Hypergroups Associated With Graphs And Automata

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Abstract. The paths in a tree define a hypercomposition in the set of its vertices and so a hypergroup is associated to each tree. Hypergroups are also associated to graphs through their spanning trees. Furthermore hypergroups are associated to automata using hypercompositions defined through strings.

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INTRODUCTION

To make this paper self-contained, we are beginning by mentioning some definitions from the theory of hypercompositional structures. A **hypercomposition** in a non empty set H, is a function from H×H to the power set $\mathcal{P}(H)$ of H. A set H endowed with a hypercomposition "·" is called **hypergroupoid** if $xy\neq\varnothing$ for all x, y in H, otherwise it is called **partial hypergroupoid**. If A and B are subsets of H, then AB signifies the union $\bigcup_{(a,b)\in A\times B} ab$. Ab and aB will have the same meaning as A{b} and {a}B. A **hypergroup** [4] is a hypergroupoid that satisfies the axioms:

i. $a(bc) = (ab)c$ for every $a,b,c \in H$ (associativity)
ii. $aH = Ha = H$ for every $a \in H$ (reproductivity)

If only (i) is valid then (H, ·) is called **semihypergroup**, while it is called **quasi-hypergroup** if only (ii) holds.

Two **induced hypercompositions** (the left and the right division) derive from the hypercomposition of the hypergroup, i.e.

$$a/b = \{x \in H \mid a \in xb\} \quad \text{and} \quad b\backslash a = \{y \in H \mid a \in by\}$$

When "·" is commutative, $a/b = b\backslash a$. Consequences of the axioms (i) and (ii) are:

i. $ab \neq \varnothing$, for all $a,b \in H$.
ii. $a/b \neq \varnothing$ and $a\backslash b \neq \varnothing$, for all $a,b \in H$.
iii. the nonempty result of the induced hypercompositions is equivalent to the reproductive axiom.
iv. $(a/b)/c = a/(c\cdot b)$, $c\backslash(b\backslash a) = (b\cdot c)\backslash a$, $(b\backslash a)/c = b\backslash(a/c)$, for all $a,b,c \in H$ (mixed associativity) [5]

A **transposition hypergroupoid** is a hypergroupoid which satisfies the axiom [3]:

$$b\backslash a \cap c/d \neq \varnothing \text{ implies } ad \cap bc \neq \varnothing$$

A commutative transposition hypergroup is called **join hypergroup** or **join space** [3], [6].

TREES AND HYPERGROUPS

In general **graph** is a set of points called **vertices** connected by lines called **edges**. A **path** in a graph is a sequence of no repeated vertices $v_1, v_2, \ldots, v_n$, such that $\overline{v_1v_2}, \overline{v_2v_3}, \ldots, \overline{v_{n-1}v_n}$, are edges in the graph. A graph is said to be **connected** if every pair of its vertices is connected by a path. A **tree** is a connected graph with no cycles. Let $\mathcal{T}$ be a tree. In the set V of its vertices a hypercompostion "·" can be introduced as follows: for each two vertices x, y in V, xy is the set of all vertices which belong to the path that connects vertex x with vertex y.

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Proposition 2.1. *If V is the set of the vertices of a tree $\mathcal{T}$, then $V = x/x$, for each x in V.*

Proof. Since $x \in xy$, for each y in V, it derives that $y \in x/x$, for each y in V, therefore $V = x/x$.

Definition 2.1. The set $\langle x,y\rangle = x/y \cup xy \cup y/x$, where $x \neq y$ are two vertices of $\mathcal{T}$, is called the **line** of $\mathcal{T}$ which is defined by x, y.

Definition 2.2. A subset S of V is called **convex**, if it holds $xy \subseteq S$, for each x, y in S.

Proposition 2.2. *If r belongs to x/y, then $ry \subseteq \langle x,y\rangle$ and if r belongs to y/x, then $rx \subseteq \langle x,y\rangle$.*

Proof. Let $r \in x/y$. Then $x \in ry$, and so $ry = rx \cup xy$. Suppose that s is an element of rx. Then $sx \subseteq rx$ and $sx \cup xy = sy$. Thus $x \in sy$, hence $s \in x/y$. Similar is the proof of the other part of the Proposition.

Proposition 2.3. *If r_1 and r_2 are elements of $\langle x,y\rangle$, then $r_1r_2 \subseteq \langle x,y\rangle$.*

Proof. Obviously, if $r_1, r_2 \in xy$, the Proposition is valid.

Let $r_1, r_2 \in x/y$, then, $x \in r_1y$ and $x \in r_2y$. If $r_1y \cap r_2y$ is equal to r_1y or r_2y, then r_1r_2 is a subset of r_1y or r_2y respectively which, according to Proposition 2.2, are subsets of $\langle x,y\rangle$. If $r_1y \cap r_2y \neq r_1y, r_2y$, then, since $\mathcal{T}$ contains no cycles there exists one and only one element s such that $s \in [r_1y \cap r_2y] \cap r_1r_2$. So $r_1r_2 = r_1s \cup sr_2$. But $r_1s \subseteq r_1y$ and $r_2s \subseteq r_2y$, and, according to Proposition 2.2, r_1y, r_2y are subsets of $\langle x,y\rangle$, hence $r_1r_2 \subseteq \langle x,y\rangle$. Similar is the case when $r_1, r_2 \in y/x$.

Next let $r_1 \in x/y$ and $r_2 \in xy$, then, $x \in r_1y$. So $r_2 \in r_1y$ and therefore $r_1r_2 \subseteq \langle x,y\rangle$. Similar is the case when $r_1 \in xy$ and $r_2 \in y/x$.

Finally if $r_1 \in x/y$ and $r_2 \in y/x$, then $r_1y = r_1x \cup xy$ and $r_2x = xy \cup yr_2$. Thus $r_1r_2 = r_1x \cup xy \cup yr_2 \subseteq \langle x,y\rangle$ and so the Proposition.

Corollary 2.1. *The lines are convex sets.*

Proposition 2.4. *If V is the set of the vertices of a tree $\mathcal{T}$, then $(V, \cdot)$ is a join space.*

Proof. Since $\{x, y\} \subseteq xy$, it derives that $xV=V$ for each x in V and therefore the reproductive axiom is valid. Also since $\mathcal{T}$ is undirected graph, the hypecomposition is commutative. Next, let x, y, z be vertices of $\mathcal{T}$. If any of these three vertices, e.g. z, belongs to the path that the two other define, then $(xy)z=x(yz)=xy$. If x, y, z do not belong to the same path, then there exists only one vertex v in xy such that $vz \cap xy=\{v\}$. Indeed if there existed a second vertex w such that $wz \cap xy=\{w\}$, then the tree $\mathcal{T}$ would have a cycle, which is absurd. So $(xy)z=xy \cup vz$ and $x(yz)= xv \cup yz$. Since $xy \cup vz = xv \cup yz$, it derives that $(xy)z=x(yz)$. Now for the transposition axiom suppose that x, y, z, w are vertices of $\mathcal{T}$ such that $x/y \cap z/w \neq \varnothing$. If x, y, z, w are in the same path, then considering all their possible arrangements in their path, it derives that $xw \cap yz \neq \varnothing$. Next suppose that the four vertices do not belong to the same path. Thus suppose that z does not belong to the path defined by y, w. Then $z \notin yw$. Consider zy and zw. As indicated above, since there are no cycles in $\mathcal{T}$, there exists only one vertex v in xy such that $zy=yv \cup vz$ and $zw=wv \cup vz$. Now we distinguish the cases:

(i) if x, y, w do not belong to the same path, then for the same reasons as above there exists only one s in xy such that $xy=ys \cup sx$ and $sw=ws \cup sx$. Since $x/y \cap z/w \neq \varnothing$, there exists r in V such that $x \in ry$ and $z \in rw$. Thus, since $\mathcal{T}$ contains no cycles, and in order for srv not to be a cycle, s and v must coincide. Hence $v \in xw \cap yz$ and therefore $xw \cap yz \neq \varnothing$.

(ii) if x belongs to the same path with y and w, then:

(ii_a) if $x \in yw$, then $yw=yx \cup xw$ and $xw \subseteq x/y$. Hence $v=x$, $x \in xw \cap yz$ and therefore $xw \cap yz \neq \varnothing$.

(ii_b) if $x \notin yw$, then $x/y \cap z/w=\varnothing$

A **spanning tree** of a connected graph is a tree whose vertex set is the same as the vertex set of the graph, and whose edge set is a subset of the edge set of the graph. Any connected graph has at least one spanning tree and there exist algorithms which find such trees. Hence any graph can be endowed with the join space structure through its spanning trees.

Proposition 2.5. *Let $\mathcal{G}$ be a graph and $\mathcal{T}$ a spanning tree of $\mathcal{G}$. The set of the vertices of the graph becomes a join space if for all vertices x, y of $\mathcal{G}$, the hypercomposition $x \bullet_{\mathcal{T}} y$ is the set of all vertices which belong to the path that connects vertex x with vertex y in $\mathcal{T}$.*

Since a graph may have more than one spanning trees, more than one join spaces can be associated to a graph.

The connection of graphs with hypercompositonal structures was studied also by M. Gionfriddo [2], J. Nieminen [13], [14], P. Corsini [1] and I. Rosenberg [15].

AUTOMATA AND HYPERGROUPS

An **automaton** $\mathcal{A}$ is a collection of five objects (Σ, S, δ, s_o, F) where Σ is the **alphabet** of input letters (a finite nonempty set of symbols), S is a finite nonvoid set of **states**, s_o is the **start** (or **initial**) state, an element of S, F is the set of the **final** (or **accepting**) states, a (possibly empty) subset of S and δ is the **state transition function** with domain $S\times\Sigma$ and range S, in the case of a **deterministic** automaton (DFA), or $\mathcal{P}$ (S), in the case of a **nondeterministic** automaton (NDFA). Σ^* denotes the set of **words** (or **strings**) formed by the letters of Σ –closure of Σ– and $\lambda\in\Sigma^*$ signifies the empty word. Σ^* under the concatenation of words is a monoid, with neutral element λ, since $\lambda x=x\lambda=x$ for all x in Σ^*. Moreover Σ^* becomes a hyperingoid under the b-hyperoperation: $x+y=\{x, y\}$ for all x, y in Σ^* [12]. Given a DFA $\mathcal{A}$, the **extended state transition function** for $\mathcal{A}$, denoted δ^*, is a function with domain $S\times\Sigma^*$ and range S defined recursively as follows:

i. $\delta^*(s,a) = \delta(s,a)$ for all s in S and a in Σ
ii. $\delta^*(s,\lambda) = s$ for all s in S
iii. $\delta^*(s,ax) = \delta^*(\delta(s,a),x)$ for all s in S, x in Σ^* and a in Σ.

Let x be a word in Σ^*, then:

$$\text{Prefix}(x)=\{y\in\Sigma^* \mid yz=x \text{ for some } z\in\Sigma^*\} \text{ and } \text{Suffix}(x)=\{z\in\Sigma^* \mid yz=x \text{ for some } y\in\Sigma^*\}$$

Let s be an element of S. Then

$$I_s= \{x\in\Sigma^* \mid \delta^*(s_o,x)=s\} \text{ and } P_s= \{s_i\in S \mid s_i=\delta^*(s_o,y), y\in\text{Prefix}(x), x\in I_s\}$$

Obviously the states s_o and s are in P_s.

Lemma 3.1. *If $r\in P_s$, then $P_r\subseteq P_s$.*

Proof. $P_r= \{s_i\in S \mid s_i=\delta^*(s_o,y), y\in\text{Prefix}(v), v\in I_r\}$ and since $r\in P_s$, it holds that $\delta^*(r,z)=s$, for some z in Suffix(x), $x\in I_s$. Thus $\delta^*(s_i,y_i)=s$, $y_i\in$ Suffix and so the Lemma.

In the set of the states of an automaton the structure of the hypergroup was introduced in many ways (see [7], [8], [9], [10], [11]). Hereafter, new hypercompositions are introduced in S. The first one is defined as follows:

$$s + q = P_s \cup P_q \quad \text{for all } s, q \in S \qquad (1)$$

This hypercomposition is commutative, thus the two induced hypercompositions coincide and so we have:

$$s/q = q\backslash s = \begin{cases} S, & \text{if } s \in P_q \\ \{r \in S \mid P_s \subseteq P_r\} & \text{if } s \notin P_q \end{cases}$$

Proposition 3.2. *The set S endowed with the hypercomposition (1) is a join hypergroup.*

Proof. First notice that $s+S = \bigcup_{q\in S}(P_s \cup P_q) = S$. Hence, the reproductive axiom is valid. Next for the associative low it holds: $s+(q+r) = s + (P_q \cup P_r) = P_s \cup (\bigcup_{u\in P_q\cup P_r} P_u)$, which, because of lemma 1.1, is equal to $P_s \cup (P_q \cup P_r)$. But $P_s \cup (P_q \cup P_r) = (P_s \cup P_q) \cup P_r$

Using again lemma 1.1, we get the equality $(P_s \cup P_q) \cup P_r = (\bigcup_{v\in P_s\cup P_q} P_v) \cup P_r$. Thus $(\bigcup_{v\in P_s\cup P_q} P_v) \cup P_r = (P_s \cup P_q) + r =$ $= (s+q)+r$ and so the associative low is valid. Finally suppose that $s/q \cap p/r \neq \varnothing$. Then $(s+r) \cap (q+p) =$ $= (P_s \cup P_r) \cap (P_q \cup P_p)$ which is non empty, since it contains s_o. Hence the transposition axiom is valid and so the Proposition.

Another hypercomposition is the following:

$$s + q = P_s \cap P_q \quad \text{for all } s, q \in S \qquad (2)$$

Since $s_o \in P_r$ for all $r \in S$, the results of hypercomposition (2) are always non void sets.

The above hypercomposition is commutative, thus the two induced hypercompositions coincide and so we have:

$$s/q = q\backslash s = \begin{cases} S, & \text{if } s \in P_q \\ \varnothing, & \text{if } s \notin P_q \end{cases}$$

Proposition 3.3. *The set S endowed with the hypercomposition (2) is a join semihypergroup.*

Proof. Since s/q, with s, q in S, is not always nonvoid, it derives that the reproductive axiom is not valid. Next let $v \in P_s \cap P_q$. Then $P_v \subseteq P_s \cap P_q$, thus $\bigcup_{v \in P_s \cap P_q} P_v = P_s \cap P_q$. Hence it holds:

$(s+q)+r = (P_s \cap P_q) + r = \bigcup_{v \in P_s \cap P_q} (P_v \cap P_r) = (\bigcup_{v \in P_s \cap P_q} P_v) \cap P_r = (P_s \cap P_q) \cap P_r = P_s \cap (P_q \cap P_r)$. Similarly $s+(q+r) =$
$= P_s \cap (P_q \cap P_r)$ and so the associative low is valid. Finally if $s/q \cap p/r \neq \varnothing$, then $s/q \cap p/r = S$ and $(s+r) \cap (q+p) =$
$= (P_s \cap P_r) \cap (P_q \cap P_p)$ is non empty, since it contains s_o.

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