Identities in Multivalued Algebraic Structures

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Abstract. In the hypecompositional structures, unlike the classical ones, there exist several types of identities. This paper deals with two main types of identities, the scalar and strong ones and with the hypergroups in which they appear. Certain such hypergroups e.g. the fortified join hypergroup and the join polysymmetric hypergroup derived from the theory of formal languages and automata.

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INTRODUCTION

In a multivalued algebraic structure the result of the operation between two elements is not a single element, but a set of elements. So the notion of the hypercomposition is introduced. A **hypercomposition** in a non empty set H, is a function from $H \times H$ to the power set $\mathcal{P}(H)$ of H. A set H endowed with a hypercomposition "·" is called **hypergroup** if satisfies the axioms:

i. $a(bc) = (ab)c$ for every $a, b, c \in H$ (associativity)
ii. $aH = Ha = H$ for every $a \in H$ (reproductivity)

If A and B are subsets of H, then AB signifies the union $\bigcup\limits_{(a,b)\in A\times B} ab$. Ab and aB will have the same meaning as $A\{b\}$ and $\{a\}B$. Two **induced hypercompositions** (the left and the right division) derive from the hypercomposition of the hypergroup, i.e.

$$a / b = \{x \in H \mid a \in xb\} \quad \text{and} \quad b \backslash a = \{x \in H \mid a \in bx\}$$

When "·" is commutative, $a / b = b \backslash a$. Consequences of the axioms (i) and (ii) are:

i. $xy \neq \varnothing$ for all x, y in H.
ii. $a / b \neq \varnothing$ and $a \backslash b \neq \varnothing$, for all x, y in H.
iii. the nonempty result of the induced hypercompositions is equivalent to the reproductive axiom.

An element e is called **right identity** if $x \in x \cdot e$, for all x in H. If $x \in e \cdot x$, for all x in H, then x is called **left identity**, while it is called **identity** if it both right and left identity. If $x = x \cdot e = e \cdot x$, for all x in H, then e is called **scalar identity**. When in H exists a scalar identity, then it is unique. An identity e is called **strong identity** if $x \in x \cdot e = e \cdot x = \{e, x\}$, for all x in H. The strong identity need not be unique [2]. A subset h of H is called **sybhypergroup** of H if $xh = hx = h$ for every $x \in h$. A subhypergroup h of H is **central** if xy=yx for each x∈h and y∈H.

Proposition 1.1. *If H is is a hypergroup with strong identities, then the set E of these identities is a central subhypergroup of H.*

The equivalence relation ε on H, whose equivalence classes are given by x_ε=E, if x∈E and x_ε=x otherwise, is a regular one, since $x_\varepsilon y_\varepsilon \subseteq (xy)_\varepsilon$. Hence

Proposition 1.2. *If H is a hypergroup with strong identities, then a hypergroup with unique strong identity can be obtained from H by factoring the set E of the identities out.*

An element x' is called **right inverse** of x, if a right identity e exists such that $e \in x \cdot x'$. Analogous is the definition of the **left inverse**, while x' is called **inverse** of x, if it is both right and left inverse.

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TRANSPOSITION HYPERGROUPS – CANONICAL HYPERGROUPS

A **transposition hypergroup** is a hypergroup which satisfies the axiom:

$$b \backslash a \cap c / d \neq \varnothing \text{ implies } ad \cap bc \neq \varnothing$$

A commutative transposition hypergroup is called **join hypergroup** or **join space** [1,3]. A **fortified transposition hypergroup** is a transposition hypergroup H, having an identity element e such that $ee=e$, $x \in ex=xe$, for all $x \in H$ and also for every $x \in H-\{e\}$ there exists a unique element $x' \in H-\{e\}$ such that $e \in xx'$, and, furthermore, x' satisfies $e \in x'x$ [2]. x' is called the symmetric (or inverse) of x. If H is also commutative, then it is a **fortified join hypergroup** [9,10]. This last hypergroup resulted from the theory of Formal Languages [6,7,8].

Example 2.1. (i) Let H be a set totally ordered, dense as for the order and symmetrical around a center denoted by 0, as for which a partition $H = H_1 \cup \{0\} \cup H_2 = H^- \cup \{0\} \cup H^+$ can be defined, such that: $x < 0 < y$ for every $x \in H^-$ and $y \in H^+$. $x \leq y$ implies that $-y \leq -x$ for every $x,y \in H$ (where -x is the symmetric of x with regard to 0). Then the following hypercomposition is defined in H:

$$x + y = \begin{cases} \{x, y\}, & \text{if } y \neq -x \\ [0, |x|] \cup \{-x\} & \text{if } y = -x \end{cases}$$

The deriving structure is a fortified join hypergroup in which

$$x-x \neq -(x-x), \text{ for every } x \neq 0,$$

since $x-x = [0, |x|] \cup \{-x\}$, while $-(x-x) = [-|x|, 0] \cup \{x\}$.

(ii) If the hypercomposition in H is defined as follows:

$$x + y = \begin{cases} \{x, y\}, & \text{if } y \neq -x \\ [-|x|, |x|], & \text{if } y = -x \end{cases}$$

then, the deriving structure is again a fortified join hypergroup

Proposition 2.1. *A fortified transposition hypergroup has a unique strong identity.*

A **quasicanonical hypergroup** or **polygroup** is a transposition hypergroup H, that contains an identity element e such that $ee=e$, $x=ex=xe$, for all $x \in H$ and also for every $x \in H-\{e\}$ there exists a unique element $x' \in H-\{e\}$ such that $e \in xx'$, and, furthermore, x' satisfies $e \in x'x$ [4]. If H is also commutative, then it is a **canonical hypergroup** [12]**.** Thus a fortified transposition hypergroup is a transposition hypergroup with a strong identity while a quasicanonical hypergroup is a transposition hypergroup with a scalar identity.

Example 2.2. Let $(H, \cdot)$ be a quasicanonical hypergroup. If a new hypercomposition $\odot$ is defined in H as follows:

$$a \odot b = a \cdot b \cup \{a, b\}, \text{ for each } a,b \in H$$

then $(H, \odot)$ is a fortified transposition hypergroup.

Let H be a hypergroup having an identity element e. An element x of H, will be called **attractive** if $e \in xe \cap ex$, while x will be called **non attractive** if $e \notin xe \cap ex$. If A is the set of the attractive elements and C the set of non attractive elements, then $H = A \cup C$ and $A \cap C = \varnothing$. If H is a fortified transposition hypergroup, then

$$A = \{x \in H \mid ex = xe = \{x, e\}\} \quad C = \{x \in H-\{e\} \mid ex = xe = x\}$$

The members of C are called **canonical** elements.

In a transposition hypergroup with a scalar identity e, each element has a unique inverse. This does not happen though in a transposition hypergroup with a strong identity. The following example shows that in a transposition hypergroup with a strong identity each element has not a unique inverse.

Example 2.3. Let H be a set totally ordered, and symmetric around a center, denoted by $0 \in H$. A hypercomposition is introduced in H as follows:

$$x+y = \{x, y\} \quad \text{if } 0 \notin [x,y]$$

$$x+y = \{x, y, 0\} \quad \text{if } 0 \in [x,y]$$

Then (H,+) is a transposition hypergroup with strong identity the element 0. Obviously if x belongs to the positive cone, then every element of the negative cone of H is opposite of x and similarly if x belongs to the negative cone, then S(x) is the positive cone.

Examples as the above one and the study of automata theory with the use of multivalued algebra led to the introduction of the **transposition polysymmetric hypergroup** [6,7,8], i.e. a transposition hypergroup H, having an identity element e such that $ee=e$, $x \in ex=xe$, for all $x \in H$ and also for each the **symmetric set** of x. A commutative transposition polysymmetrical hypergroup is called a **join polysymmetric hypergroup**.

Example 2.4. Let K be a field and G a subgroup of its multiplicative group. In K a hypercomposition $+$ is defined as follows:

$$x + y = \{ z \in K \mid z = xp+yq,\ p,q \in G \}$$

Then (K, +) becomes a join polysymmetric hypergroup having the 0 of K as its neutral element since

$$x \in x + 0 = \{ z \in K \mid z = xp+0q = xp,\ p,q \in G \}$$

The symmetric set of an element x of K is $S(x) = \{-xp \mid p \in G \}$.

The neutral element of the above join polysymmetric hypergroup is not strong. In the following two examples the join polysymmetric hypergroups which are presented have strong identity.

Example 2.5. Let the hypergroups $(A_i, \cdot)$, $i \in I$. We consider the union $T = \bigcup_{i\in I} Ai$. In this set the following hyperoperation is defined :

$a \odot b = ab$ if a,b are elements of the same hypergroup A_i

$a \odot b = A_i \cup A_j$ if $a \in A_i, b \in A_j$ and $i \neq j$

Then $(T, \odot)$ becomes a hypergroup and, if A_i, $i \in I$ are transposition hypergroups, then $(T, \odot)$ is also transposition. Now let A_i, $i\in I$ be a family of fortified transposition hypergroups which consist only of attractive elements and assume that the hypergroups A_i, $i\in I$ have their identity e, common.

If the hypercomposition is defined as follows:

$a \circledcirc b = ab$ if a,b are elements of the same hypergroup A_i

$a \circledcirc b = A_i \cup A_j$ if $a \in A_i - \{e\}$, $b \in A_j - \{e\}$ and $i \neq j$

then $(T,\circledcirc)$ becomes a transposition polysymmetrical hypergroup, which has e as strong identity. Obviously if $a\in A_i$, then $S(a) = (T-A_i) \cup \{a'\}$, where a' is the inverse of a in A_i.

Example 2.6. Let $(A_i, \cdot)$, $i \in I$, be a family of hypergroups, which have the property that the result of the hypercomposition of every two of their elements contains these two (participating) elements. We consider the union $T = \bigcup_{i\in I} Ai$ and in this set we introduce a hypercomposition «*» defined as follows:

$a * b = ab$ if a, b are elements of the same hypergroup A_i

$a * b = \{a, b\}$ if $a \in A_i, b \in A_j$ and $i \neq j$

Then (T, *) is a hypergroup and moreover, if A_i, $i \in I$ are transposition hypergroups, then (T, *) is also transposition. Now let A_i, $i \in I$ be a family of fortified transposition hypergroups which consist only of attractive elements and suppose that the hypergroups A_i, $i \in I$ have their identity e, common. Then (T, *) is also a fortified transposition hypergroup. If though, we modify slightly the hypercompositions «*» in the following way:

$a \circledast b = ab$ if a,b are elements of the same hypergroup A_i

$a \circledast b = \{a, e, b\}$ if $a \in A_i - \{e\}$, $b \in A_j - \{e\}$ and $i \neq j$

then $(T, \circledast)$ is a transposition polysymmetrical hypergroup, which has e as strong identity. Obviously if $a \in A_i$, then $S(a) = (T-A_i) \cup \{a'\}$, where a' is the inverse of a in A_i. Note that if A_i, $i \in I$ are transposition polysymmetrical hypergroups, then (T, *) is also a transposition polysymmetrical hypergroup.

Proposition 2.2. *If a polysymmetric transposition hypergroup has a strong identity e, then it is unique.*

Indeed assume that u is identity distinct from e and let $S_u(e)$ be the set of the symmetric elements of e with regard to u. Then, there would exist an element $e'\in S_u(e)$, distinct from u, such that $u \in ee'$. But $ee'=\{e,e'\}$. Thus, $u\in\{e,e'\}$, which contradicts the assumption.

Proposition 2.3. *If H is a polysymmetric transposition hypergroup with a strong identity e, which consists only of attractive elements and x is a non identity element of H, then*

i. $e/x = eS(x) = \{e\} \cup S(x) = S(x)e = x\backslash e$

ii. $x/e = e\backslash x = x$

Indeed, for (i), since e is strong identity it is straight forward that the equalities $eS(x) = \{e\} \cup S(x) = S(x)e$ hold. Next $t\in e/x$ if and only if $e\in tx$, which means that either t equals to e or t belongs to S(x) and so $e/x = \{e\} \cup S(x)$. Duality yields the rest. Also for (ii) it holds: $t \in x/e$ if and only if $x \in te \subseteq \{t,e\}$. Since $x \neq e$, clearly $t = x$. So $x/e = x$. The rest follows by duality.

Corollary 2.2. *If x is a non identity element of a fortified transposition hypergroup and x′ is its symmetric, then*

i. $e/x = x\backslash e \subseteq \{e, x'\}$

ii. $x/e = e\backslash x = x$

Proposition 2.4. *If A is the set of the attractive elements of a transposition hypergroup with a strong identity e, then* $A= e/e =e\backslash e$.

Proposition 2.5. *If H is a transposition hypergroup with a strong identity, which consists only of attractive elements and x, y are elements of H, then*

i. $\{x, y\} \subseteq xy$

ii. $x \in x/y$ *and* $x \in y\backslash x$

iii. $x/x = x\backslash x = H$

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