative hypercompositions are self-dual. Two hypercompositional structures are dual if they have dual hypercompositions. Thus there are 3 pairs of dual H_V -groups: (H_{V1}, H_{V2}) , (H_{V5}, H_{V6}) and (H_{V7}, H_{V8}) . H_{V3} and H_{V4} are self-dual.

Lemma 7. *The transposition axiom is valid in H_{V1} and H_{V2} .*

Proof. The induced hypercompositions in H_{V1} are the following:

$$b \setminus a = \begin{cases} H, & \text{if } a \neq b \\ a, & \text{if } a = b \end{cases} \quad \text{and} \quad a/b = \begin{cases} H - \{a\}, & \text{if } a \neq b \\ H, & \text{if } a = b \end{cases}$$

Next, suppose that $b \setminus a \cap c/d \neq \emptyset$. Then, $ad \cap bc \neq \emptyset$, since $ad \cap bc$ always contains the non-void intersection $[H - \{a\}] \cap [H - \{b\}]$. By duality, the Lemma is true for H_{V2} as well.

Lemma 8. *The transposition axiom is valid in H_{V3} .*

Proof. The hypercomposition is commutative. Therefore, the two induced hypercompositions coincide, i.e. $b \setminus a = a / b$. Therefore, the induced hypercomposition in H_{V3} is the following:

$$a / b = \begin{cases} H - \{a\}, & \text{if } a \neq b \\ a, & \text{if } a = b \end{cases}$$

Next, suppose that $b \setminus a \cap c / d \neq \emptyset$. Then, $ad \cap bc \neq \emptyset$, since $ad \cap bc$ always contains the non-void intersection $[H - \{a, d\}] \cap [H - \{b, c\}]$.

Lemma 9. *The transposition axiom is not valid in H_{V4} .*

Proof. The hypercomposition is commutative. Therefore, the two induced hypercompositions coincide, i.e. $b \setminus a = a / b$. Therefore, the induced hypercomposition in H_{V4} is the following:

$$a / b = \begin{cases} H - \{a\}, & \text{if } a \neq b \\ a, & \text{if } a = b \end{cases}$$

Next, let $a \neq b$; then, $b / a \cap a / a = [H - \{b\}] \cap \{a\} \neq \emptyset$, while $aa \cap ab = \{a\} \cap [H - \{a, b\}] = \emptyset$.

Lemma 10. *The transposition axiom is not valid in H_{V5} and H_{V6} .*

Proof. The induced hypercompositions in H_{V5} are the following:

$$b \setminus a = \begin{cases} H - \{b\}, & \text{if } a \neq b \\ a, & \text{if } a = b \end{cases} \quad \text{and} \quad a / b = \begin{cases} H - \{b\}, & \text{if } a \neq b \\ H, & \text{if } a = b \end{cases}$$

Next, let $a \neq b$; then, $a \setminus a \cap a / b = \{a\} \cap [H - \{b\}] \neq \emptyset$, while $aa \cap ab = \{a\} \cap [H - \{a\}] = \emptyset$. The rest follows through the duality of H_{V6} with H_{V5} .

Lemma 11. *The transposition axiom is not valid in H_{V7} and H_{V8} .*

Proof. The induced hypercompositions in H_{V7} are the following:

$$b \setminus a = \begin{cases} b, & \text{if } a \neq b \\ H, & \text{if } a = b \end{cases} \quad \text{and} \quad a / b = \begin{cases} \{a, b\}, & \text{if } a \neq b \\ a, & \text{if } a = b \end{cases}$$

Next, let a, b, c be three elements of H , each not equal to the other two, then $b \setminus a \cap c / b = \{b\} \cap \{b, c\} \neq \emptyset$, while $bc \cap ab = \{b\} \cap \{a\} = \emptyset$. The Lemma is also true for the dual structure H_{V8} .

The following theorem results from lemmas 7-11 above:

Theorem 2. H_{V1} and H_{V2} are transposition H_V -groups, while H_{V3} is a join H_V -group. The transposition axiom is not valid for the remaining rigid H_V -groups.

Remark. H_{V4} is a quasi-hypergroup if $\text{card } H = 3$, while it is an H_V -group, if $\text{card } H > 3$ [1].

In [6], the complement hyperoperation of a hyperoperation is defined as follows: If $\langle \circ \rangle$ is a hyperoperation in a set H , then the hyperoperation $a * b = H - a \circ b$ is called complement hyperoperation of $\langle \circ \rangle$. The authors of [6] have elaborated this notion in hypergroup H_3 , which resulted from the constructions of non-quotient hyperfields over a set H with $\text{card } H > 3$ [11, 12, 15]. The complement hyperoperation of H_3 produces the H_V -group H_{V4} . This hyperoperation is essentially the one used by A. Nakassis, in order to prove the existence of non-quotient hyperrings [28]. As defined in [21], such hypercompositional structures are named *complementary*. Thus, H_{V4} is *complementary* to H_3 and vice-versa. If we seek the complement of H_4 , H_5 and H_6 , then it turns out that this is a partial hypergroupoid. This occurs when equality $xy = H$ is valid in a hypercompositional structure for some $x, y \in H$. But the complement of H_2 , H_{V5} and H_{V6} is a non-partial rigid hypergroupoid. More precisely, with the exception of the 6 rigid hypergroups and the 8 rigid H_V -groups, there exist in additional 7 rigid hypergroupoids [1]. Assuming that $\text{card } H > 2$, these are the following:

| | |
|--|--|
| (h ₁) $ab = H - \{a\}$, for all $a, b \in H$ | (h ₂) $ab = H - \{b\}$, for all $a, b \in H$ |
| (h ₃) $ab = a$, for all $a, b \in H$ | (h ₄) $ab = b$, for all $a, b \in H$ |
| (h ₅) $ab = \begin{cases} a, & \text{if } a \neq b \\ H - \{a\}, & \text{if } a = b \end{cases}$ | (h ₆) $ab = \begin{cases} b, & \text{if } a \neq b \\ H - \{b\}, & \text{if } a = b \end{cases}$ |
| (h ₇) $ab = H - \{a, b\}$, for all $a, b \in H$ | |

Proposition 1. H_2 is the complement hypergroup of the hypergroupoid h_7 ; H_{V_5} and its dual H_{V_6} are the complement H_V -groups of the dual hypergroupoids h_5 and h_6 respectively; h_1 and its dual h_2 are the complement hypergroupoids of the duals h_3 and h_4 respectively.

Proposition 2. h_3 and h_4 are semi-hypergroups while for their complement hypergroupoids h_1 and h_2 respectively the weak associativity is valid. The weak associativity is also valid in h_7 when $\text{card } H > 3$.

Proof. In h_3 for any $a, b, c \in H$ it holds $(ab)c = ac = a$ and $a(bc) = ab = a$ that is $(ab)c = a(bc)$. On the other hand in h_1 it holds $(ab)c = [H - \{a\}]c = H$ and $a(bc) = a[H - \{b\}] = H - \{a\}$, that is $(ab)c \cap a(bc) \neq \emptyset$. In h_5 the associativity, either weakly or not, fails to hold true since $(ab)a = aa = H - \{a\}$ while $a(ba) = ab = a$. Dually the same results are valid for h_4 , h_2 and h_6 respectively. Finally for h_7 it holds: $(ab)c = [H - \{a, b\}]c \supseteq \{a, b\}$ and $a(bc) = a[H - \{b, c\}] \supseteq \{b, c\}$. Hence $(ab)c \cap a(bc) \neq \emptyset$.

Lemma 12. The transposition axiom is valid in h_1 and h_2 .

Proof. The induced hypercompositions in h_1 are the following:

$$b \setminus a = \begin{cases} H, & \text{if } a \neq b \\ \emptyset, & \text{if } a = b \end{cases} \quad \text{and} \quad a / b = H - \{a\}, \text{ for all } a, b \in H$$

Next, let $b \setminus a \cap c / d \neq \emptyset$. Then, $a \neq b$. Hence, $ad \cap bc = [H - \{a\}] \cap [H - \{b\}] \neq \emptyset$.

Lemma 13. The transposition axiom is valid in semi-hypergroups h_3 and h_4 .

Proof. The induced hypercompositions in h_3 are:

$$b \setminus a = \begin{cases} H, & \text{if } a = b \\ \emptyset, & \text{if } a \neq b \end{cases} \quad \text{and} \quad a / b = a, \text{ for all } a, b \in H.$$

Therefore, $b \setminus a \cap c / d = \emptyset$ if $a \neq b$, while $a \setminus a \cap c / d = \{c\}$ for all $a, c, d \in H$. From the latter, it follows that $ac \cap ad = \{a\}$. Hence, the transposition axiom is valid in h_3 . Dually, the same is true for h_4 .

Lemma 14. The transposition axiom is valid in h_7 , if $\text{card } H > 4$.

Proof. The hypercomposition is commutative. Therefore, the two induced hypercompositions coincide, i.e. $b \setminus a = a / b$. Therefore, the induced hypercomposition in h_7 is:

$$a / b = \begin{cases} H - \{a\}, & \text{if } a \neq b \\ \emptyset, & \text{if } a = b \end{cases}$$

Next, suppose that $a / b \cap c / d \neq \emptyset$. Then, $a \neq b$. Hence, $ad \cap bc = [H - \{a, d\}] \cap [H - \{b, c\}] \neq \emptyset$.

Lemma 15. The transposition axiom is not valid in h_5 and h_6 .

Proof. The induced hypercompositions in h_5 are the following:

$$b \setminus a = \begin{cases} b, & \text{if } a \neq b \\ H - \{a\}, & \text{if } a = b \end{cases} \quad \text{and} \quad a / b = \begin{cases} \{a, b\}, & \text{if } a \neq b \\ \emptyset, & \text{if } a = b \end{cases}$$

Let $a \neq b$. Then, $a \setminus a \cap a / b = [H - \{a\}] \cap \{a, b\} = \{b\} \neq \emptyset$, while $ab \cap aa = \{a\} \cap H - \{a\} = \emptyset$.

Theorem 3. The transposition axiom is valid in hypergroupoids h_1 , h_2 and in semi-hypergroups h_3 and h_4 ; it is also valid in hypergroupoid h_7 , if $\text{card } H > 4$. It is not valid in hypergroupoids h_5 and h_6 .

From Theorems 1-3 it follows that:

Theorem 4. There are 21 rigid hypergroupoids which are classified as follows:

| | non transposition | transposition | |
|------------------|-------------------|-----------------|--------------------|
| | | non commutative | Commutative (join) |
| Hypergroups | 1 | - | 5 |
| H_V -groups | 5 | 2 | 1 |
| Semi-hypergroups | - | 2 | - |
| Hypergroupoids | 2 | 2 | 1 |

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