Isomorphism Theorems in Fortified Transposition Hypergroups

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Abstract. The isomorphism theorems of both groups as well as hypergroups are a significant tool for the study of these structures. C. G. Massouros proved the isomorphism theorems for quasicanonical hypergroups (or polygroups) and afterwards J. Jantosciak generalised them for transposition hypergroups using quotients modulo closed subhypergroups. But, there also exist transposition hypergroups which do not have proper closed subhypergroups. Such are the fortified transposition hypergroups of attractive elements. The isomorphism theorems of these hypergroups are studied here.

Keywords: hypergroups, quasicanonical hypergroups, transposition hypergroups, join spaces.
PACS: 02.10.De, 02.20.Bb AMS-Subject classification: 20N20

INTRODUCTION

An *operation* or *composition* in a non void set *H* is a function from $H \times H$ to *H*, while a *hyperoperation* or *hypercomposition* is a function from $H \times H$ to the powerset *P*(*H*) of *H*. An algebraic structure that satisfies the axioms

i. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in H$ (associativity),

ii. $a \cdot H = H \cdot a = H$ for all $a \in H$ (reproduction),

is called *group* if "$\cdot$" is a composition [10] and *hypergroup* if "$\cdot$" is a hypercomposition [6]. Generally, the singleton $\{a\}$ is identified with its member a. If A and B are subsets of H, then $A \cdot B$ signifies the union $\bigcup_{(a,b)\in A\times B} a \cdot b$. Since $A\times B=\varnothing \Leftrightarrow A=\varnothing$ or $B=\varnothing$, when $A=\varnothing$ or $B=\varnothing$, then $A\cdot B=\varnothing$ and vice versa. In a hypergroup, the result of the hypercomposition is always a non-void set [9, 10]. An element $e \in H$, such that $ex = xe = x$, for all $x \in H$ is called *scalar identity* while it is called *strong identity* if $xe = ex \subseteq \{e, x\}$ for all $x \in H$ Two *induced hypercompositions* (the left and the right division) derive from the hypercomposition of the hypergroup [6], i.e.

$$a / b = \{x \in H \mid a \in xb\} \quad \text{and} \quad b \backslash a = \{x \in H \mid a \in bx\}$$

A hypergroup enriched with the axiom: $b \backslash a \cap c / d \neq \varnothing$ implies $ad \cap bc \neq \varnothing$, for all $a, b, c, d \in H$, is called *transposition hypergroup* [4]. A transposition hypergroup with scalar identity is called *quasicanonical hypergroup* [1, 7] or *polygrou*p [2]. A transposition hypergroup with strong identity is called *fortified transposition hypergroup* if for every $x \in T - \{e\}$ there exists a unique $x^{-1} \in T - \{e\}$ such that $e \in xx^{-1}$ and $e \in x^{-1}x$. As it is shown in [5, 12] the elements of fortified transposition hypergroups are separated into two classes: the set $A = \{x \in T \mid x \in ex = xe\}$, including e, of *attractive elements* and the set $C = \{x \in T - e \mid ex = xe = x\}$ of *canonical elements* (see [13] for the origin of the terminology). A fortified transposition hypergroup is isomorphic to the expansion of a quasicanonical hypergroup $C \cup \{e\}$ by a transposition hypergroup *A* of attractive elements through the idempotent e [5]. Thus the study of fortified transposition hypergroups separates into two parts, (i) the study of quasi-canonical hypergroups and (ii) the study of fortified transposition hypergroups composed of attractive elements only.

The isomorphism theorems for quasicanonical hypergroups (or polygroups) have been presented in [7]. Unfortunately, the contents of [7] were reproduced in [3]. Furthermore, the repetition in [3], of the theorems contained in [7] was pointless, after the publication of [4]. Indeed, 7 years after [7] and 13 years before [3], J. Jantosciak presented in [4] the isomorphism theorems for transposition hypergroups. In these theorems, Jantosciak uses quotients of transposition hypergroups modulo closed subhypergroups. It is known that the quasicanonical hypergroups (or polygroups) are transposition hypergroups and that their quasicanonical subhypergroups are closed

11th International Conference of Numerical Analysis and Applied Mathematics 2013
AIP Conf. Proc. 1558, 2059-2062 (2013); doi: 10.1063/1.4825940

subhypergoups. Thus the isomorphism theorems proved by Jantosciak generalize the corresponding theorems of quasicanonical hypergroups. There exist though transposition hypergroups that do not have proper closed subhypergoups, as for example the transposition hypergroups which consist of attractive elements only. In such cases, the theory developed in [4] is not applicable. For the case of the fortified transposition hypergroups, the gap is covered with the isomorphism theorems that are proved in this paper.

COSETS

K is a *subhypergroup* of H if it satisfies the axiom of reproduction, i.e. if the equality $xK = Kx = K$ is valid for all $x \in K$. Although the non-void intersection of two subhypergroups is stable under the hypercomposition, it usually is not a subhypergroup since the reproduction fails to be valid for it. A subhypergroup K of H is called *closed* if $a/b \subseteq K$ and $a \backslash b \subseteq K$, for all $a,b \in K$. The non-void intersection of two closed subhypergroups is a closed subhypergroup. In fortified transposition hypergroups it has been proved that the set A of the attractive elements is the minimum, in the sense of inclusion, closed subhypergroup [5, 12]. But in A there exist not closed subhypergroups, which when they intersect, they give subhypergroups [5, 12]. These subhypergroups are the symmetric ones. A subhypergroup K is called *symmetric* if $x \in K$ implies $x^{-1} \in K$. It is proven that the lattice of the closed subhypergroups is a sublattice of the lattice of the symmetric ones [13]. The symmetric subhypergroups can define partitions in A. So if $x \in A$ and K is a non-empty symmetric subhypergroup of A, then the *left coset* of K determined by x, $x_{\overleftarrow{K}}$ and dually, $x_{\overrightarrow{K}}$ i.e. the *right coset* of K determined by x are given by:

$$x_{\overleftarrow{K}} = \begin{cases} K, & if \quad x \in K \\ x/K, & if \quad x \notin K \end{cases} \qquad and \qquad x_{\overrightarrow{K}} = \begin{cases} K, & if \quad x \in K \\ K \backslash x, & if \quad x \notin K \end{cases}$$

For $Q \subseteq T$, $Q_{\overleftarrow{h}}$ and $Q_{\overrightarrow{h}}$ denote the unions $\cup\left\{x_{\overleftarrow{K}} \mid x \in Q\right\}$ and $\cup\left\{x_{\overrightarrow{K}} \mid x \in Q\right\}$ respectively. It is proven that distinct left cosets and right cosets, are disjoint [5]. The *double coset* of K determined by x can be defined by:

$$x_K = \begin{cases} K, & if\ x \in K \\ K \backslash (x/K) = (K \backslash x)/K, & if \quad x \notin K \end{cases}$$

Following the above notation, if Q is a non-void subset of A, then Q_K denotes the union $\cup\left\{x_K : x \in Q\right\}$.

A subhypergroup N of a hypergroup H is called *normal* if $xN = Nx$ for all $x \in H$ [4, 7], while it is called *reflexive* if $x \backslash N = N/x$ for all $x \in H$ [4]. A direct consequence of the above definition is that N is normal if and only if $N \backslash x = x/N$ for all $x \in H$.

As it is shown in [5], in a fortified transposition hypergroup of attractive elements *T*, if *K* is a non-empty symmetric subhypergroup, then, each one of the families $T:\overleftarrow{K}$ of left cosets and $T:\overrightarrow{K}$ of right cosets are partitions of *T*, but do not necessarily form a hypergroup, as associativity may fail for the induced hyperoperation. On the other hand the family of double cosets $T:K$ forms a partition of H and moreover if "$\circ$" is the induced hyperoperation on $T:K$, i.e. $a_K \circ b_K = \left\{x_K \mid x \in a_K b_K\right\}$, then:

Theorem 1. [5] $\left(T:K,\circ\right)$ *is a fortified transposition hypergroup with strong identity K for which every member of* $T:K$ *is attractive.*

$T:K$ is called the *quotient hypergroup* of T by *K*. Next, if *N* is normal, then the equality $N \backslash a = a/N$ is valid. Moreover, this equality yields $\left(N \backslash a\right)/N = \left(a/N\right)/N$, which, because of mixed associativity, gives $\left(a/N\right)/N = a/NN = a/N$. Therefore:

Theorem 2. *If N is normal symmetric subhypergroup of T, then the families of double cosets, right consets and left cosets coincide and the quotient hypergroup* $T:N$ *is a fortified transposition hypergroup.*

ISOMORPHISM THEOREMS

If *T* and *T′* are two hypergroups, a mapping $\varphi: T \to P(T')$ is called *homomorphism* if $\varphi(xy) \subseteq \varphi(x)\varphi(y)$ for all $x, y \in T$. A mapping $\varphi: T \to T'$ is called *strict homomorphism* if $\varphi(xy) \subseteq \varphi(x)\varphi(y)$ for all $x, y \in T$, while it is called *normal* if $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in T$. A homomorphism is called *complete* [8], if $x^{-1} \in ker\varphi$ for each $x \in ker\varphi$.

Proposition 1. *If* φ *is a complete homomorphism, then* $ker\varphi$ *is a symmetric subhypergroup of* *T.*

P r o o f. $x \in \ker\varphi$ implies that $x\ker\varphi \subseteq \ker\varphi$, since $\ker\varphi$ is a semisubhypergroup of T. Next, let y be an arbitrary element of $\ker\varphi$. Then, $y \in (xx^{-1})y = x(x^{-1}y) \in x\ker\varphi$. Thus, $\ker\varphi \subseteq x\ker\varphi$ and therefore $\ker\varphi = x\ker\varphi$. Dually, $(\ker\varphi)x = \ker\varphi$, and therefore, $\ker\varphi$ is a subhypergroup of *T*. In addition, $\ker\varphi$ is a symmetric subhypergroup of T, since, $x^{-1} \in \ker\varphi$ when $x \in \ker\varphi$.

Proposition 2. *The kernel of a complete and normal homomorphism φ from T to T' is a normal symmetric subhypergroup of T.*

P r o o f. According to Proposition 1, $\ker\varphi$ is a symmetric subhypergroup. Thus if $x \in \ker\varphi$, then $x\ker\varphi = (\ker\varphi)x = \ker\varphi$. Now let $x \in T$, $x \notin \ker\varphi$ and $t \in (\ker\varphi)x$. Then $t \in sx$, for some $s \in \ker\varphi$. The next statements are equivalent: $\varphi(t) \in \varphi(sx)$; $\varphi(t) \in \varphi(s)\varphi(x)$; $\varphi(t) \in \varphi(e)\varphi(x)$; $\varphi(t) \in \varphi(ex)$; $\varphi(t) \in \varphi\{e,x\}$; $\varphi(t) \in \{\varphi(e),\varphi(x)\}$. Hence $t \in \ker\varphi$ or $\varphi(t) = \varphi(x)$. If $t \in \ker\varphi$, then Theorem 1.i [11] applies and $t \in x\ker\varphi$. If $\varphi(t) = \varphi(x)$, then $\varphi(x^{-1}t) = \varphi(x^{-1})\varphi(t) = \varphi(x^{-1})\varphi(x) = \varphi(x^{-1}x)$. Thus $x^{-1}t \cap \ker\varphi \neq \varnothing$, therefore $t \in x^{-1} \setminus \ker\varphi$ and since, $x^{-1} \in x \setminus e$ it holds that $t \in (x \setminus e) \setminus \ker\varphi$. So $t \in e \setminus x\ker\varphi$. Since φ is complete and $x \notin \ker\varphi$ it derives that $x^{-1} \notin \ker\varphi$. Hence $e \notin x\ker\varphi$ and therefore Theorem 1.i [11] gives that $t \in x\ker\varphi$. Hence $(\ker\varphi)x \subseteq x\ker\varphi$. In a similar way $x\ker\varphi \subseteq (\ker\varphi)x$ and therefore $x\ker\varphi = (\ker\varphi)x$.

Proposition 3. *If a normal homomorphism φ from T to T' is surjective, then it is complete and $\varphi(e) = e'$*

P r o o f. First it will be proved that $\varphi(e) = e'$. Since φ is epimorphism, for the inverse of $\varphi(e)$ exists $x \in T$ such that $\varphi(x) = \varphi(e)^{-1}$. Thus $e' \in \varphi(e)\varphi(e)^{-1} = \varphi(e)\varphi(x) = \varphi(ex) = \varphi\{e,x\} = \{\varphi(e),\varphi(x)\}$. So either $\varphi(e) = e'$ or $\varphi(x) = e'$. If $\varphi(x) = e'$, then $\varphi(e)^{-1} = e'$, thus $\varphi(e) = e'$. Next suppose that for an element *x* in *T,* it holds that $\varphi(x) = e'$ and $\varphi(x^{-1}) \neq e'$. Then there exists $z \notin \ker\varphi$ such that $\varphi(z) = \varphi(x^{-1})^{-1}$. Thus $e' \in \varphi(x^{-1})\varphi(x^{-1})^{-1} = \varphi(x^{-1})\varphi(z) = \varphi(x^{-1}z)$. So $x^{-1}z \cap \ker\varphi \neq \varnothing$. Hence $z \in x^{-1} \setminus \ker\varphi$, which, per Theorem 23 [5], gives that $z \in (x^{-1})^{-1}\ker\varphi = x\ker\varphi$. Since $x \in \ker\varphi$, it derives that $z \in \ker\varphi$, which contradicts the supposition for *z*. Hence $\varphi(x^{-1}) = e'$ and so φ is complete.

Proposition 4. *A normal epimorphism φ from T to T' with* $\ker\varphi = \{e\}$, *is an isomorphism.*

P r o o f. Per Proposition 3, φ is compete and $\varphi(e) = e'$. Thus $e' \in \varphi(xx^{-1}) = \varphi(x)\varphi(x^{-1})$. Since $\ker\varphi = \{e\}$, the elements x, x^{-1} are not in $\ker\varphi$. Thus $\varphi(x)$, $\varphi(x^{-1}) \neq e$ and therefore $\varphi(x^{-1}) = \varphi(x)^{-1}$. Next if $\varphi(x) = \varphi(y)$, then: $e = \varphi(x)\varphi(y)^{-1} = \varphi(x)\varphi(y^{-1}) = \varphi(xy^{-1})$ and since $\ker\varphi = \{e\}$ it derives that $e \in xy^{-1}$. Hence $x = y$.

Theorem 3. *If φ is a complete normal epimorphism from T to T', then* $T:\ker\varphi \cong T'$

P r o o f. According to Proposition 2, $\ker\varphi$ is a normal symmetric subhypergroup, therefore, per Theorem 2, $T:\ker\varphi$ is a fortified transposition hypergroup. Let $\psi : T:\ker\varphi \to T'$ be given by $\psi(a_{\ker\varphi}) = \varphi(a)$. Then ψ is well defined, one-to-one onto T', and $\psi(a_{\ker\varphi}b_{\ker\varphi}) = \{\psi(x_{\ker\varphi}) \mid x \in ab\} = \{\varphi(x) \mid x \in ab\} = \varphi(ab) =$ $= \varphi(a)\varphi(b) = \psi(a_{\ker\varphi})\psi(b_{\ker\varphi})$. Therefore ψ is an isomorphism.

Lemma 1. *If N is a normal symmetric subhypergroup of T and K a symmetric subhypergroup of T, then* $(KN)_N = K_N$.

P r o o f. Let $t_N \in (KN)_N$. Then there exist $k \in K$, $n \in N$ such that $t_N \in (kn)_N$. According to Theorem 29 [5] it holds that $(kn)_N \subseteq k_N n_N \cup N$. So $t_N \in k_N n_N \cup N$. But $n_N = N$, since $n \in N$. Thus $t_N \in \{k_N, N\} \cup N = \{k_N, N\}$.

The second isomorphism Theorem comes next:

Theorem 4. *If N and K are symmetric subhypergroup of T and N is normal in $N \vee K$, then* $(N \vee K):N \cong K:(N \cap K)$.

P r o o f. Because of Proposition 6 [11] $N \vee K = NK$ and per Lemma 4.1 $NK : N = \{k_N \mid k \in K\}$. Moreover, per Proposition 7 [11], $N \cap K$ is normal in *K* and therefore the quotient $K : (N \cap K)$ is a fortified transposition hypergroup of attractive elements. Let k_n, $k \in K$ be arbitrary in $NK : N$ and let $\varphi : NK : N \to K : N \cap K$ be given by $\varphi(k_N) = k_{N \cap K}$. φ is well defined. Indeed suppose that $x_N = y_N$, $x, y \in K$ and $x, y \notin N \cap K$. Then $x / N = y / N$ or $x / N \cup N = y / N \cup N$. The last equality, per Theorem 20 [5], yields that $xN = yN$ and therefore $xN \cap K = yN \cap K$. Since *K* is a subhypergroup, the equality $xK = yK = K$ is valid. Hence $xN \cap xK = yN \cap yK$. Therefore $x(N \cap K) = y(N \cap K)$, or equivalently, per Theorem 20 [5], $[x / (N \cap K)] \cup [N \cap K] =$ $= [y / (N \cap K)] \cup [N \cap K]$. Since $x, y \notin N \cap K$, from Theorem 20 [5], results that $[x / (N \cap K)] \cap [N \cap K] = \varnothing$ and $[y / (N \cap K)] \cap [N \cap K] = \varnothing$. Thus $x / (N \cap K) = y / (N \cap K)$ and so $x_{N \cap K} = y_{N \cap K}$. If $x_N = y_N$ and $x \in K \cap N$, then $x_N = y_N = N$ and $\varphi(x_N) = \varphi(y_N) = N \cap K$. Obviously φ is onto $K : (N \cap K)$. Next suppose that $\varphi(x_N) = \varphi(y_N)$, hence $x_{N \cap K} = y_{N \cap K}$. Then $x / (N \cap K) = y / (N \cap K)$ which implies that $x / N \cap y / N \neq \varnothing$ and so, according to Theorem 22 [5], the equality $x / N = y / N$ is valid. Thus φ is one-to-one. Finally $\varphi(x_N \circ y_N) = \{\varphi(t_N) \mid t \in xy\} = \{t_{N \cap K} \mid t \in xy\} = x_{N \cap K} \circ y_{N \cap K} = \varphi(x_N) \circ \varphi(y_N)$. Therefore φ is an isomorphism.

Finally the third isomorphism Theorem appears:

Theorem 5. *If* N *and K are normal symmetric subhypergroup of T and* $K \subseteq N$*, then* $T : N \cong (T : K) : (N : K)$.

P r o o f. Let $\varphi : T : K \to T : N$ be given by $\varphi(x_K) = x_N$. Then φ is well defined. Indeed, suppose that $x_K = y_K$. If either *x* or *y* is in *K*, then both of them are in *K* and $x_K = y_K = K$. Therefore $x_N = y_N = N$. Next let $x, y \notin K$. Then the Theorems 20 and 22 of [5] and the sequence of implications: $x / K = y / K \Rightarrow x / K \cup K =$ $= y / K \cup K \Rightarrow xK = yK \Rightarrow (xK)N = (yK)N \Rightarrow x(KN) = y(KN) \Rightarrow xN = yN \Rightarrow x / N \cup N = y / N \cup N$ lead to the equality $x / N = y / N$. Hence $x_N = y_N$. Obviously, φ is onto $T : N$. Moreover $\varphi(x_K \circ y_K) =$ $= \{\varphi(t_K) \mid t \in xy\} = \{t_N \mid t \in xy\} = x_N \circ y_N = \varphi(x_N) \circ \varphi(y_N)$. Hence φ is a normal epimorphism. By Theorem 2, $T : N$ is a fortified transposition hypergroup with strong identity N. Thus $\ker\varphi = \varphi^{-1}(N) = \{x_k \mid x \in N\} = N : K$. Now according to Proposition 2 $\ker\varphi = N : K$ is a normal symmetric subhypergroup in $T : K$. Hence, because of Theorem 3, $(T : K) : \ker\varphi \cong T : N$ or equivalently $(T : K) : (N : K) \cong T : N$.

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