On Subhypergroups of Fortified Transposition Hypergroups

Ch. G. Massouros and G. G. Massouros

Department of Applied Sciences, Technological Institute of Chalkis, Evia, GR-344 00 GREECE
e-mail: Ch.Massouros@gmail.com, masouros@teihal.gr
URL: http://www.teihal.gr/gen/profesors/massouros/index.htm

Abstract. Fortified Transposition Hypergroups appeared during the study of the theory of Formal Languages and Automata from the point of view of the hypercompositional structures theory. This paper studies the subhypergroups of these hypergroups and presents properties of their closed, symmetric, normal and reflexive subhypergroups.

Keywords: hypergroups, transposition hypergroups, join spaces.
PACS: 02.10.De, 02.20.Bb AMS-Subject classification: 20N20

INTRODUCTION

An *operation* or *composition* in a non void set H is a function from $H \times H$ to H, while a *hyperoperation* or *hypercomposition* is a function from $H \times H$ to the powerset $P(H)$ of H. An algebraic structure that satisfies the axioms

i. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in H$ (associativity),

ii. $a \cdot H = H \cdot a = H$ for all $a \in H$ (reproduction),

is called *group* if "$\cdot$" is a composition [see 12] and *hypergroup* if "$\cdot$" is a hypercomposition [5]. Generally, the singleton $\{a\}$ is identified with its member a. If A and B are subsets of H, then $A \cdot B$ signifies the union $\bigcup_{(a,b) \in A \times B} a \cdot b$. Since $A \times B = \varnothing \Leftrightarrow A = \varnothing$ or $B = \varnothing$, when $A = \varnothing$ or $B = \varnothing$, then $A \cdot B = \varnothing$ and vice versa. The hypergroup was introduced in 1934 by F. Marty, in order to study problems in non commutative algebra, such as cosets determined by non invariant subgroups. In [5] F. Marty also defined the two induced hypercompositions (right and left division) that derive from the hypercomposition of the hypergroup, i.e.

$$\frac{a}{|\,b} = \{x \in H : a \in xb\} \quad \text{and} \quad \frac{a}{b\,|} = \{x \in H : a \in bx\}.$$

It is obvious that if the hypergroup is commutative, then the two induced hypercompositions coincide. For the sake of notational simplicity, W. Prenowitz denoted division in commutative hypergroups by a / b and, later on, J. Jantosciak used the notation a / b for right division and $b \setminus a$ for left division [3]. Notations $a : b$ and $a..b$ have also been used correspondingly for the above two types of division [e.g. 6, 8]. Also W. Prenowitz enriched commutative hypergroups with the following axiom: $a / b \cap c / d \neq \varnothing$ implies $ad \cap bc \neq \varnothing$, for all $a, b, c, d \in H$, in order to use them in the study of geometry. He named this axiom *transposition axiom* and the new hypergroup that derived *join space* [19]. Later on, J. Jantosciak generalized the above axiom in an arbitrary hypergroup as follows:

$$b \setminus a \cap c / d \neq \varnothing \text{ implies } ad \cap bc \neq \varnothing, \text{ for all } a, b, c, d \in H.$$

He named this hypergroup *transposition hypergroup* [3]. Hypergroups equipped with the transposition axiom are widely used in geometry [e.g. 2, 8, 19, 20 etc], in the theory of languages and automata [e.g. 13, 14, 18 etc] and elsewhere [e.g. 1]. Especially in the theory of languages and automata the hypergroups are fortified through the introduction of a *strong identity* [15, 16]. Thus the *fortified join hypergroup* and the *fortified transposition hypergroup* came into being. A transposition hypergroup T is *fortified* if it contains an element e which satisfies the axioms:

(i) $ee = e$, (ii) $x \in ex = xe$, for all $x \in T$ and (iii) for every $x \in T - \{e\}$ there exists a unique $x^{-1} \in T - \{e\}$ such that $e \in xx^{-1}$ and $e \in x^{-1}x$.

11th International Conference of Numerical Analysis and Applied Mathematics 2013
AIP Conf. Proc. 1558, 2055-2058 (2013); doi: 10.1063/1.4825939

It is proven that $xe = ex \subseteq \{e, x\}$ for all $x \in T$ [4, 16]. e is called *strong identity*. If the commutativity holds as well in T, then T is named *fortified join hypergroup* [13, 14, 18]. As it is shown in [4, 16] the elements of fortified transposition hypergroups are separated into two classes: the set $A = \{x \in T \mid x \in ex = xe\}$, including e, of *attractive elements* and the set $C = \{x \in T - e \mid ex = xe = x\}$ of *canonical elements* (see [17] for the origin of the terminology). A fortified transposition hypergroup is isomorphic to the expansion of a quasicanonical hypergroup $C \cup \{e\}$ by a transposition hypergroup A of attractive elements through the idempotent e [4]. Thus the study of fortified transposition hypergroups separates into two parts, (i) the study of quasicanonical hypergroups and (ii) the study of fortified transposition hypergroups composed of attractive elements only.

THE ALGEBRA IN FORTIFIED TRANSPOSITION HYPERGROUPS

Let H be a hypergroup. Consequences of axioms (i) and (ii) of the hypergroup are [6, 11, 12]:

i. $ab \neq \varnothing$, $a / b \neq \varnothing$ and $a \backslash b \neq \varnothing$, for all a, b in H,
ii. $H = H / a = a / H$ and $H = a \backslash H = H \backslash a$, for all a in H,
iii. the non-empty result of the induced hypercompositions is equivalent to the reproductive axiom.

In [3] and then in [4], a principle of duality is established in the theory of hypergroups. More precisely, two statements of the theory of hypergroups are dual statements, if each results from the other by interchanging the order of the hypercomposition, i.e. by interchanging any hypercomposition *ab* with the hypercomposition *ba*. One can observe that the associativity axiom is self-dual. The left and right divisions have dual definitions, thus they must be interchanged in a construction of a dual statement. Therefore, the following principle of duality holds:

Given a theorem, the dual statement resulting from interchanging the order of hypercomposition "·" (and, necessarily, interchanging of the left and the right divisions), is also a theorem.

As it is proven in [3, 6] in any hypergroup *H* the following properties are valid:

Proposition 1. *In any hypergroup H, if* $a, b, c \in H$

i. $(a / b) / c = a / (cb)$ *and* $c \backslash (b \backslash a) = (bc) \backslash a$ *(mixed associativity),*
ii. $(b \backslash a) / c = b \backslash (a / c)$,
iii. $b \in (a / b) \backslash a$ *and* $b \in a / (b \backslash a)$.

Moreover if H is a transposition hypergroup, then it has been proven in [3, 6, 8] that:

Proposition 2. *The following are true in any transposition hypergroup*:

i. $a(b / c) \subseteq ab / c$ *and* $(c \backslash b)a \subseteq c \backslash ba$,
ii. $a / (c / b) \subseteq ab / c$ *and* $(b \backslash c) \backslash a \subseteq c \backslash ba$.
iii. $(b \backslash a)(c / d) \subseteq (b \backslash ac) / d = b \backslash (ac / d)$,
iv. $(b \backslash a) / (c / d) \subseteq (b \backslash ad) / c = b \backslash (ad / c)$,
v. $(b \backslash a) \backslash (c / d) \subseteq (a \backslash bc) / d = a \backslash (bc / d)$.

An extensive study of fortified transposition hypergroups is given in a series of papers by G. Massouros, Ch. Massouros, J. Mittas and J. Jantosciak. The algebraic calculus which is developed in these papers is summarized in the following Theorem:

Theorem 1. *Let A be the set of the attractive elements and C the set of the canonical elements of a fortified transposition hypergroup T. Then the following are true*:

i. $\{a, b\} \subseteq ab$ *for all* $a, b \in A$
ii. $a \in a / b$ *and* $a \in b \backslash a$ *for all* $a, b \in A$
iii. $a / a = a \backslash a = A$ *for all* $a \in A$
iv. $ab \subseteq A$, $a / b \subseteq A$ *and* $b \backslash a \subseteq A$ *for all* $a, b \in A$
v. $a / e = e \backslash a = a$ *for all* $a \in T - \{e\}$
vi. $e / a = ea^{-1} = \{a^{-1}, e\} = a^{-1}e = a \backslash e$ *for all* $a \in T - \{e\}$
vii. *if* $a \in A$ *and* $c \in C$, *then* $ac = ca = c$
viii. $a^{-1} \in A$, *for all* $a \in A$ *and* $c^{-1} \in C$, *for all* $c \in C$
ix. $A \subseteq cc^{-1} = c^{-1}c$, *for all* $c \in C$
x. *if* $a, b \in A$ *and* $a \neq b$, *then* $ab^{-1} = a / b \cup \{b^{-1}\}$ *and* $b^{-1}a = b \backslash a \cup \{b^{-1}\}$
xi. *if* $a, b \in C$, *then* $ab^{-1} = a / b$ *and* $b^{-1}a = b \backslash a$
xii. *if* $a, b \in A$ *and* $a \neq b^{-1}$, *then* $(ab)^{-1} = b^{-1}a^{-1}$

xiii. $(ab)^{-1} = b^{-1}a^{-1}$, *for all* $a, b \in C$

xiv. $(a/b)^{-1} \cup \{b\} = b/a \cup \{a^{-1}\}$ *and* $(b \backslash a)^{-1} \cup \{b\} = a \backslash b \cup \{a^{-1}\}$

THE SUBHYPERGROUPS OF FORTIFIED TRANSPOSITION HYPERGROUPS

A non-empty subset K of H is called semisubhypergroup when it is stable under the hypercomposition, i.e. it has the property $xy \subseteq K$ for all $x, y \in K$. K is a subhypergroup of H if it satisfies the axiom of reproduction, i.e. if the equality $xK = Kx = K$ is valid for all $x \in K$. This means that when K is a subhypergroup and $a, b \in K$, the relations $a \in bx$ and $a \in yb$ always have solutions in K. Although the non-void intersection of two subhypergroups is stable under the hypercomposition, it usually is not a subhypergroup since the reproduction fails to be valid for it. This led, from the very early steps of hypergroup theory, to the consideration of more special types of subhypergroups. One of them is the *closed subhypergroup*. A subhypergroup K of H is called *left closed* with respect to H if for any two elements a and b in K, all the solutions of the relation $a \in yb$ lie in K. This means that K is left closed if and only if $a/b \subseteq K$, for all $a, b \in K$ [6]. Similarly K is *right closed* when all the solutions of the relation $a \in bx$ lie in K or equivalently if $b \backslash a \subseteq K$, for all $a, b \in K$ [6]. Finally K is *closed* when it is both right and left closed. The non-void intersection of two closed subhypergroups is a closed subhypergroup. In fortified transposition hypergroups it has been proved that the set A of the attractive elements is the minimum, in the sense of inclusion, closed subhypergroup [4, 17]. But in A there exist not closed subhypergroups, which when they intersect, they give subhypergroups [4, 17]. These subhypergroups are the symmetric ones. A subhypergroup K is called *symmetric* if $x \in K$ implies $x^{-1} \in K$.

Proposition 3. *The intersection of any two symmetric subhypergroups of a fortified transposition hypergroup H is a symmetric subhypergroup of H.*

Proposition 4. *To any two symmetric subhypergroups K and M of a fortified transposition hypergroup H there is a least symmetric subhypergroup* $K \vee M$ *containing them both.*

P r o o f. Let *U* be the set of all symmetric subhypergroups *R* of *H* which contain both *K* and *M*. The intersection of all these symmetric subhypergroups *R* of *H* is a symmetric subhypergroup with the desired property.

A subhypergroup N of a hypergroup H is called *normal* if $xN = Nx$ for all $x \in H$ [2, 7], while it is called *reflexive* if $x \backslash N = N/x$ for all $x \in H$ [2]. A direct consequence of the above definition is that N is normal if and only if $N \backslash x = x/N$ for all $x \in H$.

Proposition 5. *In fortified transposition hypergroups of attractive elements the normal subhypergroups are reflexive and vice versa.*

P r o o f. Let T be a fortified transposition hypergroup and N a normal subhypergroup of T. Suppose that $x \in N$. Then, according to Theorem 1.iii, $T = x/x \subseteq N/x$ and $T = x \backslash x \subseteq x \backslash N$ are valid. Therefore $N/x = x \backslash N = T$. Next suppose that $x \notin N$. Then Theorem 1.x yields $Nx^{-1} = N/x \cup \{x^{-1}\}$. But $e \in N$ and $x^{-1} \in e/x$. Thus $Nx^{-1} = N/x$. Dually $x^{-1}N = x \backslash N$. Since N is normal $Nx^{-1} = x^{-1}N$ is valid. Hence $N/x = x \backslash N$. Next suppose that *N* is reflexive. Then $x \backslash N = N/x$, for all $x \in T$. If $x \in N$, then $x \backslash N = x/N = T$ and $N/x = x/N = T$. Thus $x/N = N \backslash x$. Let $x \notin N$. Then $Nx = N/x^{-1} \cup \{x\}$. Since $x \in e/x^{-1}$ and $e \in N$, it results that $Nx = N/x^{-1}$. Similarly $xN = x^{-1} \backslash N$. Hence $xN = Nx$.

Proposition 6. *If* N *is a normal symmetric subhypergroup in a fortified transposition hypergroup H and* K *a symmetric subhypergroup in* H*, then* $N \vee K = NK = KN$.

P r o o f. $N \subseteq Ne \subseteq NK$ and $K \subseteq Ke \subseteq KN$. Since N is normal in H, NK is stable under hypercomposition because $(NK)(NK) = (NN)(KK) = NK$. Also NK is stable under inverse. Indeed let $x \in NK$. Then there exist $n \in N$ and $y \in K$ such that $x \in ny$. If $n \neq y^{-1}$, then $x^{-1} \in (ny)^{-1} = y^{-1}n^{-1}$. If $n = y^{-1}$, then $x \in N \cap K$, and therefore, per Proposition 3, $x^{-1} \in N \cap K \subseteq NK$. Moreover, $e \in NK$. Hence NK is a symmetric subhypergroup, it contains both N and K, and is contained in any symmetric subhypergroup containing them both. Therefore $NK = N \vee K$.

Proposition 7. *Let* T *be a fortified transposition hypergroup of attractive elements and* K*,* N *symmetric subhypergroups of* T*. If* N *is normal in* T*, then* $N \cap K$ *is normal in* K.

P r o o f. Let $x \in K$. Suppose $z \in x(N \cap K)$. Then $z \in K$ and there exists $t \in N \cap K$ such that $z \in xt$. Since N is normal $xN = Nx$ is valid, thus there exists $t' \in N$ such as $z \in t'x$. Hence $t' \in z/x$. But $z/x \subseteq zx^{-1}$.

Thus $z/x \subseteq K$. So $t' \in K \cap N$ and therefore $x(N \cap K) \subseteq (N \cap K)x$. $(N \cap K)x \subseteq x(N \cap K)$ follows by duality, and so the Proposition holds.

Lemma 1. *If N is a normal subhypergroup of a transposition hypergroup, then:*

$$(a/N)(b/N) \subseteq ab/N \text{ and } (N \setminus a)(N \setminus b) \subseteq N \setminus ab$$

P r o o f. Normality of N, Proposition 2.iii and mixed associativity give:

$$(a/N)(b/N) = (N \setminus a)(b/N) \subseteq (N \setminus ab)/N = (ab/N)/N = ab/NN = ab/N .$$

Proposition 8. *If N is a normal symmetric subhypergroup of a fortified transposition hypergroup, and $\{a,b\} \cap N = \varnothing$, then: $(a/N)(b/N) = ab/N$ and $(N \setminus a)(N \setminus b) = N \setminus ab$*

P r o o f. Theorem 25 of [4] gives $ab/N \subseteq (a/N)(b/N) \cup N$. Suppose that $a/N = b^{-1}/N$. According to Theorem 23 (b) of [4], b^{-1}/N is equal to $(N \setminus b)^{-1}$. Since N is normal $N \setminus b = b/N$ is valid. Hence $a/N = (b/N)^{-1}$. Therefore N is in $(a/N)(b/N)$ and so $ab/N \subseteq (a/N)(b/N)$. According to Lemma 1, $(a/N)(b/N) \subseteq ab/N$ and so the equality. Now suppose that $a/N \neq b^{-1}/N$. Let $t \in ab \cap N$, then the sequence of implications

$$t \in ab \;\; ; \;\; a \in t/b \;\; ; \;\; a \in tb^{-1} \;\; ; \;\; a \in N\, b^{-1} \;\; ; \;\; Na = N\, b^{-1} \;\; ; \;\; aN = b^{-1}N \;\; ; \;\; a/N \cup N = b^{-1}/N \cup N$$

leads to the contradiction $a/N = b^{-1}/N$. Hence $ab \cap N = \varnothing$, therefore $ab/N \subseteq (a/N)(b/N)$. Next Lemma 1 applies and the first assertion is established. The second assertion follows by duality.

REFERENCES

1. S. Hoskova, J. Chvalina, P. Rackova, "Transposition hypergroups of Fredholm integral operators and related hyperstructures (Part I)", *Journal of Basic Science*, **4**, no. 1 (2008) pp. 43–54.
2. J. Jantosciak, "Classical geometries as hypergroups", Atti del Convegno su Ipergruppi altre Structure Multivoche et loro Applicazioni, Udine 15-18 Octobr. 1985, pp. 93-104.
3. J. Jantosciak, "Transposition hypergroups, Noncommutative Join Spaces", *J. of Algebra*, **187** (1997), pp. 97-119.
4. J. Jantosciak and Ch. G. Massouros, "Strong Identities and fortification in Transposition hypergroups", *Journal of Discrete Mathematical Sciences & Cryptography* **6**, no 2-3 (2003), pp. 169-193.
5. F. Marty, "Sur un généralisation de la notion de groupe", Huitième Congrès des math. Scand., Stockholm 1934, pp. 45-49.
6. Ch. G. Massouros, "Hypergroups and convexity", *Riv. di Mat. pura ed applicata*, **4** (1989) pp. 7-26.
7. Ch. G. Massouros, "Quasicanonical Hypergroups", Proceedings of the 4th Internat. Cong. on Algebraic Hyperstructures and Applications, World Scientific (1990) pp. 129-136.
8. Ch. G. Massouros, "Hypergroups and Geometry", *Mem. Academia Romana, Mathematics, special issue, Seria IV*, Tom. XIX, (1996) pp. 185-191 and Scripta Scientiarum Mathematicarum, Tom. I (1997) pp.143-152.
9. Ch. G. Massouros and G. G. Massouros, "Hypergroups associated with Graphs and Automata", ICNAAM 2009 Crete Greece, American Institute of Physics (AIP) Conference Proceedings, pp. 164-167.
10. Ch. G. Massouros and G. G. Massouros, "Identities in Multivalued Algebraic Structures", ICNAAM 2010 Rhodes Greece; American Institute of Physics (AIP) Conference Proceedings, pp. 2065-2068.
11. Ch. G. Massouros and G. G. Massouros, "The Transposition Axiom in Hypercompositional Structures", *Ratio Mathematica*, **21** (2011) pp. 75-90.
12. Ch. G. Massouros and G. G. Massouros, "On certain fundamental properties of hypergroups and fuzzy hypergroups – Mimic fuzzy hypergroups", *International Journal of Risk Theory* **2**, No 2 (2012) pp. 71-82
13. G. G. Massouros and J. Mittas, "Languages - Automata and hypercompositional structures", Proceedings of the 4th Internat. Cong. on Algebraic Hyperstructures and Applications, World Scientific 1991, pp. 137-147.
14. G. G. Massouros, "Hypercompositional Structures in the Theory of the Languages and Automata", *An. stiintifice Univ. Al. I. Cuza, Iasi, Informatica*, t. iii, (1994) pp. 65-73.
15. G. G. Massouros, "Fortified Join Hypergroups and Join Hyperrings", *An. Sti. Univ. Al. I. Cuza, Iasi, Sect. I, Mat.* , **XLI**, fasc. 1, pp. 37-44 (1995).
16. G. G. Massouros, Ch. G. Massouros and J. D. Mittas, "Fortified Join Hypergroups", *Annales Matematiques Blaise Pascal* **3**, no 2 (1996) pp. 155-169.
17. G. G. Massouros, "The subhypergroups of the Fortified Join Hypergroup", *Ital. J. of Pure & Applied Math*. **2** (1997) 51-63.
18. G. G. Massouros, "Hypercompositional Structures from the Computer Theory", *Ratio Matematica*, **13** (1999) pp. 37-42.
19. W. Prenowitz, "A Contemporary Approach to Classical Geometry", *Amer. Math. Month.* **68**, No 1, part II (1961) pp. 1-67.
20. W. Prenowitz and J. Jantosciak, "Geometries and Join Spaces", *J. Reine Angew. Math.* **257** (1972) pp. 100-128.